ON THE RESULTANT OF THREE TERNARY QUADRATIC FORMS

MARKUS ROST

Contents

1. Introduction

Let f, g, h be 3 homogeneous quadratic forms in 3 variables. The resultant $\operatorname{Res}(f, g, h)$ is the first non-trivial case of a resultant beyond the well known theory of resultants of 2 homogeneous forms in 2 variables (basic references for resultants are [2], [5]). First descriptions were given by Cayley [1, p. 119] and Sylvester [8], [5, p. 118]. Eisenbud, Schreyer and Weyman presented in [3, Introduction] a Bezout formula which describes $\operatorname{Res}(f, g, h)$ as the Pfaffian of a certain alternating 8×8-matrix whose entries are linear in the Plücker coordinates of $f \wedge g \wedge h$ (the matrix is reproduced in Section 7).

In this text we describe a comparatively simple presentation of $\operatorname{Res}(f, g, h)$. After an appropriate choice of basis, the resulting expression coincides with that of [3, Introduction].

Let V be a locally free module of rank 3 over a ring R. Let further

$$U = \frac{V \otimes \Lambda^2 V}{\Lambda^3 V}$$

Date: December 4, 2018.

Here we consider $\Lambda^3 V$ as subspace of $V \otimes \Lambda^2 V$ via the natural embeddings $\Lambda^k V \subset V^{\otimes k}$. Another way to present U is as the Lie algebra of $\mathrm{PGL}(V)$ tensored with the line bundle $\Lambda^3 V$:

$$U = \frac{\mathrm{End}(V)}{R \cdot \mathrm{id}_V} \otimes \Lambda^3 V$$

One has $\operatorname{rank} U = 8$. Let

$$\mathrm{Pf}\colon \Lambda^2 U \to \Lambda^8 U = (\Lambda^3 V)^{\otimes 8}$$

denote the Pfaffian characterized by

$$\mathrm{Pf}(u_1 \wedge u_2 + u_3 \wedge u_4 + u_5 \wedge u_6 + u_7 \wedge u_8) = u_1 \wedge \cdots \wedge u_8$$

For $\omega \in \Lambda^2 U$ the square of $\mathrm{Pf}(\omega)$ is the determinant of an alternating matrix representing ω. Moreover $4!\,\mathrm{Pf}(\omega) = \omega^4$.

Here are the main results.

Proposition 1. *There exists a unique morphism of* $\mathrm{gl}(V)$*-modules*

$$\Phi\colon \Lambda^3 S^2 V \to \Lambda^2 U$$

such that

$$\Phi(xy \wedge yz \wedge zx) = [x \otimes y \wedge z] \wedge [y \otimes x \wedge z]$$

for x, y, $z \in V$.

Let

$$F(f,g,h) = \mathrm{Pf}\big(\Phi(f \wedge g \wedge h)\big) \qquad (f,g,h \in S^2 V)$$

Then

$$F(f,g,h) = 0$$

whenever f, g, h *have a common zero. Moreover*

$$F(x^2, y^2, z^2) = (x \wedge y \wedge z)^{\otimes 8}$$

Corollary. *For* f, g, $h \in S^2 V$ *one has*

$$\mathrm{Res}(f,g,h) = F(f,g,h)$$

Moreover one has:

Proposition 2. *With respect to a basis of* V *and an appropriate basis of* U*, the alternating* 8×8*-matrix corresponding to* Φ *(with entries from the dual space of* $\Lambda^3 S^2 V$*) is exactly the one presented in* [3, Introduction].

I don't have a heuristic argument why the morphism Φ does the job. Maybe one should try to follow the methods in [3].

The starting point was a rather naive ad hoc search. Looking for a Bezout formula (an expression of the resultant in terms of Plücker coordinates) means to find an invariant quartic form on

$$\Lambda^3 S^2 V$$

which yields the resultant. Over $\mathbf{Q}$ the space of invariant quartic forms on $\Lambda^3 S^2 V$ is 6-dimensional and in principle one should be able to write down the forms in a coordinate free way over $\mathbf{Z}$. The search was greatly encouraged and helped by the presentation of the 8×8-matrix in [3, Introduction]. Eventually the morphism Φ showed up.

The text contains a lot of explicit computations. Most of them are not really necessary to recognize F as the resultant. However they are used to get the 8×8-matrix. Anyway, we find them illustrative and useful.

