THE REPRESENTATION RING OF THE STRUCTURE GROUP OF THE RELATIVE FROBENIUS MORPHISM

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Introduction

Let k be a field of prime characteristic p. For a smooth k-variety X of dimension n, the r-th relative Frobenius morphism

$$F_X^r : X \to X^{(r)}$$

is an fppf-fiber bundle with fibers

$$R(n,r) := k[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r})$$

over k. The aim of this thesis is to study the automorphism group $G(n,r)$ of this fiber which will be considered as an algebraic group or group scheme over k. One can associate to each representation of $G(n,r)$ a natural vector bundle over the r-th Frobenius twist of X by twisting the $G(n,r)$-torsor F_X^r. That is, by computing the representation ring of $G(n,r)$ one obtains a description of these natural bundles. This topic is based on a correspondence between Markus Rost and Pierre Deligne where Deligne suggested to study this representation ring for $r = n = 1$. In particular, he gave a computation in this case. This thesis generalizes this computation and concentrates on the computation of this representation ring for arbitrary r and n.

The Lie algebra of $G(n,r)$ computes as $\mathrm{Der}_k(R(n,r))$, the endoderivations of $R(n,r)$. That is, for $r = 1$, this algebraic group is of *Cartan type* as its Lie algebra is isomorphic to the Jacobson-Witt algebra $W(n,(1,\ldots,1))$. The aim is to provide a parametrization and computation of all irreducible $G(n,r)$-representations. In fact, for $r = 1$, this can be deduced from the description of the irreducible p-representations of $W(n,(1,\ldots,1))$ which is given in [Nak92]. We will apply this to compute the representation ring of $G(n,r)$ since the classes of the irreducible representations provide a $\mathbb{Z}$-basis.

The parametrization works as follows: The action of GL_n on the generators $x_1, \ldots, x_n \in R(n,r)$ provides a subgroup G^0 of $G(n,r)$ which is isomorphic to GL_n. Moreover, there are two subgroups G^-, G^+ such that the multiplication map

$$m : G^+ \times G^0 \times G^- \to G(n,r)$$

is an isomorphism of k-schemes. In analogy to the theory for reductive groups, the subgroup G^0 plays the role of a maximal torus, and $G^- \rtimes G^0$ as well $G^+ \rtimes G^0$ the roles of Borel subgroups. That is, the irreducible representations of $G(n,r)$ are parametrized by the irreducible G^0-representation: On one hand, the G^--invariants of an irreducible $G(n,r)$-representation are an irreducible G^0-representation. On the other hand each irreducible G^0-representation uniquely arises in this way: There is an exact functor

$$\mathrm{I} : G^0\text{−rep} \to G(n,r)\text{−rep}$$

with the property that for an irreducible G^0-representation L the socle of $\mathrm{I}(L)$ is irreducible and its G^--invariants are isomorphic to L.

The computation works as follows: As $G^0 \cong \mathrm{GL}_n$, its irreducible representations are parametrized by the dominant weights. Further we want to study the dominant weights mod p. That is, we study their mod p-residues. If this residue is a fundamental weight, the associated irreducible $G(n,r)$-representation arises as an image of a differential map in a twist of the deRham-complex $\Omega_r^\bullet$ where $\Omega_r^i := \Omega^i_{R(n,r),k}$ are the Kähler-differentials. If the residue is 0 and $r = 1$, the associated irreducible $G(n,1)$-representation arises as a pullback of an irreducible $(G^0)^{(1)}$-representation along the group homomorphism

$$L_1 : G(n,1) \to (G^0)^{(1)}$$

This is induced as follows: Take the twist of the $G(n,1)$-representation $\Omega_{R(n,1),k}$ by the p-th power map $(-)^p : R(n,1) \to k$ which induces L_1. If the residue is 0 and $r \geq 2$ the associated irreducible $G(n,r)$-representation arises as a pullback of an irreducible $G(n,r-1)^{(1)}$-representation along the group homomorphism

$$T_r : G(n,r) \to G(n,r-1)^{(1)}$$

This is induced as follows: We restrict an $R(n,r)$-automorphism to the subalgebra generated by $x_1^p, \dots, x_n^p$ which is isomorphic to $R(n,r-1)^{(1)}$. If the residue is neither 0 nor a fundamental weight and $p \neq 2$, the socle of $\mathrm{I}(L)$ coincides with $\mathrm{I}(L)$. The computations for this generalize the computations of [Nak92].

