DEGREE FORMULA

(poorly ordered collection of Markus' e-mail messages[1])

1. The ideal I

For an algebraic variety X over F denote by $I(X)$ the ideal in $\mathbb{Z}$ generated by $\deg(x)$ for all $x \in X_{(0)}$.

Example 1.1. 1. Let X be anisotropic smooth projective quadric. Then $I(X) = 2\mathbb{Z}$.

2. Let X be the Severi-Brauer variety of a central simple F-algebra A. Then $I(X) = \operatorname{ind}(A) \cdot \mathbb{Z}$.

For a field extension L/F, denote by $I(L)$ the ideal in $\mathbb{Z}$ generated by the degrees $[F(v) : F]$ for all valuation v on L over F such that the residue field $F(v)$ is a finite extension of F.

Proposition 1.2. (1) *If for a variety X over F, $X(F) \neq \emptyset$, then $I(X) = \mathbb{Z}$;*
(2) *If there is a morphism $f : Y \to X$, then $I(Y) \subset I(X)$;*
(3) *Let X be an irreducible variety over F and $L = F(X)$.*
(a) If X is proper then $I(L) \subset I(X)$.
(b) If X is smooth then $I(X) \subset I(L)$.
(c) If X is proper and smooth, then $I(X) = I(L)$. In particular, $I(X)$ is a birational invariant of a proper smooth variety X.
(4) *Let morphism $f : Y \to X$ be a birational isomorphism of irreducible proper varieties with X smooth, then $I(Y) = I(X)$.*
(5) *If $f : Y \to X$ is a rational morphism of proper varieties with Y smooth, then $I(Y) \subset I(X)$.*

Proof. (3a) Let v be a valuation of L over F. Since X is proper, the valuation ring of v dominates a point $x \in X$. In particular, $F(x)$ is a subfield of $F(v)$ over F. Hence x is a closed point and $\deg x$ divides $[F(v) : F]$.

(3b) Let $x \in X$ be a closed point. Since x is a smooth point, there is a valuation v of L over F with the residue field $F(x)$.

(4) Let $L = F(X) = F(Y)$. By (2), $I(Y) \subset I(X)$ and by (3a,c), $I(X) = I(L) \subset I(Y)$.

(5) Let Y_1 be the graph of f in the product $Y \times X$. We have two projections $f_1 : Y_1 \to Y$ and $g : Y_1 \to X$ with f_1 a birational isomorphism. By (4), $I(Y_1) = I(Y)$ and by (2), $I(Y_1) \subset I(X)$.

□

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[1]and of Sasha's considerations.

2. Definition of the invariant $\eta_p(X)$

Let p be a prime number, F a field. We assume that $\mathrm{char}(F) \neq p$ and $\mu_p \subset F$. We fix a primitive p-th root of unity ξ.

Let X be quasi-projective variety over F. The group $G = \mathbb{Z}/p\mathbb{Z}$ acts by cyclic permutations on the product

$$X^p = X \times X \times \cdots \times X.$$

The factor variety X^p/G we denote by C^pX. The image $\overline{X}$ of the diagonal $X \subset X^p$ under the natural morphism $X^p \to C^pX$ is a closed subvariety in C^pX, isomorphic to X. In particular, $I(\overline{X}) = I(X)$.

Consider a G-action on the trivial linear bundle $X^p \times \mathbb{A}^1$ over X^p by

$$(x_1, x_2, \dots, x_p, t) \mapsto (x_2, \dots, x_p, x_1, \xi t).$$

The projection $X^p \setminus X \to C^pX \setminus \overline{X}$ is unramified, hence the restriction of the factor vector bundle $(X^p \times \mathbb{A}^1)/G$ to $C^pX \setminus \overline{X}$ is a linear bundle over $C^pX \setminus \overline{X}$. Denote it by L_X.

Let $d = \dim X$. The image of the zero section of the vector bundle $L_X^{\oplus pd}$ of rank pd over $C^pX \setminus \overline{X}$ defines an element

$$l_X \in \mathrm{CH}_{pd}(L^{\oplus pd}).$$

By homotopy invariance,

$$\mathrm{CH}_{pd}(L_X^{\oplus pd}) = \mathrm{CH}_0(C^pX \setminus \overline{X}),$$

so that we will also assume that $l_X \in \mathrm{CH}_0(C^pX \setminus \overline{X})$. If X is projective, we have the degree homomorphism

$$\deg : \mathrm{CH}_0(C^pX \setminus \overline{X}) \longrightarrow \mathbb{Z}/I(X).$$

Thus, the image of l_X defines an element

$$\eta_p(X) \in \mathbb{Z}/I(X).$$

Note that X is projective but not necessarily smooth variety over F.

Unfortunately, the invariant $\eta_p(X)$ is always of exponent p, even when $\mathbb{Z}/I(X)$ is large.

Lemma 2.1. $p \cdot l_X = 0 \in \mathrm{CH}_0(C^pX \setminus \overline{X})$. *In particular,*

$$p \cdot \eta(X) = 0 \in \mathbb{Z}/I(X).$$

Proof. The class l_X in $\mathrm{CH}_{pd}(L^{\oplus pd})$ is the direct image of the image of the zero section of the trivial vector bundle under

$$(X^p \setminus X) \times \mathbb{A}^{pd} \to L^{\oplus pd}$$

of degree p. $\square$

3. Degree formula

Theorem 3.1. (Regular Degree Formula) *Let $f : Y \to X$ be a morphism of projective varieties of dimension d. Then $I(Y) \subset I(X)$ and*

$$\eta_p(Y) = \deg(f) \cdot \eta_p(X) \in \mathbb{Z}/I(X).$$

Proof. Denote by $\widetilde{Y}$ the inverse image of $\overline{X}$ under $C^p f : C^p Y \to C^p X$, so that $\overline{Y} \subset \widetilde{Y}$. In particular,

$$I(Y) = I(\overline{Y}) \subset I(\widetilde{Y}) \subset I(\overline{X}) = I(X).$$

We have an open embedding

$$j : C^p(Y) \setminus \widetilde{Y} \hookrightarrow C^p(Y) \setminus \overline{Y}$$

and a proper morphism

$$i : C^p(Y) \setminus \widetilde{Y} \longrightarrow C^p(X) \setminus \overline{X}.$$