Naturally, an understanding of the $\mathrm{GL}(V)$-module $\Lambda^3 S^2 V$ and its variant

$$\Lambda^3 S_2 V = \left(\Lambda^3 S^2(V^\#)\right)^\#$$

is in order ($W^\#$ denotes the dual of W). Section 5 contains some related remarks. There are the two morphisms

$$J, \eta\colon \Lambda^3 S_2 V \to \Lambda^3 S^2 V$$
$$J\colon [x]_2 \wedge [y]_2 \wedge [z]_2 \ \mapsto\ x^2 \wedge y^2 \wedge z^2$$
$$\eta\colon [x]_2 \wedge [y]_2 \wedge [z]_2 \ \mapsto\ xy \wedge yz \wedge zx$$

The morphism J is induced from the standard morphism

$$S_2 V \to S^2 V$$

(passage from symmetric bilinear forms to quadratic forms) and is not an isomorphism in characteristic 2. The morphism η however is an isomorphism for $\operatorname{rank} V = 3$. Once the bijectivity of η is established, the construction of Φ becomes simple (see Section 5.1).

The first construction of Φ in Section 3 however bypasses η and the material of Section 5 is not used elsewhere.

2. Preliminaries

2.1. **Basic notations.** Let V be a locally free R-module of finite rank. The dual module is denoted by

$$V^\# = \mathrm{Hom}_R(V, R)$$

and the symmetric resp. exterior powers are denoted as usual by $S^k V$, $\Lambda^k V$. Moreover let

$$S_k V = (V^{\otimes k})^{\Sigma_k} \subset V^{\otimes k}$$

be the module of symmetric k-tensors. One has

$$(S^k V)^\# = S_k(V^\#)$$
$$(\Lambda^k V)^\# = \Lambda^k(V^\#)$$

The module $S_\bullet V$ is the divided power algebra of V, see e.g. [9]. For elements in $S_k V$ we use the notations

$$[x]_k = x \otimes \cdots \otimes x \in S_k V \subset V^{\otimes k}$$

with $x \in V$ and the product is denoted by

$$S_k V \otimes S_h V \to S_{k+h} V$$
$$\alpha \otimes \beta \ \mapsto\ \alpha * \beta$$

For instance

$$[x]_k * [x]_h = \binom{k+h}{k} [x]_{k+h}$$
$$x * y = x \otimes y + y \otimes x = [x+y]_2 - [x]_2 - [y]_2$$

2.2. **Conventions for a basis.** We assume $\operatorname{rank} V = 3$.

Given a basis e_i $(i = 0, 1, 2)$, we denote the dual basis by f_i. Thus

$$V = Re_0 \oplus Re_1 \oplus Re_2$$
$$V^{\#} = Rf_0 \oplus Rf_1 \oplus Rf_2$$

with $f_i(e_j) = \delta_{ij}$.

The elements

$$\theta_{ij} = e_i \otimes f_j$$

form a basis of $\mathrm{gl}(V) = \mathrm{End}(V) = V \otimes V^{\#}$.

We write

$$\epsilon_i = [\theta_{ii}] \in \mathrm{pgl}(V) = \frac{\mathrm{End}(V)}{R \cdot \mathrm{id}_V}$$

for the image of $\theta_{ii} = e_i \otimes f_i$ in $\mathrm{pgl}(V)$.

Then

$$\epsilon_0 + \epsilon_1 + \epsilon_2 = 0$$

and the elements

$$\epsilon_1, \quad \epsilon_2, \quad \theta_{ij} \ (i \neq j)$$

form a basis of $\mathrm{pgl}(V)$.

Here are basis elements of some line bundles:

$$e_0 \wedge e_1 \wedge e_2 \in \Lambda^3 V$$
$$e_0^2 \wedge e_1^2 \wedge e_2^2 \wedge e_0e_1 \wedge e_1e_2 \wedge e_2e_0 \in \Lambda^6 S^2 V$$
$$[e_0]_2 \wedge [e_1]_2 \wedge [e_2]_2 \wedge e_0{*}e_1 \wedge e_1{*}e_2 \wedge e_2{*}e_0 \in \Lambda^6 S_2 V$$

We use them to identify the line bundles with R or with each other.