Finally, we conclude with the computation of the representation ring for $p \neq 2$: For $r = 1$, the functor I and the group homomorphism L_1 induce a surjective map

$$\mathrm{Rep}(\mathrm{GL}_n) \oplus \mathrm{Rep}(\mathrm{GL}_n^{(1)}) \xrightarrow{\mathrm{I}+L_1^*} \mathrm{Rep}(G(n,1))$$

We also compute the kernel. For $r \geq 2$, the functor I and the group homomorphism T_r provide a surjection

$$\mathrm{Rep}(\mathrm{GL}_n) \oplus \mathrm{Rep}(G(n,r-1)^{(1)}) \xrightarrow{\mathrm{I}+T_r^*} \mathrm{Rep}(G(n,r))$$

which establishes a recursive description. We compute the kernel of this map indirectly. In all cases, the proof of the surjectivity involves Cartier's Theorem about the cohomology of the deRham-complex, namely for $r = 1$

$$H^i(\Omega_1^\bullet) \cong \Lambda^i L_1$$

where we consider L_1 as a representation. Furthermore, for $r \geq 2$

$$H^i(\Omega_r^\bullet) \cong T_r^*((\Omega_{r-1}^i)^{(1)})$$

Organization

In Section 1 we will introduce the language for algebraic groups. In particular, we will describe their representations, Lie algebras, as well as Frobenius twists and Frobenius morphisms.

In Section 2 we will introduce the algebraic group $G(n,r)$ which we are going to study in this thesis. In particular, we will compute its Lie algebra and describe important subgroups.

In Section 3 we will introduce the concept of triangulated groups and triangulated morphisms. This will be the key theory in order to obtain the parametrization of the irreducible $G(n,r)$-representations by those of its subgroup G^0 which is isomorphic to GL_n.

In Section 4 we will outline the parametrization of irreducible representations of reductive groups by dominant weights. In particular, we will compute the representation ring of GL_n and prepare the computation of the representation ring of $G(n,r)$.

In Section 5 we will extend the theory for triangulated groups to those of r-triangulated groups. This will allow us to obtain a mod p^r-periodicity for the computation of the irreducible $G(n,r)$-representations as well as a reduction to the r-th Frobenius kernel of $G(n,r)$. For $r = 1$, this will provide a computation for the irreducible $G(n,1)$-representations by the computation of the irreducible p-representations of the Jacobson-Witt algebra $W(n,(1,\ldots,1))$.

In Section 6 we will introduce several transfer morphisms between the $G(n,r)$, GL_n, and their Frobenius twists respectively. These will be heavily used in the computation of the irreducible $G(n,r)$-representations.

In Section 7 we will introduce Kähler-differentials as an important example of $G(n,r)$-representations. These fit into the deRham-complex whose cohomology is computed by Cartier's Theorem. Furthermore we will need to generalize to twisted deRham-complexes.

In Section 8 we will compute the irreducible $G(n,r)$-representations with respect to their associated dominant weights of GL_n by using transfer homomorphisms, twisted deRham-complexes, as well as an extensive computation.

In Section 9 we will provide a computation of the representation ring of $G(n,r)$ by using the preparation of section 4, the computation of the irreducible $G(n,r)$-representations, and Cartier's Theorem.

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1. Basic Notions and Results for Algebraic Groups

The aim of this section is to introduce the language we are using for algebraic groups as well as some basic results. This is taken from [DG80] and [Jan03].

We consider arbitrary fields k of prime characteristic p. Denote by $k-$Alg the category of commutative k-Algebras. Then a *k-group functor G* is a functor

$$G : k-\mathrm{Alg} \longrightarrow \mathrm{Groups}$$

For a $A \in k-$Alg the group $G(A)$ are the *A-rational points of G*. Now an *affine k-group* is a k-group functor G which is represented by a k-algebra $k[G]$.