Denote by W the restriction of the vector bundle $L_Y^{\oplus pd}$ on $C^p Y \setminus \widetilde{Y}$. Clearly, W is the inverse image of $L_X^{\oplus pd}$ with respect to i. We have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}_{pd}(L_Y^{\oplus pd}) & \xrightarrow{j^*} & \mathrm{CH}_{pd}(W) & \xrightarrow{i_*} & \mathrm{CH}_{pd}(L_X^{\oplus pd}) \\
\| & & \| & & \| \\
\mathrm{CH}_0(C^p Y \setminus \overline{Y}) & \xrightarrow{j^*} & \mathrm{CH}_0(C^p Y \setminus \widetilde{Y}) & \xrightarrow{i_*} & \mathrm{CH}_0(C^p X \setminus \overline{X}) \\
\Big\downarrow{\scriptstyle \deg} & & \Big\downarrow{\scriptstyle \deg} & & \Big\downarrow{\scriptstyle \deg} \\
\mathbb{Z}/I(Y) & \longrightarrow & \mathbb{Z}/I(\widetilde{Y}) & \longrightarrow & \mathbb{Z}/I(X).
\end{array}$$

The image $j^*(l_Y) \in \mathrm{CH}_{pd}(W)$ is the class l' of the image of the zero section of W. Then i_* maps l' to the image of the zero section of $L_X^{\oplus pd}$ and the degree of $i|_Z$ is equal to $\deg(C^p f) = \deg(f)^p$, hence

$$i_* j^*(l_Y) = i_*(l') = \deg(f)^p \cdot l_X.$$

Finally, we can replace $\deg(f)^p$ by $\deg(f)$ in view of Lemma 2.1. □

Remark 3.2. In fact, we have proved a stronger version of the degree formula on the level of Chow groups:

$$i_* j^*(l_Y) = \deg(f) \cdot l_X \in \mathrm{CH}_0(C^p X \setminus \overline{X}).$$

Theorem 3.3. (Rational Degree Formula) *Let $f : Y \to X$ be a rational morphism of projective varieties of dimension d with Y smooth. Then $I(Y) \subset I(X)$ and*

$$\eta_p(Y) = \deg(f) \cdot \eta_p(X) \in \mathbb{Z}/I(X).$$

Proof. The inclusion $I(Y) \subset I(X)$ holds by Proposition 1.2(5). Let Y_1 be the graph of f in $Y \times X$. Applying Theorem 3.1 to the birational isomorphism $Y_1 \to Y$ (of degree 1), we get

$$\eta_p(Y_1) = \eta_p(Y) \in \mathbb{Z}/I(Y) = \mathbb{Z}/I(Y_1).$$

On the other hand, applying Theorem 3.1 to the projection $Y_1 \to X$(of degree $= \deg(f)$), we get

$$\eta_p(Y_1) = \deg(f) \cdot \eta_p(X) \in \mathbb{Z}/I(X).$$

□

Corollary 3.4. *The class $\eta_p(X) \in \mathbb{Z}/I(X)$ is a birational invariant of a smooth projective variety X.*

Problem 3.5. How to define the invariant $\eta_p(X) \in \mathbb{Z}/I(X)$ out of the function field $E = F(X)$? (Note that $I(X) = I(E)$.)

Here is the main application:

Theorem 3.6. *Let X and Y be two irreducible projective varieties with Y smooth. Assume that X has a rational point over $F(Y)$. Then $I(Y) \subset I(X)$ and if $\eta_p(Y) \neq 0 \in \mathbb{Z}/I(X)$, the following holds:*
(1) $\dim(X) \geq \dim(Y)$.
(2) *If* $\dim(X) = \dim(Y)$, *then Y has a closed point over $F(X)$ of degree prime to p.*

Proof. There exists a rational morphism $f : Y \to X$. By Proposition 1.2(5), $I(Y) \subset I(X)$.

(1) Assume that $n = \dim(Y) - \dim(X) > 0$. Consider the composition

$$g : Y \xrightarrow{f} X \hookrightarrow X \times \mathbb{P}^n_F.$$

Clearly, $\deg(g) = 0$ and $I(X \times \mathbb{P}^n_F) = I(X)$. By Theorem 3.3, applied to g, $\eta_p(Y) = 0 \in \mathbb{Z}/I(X)$, a contradiction.

(2) By the degree formula, applied to f, and Lemma 2.1, the degree $\deg(f)$ is not divisible by p. Hence the generic point of Y determines a point over $F(X)$ of degree $= \deg(f)$. □

Remark 3.7. The first statement of Theorem 3.6(1) shows that a variety Y cannot be "compressed" to a variety X of smaller dimension if $\eta_p(Y) \neq 0 \in \mathbb{Z}/I(X)$.

One can give another definition of $\eta_p(X)$ by using Chern class operations as defined in [2, Ch. 3] for arbitrary (not necessarily smooth) varieties. For any vector bundle E over a variety V we have Chern class operations

$$c_p(L)\cap ? : \mathrm{CH}_k(V) \longrightarrow \mathrm{CH}_{k-p}(V), \quad u \mapsto c_p(E)\cap u.$$

In particular, the operation $c_1(L)\cap ?$ for a linear vector bundle L over V is the intersection with the Cartier divisor associated to L.

By [2, Ex. 3.3.2], the class of l_X in $\mathrm{CH}_0(C^pX \setminus \overline{X})$ is equal to

$$c_{pd}(L_X^{\oplus pd}) \cap [C^pX \setminus \overline{X}].$$

By Whitney formula [2, Th. 3.2 (e)],

$$c_{pd}(L_X^{\oplus pd}) = c_1(L_X)^{pd}.$$

Thus, we can give an equivalent definition:

$$\eta_p(X) = \deg\big(c_1(L_X)^{pd} \cap [C^pX \setminus \overline{X}]\big) \in \mathbb{Z}/I(X).$$

One can give another proof of Theorem 3.1 using standard properties of Chern class operations (projection formula, inverse image under a flat morphism) given in [2, Th. 3.2].

4. Computation of η_2 for a quadric

Let $Q = Q(V,q)$ be anisotropic smooth projective quadric of dimension d, so that $I(Q) = 2\mathbb{Z}$.

Proposition 4.1.

$$\eta_2(Q) = \begin{cases} 1 + 2\mathbb{Z}, & \text{if } d = 2^k - 1 \text{ for some } k, \\ 2\mathbb{Z}, & \text{otherwise.} \end{cases}$$

Proof. Each pair of distinct points in Q determines a line in $\mathbb{P}(V)$, i.e. a plane in V. Thus we have a rational morphism

$$\alpha : C^2Q \longrightarrow \mathrm{Gr}(2,V).$$

Clearly, α is a birational isomorphism! Indeed, let U be the open subvariety in C^2Q consisting of all pairs of points $([v],[u])$ such that $[v] \neq [u]$ and the restriction of the quadratic form q on the 2-dimensional subspace generated by v and u is nondegenerate. Then the restriction $\alpha|_U$ is an open immersion identifying U with the open subvariety of $\mathrm{Gr}(2,V)$ consisting of all planes $W \subset V$ such that the restriction $q|_W$ is nondegenerate. The inverse rational morphism α^{-1} takes a plane $W \subset V$ to the intersection $\mathbb{P}(W) \cap Q$. If $q|_W$ is nondegenerate, this intersection is an effective 0-cycle of degree 2, i.e. is a point of C^2X.