3. Definition of Φ

3.1. **The morphism Ψ.** We start with the morphism

$$\Psi_1 \colon \Lambda^2 S_2 V \otimes S_2 V \to \Lambda^2(V \otimes \Lambda^2 V)$$
$$[x]_2 \wedge [y]_2 \otimes [z]_2 \ \mapsto \ (x \otimes y \wedge z) \wedge (y \otimes x \wedge z)$$

Remark. The term on the right is a homogeneous polynomial of degree 2 in each of x, y, z. By definition such a polynomial is a linear morphism

$$S_2 V \otimes S_2 V \otimes S_2 V \to \Lambda^2(V \otimes \Lambda^2 V)$$

In fact it defines a morphism of strict polynomial functors (see [4, §2], [7, §2, pp. 702]) over $R = \mathbf{Z}$. By the skew symmetry in x, y, it factors through $\Lambda^2 S_2 V \otimes S_2 V$.

Consider the natural inclusion

$$\Lambda^3 V \to V \otimes \Lambda^2 V$$
$$x_0 \wedge x_1 \wedge x_2 \mapsto \sum_i x_i \otimes x_{i+1} \wedge x_{i-1}$$

with the indices taken mod 3. Put

$$U = \frac{V \otimes \Lambda^2 V}{\Lambda^3 V}$$

After passing to U, Ψ_1 becomes entirely alternating (if $u_0 + u_1 + u_2 = 0$, then $u_0 \wedge u_1 = u_1 \wedge u_2$) and yields the morphism

$$\begin{aligned} \Psi \colon \Lambda^3 S_2 V &\to \Lambda^2 U \\ [x]_2 \wedge [y]_2 \wedge [z]_2 &\mapsto [x \otimes y \wedge z] \wedge [y \otimes x \wedge z] \end{aligned}$$

Remark. One may write Ψ in a different way using the exact complex

$$0 \to \Lambda^3 V \to V \otimes \Lambda^2 V \xrightarrow{\kappa} S^2 V \otimes V \xrightarrow{\mu} S^3 V \to 0$$

where

$$\kappa(x \otimes y \wedge z) = xy \otimes z - xz \otimes y$$

and μ is the multiplication. The morphism κ identifies U with a subbundle of $S^2 V \otimes V$ and so no essential information gets lost when composing with κ. One has

$$\begin{aligned} \Lambda^2 \kappa \circ \Psi \colon \Lambda^3 S_2 V &\to \Lambda^2 (S^2 V \otimes V) \\ [x_0]_2 \wedge [x_1]_2 \wedge [x_2]_2 &\mapsto \sum_i (x_i x_{i+1} \otimes x_{i-1}) \wedge (x_i x_{i-1} \otimes x_{i+1}) \end{aligned}$$

I haven't looked at the corresponding presentation $\Lambda^2 \kappa \circ \Phi$ of Φ in detail.

3.2. **Duality for rank 3.** From now on we assume $\operatorname{rank} V = 3$.

One has

$$\begin{aligned} \Lambda^2 V &= V^{\#} \otimes \Lambda^3 V \\ V \otimes \Lambda^2 V &= \operatorname{End}(V) \otimes \Lambda^3 V \end{aligned}$$

Moreover

$$U = \operatorname{pgl}(V) \otimes \Lambda^3 V, \qquad \operatorname{pgl}(V) = \frac{\operatorname{End}(V)}{R \cdot \operatorname{id}_V}$$

and Ψ becomes a morphism

$$\Psi \colon \Lambda^3 S_2 V \to \Lambda^2 \operatorname{pgl}(V) \otimes (\Lambda^3 V)^{\otimes 2}$$

In coordinates one has

$$\Psi([e_0]_2 \wedge [e_1]_2 \wedge [e_2]_2) = -\epsilon_0 \wedge \epsilon_1 = \epsilon_2 \wedge \epsilon_1$$

The non-degenerate pairing

$$\Lambda^3 S^2 V \otimes \Lambda^3 S^2 V \to \Lambda^6 S^2 V = (\Lambda^3 V)^{\otimes 4}$$

induces an isomorphism

$$H \colon \Lambda^3 S^2 V \to \left(\Lambda^3 S^2 V\right)^{\#} \otimes \Lambda^6 S^2 V = \Lambda^3 S_2 (V^{\#}) \otimes (\Lambda^3 V)^{\otimes 4}$$

In coordinates one finds (with appropriate sign in the identification $\Lambda^6 S^2 V = R$)

$$H(e_0 e_1 \wedge e_1 e_2 \wedge e_2 e_0) = [f_0]_2 \wedge [f_1]_2 \wedge [f_2]_2$$