Remark 1.1. Note that for an affine k-group G the k-algebra $k[G]$ carries the structure of a commutative Hopf algebra. That is, there is a *comultiplication*

$$\Delta_G : k[G] \to k[G] \otimes_k k[G]$$

a *coinverse*

$$\sigma_G : k[G] \to k[G]$$

and a *counit*

$$\epsilon_G : k[G] \to k$$

These three maps uniquely determine the group structure of $G(A)$ for all $A \in k-$Alg by the Yoneda-Lemma. Confer also [Jan03, I.2.3].

Note that by the Yoneda-Lemma a morphism of affine k-groups

$$f : G \to H$$

corresponds uniquely to a morphism of Hopf algebras

$$f^{\#} : k[H] \to k[G]$$

Definition 1.2. An *algebraic k-group* is an affine k-group G such that $k[G]$ is a finitely presented k-algebra.

We are going to give some of the main examples.

Example 1.3. The *additive group* $\mathbb{G}_a$ defined by

$$\mathbb{G}_a(A) = (A, +)$$

is an algebraic k-group with Hopf algebra

$$k[\mathbb{G}_a] = k[X]$$

the polynomial ring in one variable. The comultiplication of $k[X]$ is given by $\Delta(X) = X \otimes 1 + 1 \otimes X$, the coinverse by $\sigma(X) = -X$ and the counit by $\epsilon(X) = 0$.

More generally, let V be a finite dimensional k-vector space. Then set

$$\mathbb{G}_a(V) := \underline{\mathrm{Hom}}_k(V, k)$$

That is,

$$\mathbb{G}_a(V)(A) := \mathrm{Hom}_A(V \otimes_k A, A) \cong \mathrm{Hom}_k(V, A)$$

The group structure is induced by $(A, +)$. This is an algebraic k-group with representing Hopf algebra

$$k[\mathbb{G}_a(V)] = S^\bullet V$$

the symmetric algebra of V.

Note that by a choice of a basis of V with $n = \dim(V)$, we get

$$\mathbb{G}_a(V) \cong (\mathbb{G}_a)^n$$

and

$$k[\mathbb{G}_a(V)] \cong k[X_1, \dots, X_n]$$

the polynomial ring in n variables.

Example 1.4. The *multiplicative group* $\mathbb{G}_m$ defined by

$$\mathbb{G}_m(A) = (A^\times, \cdot)$$

is an algebraic k-group with Hopf algebra

$$k[\mathbb{G}_m] = k[X, X^{-1}]$$

the Laurent polynomial ring in one variable. The comultiplication is given by $\Delta(X) = X \otimes X$, the coinverse by $\sigma(X) = X^{-1}$ and the counit by $\epsilon(X) = 1$.

More generally, the *general linear group* $\mathrm{GL}(V)$ for a finite dimensional k-vector space V is an algebraic k-group: We start by defining the k-semigroup functor $\underline{\mathrm{End}}_k(V)$ by

$$\underline{\mathrm{End}}_k(V)(A) = \mathrm{End}_A(V \otimes_k A)$$

It is represented by the k-algebra

$$k[\underline{\mathrm{End}}_k(V)] = S^\bullet(\mathrm{End}(V)^\vee)$$

the symmetric algebra of the dual space of $\mathrm{End}(V)$. Now $\mathrm{GL}(V) \subset \mathrm{End}(V)$ is just defined by

$$\mathrm{GL}(V)(A) = \mathrm{GL}_A(V \otimes_k A) \subset \underline{\mathrm{End}}_k(V)(A)$$

The representing Hopf algebra is

$$k[\mathrm{GL}(V)] = k[\underline{\mathrm{End}}_k(V)][\det{}^{-1}]$$

where the element det is understood as follows: The determinant defines a morphism

$$\underline{\mathrm{End}}_k(V) \xrightarrow{\det} (\mathbb{A}^1, \cdot)$$

of k-semigroup functors which corresponds to a k-algebra morphism

$$k[X] \to k[\underline{\mathrm{End}}_k(V)]$$

Hence it uniquely defines an element $\det \in k[\underline{\mathrm{End}}_k(V)]$ by the image of X. Now the comultiplication is induced by the composition

$$\mathrm{End}(V) \otimes_k \mathrm{End}(V) \xrightarrow{\circ} \mathrm{End}(V)$$

The coinverse is induced by the inverse map

$$\mathrm{End}(V)^\times \xrightarrow{(-)^{-1}} \mathrm{End}(V)^\times$$

and the counit is induced by the inclusion

$$k \to \mathrm{End}(V)$$

which maps 1 to id_V.