Remark 4.2. If $\dim(Q) = 1$, i.e. if Q is a conic, α is an isomorphism between C^2Q and $\mathrm{Gr}(2,V) = \mathbb{P}(V^*)$. In the split case this isomorphism looks as follows: $C^2\mathbb{P}^1_F \simeq \mathbb{P}^2_F$.

Let E be the canonical linear bundle over $\mathrm{Gr}(2,V)$ (the second exterior power of rank 2 tautological vector bundle). Denote by L' the restriction of the linear bundle L_Q to the open subvariety $U \subset C^2Q$.

Lemma 4.3. $(\alpha|_U)^*(E) \simeq L'$.

Proof. Let U' be the inverse image of U under the natural morphism $Q^2 \to C^2Q$. We have the following morphism of vector bundles:

$$\beta : U' \times \mathbb{A}^1_F \to E, \quad ([v],[u],t) \mapsto (\langle v,u\rangle, t\,\frac{v \wedge u}{b(v,u)})$$

where b is the polar form of q. The action of $G = \mathbb{Z}/2\mathbb{Z}$ on $U' \times \mathbb{A}^1_F$ by $([v],[u],t) \mapsto ([u],[v],-t)$ commutes with the trivial action of G on E. Hence β induces an isomorphism $L' \to E$ over $\alpha|_U$. □

Clearly, $2\mathbb{Z} = I(Q) = I(C^2Q \setminus U)$. Lemma 4.3 and the commutativity of the diagram

$$\begin{tikzcd} \mathrm{CH}_0\big(\mathrm{Gr}(2,V)\big) \arrow[r, "\alpha|_U^*"] \arrow[d, "\deg"'] & \mathrm{CH}_0(U) \arrow[d, "\deg"'] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

imply that

$$\eta_2(Q) = \deg(c_1E)^{2d} + 2\mathbb{Z}.$$

Let

$$i : \mathrm{Gr}(2,V) \hookrightarrow \mathbb{P}(\wedge^2 V)$$

be the canonical closed embedding. The vector bundle E on $\mathrm{Gr}(2,V)$ is the inverse image under i of the tautological vector bundle on $\mathbb{P}(\wedge^2 V)$. Thus, the degree $\deg(c_1E)^{2d}$ is equal to the degree of the subvariety $i(\mathrm{Gr}(2,V))$ in the projective space $\mathbb{P}(\wedge^2 V)$. It is the Catalan number [2, Ex. 14.7.11]

$$\frac{1}{d+1}\binom{2d}{d}.$$

One easily checks that this number is odd iff $d = 2^k - 1$ for some k.

□

Theorem 4.4. *Let $Q = Q(V,q)$ be a smooth projective quadric of dimension $d \geq 2^k - 1$ and let X be a variety over F such that $I(X) \subset 2\mathbb{Z}$ and X has a point over $F(Q)$. Then*
(1) $\dim(X) \geq 2^k - 1$.
(2) *If* $\dim(X) = 2^k - 1$, *then Q has a point over $F(X)$.*

Proof. Let Q' be a subquadric in Q of dimension $2^k - 1$. Clearly, there is a rational morphism $f : Q' \to X$.

(1) Since $\eta_2(Q') = 1 + I(Q') \neq 0 \in \mathbb{Z}/I(X)$, by Theorem 3.6(1),

$$\dim(X) \geq \dim(Q') = 2^k - 1.$$

(2) By Theorem 3.6(2), Q' has a point of odd degree over $F(X)$. By Springer's Theorem, Q' and hence Q has a point over $F(X)$. □

Remark 4.5. One can replace the condition that X has a point over $F(Q)$ by the weaker one: X has an odd degree 0-cycle over $F(Q)$.

Corollary 4.6. *(Hoffmann) Let Q_1 and Q_2 be two anisotropic quadrics. If $\dim(Q_1) \geq 2^k - 1$ and Q_2 is isotropic over $F(Q_1)$, then $\dim(Q_2) \geq 2^k - 1$.*

Corollary 4.7. *(Izhboldin) Let Q_1 and Q_2 be two anisotropic quadrics. If $\dim(Q_1) = 2^k - 1$ and Q_2 is isotropic over $F(Q_1)$, then Q_1 is isotropic over $F(Q_2)$.*

5. Computing η_2 for smooth varieties

Let X be a smooth proper variety. Denote by c' the total characteristic class opposite to the Chern class c, i.e.

$$1 + c'_1 + c'_2 + \cdots = (1 + c_1 + c_2 + \dots)^{-1}.$$

In other words, $c'_d(E) = c_d(-E)$ for any vector bundle over X. The class c'_d is known as the *Segre class* (denoted s_d in [2, Ch. 3]; by the letter s_d we denote the additive class of degree d).

Denote by T_X the tangent bundle over X.

Theorem 5.1. *Let X be a proper smooth variety of dimension d. Then the degree of the 0-cycle $c'_d(T_X) = c_d(-T_X)$ is even and*

$$\eta_2(X) = \frac{\deg c'_d(T_X)}{2} \in \mathbb{Z}/I(X).$$

Proof. We would like first to compactify smoothly $C^2X \setminus \overline{X}$ and extend the line bundle L_X to the compactification. Note that C^2X is smooth only if $\dim(X) = 1$.

Let W be the blow up of the diagonal X in X^2. The G-action on X^2 extends to one on W. The subvariety W^G coincides with the exceptional divisor

$$\mathbb{P}\big((T_X \oplus T_X)/T_X\big) = \mathbb{P}(T_X).$$

Since W^G is of codimension 1 in W, W/G is smooth and therefore can be taken as a smooth compactification of $C^2X \setminus \overline{X}$.

Now we would like to construct a canonical extension L' of L_X to the whole W/G. There is a canonical linear bundle $O_W(1)$ over the blow up W with the induced G-action. The group G acts by -1 on the restriction $O_{\mathbb{P}(T_X)}$ of $O_W(1)$ on the exceptional divisor $\mathbb{P}(T_X)$. The restriction of $O_W(1)$ to the complement of the exceptional divisor $W \setminus \mathbb{P}(T_X) \simeq X^2 \setminus X$ is the trivial vector bundle with the trivial G-action. Now we modify the G-action on $O_W(1)$ by -1. New G-action of the restriction of $O_W(1)$ on the exceptional divisor $\mathbb{P}(T_X)$ is trivial, hence $O(1)$ descends to a linear vector bundle L' on W/G which is a desired extension of L_X on $C^2X \setminus \overline{X}$.

