NOTES ON SOME EXAMPLES FOR $\mathrm{GL}_2(\mathbf{Z}[t^{\pm 1}])$

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Contents

1. Basic Notions

Let $\mathcal{R}$ be a ring and let $G = \mathrm{GL}_2(\mathcal{R})$.

A row (P, Q) with P, $Q \in \mathcal{R}$ is called *unimodular* if there exists R, $S \in \mathcal{R}$ with $PR + QS = 1$. A row is unimodular if and only if it is a row of an element of G.

An element of G is called *elementary* if it is in the subgroup generated by upper and lower triangular matrices.

Two rows are called (elementary) *equivalent* if one can be obtained from the other by multiplication with an elementary element of G.

A row is called *elementary* if it is equivalent to $(1, 0)$. A row is elementary if and only if it is a row of an elementary element of G.

Similar notions are understood for columns.

An element of G is elementary if and only if any of its rows or columns is elementary.

Every element of $\mathrm{GL}_2(\mathbf{Z})$ is elementary.

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2. Introduction

It is not known whether every element of $\mathrm{GL}_2(\mathbf{Z}[t^{\pm 1}])$ is elementary [1].

These notes present some considerations for rows consisting of linear polynomials.

In section 3 a simple criterion for unimodularity of such rows is given.

A first challenge was the question whether the row

$$(13 + 11t, 11^2)$$

is elementary.

It was shown by Matt Zaremsky that this indeed the case (thanks to Kai-Uwe Bux for showing me Zaremsky's notes).

We extend Zaremsky's calculations to some further examples, see Corollary 4.7, Corollary 5.1 and Remark 6.3.

The material from section 6 on can be considered as an appendix. It contains some musings about the method, driven mainly by curiosity about an underlying identity, see Remarks 6.2 and 7.2.

3. Unimodular rows given by linear polynomials

All statements are understood for the group $\mathrm{GL}_2(\mathbf{Z}[t^{\pm 1}])$.

We consider rows of the form $(a + bt, c + dt)$ with a, b, c, $d \in \mathbf{Z}$ and give a simple criterion for unimodularity.

I don't know whether all unimodular rows of this type are elementary.

Note that such a row is equivalent to one of the form $(a + bt, c)$ (use multiplication with a matrix from $\mathrm{GL}_2(\mathbf{Z})$).

Lemma 3.1. *Let a, b, $c \in \mathbf{Z}$. The row $(a + bt, c)$ is unimodular if and only if every prime divisor of c divides either a or b.*

Proof. The row is $(a + bt, c)$ is unimodular if and only if there exists P, $Q \in \mathbf{Z}[t^{\pm 1}]$ with

$$(a + bt)P = 1 + cQ$$

which means that the image of $a + bt$ in the ring

$$(\mathbf{Z}/c\mathbf{Z})[t^{\pm 1}]$$

is invertible.

The claim holds[*] for $c = 0$ and we assume $c \neq 0$.

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[*] The group of units of $\mathbf{Z}[t^{\pm 1}]$ is $\pm t^{\mathbf{Z}}$. Further, 0 is the only integer divisible by infinitely many prime numbers and ± 1 are the only integers not divisible by any prime number.

To detect invertibility one may pass to the reduction (quotient by the nilradical). Since

$$(\mathbf{Z}/c\mathbf{Z})_{\mathrm{red}} = \prod_{p|c} \mathbf{F}_p$$

where p runs through the prime divisors of c and since

$$(\mathbf{F}_p[t^{\pm 1}])^\times = \mathbf{F}_p^\times t^{\mathbf{Z}}$$

the claim follows. □

Corollary 3.2. *Let a, b, c, $d \in \mathbf{Z}$. The row*

$$(a + bt, c + dt)$$

is unimodular if and only if

(1) $\gcd(a, b, c, d) = 1$
(2) *Every prime divisor p of $ad - bc$ divides* $\gcd(a, c)\gcd(b, d)$.

Proof. All statements are invariant under row changes

$$\begin{aligned}(P, Q) &\mapsto (Q, P)\\ (P, Q) &\mapsto (P + nQ, Q), \quad n \in \mathbf{Z}\end{aligned}$$

Hence we may assume $d = 0$.

