NOTES ON MORLEY'S THEOREM

MARKUS ROST

Contents

Introduction

Morley's theorem states that

The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle.

This theorem is very curious. A standard source seems to be [4]. Among the many existing proofs we mention here D. J. Newman's proof [10] (also in [6, Ch. 20, p. 163] and on the web) and the article [9]. For more sources see the end of the text.

These notes evolved from a study of the fairly recent proof of Connes ([2]; see also [7], [1]).

We briefly discuss the relation of Connes' point of view of affine transformations with triangles and quadrangles. Then we give a proof of Morley's theorem a la Connes [2]. Finally we consider a purely group theoretic lemma (Lemma 4) which implies Connes' lemma on affine transformations.

In the context of Morley's angle trisector theorem we found it useful to look also—as a toy model—at the fact that the angle bisectors of any triangle meet in

Date: August 26, 2003.

one point. We call this for short the incenter theorem since the point of intersection is the center of the incircle of the triangle. We have complemented many considerations with the corresponding incenter variants.

Were we to give up, forever, understanding the Morley Miracle?
— D. J. Newman

1. Affine transformations and triangles

Let F be a field and let $\mathrm{Aff}(1,F)$ denote the group of affine transformations of the affine line over F. Elements $f \in \mathrm{Aff}(1,F)$ will be denoted by

$$f(t) = at + b \qquad \text{or} \qquad f = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

If $a = 1$, then f is a translation. Otherwise f has the unique fixed point

$$\mathrm{Fix}(f) = \frac{b}{1-a}$$

Note that if $a \neq 1$, but $a^n = 1$ for some $n > 1$, then $f^n = 1$. Moreover, if f, $g \in \mathrm{Aff}(1,F)$ commute, then f and g are both translations or have a common fixed point.

Lemma 1. *Let f_0, f_1, $f_2 \in \mathrm{Aff}(1,F)$. Suppose that the f_i have no common fixed point and that none of them is a translation. Let $x_i = \mathrm{Fix}(f_i)$.*

Then $f_0 f_1 f_2 = 1$ if and only if there exists $c \in F$ such that

$$f_i(t) = \frac{x_{i-1} - c}{x_{i+1} - c}(t - x_i) + x_i \tag{1}$$

(with the indices reduced mod 3*). The element c is uniquely determined by f_0, f_1, f_2.*

Proof. Let $d_i = \det(f_i)$. We may assume $x_0 = 0$ and $x_1 = 1$. The condition $f_0 f_1 f_2 = 1$ is equivalent to the conditions

$$d_0 d_1 d_2 = 1, \qquad x_2 = \frac{d_0(1-d_1)}{(1-d_0 d_1)} = \frac{1-d_0 d_2}{1-d_2}$$

Consider the change of variables

$$c = \frac{1}{1-d_2}, \qquad d_2 = \frac{1-c}{-c}$$

Then our conditions give indeed

$$d_0 = \frac{1-(1-d_2)x_2}{d_2} = \frac{x_2 - c}{1-c}, \qquad d_1 = \frac{1}{d_0 d_2} = \frac{-c}{x_2 - c}$$

□

Example 1. Consider an Euclidean triangle with vertices x_0, x_1, $x_2 \in \mathbf{C}$ and let c be its circumcenter. Then the affine transformation f_i given by (1) is the rotation with fixed point x_i and angle twice the angle at x_i (with appropriate orientation) of the triangle.

This way Euclidean triangles appear as a special case of triples f_0, f_1, $f_2 \in \mathrm{Aff}(1,F)$ with $f_0 f_1 f_2 = 1$. This is the view point of Connes in his proof of Morley's theorem. We state Connes' generalization of Morley's theorem [2]:

Lemma 2 (Connes)**.** *Let* t_0, t_1, $t_2 \in \mathrm{Aff}(1, F)$. *Suppose none of* t_0t_1, t_1t_2, t_2t_0, $t_0t_1t_2$ *is a translation and that* $t_0^3t_1^3t_2^3 = 1$. *Let* $\zeta = \det(t_0t_1t_2)$.