Now we compute $\deg c_1(L')^{2d}$. At one hand, since L_X is the restriction of L' on $C^2X \setminus \overline{X}$, we have

$$\deg c_1(L')^{2d} = \deg c_1(L_X)^{2d} = \eta_2(X) \in \mathbb{Z}/I(X).$$

On the other hand, let

$$i : \mathbb{P}(T_X) \hookrightarrow W$$

be the closed embedding and

$$p : W \longrightarrow W/G$$

the natural projection. We have

$$p^* c_1(L') = c_1\big(O_W(1)\big) = [\mathbb{P}(T_X)] = i_*(1) \in \mathrm{CH}^1(W).$$

By the projection formula,

$$c_1\big(O_W(1)\big)^{2d} = i_*(1) \cdot c_1\big(O_W(1)\big)^{2d-1} = i_*\Big(c_1\big(O_{\mathbb{P}(T_X)}(1)\big)^{2d-1}\Big) \in \mathrm{CH}_0(W).$$

Let $q : \mathbb{P}(T_X) \to X$ be the natural morphism. The class

$$q_*\Big(c_1\big(O_{\mathbb{P}(T_X)}(1)\big)^{2d-1}\Big) \in \mathrm{CH}_0(X)$$

is the Segre class $c'_d(T_X)$ [2, Ch. 3]. Finally,

$$2 \deg c_1(L')^{2d} = \deg c_1\big(O_W(1)\big)^{2d} = \deg c_1\big(O_{\mathbb{P}(T_X)}(1)\big)^{2d-1} = \deg c'_d(T_X).$$

□

Remark 5.2. If $p > 2$, the blow up W of the diagonal in X^p does not have smooth orbit space W/G and the linear bundle L_X cannot be extended to a linear bundle on W/G.

Remark 5.3. The number $\deg c_d(-T_X)$ does not change under field extensions and hence can be computed over algebraically closed fields.

Remark 5.4. The Theorem shows that the degree m of the cycle $c_d(-T_X)$ is always even. Hence this cycle defines a cycle on the symmetric square C^2X of degree $\frac{m}{2}$. In the proof we construct this class "canonically" (it is $c_1(L_X)^{2d}$). The degree formula follows from "canonical" nature of this class. One should study other cases of divisibility of characteristic numbers.

Example 5.5. Let Q be again a smooth projective quadric $Q(V, q)$ and let $i : Q \hookrightarrow \mathbb{P}(V)$ be the embedding. We have the following exact sequence of vector bundles

$$0 \longrightarrow T_Q \longrightarrow i^* T_{\mathbb{P}(V)} \longrightarrow i^* O_{\mathbb{P}(V)}(2) \longrightarrow 0.$$

A nasty calculation gives the Catalan number again:

$$\frac{1}{2}\, c_d(-T_Q) = \frac{(-1)^d}{d+1}\binom{2d}{d}.$$

Can one get this computation using $\mathrm{Gr}(2, V)$?

6. Computing η_2 for projective spaces

Proposition 6.1. *Let $X = \mathbb{P}^d_F$. Then*

$$\deg c_d(-T_X) = (-1)^d \binom{2d}{d}.$$

Proof. It is known that $[-T_X] = -(d+1)[O_X(1)] + [O_X]$. We can compute the total Chern class:

$$c(-T_X) = \frac{1}{c\big(O_X(1)\big)^{d+1}} = \frac{1}{(1+t)^{d+1}}.$$

Hence

$$\deg c_d(-T_X) = \binom{-d-1}{d} = (-1)^d \binom{2d}{d}.$$

□

Let v_p be the p-adic discrete valuation and $s_p(a)$ be the sum of digits of a written in base p.

Corollary 6.2. *Let X be a Severi-Brauer variety of dimension d. Then*

$$v_2\big(\deg c_d(-T_X)\big) = s_2(d), \quad \eta_2(X) = 2^{s_2(d)-1}.$$

Corollary 6.3. *(Karpenko) Let A be a division algebra with orthogonal involution σ, Y the Severi-Brauer variety of A. Then the involution σ is anisotropic over $F(Y)$.*

Proof. Let $X = I(A,\sigma) \subset Y$ be the involution variety, $\deg(A) = 2^k$. Then $I(X) \subset I(Y) = 2^k\mathbb{Z}$. If σ is isotropic over $F(Y)$ then there is a rational morphism $Y \to X$. By Corollary 6.2,

$$v_2\big(\deg c_d(-T_Y)\big) = s_2(2^k - 1) = k,$$

hence $\eta_2(Y) = 2^{k-1}$ is nontrivial in $\mathbb{Z}/I(X)$. A contradiction by Theorem 3.6(1). □

Remark 6.4. In fact, we proved that the Severi-Brauer variety of a division algebra of degree 2^k cannot be compressed to a variety X of smaller dimension with $I(X) = 2^k\mathbb{Z}$.

Remark 6.5. Let p be any prime integer. We will see later (Corollary 10.6) that for a Severi-Brauer variety X of dimension d divisible by $p-1$,

$$\eta_p(X) = p^{\frac{s_p(d)}{p-1}-1} \in \mathbb{Z}/I(X).$$

As in Corollary 6.3 one can prove that the Severi-Brauer variety of a division algebra of degree p^k cannot be compressed to a variety X of smaller dimension with $I(X) = p^k\mathbb{Z}$.

7. Todd class

The total Todd class td is the rational characteristic class

$$td = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1c_2) + \dots.$$

Lemma 7.1. [3, Lemma 1.7.3] *The denominator of the Todd class td_d is equal to*

$$\prod_{p\ prime} p^{[\frac{d}{p-1}]}.$$

It follows from the RR Theorem that for a smooth proper variety X of dimension d,

$$td_d(X) \overset{def}{=} \deg td_d(T_X) = \chi(X) = \sum_{i=1}^{d}(-1)^i \dim H^i(X, \mathcal{O}_X) \in \mathbb{Z}$$

is the Euler characteristic of X.

The structure morphism $p^X : X \to \operatorname{Spec} F$ induces the direct image

$$p_*^X : K_0(X) \longrightarrow K_0(F) = \mathbb{Z}.$$

We have $p_*([\mathcal{O}_X]) = \chi(X)$.

Denote by $J(X)$ the image of $K_0(X)^{(1)}$ under p_*. Clearly, $I(X) \subset J(X)$ and this is an equality if $\dim X = 1$.