Then condition (2) reads as

$$p \mid bc \Rightarrow p \mid \gcd(a, c)b$$

which is the same as

$$p \mid c \Rightarrow p \mid ab$$

Under condition (1) this is the criterion of Lemma 3.1 □

Example 3.1. The row $(2 + 3t, 12)$ is unimodular. Indeed

$$(2 + 3t)(6 - 9t - 4t^2) - 12(1 - 3t^2 - t^3) = t^2$$

It is elementary, see Example 5.1.

Example 3.2. Let a, b be coprime integers. The rows

$$(a + bt, a^2),\ (a + bt, ab),\ (a + bt, b^2)$$

are unimodular by Lemma 3.1. They are equivalent, as one can see from

$$\begin{pmatrix} 1 & 0 \\ c & -t \end{pmatrix} \begin{pmatrix} a + bt \\ bc \end{pmatrix} = \begin{pmatrix} a + bt \\ ac \end{pmatrix}$$

Example 3.3 (Zaremsky). The row $(4t + 7, 16)$ is elementary: one has

$$\begin{pmatrix} 1 & -t^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4t + 7 \\ 16 \end{pmatrix} = \begin{pmatrix} -1 + 2t^{-1} \\ 4t^{-1} \end{pmatrix}$$

and may then proceed as in Example 3.2.

4. The matrices A_n and B_n

For a column

$$\alpha = \begin{pmatrix} a \\ b \end{pmatrix}$$

we use the notation[†]

$$\alpha^{\#} = \begin{pmatrix} b & -a \end{pmatrix}$$

Clearly $\alpha^{\#}\alpha = 0$ and the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha\alpha^{\#} = \begin{pmatrix} 1+ab & -a^2 \\ b^2 & 1-ab \end{pmatrix}$$

fixes α and has determinant 1. Moreover

$$\begin{pmatrix} \beta^{\#} \\ -\alpha^{\#} \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} = \det \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \alpha\beta^{\#} - \beta\alpha^{\#}$$

We work in the ring $\mathbf{Z}[t^{\pm 1}]$. Let

$$\theta = t - 1$$

For $n \in \mathbf{Z}$ let

$$\begin{aligned}
\varphi_n &= \begin{pmatrix} \theta \\ n \end{pmatrix}, \quad \varphi_n^{\#} = \begin{pmatrix} n & -\theta \end{pmatrix} \\
T_n &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varphi_n \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} t & 0 \\ n & 1 \end{pmatrix} \\
A_n &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varphi_n \varphi_n^{\#} \\
&= \begin{pmatrix} 1+n\theta & -\theta^2 \\ n^2 & 1-n\theta \end{pmatrix} = \begin{pmatrix} 1-n+nt & -1+2t-t^2 \\ n^2 & 1+n-nt \end{pmatrix} \\
B_n &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varphi_{n-1} \varphi_n^{\#} \\
&= \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \varphi_n \varphi_{n-1}^{\#} \\
&= \begin{pmatrix} 1+n\theta & -\theta^2 \\ n(n-1) & 1-(n-1)\theta \end{pmatrix} = \begin{pmatrix} 1-n+nt & -1+2t-t^2 \\ n(n-1) & n-(n-1)t \end{pmatrix}
\end{aligned}$$

[†] If M is a free module of rank 2, then $M \simeq M^{\vee} \otimes \Lambda^2 M$ under $v \mapsto (w \mapsto w \wedge v)$. Let M have the basis e_i ($i = 1, 2$) with dual basis f_i. Then e_1 corresponds to $-f_2(e_1 \wedge e_2)$ and e_2 corresponds to $f_1(e_1 \wedge e_2)$.

In particular

$$A_0 = \begin{pmatrix} 1 & -\theta^2 \\ 0 & 1 \end{pmatrix}$$
$$B_0 = \begin{pmatrix} 1 & -\theta^2 \\ 0 & t \end{pmatrix}$$
$$B_1 = \begin{pmatrix} t & -\theta^2 \\ 0 & 1 \end{pmatrix}$$

One has the following relations.