3.3. **The morphism Φ.** We denote by $\Psi^{V^\#}$ the morphism Ψ with V replaced by $V^\#$ and define

$$\Phi = \Psi^{V^\#} \circ H$$

as the composite of

$$\Lambda^3 S^2 V \xrightarrow{H} \Lambda^3 S_2(V^\#) \otimes (\Lambda^3 V)^{\otimes 4} \xrightarrow{\Psi^{V^\#}} \Lambda^2 \operatorname{pgl}(V) \otimes (\Lambda^3 V)^{\otimes 2}$$

In coordinates, Φ is the morphism

$$\Lambda^3 S^2 V \to \Lambda^2 \operatorname{gl}(V)$$

with

$$e_0 e_1 \wedge e_1 e_2 \wedge e_2 e_0 \mapsto \epsilon_2 \wedge \epsilon_1$$

Remark. The element $\epsilon_2 \wedge \epsilon_1$ is a generator of $\Lambda^2 \mathcal{C}$, where $\mathcal{C} \subset \operatorname{pgl}(V)$ is the Cartan subalgebra corresponding to the basis. It follows that the image of Φ is in the kernel of the (lifted) Lie bracket

$$[\ ,\ ] \colon \Lambda^2 \operatorname{pgl}(V) \to \operatorname{sl}(V)$$

More precisely, there is the short exact sequence

$$0 \to \Lambda^3 S^2 V \otimes (\Lambda^3 V^\#)^{\otimes 2} \xrightarrow{\Phi} \Lambda^2 \operatorname{pgl}(V) \xrightarrow{[\ ,\ ]} \operatorname{sl}(V) \to 0$$

Indeed, the formulas in Section 6 (or an inspection of the 8×8-matrix in Section 7) show that the image of Φ is a subbundle (the dual of Φ is an epimorphism) and the claim follows from rank reasons.

4. Identifying the resultant

We assume $\operatorname{rank} V = 3$. Let us recall a characterization of the resultant, for the special case of three forms $g_i \in S^2 V$ $(i = 0, 1, 2)$.

As definition of the resultant we take [2, Définition 3, pp. 348]. The following claim follows then from [2, Corollaire, pp. 346] and degree reasons.

Lemma. *Assume $R = \mathbf{Z}$. Let $F(g_0, g_1, g_2)$ be a homogeneous polynomial in the g_i of degree 12. If $F(g_0, g_1, g_2) = 0$ whenever the g_i have a common non-trivial zero (over say algebraically closed fields), then $F(g_0, g_1, g_2)$ is a scalar multiple of the resultant* $\operatorname{Res}(g_0, g_1, g_2)$. □

Remark. To give a point (=section) in the projective space

$$\mathbf{P}(V) = \operatorname{Proj} S^\bullet V$$

means to give a codimension 1 subbundle W of V. Then $L = V/W$ is a line bundle. This way a point in $\mathbf{P}(V)$ is given by a short exact sequence

$$0 \to W \to V \xrightarrow{\lambda} L \to 0$$

with $\operatorname{rank} L = 1$.

Let $g_i \in S^2 V$ $(i = 0, 1, 2)$ and assume that there is a common zero in $\mathbf{P}(V)$. This means that there is a line bundle L and an epimorphism

$$\lambda \colon V \to L$$

such that

$$S^2\lambda(g_i) = 0 \qquad (i = 0, 1, 2)$$

$(S^2\lambda(g) \in L^{\otimes 2}$ is the evaluation of g at the point λ.)

Let

$$W = \ker \lambda$$

The morphism λ induces a morphism $\tilde{\lambda}$ on $\mathrm{pgl}(V)$, namely

$$\begin{aligned} \tilde{\lambda}\colon \frac{V \otimes V^{\#}}{R \cdot \mathrm{id}_V} &\to L \otimes \frac{V^{\#}}{L^{\#}} = L \otimes W^{\#} \\ [v \otimes \alpha] &\mapsto \lambda(v) \otimes (\alpha|W) \end{aligned}$$

$\tilde{\lambda}$ is an epimorphism and $\ker \tilde{\lambda}$ has rank 6.

Lemma.

$$\Phi\big(\Lambda^3(\ker S^2\lambda)\big) \subset \Lambda^2(\ker \tilde{\lambda}) \otimes (\Lambda^3 V)^{\otimes 2}$$

Proof. I checked by inspection of the formulas in Section 6: One takes a basis with $f_0 = \lambda$. Using that θ_{1i}, θ_{2i} leave f_0 invariant, one finds that it suffices to check that

$$\Phi(A) = \epsilon_2 \wedge \epsilon_1 \in \Lambda^2(\ker \tilde{f}_0)$$

which is obvious.