Then $1 + \zeta + \zeta^2 = 0$ *and*

$$\mathrm{Fix}(t_0t_1) + \zeta\,\mathrm{Fix}(t_1t_2) + \zeta^2\,\mathrm{Fix}(t_2t_0) = 0$$

The incenter theorem generalizes as follows:

Lemma 3. *Let* t_0, t_1, $t_2 \in \mathrm{Aff}(1, F)$. *Suppose none of* t_0t_1, t_1t_2, t_2t_0, $t_0t_1t_2$ *is a translation and that* $t_0^2t_1^2t_2^2 = 1$.

Then the transformations t_0t_1, t_1t_2, t_2t_0 *commute. In particular, their fixed points coincide.*

These lemmata will proved in Section 5.

2. Affine transformations and quadrangles

This section will not be used later on. We assume $\mathrm{char}\, F \neq 2$.

For (generic) points x_0, x_1, x_2, $x_3 \in F$ consider the affine transformations

$$f_{ijk\ell} = \frac{(x_i + x_j) - (x_k + x_\ell)}{(x_i + x_k) - (x_j + x_\ell)}(t - x_i) + x_i \tag{2}$$

where i, j, k, ℓ stand for any permutation of 0, 1, 2, 3.

If one takes in (1)

$$c = \frac{x_0 + x_1 + x_2 - x_3}{2}$$

one finds

$$f_i = f_{i,i+1,i-1,3}$$

This way Lemma 1 shows that triples f_0, f_1, $f_2 \in \mathrm{Aff}(1, F)$ with $\det(f_i) \neq 1$ and $f_0f_1f_2 = 1$ (and no common fixed point) are in characteristic different from 2 essentially just quadruples of points in F. Thus the symmetric group S_4 is a group of symmetries of such triples of affine transformations (this is true also in characteristic 2, and more generally over any commutative ring F).

Example 2. Let x_0, x_1, x_2, $x_3 \in \mathbf{C}$ be an Euclidean orthocentric quadrangle. This means that all pairs $x_i - x_j$, $x_k - x_\ell$ are orthogonal, or, equivalently, that (at least) one of the x_i is the orthocenter of the triangle formed by the other points x_j, x_k, x_ℓ.

Let c be the circumcenter and let $h = x_3$ be the orthocenter of the triangle x_0, x_1, x_2. Then

$$2c + h = x_0 + x_1 + x_2$$

(In fact, c, h and the center of mass $(x_0 + x_1 + x_2)/3$ lie on the Euler line of the triangle.)

It follows that the affine transformation $f_{ijk\ell}$ is the rotation with fixed point x_i and angle twice the angle at x_i (with appropriate orientation) of the triangle x_i, x_j, x_k.

3. Proof of Morley's theorem

Let F be an algebraically closed field with $\mathrm{char}\, F \neq 3$ and let $\zeta \in F$ be a primitive cube root of 1.

Let x_0, x_1, $x_2 \in F$. In the following the letters i, j, k stand for any permutation of 0, 1, 2. We assume that $x_i \neq 0$ and $x_i \neq x_j$.

Choose $s_{ij} \in F^*$ such that

$$s_{ij}^3 = \frac{x_j}{x_i}, \qquad s_{ij}s_{ji} = 1, \qquad s_{01}s_{12}s_{20} = \zeta$$

It is easy to see that such families s_{ij} exist and that any such family is determined by s_{01}, s_{12}. Moreover, there are exactly 9 such families which one can get by multiplying s_{01}, s_{12} by powers of ζ.

We write

$$\zeta_{ijk} = s_{ij}s_{jk}s_{ki}$$

Thus $\zeta_{ijk} = \zeta_{jki}$, $\zeta_{ijk} = \zeta_{ikj}^{-1}$ and $\zeta_{012} = \zeta$.

3.1. **The Euclidean case.** As for the proof of Morley's theorem we use the following setup.

One takes $F = \mathbf{C}$, $\zeta = e^{2\pi i/3}$ and assumes that the circumcenter of the triangle x_0, x_1, x_2 is the origin. In other words, $|x_0| = |x_1| = |x_2|$ where $|\cdot|$ is the Euclidean norm. Moreover one assumes that the triangle is positively oriented.

