Summer term 2022
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Exercises to Probability Theory II

Sheet 7
Submission before: Friday, 27.05.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 18. (A submartingale)
Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(\mathcal{A}_t)_{t\in I}$ a family of increasing sub-σ algebras (i.e. $s \leqslant t \Rightarrow \mathcal{A}_s \subset \mathcal{A}_t$). Prove that if $(M_t)_{t\in I}$ is an $(\mathcal{A}_t)_{t\in I}$-martingale, then $X_t := |M_t|$, $t \in I$, is an $(\mathcal{A}_t)$-submartingale. (4 points)

Problem 19. (Symmetric random walk on $\mathbb{Z}$)
Let (X_n) be the symmetric random walk on $\mathbb{Z}$ in the sense of Def. 7.2.2 (i) with parameter $p = 1/2$, i.e.[1]

$$X_n := \sum_{k=1}^{n} Y_k, \quad Y_k \text{ i.i.d. with } P(Y_k = 1) = P(Y_k = -1) = \frac{1}{2}.$$

Find an α such that $M_n := \exp(\alpha X_n - \lambda n)$ is an $(\mathcal{A}_n)_{n\in\mathbb{N}_0})$-martingale for $n \in \mathbb{N}_0$, $\lambda \geqslant 0$, where $\mathcal{A}_n := \sigma(X_0, \dots, X_n)$. (4 points)

Problem 20. (A martingale for 2-dimensional random walk)
Let $(Z_n)_{n\in\mathbb{N}_0} := (X_n, Y_n)_{n\in\mathbb{N}_0}$ be the 2-dimensional symmetric random walk, i.e. the Markov chain with state space $S = \mathbb{Z}^2$ and transition kernel

$$p((x, y), \cdot) := \frac{1}{4} \left(\delta_{(x+1,y)} + \delta_{(x-1,y)} + \delta_{(x,y+1)} + \delta_{(x,y-1)} \right)$$

with $(x, y) \in \mathbb{Z}^2$. Show that $M_n := |Z_n|^2 - n$, $n \in \mathbb{N}_0$ is a martingale, where $|\cdot|$ is the Euclidean norm. (4 points)

[1] The independence of the Y_k was shown in Problem 16 (i).