Theorem 7.2. *For a morphism $f : Y \to X$ of proper smooth varieties of dimension d,*

$$td_d(Y) = \deg(f) \cdot td_d(X) \in \mathbb{Z}/J(X).$$

Proof. Clearly,

$$f_*([\mathcal{O}_Y]) - \deg(f) \cdot [\mathcal{O}_X] \in K_0(X)^{(1)}.$$

We apply then p_*^X and use $p^Y = p^X \circ f$. □

Example 7.3. Let Y be the Severi-Brauer variety corresponding to a central simple algebra of prime degree p. One has $J(Y) = p\mathbb{Z}$ and $td_{p-1}(Y) = 1$. Hence Y cannot be compressed to a variety X of smaller dimension with $J(Y) = p\mathbb{Z}$.

Example 7.4. Let X be the product of n Severi-Brauer varieties corresponding to central simple algebras $A_1, A_2, \dots, A_n$ of prime degree p. Assume that these algebras are linearly independent in the Brauer group of F. Then $J(X) = p\mathbb{Z}$. Let L/F be a field extension such that

1. $\operatorname{tr.deg.}(L/F) \leq n(p-1)$;
2. L splits all the A_i.
3. $F_{\mathrm{sep}}L/F_{\mathrm{sep}}$ is purely transcendental.

Claim: The degree $\operatorname{tr.deg.}(L/F)$ is exactly $n(p-1)$, the field L can be embedded into $F(X)$ over F and for any such an embedding the degree $[F(X) : L]$ is prime to p.

Proof: Let Y be a projective variety such that $L = F(Y)$. We may assume that $\dim(Y) = n(p-1)$ by replacing Y by $Y \times \mathbb{P}^k$ for appropriate k. Since

$F(Y)$ splits all the A_i, there is a rational morphism $f : Y \to X$. Replacing Y by the graph of this morphism we may assume that f is regular. Since Y is rational over F_{sep}, $td_{n(p-1)}(Y) = 1$. By the degree formula, $\deg(f)$ is not divisible by p.

In particular, X does not contain proper subvarieties rational over F_{sep}.

Example 7.5. Let L/F be a quadratic field extension and Q be a quaternion algebra over L. Consider $X = R_{L/F}(C)$ where C is the conic over L corresponding to Q. Clearly $td_2(X) = 1$ since X is rational over F_{sep}. Assume that the algebra $A = cor_{L/F}(Q)$ of degree 4 over F is a division algebra. We show that $J(X) = 4\mathbb{Z}$. Indeed,

$$K_0(X) = K_0(F) \oplus K_0(A) \oplus K_0(Q).$$

The degree

$$K_0(X) \longrightarrow K_0(F) = \mathbb{Z}$$

is the identity on the first component and trivial on the others (look at the split case). On the other hand, the map to the generic point is given by the norms. Hence, the first component of any element in $K_0(X)^{(1)}$ is divisible by 4.

8. Complex cobordisms

Consider the formal polynomial ring $R = \mathbb{Z}[c_1, c_2, \dots]$ generated by the Chern classes. We think of c_d having degree $2d$, so that R is a graded ring. Denote by R_d the component of R of degree $2d$. It is a free abelian group generated by $c_d, \dots, c_1^d$. We call the elements of R_d the *integral characteristic classes of degree d*. The *rational characteristic classes of degree d* are the elements of $R_d \otimes \mathbb{Q}$.

Let E be a vector bundle over a variety X of dimension d. Then for any $c \in R_d \otimes \mathbb{Q}$ the value $c(E)$ is a rational 0-cycle in $\mathrm{CH}_0(X) \otimes \mathbb{Q}$. If X is proper, we can compute the degree $\deg c(E) \in \mathbb{Q}$.

Denote by R'_d the subgroup in $R_d \otimes \mathbb{Q}$ consisting of all rational characteristic classes c such that $\deg c(-T_X) \in \mathbb{Z}$ for all smooth proper varieties X of dimension d over F. Clearly, $R_d \subset R'_d$.

Example 8.1. We know that $\frac{1}{2}\, c_d \in R'_d$, where c is the Chern class, defines an element of R'_d/R_d of exponent 2. Let td' be the inverse td^{-1} of the Todd class. The class td'_d is another example of an element of R'_d.

Let $t \in R'_d$. For a smooth proper variety X of dimension d we can consider the class $t(X) = \deg t(-T_X) + I(X) \in \mathbb{Z}/I(X)$. If $t \in R_d$ is an integral class, then $t(E) \in \mathrm{CH}_0(X)$ for any vector bundle E over X, hence $\deg t(E) \in I(X)$ and therefore $t(X) = 0 \in \mathbb{Z}/I(X)$. Thus, for any smooth proper variety X of dimension d we have a well defined group homomorphism

$$R'_d/R_d \longrightarrow \mathbb{Z}/I(X), \quad t + R_d \mapsto t(X).$$

Problem 8.2. Determine all classes $t + R_d$ which admit the degree formula, i.e. for a morphism $f : Y \to X$ of smooth proper varieties,

$$t(Y) = \deg(f) \cdot t(X) \in \mathbb{Z}/I(X).$$

Probably in some cases one has to enlarge the group $I(X)$ (see the degree formula for the Todd class).

Example 8.3. The classes $\frac{1}{2}\, c_d$ and td'_d admit the degree formula.

First, we would like to determine the factor group R'_d/R_d. Let $\Omega^U_* = \pi_*(MU)$ be the complex bordism ring. The Hurewicz ring homomorphism

$$h_d : \Omega^U_{2d} \longrightarrow H_{2d}(BU, \mathbb{Z})$$

is injective [5, Cor. 20.26]. Using duality between $H_*(BU, \mathbb{Z})$ and $H^*(BU, \mathbb{Z})$ one can determine h by the formula

$$[X] \mapsto (t \mapsto \langle t(-T_X), \sigma_X \rangle)$$

where $t \in H^{2d}(BU, \mathbb{Z})$, $t(-T_X) \in H^{2d}(X, \mathbb{Z})$ and $\sigma_X \in H_{2d}(X, \mathbb{Z})$ is the fundamental class of X.

The ring $H_*(BU, \mathbb{Z})$ is a graded polynomial ring $\mathbb{Z}[b_1, b_2, \ldots]$, $\deg b_d = 2d$. The ring Ω^U_* is also a polynomial ring $\mathbb{Z}[M_1, M_2, \ldots]$ where the generators M_i can be chosen such that [4, p. 129]

$$h_i(M_i) = \begin{cases} p(b_i + \{\text{decomposable terms}\}), & \text{if } i = p^k - 1 \\ & \text{for some prime } p \text{ and integer } k > 0, \\ b_i + \{\text{decomposable terms}\}, & \text{otherwise.} \end{cases}$$

Consider the following elements in $H_*(BU, \mathbb{Z})$:

$$x_i = \begin{cases} \frac{h_i(M_i)}{p}, & \text{if } i = p^k - 1 \\ & \text{for some prime } p \text{ and integer } k > 0, \\ h_i(M_i), & \text{otherwise.} \end{cases}$$

By [4, p. 130], $H_*(BU, \mathbb{Z})$ is the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$.