Then

$$\frac{x_2}{x_1} \in \mathbf{S}^1 = \{\, s \in \mathbf{C} \mid |s| = 1 \,\}$$

is twice the angle of the triangle at x_0. We choose the unique family s_{ij} with

$$\arg s_{ij} = \frac{1}{3} \arg \frac{x_j}{x_i}$$

where $0 \leq \arg s < 2\pi$ is defined for $s \in \mathbf{S}^1$ by $s = e^{i \arg s}$.

3.2. **Trisectors.** Consider the 6 elements

$$y_{ij} \stackrel{\text{def}}{=} s_{ij}x_i = s_{ji}^2 x_j = \zeta_{ijk}s_{kj}s_{ki}^2 x_k$$

In the Euclidean case, the elements y_{ij} are points of the circumcircle. They trisect each of the arcs between the points x_i.

One has

$$y_{ij}^3 = x_i^2 x_j, \qquad y_{ij}y_{jk}y_{ki} = \zeta_{ijk}x_0x_1x_2$$

3.3. **The geometric mean.** Consider the 3 elements

$$z_i = s_{ij}s_{ik}x_i$$

One has

$$z_i^3 = z_0z_1z_2 = x_0x_1x_2$$

Moreover

$$z_{i+1} = \zeta z_i$$

which can be seen for instance from

$$\frac{z_1}{z_0} = \frac{s_{12}s_{10}x_1}{s_{01}s_{02}x_0} = s_{12}s_{20}s_{01}s_{10}^3\frac{x_1}{x_0} = \zeta$$

Hence

$$z_0 + \zeta z_1 + \zeta^{-1}z_2 = 0$$

In the Euclidean case, the elements z_i are points of the circumcenter and form an equilateral triangle.

3.4. **Morley points.** Let g_{jk} be the affine transformation with

$$g_{jk}(t) = s_{jk}(t - x_i) + x_i$$

We define the Morley points m_i by

$$g_{ij}(m_i) = g_{ik}(m_i)$$

In the Euclidean case, the transformation g_{jk} is the rotation with center x_i and angle s_{jk}. Moreover m_i is the fixed point of $g_{ik}^{-1} \circ g_{ij}$ which can be easily seen as one of the "intersections of the adjacent trisectors" in Morley's theorem. This description is due to Connes [2].

Let us compute m_i. The defining equation is

$$s_{ij}(m_i - x_k) + x_k = s_{ik}(m_i - x_j) + x_j$$

This gives

$$\begin{aligned}(s_{ij} - s_{ik})m_i &= (x_j - x_k) - (s_{ik}x_j - s_{ij}x_k) \\ &= (s_{ij}^3 - s_{ik}^3)x_i - (s_{ij}^2 - s_{ik}^2)s_{ij}s_{ik}x_i\end{aligned}$$

Hence

$$\begin{aligned}m_i &= (s_{ij}s_{ik} + s_{ij}^2 + s_{ik}^2)x_i - (s_{ij} + s_{ik})s_{ij}s_{ik}x_i \\ &= z_i + y_{ji} + y_{ki} - \zeta_{ijk}^{-1}y_{jk} - \zeta_{ikj}^{-1}y_{kj} \\ &= z_i + v_{ji} + v_{ki}\end{aligned}$$

where

$$v_{ij} = y_{ij} - \zeta_{ijk}^{-1}y_{ki}$$

Next note that

$$v_{ij} + \zeta_{ijk}v_{jk} + \zeta_{ijk}^{-1}v_{ki} = 0$$

Indeed, one has

$$(y_{10} - \zeta y_{21}) + \zeta(y_{21} - \zeta y_{02}) + \zeta^{-1}(y_{02} - \zeta y_{10}) = 0$$

and

$$(y_{20} - \zeta^{-1}y_{12}) + \zeta(y_{01} - \zeta^{-1}y_{20}) + \zeta^{-1}(y_{12} - \zeta^{-1}y_{01}) = 0$$

since all terms cancel out.

Hence

$$m_0 + \zeta m_1 + \zeta^{-1}m_2 = 0$$

which is Morley's theorem.