Note that by a choice of a basis of V with $n = \dim(V)$, we get

$$\mathrm{GL}(V)(A) = \mathrm{GL}_n(A)$$

the invertible $n \times n$-matrices over A. Then the Hopf algebra reads as

$$k[\mathrm{GL}(V)] = k[\mathrm{GL}_n] = k[a_{ij}]_{1\le i,j\le n}[\det^{-1}]$$

where det is given by the Leibniz formula. Note also that

$$\mathrm{GL}_1 = \mathbb{G}_m$$

1.1. **Representations.**

Definition 1.5. A *(linear) representation V of G* is a finite dimensional k-vector space V together with a morphism of algebraic k-groups

$$G \to \mathrm{GL}(V)$$

Note that a G-representation is nothing else than a natural linear action of $G(A)$ on each $V \otimes_k A$.

Remark 1.6. A G-representation V corresponds uniquely to a Hopf algebra map

$$S^\bullet(\mathrm{End}(V)^\vee) \to k[G]$$

which corresponds uniquely to a $k[G]$-*comodule map*

$$\Delta_V : V \to V \otimes_k k[G]$$

Confer [Jan03, I.2.8]. A morphism between G-representations V and W is a k-vector space map $f : W \to V$ such that the diagram

$$\begin{tikzcd} W \arrow[r, "\Delta_W"] \arrow[d, "f"'] & W \otimes_k k[G] \arrow[d, "f\otimes \mathrm{id}_{k[G]}"] \\ V \arrow[r, "\Delta_V"] & V \otimes_k k[G] \end{tikzcd}$$

commutes.

Definition 1.7. We call a G-representation V *irreducible* if $V \neq 0$ and for a subrepresentation $U \subset V$ we get $U = 0$ or V.

Some authors like Jantzen call these representations *simple*.

Remark 1.8. Note that for the category of finite dimensional G-representations the *Jordan-Hölder Theorem* holds: For all finite dimensional representations V there is a finite composition series

$$0 = W_1 \subset W_2 \subset \ldots \subset W_{n-1} \subset W_n = V$$

That is, all quotients W_{i+1}/W_i are irreducible. Further for two composition series of V the multiplicities of an irreducible representation L in the composition series coincide.

Now we can introduce some important notions. The first are fixed points.

Definition 1.9. Let V be a G-representation. Then the *fixed points* are given by

$$\begin{aligned} V^G &:= \{v \in V \mid g(v \otimes 1) = v \otimes 1 \ \forall g \in G(A) \ \forall A \in k\text{−Alg}\} \\ &= \{v \in V \mid \Delta_V(v) = v \otimes 1\} \end{aligned}$$

Another important notion are *weight spaces*. For this, we introduce the character group.

Definition 1.10. The *character group of an algebraic group* G is defined by

$$X(G) = \mathrm{Hom}(G, \mathbb{G}_m)$$

where Hom means morphisms of algebraic groups.

Remark 1.11. Since $\mathbb{G}_m \subset \mathbb{G}_a = \mathbb{A}^1$ as k-varieties, we get an embedding

$$X(G) \subset \mathrm{Mor}(G, \mathbb{G}_a) = \mathrm{Hom}_{k-\mathrm{Alg}}(k[X], k[G]) \cong k[G]$$

where Mor means morphisms of k-varieties. We obtain an isomorphism

$$X(G) \cong \{f \in k[G] \mid f(1) = 1, \ \Delta_G(f) = f \otimes f\}$$

Confer [Jan03, I.2.4]. Since we work over a field k, the elements of $X(G)$ are linearly independent by [DG80, II§1,2.9].

Now we come to the promised definition of weight spaces.