Lemma 4.1. *For $n \in \mathbf{Z}$ one has*

$$\begin{aligned} A_n &= t^{-1} T_n B_n \\ B_{n+1} &= A_n T_n \end{aligned}$$

Proof. The first claim is readily checked. Similar for the second claim by inspecting

$$B_{n+1} = \begin{pmatrix} -n + (n+1)t & -1 + 2t - t^2 \\ n(n+1) & n + 1 - nt \end{pmatrix}$$

□

Corollary 4.2. *For $n \in \mathbf{Z}$ one has*

$$\begin{aligned} A_{n+1} &= t^{-1} T_{n+1} A_n T_n \\ B_{n+1} &= t^{-1} T_n B_n T_n \end{aligned}$$

For $n \geq 1$ one has

$$\begin{aligned} A_n &= t^{-n} T_n \cdots T_1 A_0 T_0 \cdots T_{n-1} \\ B_n &= t^{1-n} T_{n-1} \cdots T_1 B_1 T_1 \cdots T_{n-1} \end{aligned}$$

□

Since A_0 and the T_n are triangular, we have

Corollary 4.3. *For $n \in \mathbf{Z}$ the elements A_n, B_n of $\mathrm{GL}_2(\mathbf{Z}[t^{\pm 1}])$ are elementary.*

Let

$$\varepsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Lemma 4.4. *One has*

$$\begin{aligned} T_n\varepsilon &= \varepsilon + \varphi_n \\ T_n\varphi_n &= t\varphi_n \\ A_n\varepsilon &= \varepsilon + n\varphi_n \\ A_n\varphi_n &= \varphi_n \\ B_n\varepsilon &= \varepsilon + n\varphi_{n-1} \\ B_n\varphi_{n-1} &= t\varphi_{n-1} \\ B_n\varphi_n &= \varphi_n \end{aligned}$$

Proof. Everything follows from the definitions of T_n, A_n, B_n and from $\varphi_n^{\#}\varepsilon = n$ (one may also consult Lemma 4.1). □

Corollary 4.5. *For integers a, b, n the columns*

$$\begin{pmatrix} a + b(t-1) \\ bn \end{pmatrix}, \qquad \begin{pmatrix} a + \big(b + an\big)(t-1) \\ \big(b+an\big)n \end{pmatrix}$$

are equivalent.

Proof. Apply A_n to $a\varepsilon + b\varphi_n$. □

Here a particular case is $a = -2$, $b = n$. We consider this more closely.

Let

$$\begin{gathered} \sigma\colon \mathbf{Z}[t^{\pm 1}] \to \mathbf{Z}[t^{\pm 1}] \\ \sigma(t) = t^{-1} \end{gathered}$$

be the standard involution.

Lemma 4.6. *Let*

$$\nu_n = -2\epsilon + n\varphi_n = \begin{pmatrix} -n-2+nt \\ n^2 \end{pmatrix}$$

Then

$$B_n\nu_n = -2\epsilon - n\varphi_{n-2} = \begin{pmatrix} n-2-nt \\ -n(n-2) \end{pmatrix}$$

and

$$\begin{pmatrix} t^{-1} & 0 \\ n-2 & -t \end{pmatrix} B_n\nu_n = \sigma^*\nu_{n-2}$$

Proof. The first claim follows from Lemma 4.4 and $2\varphi_{n-1} = \varphi_n + \varphi_{n-2}$. The second claim is easily checked. □

Corollary 4.7. *For $n \in \mathbf{Z}$ the columns ν_n and $\sigma^*\nu_{n-2}$ are equivalent.* □

5. Applications

Corollary 5.1. *The columns*

$$\begin{aligned} u_n &= \begin{pmatrix} (n+1)+nt \\ n^2 \end{pmatrix} \qquad (n \in \mathbf{Z}) \\ v_n &= \begin{pmatrix} (n+2)+nt \\ n^2 \end{pmatrix} \qquad (n \in 1+2\mathbf{Z}) \\ w_n &= \begin{pmatrix} (n+1)+nt \\ 2n^2 \end{pmatrix} \qquad (n \in \mathbf{Z}) \end{aligned}$$

are elementary.

Proof. The column u_n is the first column of $A_{-n}(-t)$ and therefore elementary.

Next note that for $n \in \mathbf{Z}$ the columns v_n and $\sigma^* v_{n-2}$ are equivalent. This follows from Corollary 4.7 by changing the signs of t and of the first entry.

Therefore v_{2k+1} is equivalent to v_1 (or $\sigma^* v_1$) which is elementary. Since $2w_n = v_{2n}$ the same argument shows that w_n is equivalent to the elementary element w_0. $\square$

Example 5.1. The row $(2+3t, 12)$ is elementary, since (cf. Example 3.2)

$$\begin{pmatrix} 1 & 0 \\ 4 & -t \end{pmatrix} \begin{pmatrix} 2+3t \\ 12 \end{pmatrix} = \begin{pmatrix} 2+3t \\ 8 \end{pmatrix}$$

is essentially w_2.