Also set

$$r_i = \begin{cases} p, & \text{if } i = p^k - 1 \\ & \text{for some prime } p \text{ and integer } k > 0, \\ 1, & \text{otherwise.} \end{cases}$$

For any partition α: $d = i_1 + i_2 + \cdots + i_k$, $i_1 \le i_2 \le \cdots \le i_k$ set

$$x_\alpha = x_{i_1} x_{i_2} \ldots x_{i_k} \in H_{2d}(BU, \mathbb{Z}),$$

$$r_\alpha = r_{i_1} r_{i_2} \ldots r_{i_k} \in \mathbb{Z}.$$

Proposition 8.4. ([4, Cor., page 129])

$$\operatorname{Coker} h_d = \coprod_{\alpha} (\mathbb{Z}/r_\alpha \mathbb{Z})\ x_\alpha,$$

where the coproduct is taken over all partitions α of d.

Corollary 8.5. *The exponent of the group* $\operatorname{Coker} h_d$ *coincides with the denominator of the Todd class td_d.*

Proof. Let p be a prime integer, $k = [\frac{d}{p-1}]$. Clearly, for any partition α of d, $v_p(x_\alpha) \leq k$ and there is a partition $\alpha : d = (p-1) + (p-1) + \dots$ with $v_p(x_\alpha) = k$. The statement now follows from Lemma 7.1. ☐

Now we dualize h over $\mathbb{Z}$. The group $H^{2d}(BU, \mathbb{Z})$ is canonically isomorphic to R_d since by [5, Cor. 16.11],

$$H^*(BU, \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \dots].$$

Thus, the dual of h_d is the embedding

$$R_d \hookrightarrow \operatorname{Hom}(\Omega^U_{2d}, \mathbb{Z})$$

being an isomorphism after tensoring with $\mathbb{Q}$. This map takes a characteristic class t of degree $2d$ to a homomorphism

$$\Omega^U_{2d} \longrightarrow \mathbb{Z}, \quad [X] \mapsto \langle t(-T_X), \sigma_X \rangle. \tag{1}$$

If X is a smooth proper variety and we consider characteristic classes with values in Chow groups of X the homomorphism (1) takes $[X]$ to $\deg t(-T_X)$. Thus, we can identify the group $\operatorname{Hom}(\Omega^U_{2d}, \mathbb{Z})$ with R'_d.

The pairing

$$H_{2d}(BU, \mathbb{Z}) \otimes R'_d \longrightarrow \mathbb{Q}$$

induces an isomorphism

$$R'_d/R_d \simeq \operatorname{Hom}(\operatorname{Coker} h_d, \mathbb{Q}/\mathbb{Z}) = (\operatorname{Coker} h_d)^*.$$

Thus, the structure of the finite group R'_d/R_d is similar to that in Proposition 8.4.

Example 8.6. Let $d = 1$. We have $R_1 = \mathbb{Z}c_1$, $M_1 = 2b_1$ is the generator of Ω^U_2 with $M_1 = -[\mathbb{P}^1]$ by [5, Ex. 16.46]. Hence $R'_1 = \frac{1}{2}\mathbb{Z}c_1$ and the class $td_1 = \frac{1}{2}c_1 + R_1$ generates R'_1/R_1:

$$R'_1/R_1 = (\mathbb{Z}/2\mathbb{Z})td_1 = (\mathbb{Z}/2\mathbb{Z})(\frac{1}{2}c_1) = (\mathbb{Z}/2\mathbb{Z})(\frac{1}{2}s_1) = (\mathbb{Z}/2\mathbb{Z})(\frac{1}{2}c'_1),$$

where $c_1 = s_1 = -c'_1$. The generator admits the degree formula.

Example 8.7. Let $d = 2$. We have $R_2 = \mathbb{Z}c_2 \oplus \mathbb{Z}c_1^2$. By [5, Ex. 16.46], the generators of Ω^U_4 can be chosen as follows:

$$M_2 = -[\mathbb{P}^2] = 3b_2 - 6b_1^2, \quad M_1^2 = [\mathbb{P}_1 \times \mathbb{P}_1] = 4b_1^2.$$

Hence the group R'_2/R_2 which is dual to $\operatorname{Coker} h_2$, is cyclic of order 12,

$$R'_2/R_2 = (\mathbb{Z}/12\mathbb{Z})td'_2,$$

where
$$td_2 = \frac{1}{12}(c_1^2 + c_2), \quad td_2' = \frac{1}{12}(2c_1^2 - c_2).$$
Also:
$$\frac{1}{2}c_2' + R_2 = 6td_2 + R_2,$$
$$\frac{1}{3}s_2 + R_2 = 4td_2 + R_2$$
both admit degree formulas. What about td_2?

Example 8.8. Let $d = 3$. We have $R_3 = \mathbb{Z}c_3 \oplus \mathbb{Z}c_1c_2 \oplus \mathbb{Z}c_1^3$. Let Q_3 be a smooth 3-dimensional quadric. One has
$$h(Q_3) = 6b_3 - 10b_1^3,$$
$$h(\mathbb{P}^3) = -4b_3 + 20b_1b_2 - 20b_1^3.$$
One can choose the following generators of Ω_6^U:
$$M_3 = [Q_3] + [\mathbb{P}^3], \quad M_1M_2 = [\mathbb{P}^1 \times \mathbb{P}^2], \quad M_1^3 = [\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1].$$
One finds
$$R_3'/R_3 = (\mathbb{Z}/2\mathbb{Z})(\frac{1}{2}c_3') \oplus (\mathbb{Z}/2\mathbb{Z})(\frac{1}{2}c_1^3) \oplus (\mathbb{Z}/24\mathbb{Z})td_3'.$$

9. General problem

Let X be a projective variety of dimension d over F. For any characteristic class $t \in R_d'$ denote by $I_t(X)$ the smallest subgroup in $\mathbb{Z}$ containing $I(X)$ such that for any morphism $f : Y \to X$, where Y is a projective variety of dimension d, one has
$$t(Y) \equiv \deg(f) \cdot t(X) \pmod{I_t(X)}.$$
Clearly, $I_t(X)$ depends only on t modulo R_d. Denote by o_t the l.c.m. of the orders of $t(X)$ in $\mathbb{Z}/I_t(X)$ for all projective X of dimension d. Thus, if $o_t \neq 1$, there is a nontrivial degree formula involving the class t.