Remark 1. The only thing which might be new in this deduction is that the Morley triangle appears as a superposition of three terms, the triple z_0, z_1, z_2, the triple v_{10}, v_{21}, v_{02}, and triple v_{20}, v_{01}, v_{12}, each of which is subject by itself to the equilaterality relation $X_0 + \zeta X_1 + \zeta^{-1}X_2 = 0$:

$$\begin{aligned}m_0 &= z_0 + (y_{10} - \zeta y_{21}) + (y_{20} - \zeta^{-1}y_{12}) \\ m_1 &= z_1 + (y_{21} - \zeta y_{02}) + (y_{01} - \zeta^{-1}y_{20}) \\ m_2 &= z_2 + (y_{02} - \zeta y_{10}) + (y_{12} - \zeta^{-1}y_{01})\end{aligned}$$

I don't know a geometric or algebraic interpretation of this observation.

4. The incenter

Let F be an algebraically closed field with $\operatorname{char} F \neq 2$.

Let x_0, x_1, $x_2 \in F$. In the following the letters i, j, k stand for any permutation of 0, 1, 2. We assume that $x_i \neq 0$ and $x_i \neq x_j$.

Choose $s_{ij} \in F^*$ such that

$$s_{ij}^2 = \frac{x_j}{x_i}, \qquad s_{ij}s_{ji} = 1, \qquad s_{01}s_{12}s_{20} = -1$$

It is easy to see that such families s_{ij} exist and that any such family is determined by s_{01}, s_{12}. Moreover, there are exactly 4 such families which one can get by multiplying s_{01}, s_{12} by powers of -1.

4.1. **The Euclidean case.** As for the classical fact that the angle bisectors of an Euclidean triangle meet in one point, the incenter, we use the following setup.

One takes $F = \mathbf{C}$, and assumes that the circumcenter of the triangle x_0, x_1, x_2 is the origin. In other words, $|x_0| = |x_1| = |x_2|$ where $|\cdot|$ is the Euclidean norm. Moreover one assumes that the triangle is positively oriented.

Then

$$\frac{x_2}{x_1} \in \mathbf{S}^1 = \{\, s \in \mathbf{C} \mid |s| = 1 \,\}$$

is twice the angle of the triangle at x_0. We choose the unique family s_{ij} with

$$\arg s_{ij} = \frac{1}{2} \arg \frac{x_j}{x_i}$$

where $0 \leq \arg s < 2\pi$ is defined for $s \in \mathbf{S}^1$ by $s = e^{i \arg s}$.

4.2. **Bisectors.** Consider the 3 elements

$$y_{ij} \overset{\text{def}}{=} s_{ij}x_i = s_{ji}x_j = -s_{kj}s_{ki}x_k$$

One has $y_{ij} = y_{ji}$.

In the Euclidean case, the elements y_{ij} are points of the circumcircle. They bisect each of the arcs between the points x_i.

We also write

$$z_i = y_{jk} = -s_{ij}s_{ik}x_i$$

One has

$$z_i^2 = x_jx_k, \qquad z_0z_1z_2 = -x_0x_1x_2$$

4.3. **Incenters.** We write

$$z = z_0 + z_1 + z_2$$

Let g_{jk} be the affine transformation with

$$g_{jk}(t) = s_{jk}(t - x_i) + x_i$$

and let m_i be the fixed point of $g_{ik}^{-1} \circ g_{ij}$.

In the Euclidean case, the transformation g_{jk} is the rotation with center x_i and angle s_{jk}. The fixed point m_i is therefore the intersection of the bisectors of the angles at x_j and x_k. Thus $m_1 = m_2 = m_3$ is the incenter of the triangle x_0, x_1, x_2.

In general we have

$$g_{ij}(z) = g_{ik}(z) \tag{3}$$

so that $m_1 = m_2 = m_3 = z$.