Certainly there is an intrinsic proof without explicit computations. □

Since $\Lambda^8(\ker \tilde{\lambda}) = 0$, the Pfaffian vanishes on $g_0 \wedge g_1 \wedge g_2$ if $g_i \in \ker S^2\lambda$ for $i = 0$, 1, 2.

Hence for arbitrary g_i one has

$$\mathrm{Pf}\big(\Phi(g_0 \wedge g_1 \wedge g_2)\big) = a \,\mathrm{Res}(g_0, g_1, g_2)$$

for some $a \in \mathbf{Z}$ (assuming $R = \mathbf{Z}$). The computation at the very end of Section 6 shows

$$\mathrm{Pf}\big(\Phi(e_0^2 \wedge e_1^2 \wedge e_2^2)\big) = \pm 1$$

and therefore $a = \pm 1$. (The sign is not important. It depends on some choices anyway.)

5. Alternative definition of Φ

The material of this section is not really needed elsewhere, but hopefully illustrative.

5.1. The isomorphism $\Lambda^3 S_2 V \to \Lambda^3 S^2 V$ (rank $V = 3$).

Let

$$\begin{aligned} \eta\colon \Lambda^3 S_2 V &\to \Lambda^3 S^2 V \\ [x]_2 \wedge [y]_2 \wedge [z]_2 &\mapsto xy \wedge yz \wedge zx \end{aligned}$$

Remark. For $\mathrm{rank}\, V = 3$, an explicit computation of η is provided below. For instance one has

$$\eta(e_0{*}e_1 \wedge e_1{*}e_2 \wedge e_2{*}e_0) = e_0^2 \wedge e_1^2 \wedge e_2^2 - 2e_0e_1 \wedge e_1e_2 \wedge e_2e_0$$

Lemma. *If* $\operatorname{rank} V = 3$*, then* η *is an isomorphism.*

Proof. This is evident from the explicit computations below. However there is a more conceptual proof. Namely, the inverse of η is the dual of η in the appropriate sense. More precisely, one has

$$(H \circ \eta)([e_0]_2 \wedge [e_1]_2 \wedge [e_2]_2) = [f_0]_2 \wedge [f_1]_2 \wedge [f_2]_2$$

with H as in Section 3.2. It follows that $H \circ \eta$ is an epimorphism (for any V the elements $[x]_2 \wedge [y]_2 \wedge [z]_2$ generate $\Lambda^3 S_2 V$). But then $H \circ \eta$ must be an isomorphism since both modules are locally free of the same rank. □

One may now define Φ as

$$\Phi = \Psi \circ \eta^{-1} \colon \Lambda^3 S^2 V \to \Lambda^2 U$$

Remark. The morphism η is defined for any V of arbitrary rank r. It is another example of a morphism of strict polynomial functors. If $r \leq 2$, it is easy to see that η is an isomorphism. In general, $\operatorname{coker} \eta$ is annihilated by 8 (hint: the elements $x^2 \wedge y^2 \wedge z^2$ are in the image of η). In characteristic 2 there is an epimorphism $\operatorname{coker} \eta \to \Lambda^4 V \otimes S^2 V$.

5.2. **Some explicit computations.** The following tables describe some actions of elements of $\mathrm{sl}(V)$ and yield generators of the $\mathrm{sl}(V)$-modules $\Lambda^3 S_2 V$ resp. $\Lambda^3 S^2 V$. The dim-slot shows the rank of the subspace generated by all permutations of indices.

Table 1.

$$\begin{aligned}
A &= [e_0]_2 \wedge [e_1]_2 \wedge [e_2]_2 && (\dim 1)\\
B = \theta_{12}(A) &= [e_0]_2 \wedge [e_1]_2 \wedge e_1{*}e_2 && (\dim 6)\\
\theta_{02}(B) &= [e_0]_2 \wedge [e_1]_2 \wedge e_1{*}e_0 && (\dim 3)\\
\theta_{10}(B) &= e_0{*}e_1 \wedge [e_1]_2 \wedge e_1{*}e_2 && (\dim 3)\\
C = \theta_{20}(B) &= e_2{*}e_0 \wedge [e_1]_2 \wedge e_1{*}e_2 && (\dim 6)\\
D = \theta_{01}(C) &= e_2{*}e_0 \wedge e_0{*}e_1 \wedge e_1{*}e_2 && (\dim 1)
\end{aligned}$$