Definition 1.12. Let G be an algebraic group and $\lambda \in X(G)$ a character. Then the *λ-th weight space of a G-representation* V is defined by

$$\begin{aligned} V_\lambda &:= \{v \in V \mid g(v \otimes 1) = v \otimes \lambda(g) \ \forall g \in G(A) \ \forall A \in k\text{−Alg}\} \\ &= \{v \in V \mid \Delta_V(v) = v \otimes \lambda\} \end{aligned}$$

The elements of V_λ are also called the *vectors of weight* λ.

Remark 1.13. By the linear independence of the characters $X(G)$, we get that the sum of the V_λ is direct and

$$\bigoplus_{\lambda \in X(G)} V_\lambda \subset V$$

In general, this inclusion does not have to be an equality. But it is an equality for tori $T = (\mathbb{G}_m)^n$ where we get

$$X(T) = \mathbb{Z}^n$$

Finally, we can describe subrepresentations generated by subspaces: Let V be a G-representation and $W \subset V$ a k-subspace. Then denote by

$$GW \subset V$$

the smallest subrepresentation of V which contains W, the *subrepresentation generated by* W.

Further, let

$$\Delta_V : V \to V \otimes_k k[G]$$

be the $k[G]$-comodule map which corresponds to V. Then a k-subspace $W \subset V$ is G-invariant, that is, a subrepresentation, if and only if $\Delta_V|_W$ factors through $W \otimes_k k[G]$.

Now let us choose a $k[G]$-basis $(a_i)_{i\in I}$ and write

$$\Delta = \sum_{i\in I} \Delta_i a_i$$

Then $W \subset V$ is G-invariant if and only if

$$\Delta_i(W) \subset W$$

for all $i \in I$. Furthermore the subrepresentation generated by W is just

$$GW = \sum_{i\in I} \Delta_i(W)$$

We will study subrepresentations and subrepresentations generated by subspaces of G with this criteria in mind.

1.2. **Lie Algebras.** An important tool to study algebraic groups are their *Lie algebras*. For this, consider the *dual numbers*

$$k[\epsilon] := k[T]/T^2$$

where $\epsilon = \overline{T}$. Hence $\epsilon^2 = 0$. Denote by $p : k[\epsilon] \to k$ the k-algebra projection which maps ϵ to 0. Note that

$$k[\epsilon]^\times = \{\lambda 1 + \mu\epsilon \mid \lambda \in k^\times,\ \mu \in k\}$$

This follows by $(\lambda 1 + \mu\epsilon)^{-1} = (\lambda^{-1}1 - \lambda^{-2}\mu\epsilon)$ which shows that $\lambda \in k^\times$ is sufficient for an element $\lambda 1 + \mu\epsilon$ to be invertible in $k[\epsilon]$. But it is also necessary because of the k-algebra homomorphism p.

Definition 1.14. Let G be an algebraic k-group. Then the *Lie algebra of* G is the tangent space at the unit element $1 \in G$. That is,

$$\mathrm{Lie}(G) := p_*^{-1}(\epsilon_G)$$

where

$$p_* : \mathrm{Hom}_{k-\mathrm{Alg}}(k[G], k[\epsilon]) \to \mathrm{Hom}_{k-\mathrm{Alg}}(k[G], k)$$

and ϵ_G is the counit of $k[G]$. For the Lie brackets confer [DG80, II.4].

Remark 1.15. A map $f \in \mathrm{Lie}(G)$ can uniquely be written as

$$\begin{array}{rcl} k[G] & \xrightarrow{f} & k[\epsilon] \\ x & \mapsto & \epsilon_G(x)1 + d(x)\epsilon \end{array}$$

which defines a map $d : k[G] \to k$.

The description in the Remark allows us to define a bijective map

$$\begin{array}{rcl} \mathrm{Lie}(G) & \to & \mathrm{Der}_k(k[G], k) \\ f & \mapsto & d \end{array}$$

where we consider k as a $k[G]$-algebra via the counit ϵ_G. This is a well defined map since f is a k-algebra map if and only if d is a derivation.

These two descriptions of the Lie algebra are helpfull for computations but it is not that easy to introduce the Lie brackets. Later, we will see how we can explicitly compute the Lie algebra including its brackets for closed subgroups of general linear groups.