Proof of (3). Let us compute m_i. The defining equation is

$$s_{ij}(m_i - x_k) + x_k = s_{ik}(m_i - x_j) + x_j$$

This gives

$$\begin{aligned}(s_{ij} - s_{ik})m_i &= (x_j - x_k) - (s_{ik}x_j - s_{ij}x_k) \\ &= (s_{ij}^2 - s_{ik}^2)x_i - (s_{ij} - s_{ik})s_{ij}s_{ik}x_i\end{aligned}$$

Hence

$$m_i = (s_{ij} + s_{ik} - s_{ij}s_{ik})x_i = z_k + z_j + z_i$$

□

One can set up things also this way: Choose a, u_1, u_2, u_3 with

$$x_i = au_i^2$$

Then one can take

$$s_{ij} = -\frac{u_j}{u_i}, \qquad z_i = -au_iu_j$$

and for the incenter one has

$$z = -a(u_0u_1 + u_1u_2 + u_2u_0)$$

5. A group theoretic lemma

Lemma 4. *Let t_0, t_1, t_2 be elements of a group G.*

Suppose that G is metabelian (i. e., $[G,G]$ is abelian) and

$$(t_0t_1t_2)^3 = (t_0^2t_1^2t_2^2)^3 = t_0^3t_1^3t_2^3 = 1$$

Then

$$[[t_0t_1, t_1t_2], t_2t_0] = (t_0t_1t_2)[[t_2t_0, t_0t_1], t_1t_2](t_0t_1t_2)^{-1} \tag{4}$$

Proof. We have to show

$$[[t_0t_1, t_1t_2], t_2t_0](t_0t_1t_2) \stackrel{?}{=} (t_0t_1t_2)[[t_2t_0, t_0t_1], t_1t_2]$$

One multiplies out and collects appropriate terms.

$$\begin{aligned}&(t_0t_1^2t_2t_1^{-1})[(t_0^{-1}t_2^{-1}t_1^{-1})(t_2t_0t_1)(t_2t_0t_1)(t_2^{-1}t_1^{-2}t_0^{-2}t_2^{-1})(t_0t_1t_2)] \\ &\stackrel{?}{=} (t_0t_1t_2)(t_2t_0^2t_1t_0^{-1})(t_2^{-1}t_1^{-1}t_0^{-1})[(t_1t_2t_0)(t_1t_2t_0)(t_1^{-1}t_0^{-2}t_2^{-2}t_1^{-1})]\end{aligned}$$

The terms in square brackets are commutators and therefore commute. Moreover $(t_2t_0t_1)^3 = (t_1t_2t_0)^3 = 1$. This yields

$$\begin{aligned}&(t_0t_1^2t_2t_1^{-1})[(t_1t_2t_0)^{-1}(t_1^{-1}t_0^{-2}t_2^{-2}t_1^{-1})]^{-1} \\ &\stackrel{?}{=} (t_0t_1t_2)(t_2t_0^2t_1t_0^{-1})(t_2^{-1}t_1^{-1}t_0^{-1})[(t_0^{-1}t_2^{-1}t_1^{-1})(t_2t_0t_1)^{-1}(t_2^{-1}t_1^{-2}t_0^{-2}t_2^{-1})(t_0t_1t_2)]^{-1}\end{aligned}$$

Then, using $(t_0t_1t_2)^3 = 1$,

$$\begin{aligned}&t_0t_1^2t_2^3t_0^2t_1(t_1t_2t_0) \\ &\stackrel{?}{=} (t_0t_1t_2)(t_2t_0^2t_1t_0^{-1})(t_0t_1t_2)(t_2^{-1}t_1^{-2}t_0^{-2}t_2^{-1})^{-1}(t_2t_0t_1)(t_0^{-1}t_2^{-1}t_1^{-1})^{-1}\end{aligned}$$

Finally, using $(t_0^2t_1^2t_2^2)^3 = 1$,

$$\begin{aligned}t_0t_1^2t_2^3t_0^2t_1 &\stackrel{?}{=} (t_0t_1t_2)(t_2t_0^2t_1t_0^{-1})(t_0t_1t_2)(t_2t_0^2t_1^2t_2)(t_2t_0t_1) = \\ &= (t_0t_1t_2^2)(t_0^2t_1^2t_2^2)^2(t_0t_1) = (t_0t_1t_2^2)(t_0^2t_1^2t_2^2)^{-1}(t_0t_1) = t_0t_1^{-1}t_0^{-1}t_1\end{aligned}$$

This amounts to $t_0^3t_1^3t_2^3 = 1$. □

Corollary 1. *Let t_0, t_1, t_2 be elements of* $\mathrm{Aff}(1,F)$ *with* $t_0^3t_1^3t_2^3 = 1$. *Suppose* $d_0d_1d_2 \neq 1$ *where* $d_i = \det(t_i)$.