Table 2.

$$\begin{aligned}
A &= e_0e_1 \wedge e_1e_2 \wedge e_2e_0 && (\dim 1)\\
B = \theta_{12}(A) &= e_0e_1 \wedge e_1^2 \wedge e_2e_0 && (\dim 6)\\
\theta_{02}(B) &= e_0e_1 \wedge e_1^2 \wedge e_0^2 && (\dim 3)\\
\theta_{10}(B) &= e_0e_1 \wedge e_1^2 \wedge e_1e_2 && (\dim 3)\\
C = \theta_{20}(B) &= e_0e_1 \wedge e_1^2 \wedge e_2^2 + e_1e_2 \wedge e_1^2 \wedge e_2e_0 &&\\
&= e_0e_1 \wedge e_1^2 \wedge e_2^2 + B|_{e_0 \leftrightarrow e_2} && (\dim 6)\\
D = \theta_{01}(C) &= e_0^2 \wedge e_1^2 \wedge e_2^2 + 2e_1e_2 \wedge e_0e_1 \wedge e_2e_0 &&\\
&= e_0^2 \wedge e_1^2 \wedge e_2^2 - 2A && (\dim 1)
\end{aligned}$$

Corollary. $\Lambda^3 S_2 V$ *resp.* $\Lambda^3 S^2 V$ *are as* $\mathrm{sl}(V)$*-modules generated by*

$$[e_0]_2 \wedge [e_1]_2 \wedge [e_2]_2, \qquad e_0e_1 \wedge e_1e_2 \wedge e_2e_0$$

Remark. Clearly the tables describe the isomorphism η in terms of basis elements.

5.3. **Decomposition of $\Lambda^3 S^2 V$.** We conclude with some exercises ($\operatorname{rank} V = 3$).

Lemma. *There is a short exact sequence of* $\mathrm{PGL}(V)$*-modules*

$$0 \to S_3 V \otimes \Lambda^3 V^{\#} \to \Lambda^3 S^2 V \otimes \left(\Lambda^3 V^{\#}\right)^{\otimes 2} \to S^3(V^{\#}) \otimes \Lambda^3 V \to 0$$

This is a "must know" on $\Lambda^3 S^2 V$ ($\operatorname{rank} V = 3$), albeit not needed in this text. It is the integral version of the classical decomposition $\Lambda^3 S^2 V = S^3 V \oplus S^3(V^{\#})$ of $\mathrm{SL}(3)$-modules over $\mathbf{Q}$ and related with classical constructions for plane cubics, like the Hessian curve and the invariants c_4, c_6 of elliptic curves [6, pp. 188].

The joy of proof is left to the reader. The same goes for

Lemma. *Let*

$$J \colon \Lambda^3 S_2 V \to \Lambda^3 S^2 V$$
$$[x]_2 \wedge [y]_2 \wedge [z]_2 \;\mapsto\; x^2 \wedge y^2 \wedge z^2$$

and put

$$T = J \circ \eta^{-1} \in \operatorname{End}_{GL(V)}(S^2 V)$$

Then

$$(T-4)(T+2) = 0$$

6. Computation of Φ

The purpose of the following explicit computations is to verify:

Lemma. *With respect to the basis*

$$\theta_{20},\ -\theta_{21},\ \theta_{10},\ \theta_{12},\ -\theta_{01},\ \theta_{02},\ -\epsilon_1,\ \epsilon_2$$

of $\operatorname{pgl}(V)$*, the morphism* Φ *is given by the matrix in Section 7 (which equals that of* [3, Introduction]*).*

To compute Φ on all basis elements, we apply appropriate elements of the Lie algebra $\operatorname{sl}(V)$. Actually we consider the actions of the universal enveloping algebra. For instance we understand

$$\theta_{21}\theta_{01}(Y) = \theta_{21}\big(\theta_{01}(Y)\big)$$

The action of $\operatorname{sl}(V)$ on $S^2 V$ is given by

$$\theta_{ij}(e_h e_k) = \delta_{jh} e_i e_k + \delta_{jk} e_h e_i$$

and the action of $\operatorname{sl}(V)$ on $\operatorname{pgl}(V)$ is given by commutators.