Example 1.16. For the additive group $\mathbb{G}_a$, we see that

$$\begin{aligned}\mathrm{Lie}(\mathbb{G}_a) &= \{f : k[X] \to k[\epsilon] \mid f(X) = d(X)\epsilon\} \\ &= \{f : k[X] \to k[\epsilon] \mid f(X) = \lambda\epsilon,\ \lambda \in k\}\end{aligned}$$

This translates to

$$\mathrm{Lie}(\mathbb{G}_a) \cong \mathrm{Der}_k(k[X], k) = k\left(\frac{\partial}{\partial X}\Big|_{X=0}\right)$$

which is a 1-dimensional k-vector space.

More general, for a finite dimensional k-vector space V and the group $\mathbb{G}_a(V)$, we get

$$\begin{aligned}\mathrm{Lie}(\mathbb{G}_a(V)) &= \{f : V \to k[\epsilon] \mid f(v) = d(v)\epsilon,\ f\ k\text{-linear}\} \\ &= \mathrm{Hom}_k(V, k) \\ &= V^\vee\end{aligned}$$

The identification

$$\mathrm{Hom}_k(V, k) \xrightarrow{\cong} \mathrm{Der}_k(S^\bullet V, k)$$

is just the extension as k-derivations.

If we choose a basis of V and work with the Hopf algebra $k[X_1, \ldots, X_n]$, we can consider the derivations

$$\delta_i := \frac{\partial}{\partial X_i}\Big|_{X_i=0}$$

These are in fact a k-basis of the Lie algebra. That is,

$$\mathrm{Lie}((\mathbb{G}_a)^n) = \mathrm{Der}_k(k[X_1, \ldots, X_n], k) = \bigoplus_{i=1}^{n} k\delta_i$$

Example 1.17. For the multiplicative group $\mathbb{G}_m$, we see that

$$\begin{aligned}\mathrm{Lie}(\mathbb{G}_m) &= \{f : k[X, X^{-1}] \to k[\epsilon] \mid f(X) = 1 + d(X)\epsilon\} \\ &= \{f : k[X, X^{-1}] \to k[\epsilon] \mid f(X) = 1 + \lambda\epsilon,\ \lambda \in k\})\end{aligned}$$

This translates to

$$\mathrm{Lie}(\mathbb{G}_m) \cong \mathrm{Der}_k(k[X, X^{-1}], k) = k\left(\frac{\partial}{\partial X}\Big|_{X=1}\right)$$

which is a 1-dimensional k-vector space.

More general, for the general linear group $\mathrm{GL}(V)$ we see that

$$\begin{aligned} & \mathrm{Lie}(\mathrm{GL}(V)) \\ = \ & \{f : \mathrm{End}(V)^\vee \to k[\epsilon] \mid f(x) = x(\mathrm{id}_V) + d(x)\epsilon,\ f\ k\text{-linear}\} \\ \cong \ & \{d : \mathrm{End}(V)^\vee \to k \mid d\ k\text{-linear}\} \\ = \ & (\mathrm{End}(V)^\vee)^\vee \\ \cong \ & \mathrm{End}(V)\end{aligned}$$

with the Hopf algebra $k[\mathrm{GL}(V)] = S^\bullet(\mathrm{End}(V)^\vee)[\det^{-1}]$. Note that we used that V is finite dimensional. The Lie algebra structure on $\mathrm{Lie}(G)$ corresponds to the usual one on $\mathrm{End}(V)$.

If we choose a basis and work with the Hopf algebra $k[a_{ij}][\det^{-1}]$, we can consider the derivations

$$k[a_{ij}][\det^{-1}] \xrightarrow{\frac{\partial}{\partial a_{rs}}} k[a_{ij}][\det^{-1}] \xrightarrow{a_{ij}\mapsto\delta_{ij}} k$$

for all pairs $1 \leq r, s \leq n$ where δ_{ij} is the Kronecker-δ. These are in fact a basis of the Lie algebra. That is,

$$\mathrm{Lie}(\mathrm{GL}_n) = \mathrm{Der}_k(k[a_{ij}][\det^{-1}], k) = \bigoplus_{r,s} k \left(\frac{\partial}{\partial a_{rs}}\Big|_{a_{ij}=\delta_{ij}} \right)$$

