Summer term 2022
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Exercises to Probability Theory II

Sheet 13
Submission before: Friday, 08.07.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 32. (Another characterisation of Brownian motion)
Let $(X_t)_{t\in[0,1]}$ be a family of random variables on a probability space $(\Omega, \mathcal{A}, P)$ satisfying the following properties:

(1) For $0 \leqslant t_0 < t_1 < \ldots < t_n$ are the increments $X_{t_i} - X_{t_{i-1}}$, $1 \leqslant i \leqslant n$, independent.

(2) For all $s \geqslant 0$, the distribution of $X_{t+s} - X_t$ does not depend on t ("stationary increments").

(3) $\mathbb{E}[X_t] = 0$, $\mathbb{E}[X_t^2] = t$ and $\mathbb{E}[|X_t|^3] = \frac{2^{3/2}t^{3/2}}{\sqrt{\pi}}$ for all $t \geqslant 0$.

Show that $(X_t)_{t\in[0,1]}$ is a Brownian motion if and only if for all $c > 0$ and $t \geqslant 0$, the distribution of $c^{-1}X_{c^2t}$ is equal to the distribution of X_t. (4 points)
Hint: Use the central limit theorem.

Problem 33. (Filtrations associated to Brownian motion)
Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(X_t)_{t\geqslant 0}$ be a Brownian motion on $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{R}$, starting in 0. We define its filtration by

$$\mathcal{F}_t^0 := \sigma\left(X_s \mid s \leqslant t\right), \quad t \geqslant 0.$$

Further, we make this filtration **right-continuous** (cf. part (c)* below) by setting

$$\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}_s^0, \quad t \geqslant 0.$$

(a) Let $C \subset \mathbb{R}$ be a **closed** set. Then

$$T_C(\omega) := \inf\{t \geqslant 0 \mid X_t(\omega) \in C\}$$

is an $(\mathcal{F}_t^0)_{t\geqslant 0}$-stopping time. (1 point)

This case is one of very few for which the hitting time is an $(\mathcal{F}_t^0)_{t\geqslant 0}$-stopping time. In general, we can only expect a hitting time T_A of a set A to be an $(\mathcal{F}_t)_{t\geqslant 0}$-stopping time.

In the following, let $\emptyset \neq A \subset \mathbb{R}$ be an **open** set. Consider the hitting time

$$T_A(\omega) := \inf\{t \geqslant 0 \mid X_t(\omega) \in A\}.$$

(b)* Draw pictures to illustrate that T_A is only an $(\mathcal{F}_t)_{t\geqslant 0}$-stopping time, and not an $(\mathcal{F}_t^0)_{t\geqslant 0}$-stopping time. (2 points)

The rest of the exercise is devoted to proving this rigorously.

(c)* Show that the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ is **right-continuous**, i.e. (1 point)

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s.$$

(d) Let $(\mathcal{F}_t)_{t\geqslant 0}$ be an arbitrary right-continuous filtration. Show that a random variable $T\colon \Omega \to \bar{\mathbb{R}}_+$ is an $(\mathcal{F}_t)_{t\geqslant 0}$-stopping time if and only if $\{T < t\} \in \mathcal{F}_t$. (1 point)

(e) Show that T_A is an $(\mathcal{F}_t)_{t\geqslant 0}$-stopping time. (1 point)

We now want to prove that T_A is *not* an $(\mathcal{F}_t^0)_{t\geqslant 0}$-stopping time using the following criterion:

(f) (Galmarino's test) Let $\Omega := C(\mathbb{R}_+, \mathbb{R})$ and X_t be the coordinate process, i.e. $X_t(\omega) = \omega(t)$. Let $T\colon \Omega \to \bar{\mathbb{R}}_+$ be a random variable. Then the following are equivalent: (4 points)

  (i) T is an $(\mathcal{F}_t^0)_{t\geqslant 0}$-stopping time.

  (ii) For all $t \geqslant 0, \omega \in \Omega$ and $\omega' \in \Omega$, we have

$$T(\omega) \leqslant t \text{ and } X_s(\omega) = X_s(\omega') \ \forall s \leqslant t \quad \Rightarrow \quad T(\omega) = T(\omega').$$

Hints: Show first that (ii) is equivalent to:

(ii)' For all $t \geqslant 0, \omega \in \Omega$ and $\omega' \in \Omega$, we have

$$T(\omega) \leqslant t \text{ and } X_s(\omega) = X_s(\omega') \ \forall s \leqslant t \quad \Rightarrow \quad T(\omega') \leqslant t.$$

To prove (ii)' $\Rightarrow$ (i), define the function $a_t\colon \Omega \to \Omega,\ \omega \mapsto \omega(\cdot \wedge t)$ and, show that $a_t\colon (\Omega, \mathcal{F}_t^0) \to (\Omega, \mathcal{F})$ is measurable, that $\mathcal{F}_t^0 = a_t^{-1}(\mathcal{F})$, and that (ii)' implies that $\{T \leqslant t\} = a_t^{-1}(\{T \leqslant t\})$. Here, $\mathcal{F} = \sigma\left(X_s \mid s \geqslant 0\right)$.

(g) Show using (f) that T_A is **not** an $(\mathcal{F}_t^0)_{t\geqslant 0}$-stopping time. (1 point)