Problem 9.1. Determine the function
$$R_d'/R_d \longrightarrow \mathbb{Z}, \quad t + R_d \mapsto o_t.$$

Clearly, o_t divides the order of $t + R_d$ in R_d'/R_d.

Example 9.2. Let $d = 1$. The group R_1'/R_1 is cyclic of order 2 with the generator $t = \frac{c_1}{2}$. For a conic curve X one has $t(X) = 1 + 2\mathbb{Z} \in \mathbb{Z}/2\mathbb{Z}$, hence $o_t = 2$.

Example 9.3. Let $d = 2$. The group R_2'/R_2 is cyclic of order 12 with the generator $t = td_2'$. Example of a Severi-Brauer variety of a csa of degree 3 shows that $3|o_t$ and Example 7.5 shows that $4|o_t$. Thus, $o_t = 12$.

10. Steenrod operations

Let p be a prime number. We consider the action of the Steenrod algebra modulo p on the Thom class T in the cohomology ring

$$H^*(MU, \mathbb{Z}/p\mathbb{Z}).$$

For an operation A, denote by $c(A)$ the image of $A(T)$ under the canonical isomorphism

$$H^*(MU, \mathbb{Z}/p\mathbb{Z}) \simeq H^*(BU, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}[c_1, c_2, \dots].$$

Example 10.1. Let P^i be the Steenrod power. Then $c(P^i) = (b^{-1})_{i(p-1)}$ where b is the total characteristic class defined by

$$b = (1 + t_1^{p-1})...(1 + t_n^{p-1})$$

where

$$c = (1 + t_1)...(1 + t_n)$$

is the total Chern class. In particular, $b = c$, the Chern class, if $p = 2$.

Example 10.2. Let q_i be the Quillen operation of degree $p^i - 1$. Then $c(q_i) = s_{p^i-1}$ is the additive class.

Let X be a smooth variety over a field F. In [1], P. Brosnan defines Steenrod operations

$$P^i : \mathrm{CH}_*(X) \longrightarrow \mathrm{CH}_{*-i(p-1)}(X)/p$$

which satisfy the following properties:

• (functoriality with respect to proper morphisms) Let $f : Y \to X$ be a proper morphism of smooth varieties. Then the following diagram commutes [1, Prop. 8.11]:

$$\begin{array}{ccc} \mathrm{CH}_*(Y) & \xrightarrow{P^i} & \mathrm{CH}_{*-i(p-1)}(Y)/p \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ \mathrm{CH}_*(X) & \xrightarrow{P^i} & \mathrm{CH}_{*-i(p-1)}(X)/p. \end{array}$$

• Let $\dim(X) = d$. The image of the the class $[X]$ under

$$P^i : \mathrm{CH}_d(X) \longrightarrow \mathrm{CH}_{d-i(p-1)}(X)/p$$

equals $b_{i(p-1)}(-T_X)$ by [1, Prop. 8.4].

Theorem 10.3. (1) *Let X be a smooth variety of dimension d divisible by $p - 1$. Then $\deg b_d(-T_X)$ is divisible by p, i.e. $\frac{1}{p}b_d$ is a rational class in R'_d.*
(2) *For a proper morphism $f : Y \to X$ of smooth varieties of dimension d divisible by $p - 1$,*

$$f_*\big(b_d(-T_Y)\big) \equiv \deg(f) \cdot b_d(-T_X) \pmod{p\,\mathrm{CH}_0(X)}.$$

Proof. Both statements follow from the properties above applied to the operation P^i for $i = \frac{d}{p-1}$ and morphisms $X \to \operatorname{Spec} F$ and $f : Y \to X$. □

In view of Example 10.1, this Theorem implies Theorem 5.1 in the case $p = 2$ for smooth projective varieties.

Remark 10.4. Let X be a smooth variety of dimension d divisible by $p-1$. One can show that

$$\eta_p(X) = \frac{\deg b_d(-T_X)}{p} \in \mathbb{Z}/I(X).$$

Thus, Theorem 10.3 implies Theorem 3.1 for smooth varieties.

We generalize now Proposition 6.1:

Proposition 10.5. *Let $X = \mathbb{P}^d$ where $d = i(p-1)$. Then*

$$\deg b_d(-T_X) = (-1)^i \binom{pi}{i}.$$

Corollary 10.6. *Let X be a Severi-Brauer variety of dimension d divisible by $p-1$. Then*

$$\eta_p(X) = p^{\frac{s_p(d)}{p-1}-1}.$$

Proof. Let $d = i(p-1)$. Then

$$v_p\binom{pi}{i} = \frac{s_p(d) + s_p(i) - s_p(pi)}{p-1} = \frac{s_p(d)}{p-1}.$$

□

11. General formula

We would like to define things algebraically and give topological explanations.

Let $H_*\mathbb{Z}[b_1, b_2, \ldots]$ be the polynomial ring. We assume that H_* is graded in such a way that b_i is of degree i.

Explanation: In topology, it is the homology ring $H_*(BU, \mathbb{Z}) = H_*(MU, \mathbb{Z})$ (the difference is that the degree of b_i in topology is $2d$).

For any partition α: $d = i_1 + i_2 + \cdots + i_k$, $i_1 \leq i_2 \leq \cdots \leq i_k$ set

$$b^\alpha = b_1 \ldots b_{i_1} b_{i_1+1}^2 \ldots b_{i_2}^2 b_{i_2+1}^3 \ldots b_{i_k}^k.$$

The elements b^α for a basis of the polynomial ring over $\mathbb{Z}$, and more precisely, the b_α with $|\alpha| = d$ form a basis of H_d.

Consider another polynomial ring $H^* = \mathbb{Z}[c_1, c_2, \ldots]$ with the same grading $\deg(c_i) = i$. For any partition α as above we define the symmetric polynomial

$$P_\alpha = \sum X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_k^{\alpha_k}$$

and set

$$c_\alpha = P_\alpha(c_1, c_2, \ldots, c_k).$$

The c_α with $|\alpha| = d$ form a basis of H^d.

We consider a pairing between H_d and H^d such that the b^α and c_α form dual bases.

Explanation: H_* is the cohomology ring $H^*(BU,\mathbb{Z}) = H^*(MU,\mathbb{Z})$ with the standard pairing with the homology ring.