The brackets $[ijk]$ stand for the Plücker basis with respect to the ordered basis

0	1	2	3	4	5
e_0^2	$e_0 e_1$	$e_2 e_0$	e_1^2	$e_1 e_2$	e_2^2

Here are the computations:

1 element with weights $2,2,2$ of type $xy \wedge xz \wedge yz$

$$\begin{aligned} -[124] = A &= e_0e_1 \wedge e_1e_2 \wedge e_2e_0 \\ \mapsto X &= \epsilon_2 \wedge \epsilon_1 = \epsilon_1 \wedge \epsilon_0 \end{aligned}$$

6 elements with weights $3,2,1$ of type $x^2 \wedge xy \wedge yz$

$$\begin{aligned} -[024] = \theta_{01}(A) &= e_0^2 \wedge e_1e_2 \wedge e_2e_0 \\ \mapsto \theta_{01}(X) &= -\theta_{01} \wedge \epsilon_2 \\ [234] = \theta_{10}(A) &= e_1^2 \wedge e_1e_2 \wedge e_2e_0 \\ \mapsto \theta_{10}(X) &= \theta_{10} \wedge \epsilon_2 \\ -[123] = \theta_{12}(A) &= e_0e_1 \wedge e_1^2 \wedge e_2e_0 \\ \mapsto \theta_{12}(X) &= -\theta_{12} \wedge \epsilon_0 \\ -[125] = \theta_{21}(A) &= e_0e_1 \wedge e_2^2 \wedge e_2e_0 \\ \mapsto \theta_{21}(X) &= \theta_{21} \wedge \epsilon_0 \\ [145] = \theta_{20}(A) &= e_0e_1 \wedge e_1e_2 \wedge e_2^2 \\ \mapsto \theta_{20}(X) &= -\theta_{20} \wedge \epsilon_1 \\ [014] = \theta_{02}(A) &= e_0e_1 \wedge e_1e_2 \wedge e_0^2 \\ \mapsto \theta_{02}(X) &= \theta_{02} \wedge \epsilon_1 \end{aligned}$$

3 elements with weights $3,3,0$ of type $x^2 \wedge xy \wedge y^2$

$$\begin{aligned} -[025] = \theta_{21}\theta_{01}(A) &= e_0^2 \wedge e_2^2 \wedge e_2e_0 \\ \mapsto \theta_{21}\theta_{01}(X) &= \theta_{01} \wedge \theta_{21} \\ [013] = \theta_{02}\theta_{12}(A) &= e_0e_1 \wedge e_1^2 \wedge e_0^2 \\ \mapsto \theta_{02}\theta_{12}(X) &= \theta_{12} \wedge \theta_{02} \\ [345] = \theta_{10}\theta_{20}(A) &= e_1^2 \wedge e_1e_2 \wedge e_2^2 \\ \mapsto \theta_{10}\theta_{20}(X) &= \theta_{20} \wedge \theta_{10} \end{aligned}$$

3 elements with weights $4,1,1$ of type $x^2 \wedge xy \wedge xz$

$$\begin{aligned} [012] = \theta_{02}\theta_{01}(A) &= e_0^2 \wedge e_1e_0 \wedge e_2e_0 \\ \mapsto \theta_{02}\theta_{01}(X) &= \theta_{02} \wedge \theta_{01} \\ [134] = \theta_{10}\theta_{12}(A) &= e_0e_1 \wedge e_1^2 \wedge e_2e_1 \\ \mapsto \theta_{10}\theta_{12}(X) &= \theta_{10} \wedge \theta_{12} \\ [245] = \theta_{21}\theta_{20}(A) &= e_0e_2 \wedge e_1e_2 \wedge e_2^2 \\ \mapsto \theta_{21}\theta_{20}(X) &= \theta_{21} \wedge \theta_{20} \end{aligned}$$