Then (4) *holds.*

Proof. One uses the fact that any affine transformation whose determinant is a primitive n-th root of unity has order n itself ($n > 1$).

Let $t = t_0t_1t_2$ and $d = \det(t) = d_0d_1d_2$. Then $d^3 = d_0^3d_1^3d_2^3 = 1$ and $d \neq 1$. Therefore $d^2 + d + 1 = 0$. Thus $t^3 = 1$. Similarly one finds $(t_0^2t_1^2t_2^2)^3 = 1$. By Lemma 4 the claim is clear. □

Formula (4) translates apparently the cyclic permutation $t_i \mapsto t_{i+1}$ into multiplication with a cube root of unity.

Proof of Lemma 2. One finds that

$$[[t_0t_1, t_1t_2], t_2t_0]$$

is the translation with vector

$$\left(\prod_0^2 (1 - d_id_{i+1})\right)\big(\mathrm{Fix}(t_0t_1) - \mathrm{Fix}(t_1t_2)\big)$$

where $d_i = \det(t_i)$. By (4) one gets

$$\big(\mathrm{Fix}(t_0t_1) - \mathrm{Fix}(t_1t_2)\big) = d\big(\mathrm{Fix}(t_2t_0) - \mathrm{Fix}(t_0t_1)\big)$$

with $d^2 + d + 1 = 0$. The claim is now clear. □

Corollary 2. *Morley's theorem.*

Proof. [Connes, [2]] For an Euclidean triangle with vertices x_0, x_1, $x_2 \in \mathbf{C}$ one takes for t_i the rotation with fixed point x_i and angle 2/3 the angle at x_i (with appropriate orientation) of the triangle. Then indeed $t_0^3t_1^3t_2^3 = 1$ and the fixed points $\mathrm{Fix}(t_it_{i+1})$ are the intersections of the trisectors in Morley's theorem. □

Remark 2. Lemma 4 suggests to consider the group $\widehat{G}$ generated by elements t and σ with relations

$$\sigma^3 = 1, \qquad (t\sigma)^9 = (t^2\sigma)^9 = (t^3\sigma)^3 = 1$$

and some commutation relations. Indeed if we put $t_i = \sigma^it\sigma^{-i}$, then $(t\sigma)^3 = t_0t_1t_2$ etc.

However I don't know whether this really helps. Anyway, let us note the following general formulas for elements x and σ in a group with relation $\sigma^3 = 1$:

$$[\sigma x\sigma^{-1}, \sigma^2x\sigma^{-2}] = (\sigma x)^3(x^{-1}\sigma)^3$$

and

$$[x, [\sigma x\sigma^{-1}, \sigma^2x\sigma^{-2}]] = (x\sigma)^3(\sigma x^{-1})^3(\sigma^{-1}x)^3(x^{-1}\sigma^{-1})^3$$

Remark 3. Let $s_{ij} = t_it_j$. In the situation of Lemma 4 the elements t_i are in the subgroup generated by the s_{12}, s_{20}, s_{01}. Maybe one can simplify things by using the s_{ij} as generators. Similarly for Lemma 5.

We conclude with similar (and much simpler) considerations for the incenter theorem.

Lemma 5. *Let G be a group and let t_0, t_1, t_2 be elements of G with $t_0^2t_1^2t_2^2 = 1$ and $(t_0t_1t_2)^2 = 1$. Then the elements t_0t_1, t_1t_2, t_2t_0 commute.*

Proof. By symmetry, it suffices to show that t_0t_1 and t_2t_0 commute. Indeed,

$$\begin{aligned}(t_0t_1)(t_2t_0)(t_0t_1)^{-1}(t_2t_0)^{-1} &= (t_0t_1t_2)t_0(t_1^{-1}t_0^{-1})(t_0^{-1}t_2^{-1})\\ &= (t_0t_1t_2)^{-1}t_0t_1^{-1}t_0^{-2}t_2^{-1}\\ &= t_2^{-1}t_1^{-2}t_0^{-2}t_2^{-1} = t_2(t_0^2t_1^2t_2^2)^{-1}t_2^{-1} = 1\end{aligned}$$