Example 5.2. The row $(4-3t, 9)$ is elementary, since it is essentially u_3. Here is another solution due to Bux:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -t & 1 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4-3t \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ -2-3t \end{pmatrix}$$

Example 5.3 (Zaremsky). The row

$$v_{11} = \begin{pmatrix} 13+11t \\ 11^2 \end{pmatrix}$$

is elementary. See also Remark 6.3.

6. Remarks on B_n

In this section we assume $n \geq 0$. Let

$$\begin{aligned} F_n(t) &= \frac{t^{n+1}-1}{t-1} \\ P_n(t) &= \frac{\mathrm{d}F}{\mathrm{d}t} \\ Q_n(t) &= t^{n-1}P_n(t^{-1}) \end{aligned}$$

so that

$$\begin{aligned} F_n(t) &= 1 + t + t^2 + \cdots + t^n \\ P_n(t) &= 1 + 2t + 3t^2 + \cdots + nt^{n-1} \\ Q_n(t) &= n + (n-1)t + \cdots + t^{n-1} \end{aligned}$$

The following computation is easily proved by induction:

Lemma 6.1. *One has*

$$\begin{aligned} T_nT_{n-1}\cdots T_1 &= \begin{pmatrix} t^n & 0 \\ P_n & 1 \end{pmatrix} \\ T_1T_2\cdots T_n &= \begin{pmatrix} t^n & 0 \\ Q_n & 1 \end{pmatrix} \end{aligned}$$

□

By Corollary 4.2 one has

$$t^nB_{n+1} = \begin{pmatrix} t^n & 0 \\ P_n & 1 \end{pmatrix} \begin{pmatrix} t & -(1-t)^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^n & 0 \\ Q_n & 1 \end{pmatrix} \tag{1}$$

Expanding the product on the right yields:

Corollary 6.2. *One has*

$$(1-t)^2P_n = 1 - (n+1)t^n + nt^{n+1} \tag{2}$$

and

$$t^{n+1}P_n + Q_n - (1-t)^2P_nQ_n = t^nn(n+1) \tag{3}$$

□

Remark 6.1. Here is a more symmetric variant of (1). The matrix

$$\begin{aligned} \tilde{B}_{n+1} &= \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} B_{n+1} \\ &= \begin{pmatrix} n+1-nt^{-1} & (1-t)(1-t^{-1}) \\ n(n+1) & n+1-nt \end{pmatrix} \end{aligned}$$

can be written as

$$\tilde{B}_{n+1} = \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \hat{B}_{n+1} \begin{pmatrix} 1 & 0 \\ 0 & t^{-n} \end{pmatrix}$$

where

$$\hat{B}_{n+1} = \begin{pmatrix} 1 & 0 \\ tP_n(t) & 1 \end{pmatrix} \begin{pmatrix} 1 & (1-t)(1-t^{-1}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-1}P_n(t^{-1}) & 1 \end{pmatrix}$$

Remark 6.2. Relation (2) can be easily verified directly. Namely, in

$$(1-t)F_n = 1 - t^{n+1}$$

take derivatives

$$(1-t)P_n - F_n = -(n+1)t^n$$

and multiply with $(1-t)$:

$$(1-t)^2 P_n - (1 - t^{n+1}) = -(n+1)t^n + (n+1)t^{n+1}$$

An ad hoc verification of (3) however looks tiresome. I wonder whether there is some better explanation for (3) (and for all of section 4). See also Remark 7.2.

Remark 6.3. The fact that ν_n and $\sigma^*\nu_{n-2}$ are equivalent (see Corollary 4.7) is basically due to Zaremsky. The matrices

$$\begin{pmatrix} t & -(1-t)^2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} t^n & 0 \\ Q_n & 1 \end{pmatrix}$$

appear (for $n = 10, 8, 6, 4, 2$) essentially in his notes on $(13+11t, 11^2)$. Instead of

$$L = \begin{pmatrix} t^n & 0 \\ P_n & 1 \end{pmatrix}$$

Zaremsky used the modification

$$\begin{pmatrix} 1 & 0 \\ 1-n & t \end{pmatrix} L = \begin{pmatrix} t^n & 0 \\ P_{n-1} + t^{n-1} & t \end{pmatrix}$$

which appears also in Lemma 4.6.