6 elements with weights $3,2,1$ of type $x^2 \wedge xy \wedge z^2$

$$\begin{aligned}
[045] &= \theta_{20}\theta_{01}(A) = e_0^2 \wedge e_1e_2 \wedge e_2^2 \\
&\mapsto \theta_{20}\theta_{01}(X) = -\theta_{20} \wedge \theta_{01} - \theta_{21} \wedge \epsilon_2 \\
[235] &= \theta_{21}\theta_{10}(A) = e_1^2 \wedge e_2^2 \wedge e_2e_0 \\
&\mapsto \theta_{21}\theta_{10}(X) = \theta_{21} \wedge \theta_{10} + \theta_{20} \wedge \epsilon_2 \\
-[023] &= \theta_{01}\theta_{12}(A) = e_0^2 \wedge e_1^2 \wedge e_2e_0 \\
&\mapsto \theta_{01}\theta_{12}(X) = -\theta_{01} \wedge \theta_{12} - \theta_{02} \wedge \epsilon_0 \\
[015] &= \theta_{02}\theta_{21}(A) = e_0e_1 \wedge e_2^2 \wedge e_0^2 \\
&\mapsto \theta_{02}\theta_{21}(X) = \theta_{02} \wedge \theta_{21} + \theta_{01} \wedge \epsilon_0 \\
[135] &= \theta_{12}\theta_{20}(A) = e_0e_1 \wedge e_1^2 \wedge e_2^2 \\
&\mapsto \theta_{12}\theta_{20}(X) = -\theta_{12} \wedge \theta_{20} - \theta_{10} \wedge \epsilon_1 \\
[034] &= \theta_{10}\theta_{02}(A) = e_1^2 \wedge e_1e_2 \wedge e_0^2 \\
&\mapsto \theta_{10}\theta_{02}(X) = \theta_{10} \wedge \theta_{02} + \theta_{12} \wedge \epsilon_1
\end{aligned}$$

1 element with weights $2,2,2$ of type $x^2 \wedge y^2 \wedge z^2$

$$\begin{aligned}
[035] &= \theta_{12}\theta_{20}\theta_{01}(A) = e_0^2 \wedge e_1^2 \wedge e_2^2 \\
&\mapsto \theta_{12}\theta_{20}\theta_{01}(X) = \theta_{01} \wedge \theta_{10} + \theta_{20} \wedge \theta_{02} + \theta_{12} \wedge \theta_{21} \\
&\qquad + \epsilon_2 \wedge \epsilon_1
\end{aligned}$$

7. The alternating 8×8-matrix

	θ_{20}	$-\theta_{21}$	θ_{10}	θ_{12}	$-\theta_{01}$	θ_{02}	$-\epsilon_1$	ϵ_2
θ_{20}	0	[245]	[345]	[135]	[045]	[035]	[145]	[235]
$-\theta_{21}$	−[245]	0	−[235]	[035]	[025]	[015]	[125]	−[125]+[045]
θ_{10}	−[345]	[235]	0	[134]	[035]	[034]	[135]	[234]
θ_{12}	−[135]	−[035]	−[134]	0	[023]	[013]	[123]−[034]	−[123]
$-\theta_{01}$	−[045]	−[025]	−[035]	−[023]	0	[012]	−[015]	−[024]+[015]
θ_{02}	−[035]	−[015]	−[034]	−[013]	−[012]	0	[023]−[014]	−[023]
$-\epsilon_1$	−[145]	−[125]	−[135]	−[123]+[034]	[015]	−[023]+[014]	0	−[124]+[035]
ϵ_2	−[235]	[125]−[045]	−[234]	[123]	[024]−[015]	[023]	[124]−[035]	0

References

[1] A. Cayley, *On the theory of elimination*, Cambridge and Dublin Math. J. **3** (1848), 116–120.
[2] M. Demazure, *Résultant, discriminant*, Enseign. Math. (2) **58** (2012), no. 3-4, 333–373.
[3] D. Eisenbud, F. Schreyer, and J. Weyman, *Resultants and Chow forms via exterior syzygies*, J. Amer. Math. Soc. **16** (2003), no. 3, 537–579.
[4] E. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), no. 2, 209–270.
[5] I. Gelfand, M. Kapranov, and A. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008, Reprint of the 1994 edition.
[6] G. Salmon, *A treatise on the higher plane curves: intended as a sequel to "A treatise of conic sections". Third edition.*, 1879.
[7] A. Suslin, E. Friedlander, and C. Bendel, *Infinitesimal* 1*-parameter subgroups and cohomology*, J. Amer. Math. Soc. **10** (1997), no. 3, 693–728.
[8] J. Sylvester, *On the principles of the calculus of forms*, The Collected Mathematical Papers. Volume I: (1837–1853), 1853, pp. 284–327.
[9] The Stacks Project Authors, *Stacks project*, ⟨http://stacks.math.columbia.edu⟩.

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: rost *at* math.uni-bielefeld.de
URL: www.math.uni-bielefeld.de/~rost