□

Corollary 3. *Let t_0, t_1, t_2 be elements of $\mathrm{Aff}(1, F)$ with $t_0^2t_1^2t_2^2 = 1$. Suppose $d_0d_1d_2 \neq 1$ where $d_i = \det(t_i)$. Then t_0t_1, t_1t_2, t_2t_0 have the same fixed point.*

Proof. Let $t = t_0t_1t_2$ and $d = \det(t) = d_0d_1d_2$. Then $d^2 = d_0^2d_1^2d_2^2 = 1$ and $d \neq 1$. Therefore $d + 1 = 0$. Thus $t^2 = 1$. By Lemma 5 the elements t_0t_1, t_1t_2, t_2t_0 commute. Hence their fixed points coincide. □

Corollary 4. *The bisectors of the angles of a triangle meet in one point.*

Proof. For an Euclidean triangle with vertices x_0, x_1, $x_2 \in \mathbf{C}$ one takes for t_i the rotation with fixed point x_i and angle the angle at x_i (with appropriate orientation) of the triangle. Then indeed $t_0^2t_1^2t_2^2 = 1$ and the fixed points $\mathrm{Fix}(t_it_{i+1})$ are the intersections of the bisectors. □

More sources

Here is a list of other possible sources for Morley's theorem: [3, 5, 8, 11] and, of course, the web:

`http://www.google.com/search?q=morley+triangle`
`http://www-cabri.imag.fr/abracadabri/GeoPlane/Classiques/Morley/Morley1.htm`
`http://www.cut-the-knot.org/triangle/Morley/index.shtml`
`http://mathforum.org/library/drmath/view/51789.html`

Under the last address one finds a proof of Conway.

References

[1] A. Connes, *Symétries*, Pour la Science, N° 292, 2002.
[2] ______, *A new proof of Morley's theorem*, Les relations entre les mathématiques et la physique théorique, Inst. Hautes Études Sci., Bures, 1998, pp. 43–46, MR 99m:51027.
[3] J. H. Conway and R. K. Guy, *The book of numbers*, Copernicus, New York, 1996, MR 98g:00004.
[4] H. S. M. Coxeter, *Introduction to geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989, Reprint of the 1969 edition, MR 90a:51001.
[5] H. S. M. Coxeter and S. L. Greitzer, *Zeitlose Geometrie*, Klett Studienbücher Mathematik. [Klett Textbooks in Mathematics], Ernst Klett Verlag, Stuttgart, 1983, A translation of *Geometry revisited*, Translated from the English by Rolf Müller, Herbert Rauck and Hartmut Wellstein, MR 85c:51031.
[6] D. Gale, *Tracking the automatic ant*, Springer-Verlag, New York, 1998, And other mathematical explorations, A collection of Mathematical Entertainments columns from The Mathematical Intelligencer, MR 1 661 863.
[7] H. Geiges, *Beweis des Satzes von Morley nach A. Connes*, Elem. Math. **56** (2001), no. 4, 137–142, MR 2003b:51024.
[8] H. Lebesgue, *Leçons sur les Constructions Géométriques*, Gauthier-Villars, Paris, 1950, MR 11,678d.

[9] C. Lubin, *A proof of Morley's theorem*, Amer. Math. Monthly **62** (1955), 110–112, MR 16,848c.

[10] D. J. Newman, *The Morley Miracle*, Math. Intelligencer **18** (1996), no. 1, 31–32.

[11] C. O. Oakley and J. C. Baker, *The Morley trisector theorem*, Amer. Math. Monthly **85** (1978), no. 9, 737–745, MR 80i:01012.

Department of Mathematics, The Ohio State University, 231 W 18th Avenue, Columbus, OH 43210, USA

E-mail address: rost@math.ohio-state.edu

URL: http://www.math.ohio-state.edu/~rost