7. COMPUTATIONS FOR $(x^n - y^n)/(x-y)$

In this section we look at things in homogeneous coordinates. We start from scratch.

We work in the graded ring $\mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$ with x, y of degree 1.

We fix an integer n. Let

$$f = f_n = \frac{x^n - y^n}{x-y} = \sum_{i=0}^{n-1} x^i y^{n-i}$$

By f_x, f_y we denote the derivatives of f with respect to x, y, respectively.

Lemma 7.1. *One has*

$$(x-y)f_x = nx^{n-1} - f \tag{4}$$

$$(y-x)f_y = ny^{n-1} - f \tag{5}$$

$$y^n - (x-y)^2 f_x = x^{n-1}\big(ny - (n-1)x\big) \tag{6}$$

$$x^n - (x-y)^2 f_y = y^{n-1}\big(nx - (n-1)y\big) \tag{7}$$

$$x^n f_x + y^n f_y - (x-y)^2 f_x f_y = n(n-1)x^{n-1}y^{n-1} \tag{8}$$

Proof. (4) follows by taking the derivatives with respect to x in

$$(x-y)f = x^n - y^n$$

(4) yields

$$\begin{aligned} (x-y)^2 f_x &= (x-y)(nx^{n-1} - f) \\ &= nx^n - nyx^{n-1} - x^n + y^n \\ &= y^n - x^{n-1}\big(ny - (n-1)x\big) \end{aligned}$$

which is (6).

(5) and (7) follow now from $f(y,x) = f(x,y)$.

Let g denote the left-hand side of (8). Using (6), (5) and the general identity

$$xf_x + yf_y = (n-1)f$$

for homogeneous functions of degree $n-1$, one finds

$$\begin{aligned} g &= x^n f_x + \big(y^n - (x-y)^2 f_x\big) f_y \\ &= x^n f_x + x^{n-1}\big(ny - (n-1)x\big) f_y \\ &= x^{n-1}\big(xf_x + yf_y - (n-1)(x-y)f_y\big) \\ &= x^{n-1}(n-1)\big(f + ny^{n-1} - f\big) \\ &= x^{n-1}(n-1)ny^{n-1} \end{aligned}$$

□

Corollary 7.2. *For $n \in \mathbf{Z}$ the matrix*

$$\mathbf{C}_n = \begin{pmatrix} y^{n-1}\bigl(nx-(n-1)y\bigr) & -(x-y)^2 \\ n(n-1)x^{n-1}y^{n-1} & x^{n-1}\bigl(ny-(n-1)x\bigr) \end{pmatrix}$$

in $\mathrm{GL}_2(\mathbf{Z}[x^{\pm 1}, y^{\pm 1}])$ is elementary. More specifically:

$$\mathbf{C}_n = LMR$$

with

$$L = \begin{pmatrix} 1 & 0 \\ f_x & y^{n-1} \end{pmatrix}, \quad M = \begin{pmatrix} x & -(x-y)^2 \\ 0 & y \end{pmatrix}, \quad R = \begin{pmatrix} x^{n-1} & 0 \\ f_y & 1 \end{pmatrix}$$

Proof. One has

$$\begin{aligned} MR &= \begin{pmatrix} x^n - (x-y)^2 f_y & -(x-y)^2 \\ y f_y & y \end{pmatrix} \\ &= \begin{pmatrix} y^{n-1}\bigl(nx-(n-1)y\bigr) & -(x-y)^2 \\ y f_y & y \end{pmatrix} \end{aligned}$$

using (7). From this one gets

$$LMR = \begin{pmatrix} y^{n-1}\bigl(nx-(n-1)y\bigr) & -(x-y)^2 \\ x^n f_x + y^n f_y - (x-y)^2 f_x f_y & y^n - (x-y)^2 f_x \end{pmatrix}$$

Now use (8) and (6). ☐

Remark 7.1. One also has

$$\mathbf{C}_n = L'M'R'$$

with

$$L' = \begin{pmatrix} 1 & 0 \\ f_x & y^n \end{pmatrix}, \quad M' = \begin{pmatrix} 1 & -(x-y)^2 \\ 0 & 1 \end{pmatrix}, \quad R' = \begin{pmatrix} x^n & 0 \\ f_y & 1 \end{pmatrix}$$

Remark 7.2. It is unsatisfactory to establish (8) and Corollary 7.2 by mere computations.