Let

$$\exp(x) = x + b_1x^2 + b_2x^3 + \dots$$

be the formal power series over the ring H_* and

$$\log(x) = x + m_1x^2 + m_2x^3 + \dots$$

be the formal inverse of exp. Clearly,

$$H_* = \mathbb{Z}[m_1, m_2, \dots]$$

is a graded polynomial ring. For example,

$$\begin{aligned} m_1 &= -b_1, \\ m_2 &= 2b_1^2 - b_2, \\ m_3 &= -5b_1^3 + 5b_1b_2 - b_3, \dots \end{aligned}$$

Consider the formal power series in two variables

$$\mu(x,y) = \exp(\log x + \log y)$$

over the ring $H*$. Clearly, μ is a commutative formal group law over H_*. The map exp is an isomorphism of the additive group law $x+y$ and the group law μ. Denote the coefficient of μ by $a_{ij} \in H_{i+j-1}$, i.e.

$$\mu(x,y) = x + y + \sum_{i\geq 1, j\geq 1} a_{ij}x^iy^j.$$

For example,

$$\begin{aligned} a_{11} &= 2b_1, \\ a_{12} &= 3b_2 - 2b_1^2, \\ a_{13} &= 4b_3 - 8b_1b_2 + 4b_1^3, \\ a_{22} &= 6b_3 - 6b_1b_2 + 2b_1^3, \dots \end{aligned}$$

Denote by L the graded subring in H_* generated by the coefficients a_{ij}. By definition, the group law μ is defined over L. It is the universal formal group law. The ring L is called the *Lazard ring*.

Let X be an algebraic proper smooth variety over F of dimension d. Denote by $[X]$ the element in H_d such that

$$\langle c_\alpha, [X] \rangle = \deg c_\alpha(-T_X)$$

for any partition α with $|\alpha| = d$. It turns out that $[X] \in L_d$ (algebraic proof?).

Example 11.1. $[\mathbb{P}^d] = (d+1)m_d$.

Explanation: The Lazard ring is identified with the complex cobordism ring Ω^U_*. The map $X \mapsto [X]$ is the injective Hurewicz homomorphism

$$h : \Omega^U_* \longrightarrow H_*.$$

The formula above looks as follows:

$$\langle c_\alpha, h[X]\rangle = \langle c_\alpha(-T_X), \sigma_X\rangle,$$

where σ_X is the fundamental class of X in $H_{2d}(X, \mathbb{Z})$.

Let X be an algebraic proper smooth variety over F of dimension d. We define the (graded) ideal $M(X)$ in L generated by the classes $[Y] \in L$ for all smooth proper varieties Y with $\dim(Y) < d$ and such that there exists a morphism $Y \to X$.

Conjecture 11.2. (General Degree Formula[2]) *For any morphism $f : Y \to X$ of proper smooth varieties of dimension d,*

$$[Y] = \deg(f) \cdot [X] \in L_d/M(X)_d.$$

Let $t : L_d \to \mathbb{Z}$ be a homomorphism. Applying t to the degree formula, we get a degree formula

$$t[Y] = \deg(f) \cdot t[X] \in \mathbb{Z}/t(M(X)_d).$$

In section (8) we identified such homomorphisms t with rational characteristic classes in R'_d.

Example 11.3. Let $t = \frac{1}{2}c_d$ (c is the Chern class). The group $M(X)_d$ is generated by the classes $[Y \times Z]$ of varieties of dimension d, where $k = \dim(Y) < d$ and there is a morphism $Y \to X$. Let p and q be two projections of $Y \times Z$ on Y and Z respectively. Then

$$T_{Y\times Z} = p^*(T_Y) \otimes q^*(T_Z).$$

The highest Chern class is multiplicative, hence

$$\frac{1}{2}\deg c_d(-T_{Y\times Z}) = \deg c_k(-T_Y) \cdot \frac{1}{2}\deg c_{d-k}(-T_Z).$$

Clearly, $\deg c_k(-T_Y) \in I(Y)$. Since there is a morphism $Y \to X$, $\deg c_k(-T_Y) \in I(X)$. Thus $t(M(X)_d) = I(X)$ and we get the degree formula 3.1 in the case of smooth varieties and $p = 2$.

Similar argument works for the class $t = \frac{1}{p}b_d$ to give Theorem 3.1 for any prime p.

Example 11.4. Let $t = td'_d$. Since the Todd class is multiplicative, similar argument gives $t(M(X)_d) = J(X)$ and one obtains the degree formula in Theorem 7.2.

[2]More generally one can expect that for any morphism $f : Y \to X$ of proper smooth varieties (not necessarily of the same dimension) with X irreducible one has

$$[Y] = \deg(f) \cdot [X] \in L/M(X),$$

where $\deg(f)$ denotes the class of the generic fibre.

12. Landweber-Novikov operations

Let E be a $\mathbb{C}$-oriented spectrum. We have a natural isomorphism

$$E^*(BU) \simeq E^*(MU).$$

The ring $E^*(BU)$ is the polynomial ring over $E^*(pt)$ in the "Chern classes" $c_1, c_2, \dots$. The corresponding elements in $E^*(MU)$ we denote $s_1, s_2, \dots$. For any partition α of d one can define in a standard way the elements $s_\alpha \in E^{2d}(MU)$.

Now set $E = MU$; we get the elements $s_\alpha \in MU^{2d}(MU)$ corresponding to the Conner-Floyd classes $c_\alpha \in MU^{2d}(BU)$. The class s_α can be considered as an operation

$$MU \longrightarrow S^{2|\alpha|}MU.$$

In particular, s_α acts on $L_* = MU^{-*}(pt)$ by endomorphisms

$$L_n \longrightarrow L_{n-|\alpha|}.$$

Example 12.1. The operations s_α act nicely on the classes of projective spaces:

$$s_\alpha[\mathbb{P}^n] = \langle c_\alpha, b^{-n-1}\rangle[\mathbb{P}^{n-|\alpha|}],$$

where $b = \sum_{i=0}^{\infty} b_i$.

Example 12.2. The action of s_α on L_* extends (uniquely) to an action on H_*:

$$s_\alpha(b) = \sum_{i\geq 0}\langle c_\alpha, b_i\rangle b^{i+1},$$

$$s_\alpha(xy) = \sum_{(\beta,\gamma)=\alpha}(s_\beta x)(s_\gamma y).$$

Fix a dimension n. For any partition α choose an element

$$x_\alpha \in MU^{2n-2|\alpha|}(pt) = L_{|\alpha|-n} \subset H_{|\alpha|-n}.$$

We view the multiplication by x_α as an operation

$$MU \longrightarrow S^{2n-2|\alpha|}MU.$$

We can consider the infinite sum

$$\sum_\alpha x_\alpha s_\alpha \tag{2}$$

as an operation

$$MU \longrightarrow S^{2n}MU.$$

Theorem 12.3. (Landweber-Novikov) *Any cohomology operation on the spectrum MU is of the form* (2) *for uniquely determined elements x_α.*

Conjecture 12.4. *For any smooth proper variety X, the ideal $M(X)$ in L is invariant under Landweber-Novikov operations.*[3]

[3]One may hope to prove these conjectures using Voevodsky's stable homotopy theory of schemes.

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