I wonder whether there is a geometric argument.

Corollary 7.2 perhaps indicates to look at certain vector bundles. Maybe the variant $\mathbf{B}_n$ (in section 8) is useful here.

Note further that $x^n - y^n$ defines the subscheme

$$\mu_n \subset \mathbf{G}_\mathrm{m} = \operatorname{Proj} \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$$

The element f_n defines (over $\mathbf{Z}[n^{-1}]$) the subscheme $\mu_n \setminus \{1\}$.

8. THE MATRICES A_n AND B_n IN HOMOGENEOUS FORM

Here are variants of $\mathbf{C}_n$ with entries of low degree.

$$\begin{aligned}
\mathbf{B}_n &= \begin{pmatrix} 1 & 0 \\ 0 & x^{1-n} \end{pmatrix} \mathbf{C}_n \begin{pmatrix} y^{1-n} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} nx-(n-1)y & -(x-y)^2 \\ n(n-1) & ny-(n-1)x \end{pmatrix} \\
&= \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} x-y \\ n \end{pmatrix} (n-1, y-x) \\
&= \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} x-y \\ n-1 \end{pmatrix} (n, y-x)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A}_n &= \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n & y \end{pmatrix} \mathbf{B}_n \\
&= \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} nx-(n-1)y & -(x-y)^2 \\ n^2x & -nx^2+(n+1)xy \end{pmatrix} \\
&= y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} x-y \\ n \end{pmatrix} (n, y-x)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A}'_n &= \mathbf{B}_n \begin{pmatrix} x & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} (n+1)xy-ny^2 & -(x-y)^2 \\ n^2y & ny-(n-1)x \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\
&= x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} x-y \\ n \end{pmatrix} (n, y-x)
\end{aligned}$$

In homogeneous coordinates the matrices A_n, B_n read as follows

$$\begin{aligned}
\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} A_n(xy^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} &= \begin{pmatrix} (1-n)y+nx & -(x-y)^2 \\ n^2 & (1+n)y-nx \end{pmatrix} \\
\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} B_n(xy^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} &= \begin{pmatrix} (1-n)y+nx & -(x-y)^2 \\ n(n-1) & ny+(1-n)x \end{pmatrix}
\end{aligned}$$

and one has

$$\begin{aligned}
A_n &= \mathbf{A}_n|_{x=t,y=1} \\
B_n &= \mathbf{B}_n|_{x=t,y=1}
\end{aligned}$$

9. More remarks on B_n

Let

$$\Theta = x - y$$

We invert Θ, that is we work now over

$$\mathbf{Z}[x, y][\frac{1}{xy(x-y)}]$$

Remark 9.1. Geometrically, Θ is a parameter at $1 \in \mathbf{G}_{\mathrm{m}}$. The new base ring is the homogeneous coordinate ring of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$.

Let

$$\varphi_n = \begin{pmatrix} \Theta \\ n \end{pmatrix}$$

Then (cf. Lemma 4.4)

$$\mathbf{B}_n \varphi_{n-1} = x\varphi_{n-1}$$

$$\mathbf{B}_n \varphi_n = y\varphi_n$$

The φ_n generate the subspace generated by

$$\begin{pmatrix} \Theta \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

With respect to these elements, $\mathbf{B}_n$ has the form

$$\begin{pmatrix} \Theta & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{B}_n \begin{pmatrix} \Theta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} nx - (n-1)y & y - x \\ n(n-1)(x-y) & ny - (n-1)x \end{pmatrix}$$

Here all entries are of degree 1.

Consider

$$\Omega = \begin{pmatrix} 1 & 0 \\ \Theta^{-1} & 1 \end{pmatrix}$$

One has

$$\Omega \varphi_n = \varphi_{n+1}$$

It follows that

$$\mathbf{B}_{n+1} = \Omega \mathbf{B}_n \Omega^{-1}$$

Since

$$\mathbf{B}_1 = \begin{pmatrix} x & -\Theta^2 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}^{-1}$$

one gets

$$\mathbf{B}_{n+1} = \begin{pmatrix} \Theta & 0 \\ 1 & \Theta \end{pmatrix}^{n} \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \Theta & 0 \\ 1 & \Theta \end{pmatrix}^{-n}$$

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