Definition 1.18. A *representation of the Lie algebra* $\mathrm{Lie}(G)$ is a k-vector space V together with a Lie algebra morphism

$$\mathrm{Lie}(G) \to \mathrm{End}(V)$$

If we have a morphism $f : G \to H$ of algebraic k-groups, there is an induced map

$$\mathrm{Lie}(f) : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$$

of Lie algebras. In both the dual numbers and the derivation description, it is given by precomposition with $f^\# : k[H] \to k[G]$. Note that for a closed immersion f, the induced map $\mathrm{Lie}(f)$ is injective as $f^\#$ is surjective.

For a G-representation V we obtain a Lie algebra representation

$$\mathrm{Lie}(G) \to \mathrm{End}(V)$$

This can be computed as follows: Let $f \in \mathrm{Der}_k(k[G], k)$. Then the image in $\mathrm{End}(V)$ is the composition

$$V \xrightarrow{\Delta_V} V \otimes_k k[G] \xrightarrow{\mathrm{id}_V \otimes f} V \otimes_k k \cong V$$

Now let G be a closed subgroup of the general linear group $\mathrm{GL}(V)$. Further assume, that we have an explicit description of $\mathrm{Lie}(G)$ as derivations. Then we can in fact compute the Lie algebra of G by the inclusion

$$\mathrm{Lie}(G) \hookrightarrow \mathrm{End}(V)$$

since the image is computed by the method we just described. This provides a computation of $L(G)$ as a Lie subalgebra of $\mathrm{End}(V)$ including the brackets.

Here comes an important class of examples of algebraic k-groups and its Lie algebras.

Notation 1.19. Let $R \in k{-}\mathrm{Alg}$ be finite dimensional. Denote by $\underline{\mathrm{Aut}}(R)$ the k-group functor

$$\underline{\mathrm{Aut}}(R)(A) := \mathrm{Aut}_A(R \otimes_k A)$$

of algebra automorphisms.

Note that

$$\underline{\mathrm{Aut}}(R) \subset \mathrm{GL}(R)$$

is a closed algebraic k-group and hence an algebraic k-group. This provides

$$\mathrm{Lie}(\underline{\mathrm{Aut}}(R)) \subset \mathrm{End}(R)$$

as a Lie subalgebra. The next Proposition follows from [DG80, II§4,2.3 Proposition] and computes this subset.

Proposition 1.20. *Let $R \in k-\mathrm{alg}$ be finite, then*

$$\mathrm{Lie}(\underline{\mathrm{Aut}}(R)) = \mathrm{Der}_k(R) \subset \mathrm{End}(R)$$

As we are in prime characteristic p, the Lie algebras of algebraic k-groups carry the additional structure of a *p-Lie algebra* (also called *restricted Lie algebra*). That is, there is a p-th power operation

$$\begin{array}{rcl} \mathrm{Lie}(G) & \to & \mathrm{Lie}(G) \\ x & \mapsto & x^{[p]} \end{array}$$

satisfying certain axioms (cf. [DG80, II§7,2.1,2.2,3.3]).

Example 1.21. For the general linear group $\mathrm{GL}(V)$ the Lie algebra is $\mathrm{End}(V)$ and the operation $x \mapsto x^{[p]}$ is given by the usual p-th power of endomorphisms.

There is also the obvious notion of *p-Lie algebra representations* of $\mathrm{Lie}(G)$: These are k-vector spaces V together with a p-Lie algebra morphism

$$\mathrm{Lie}(G) \to \mathrm{End}(V)$$

Further any morphism of algebraic k-groups $f : G \to H$ induces a morphism of p-Lie algebras

$$\mathrm{Lie}(f) : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$$

That is, any G-representation V induces a p-$\mathrm{Lie}(G)$-representation V. In the case that G is a closed subgroup of the general linear group $\mathrm{GL}(V)$ we get an inclusion $\mathrm{Lie}(G) \subset \mathrm{End}(V)$ of p-Lie algebras. Hence the operation $x \mapsto x^{[p]}$ on $\mathrm{Lie}(G) \subset \mathrm{End}(V)$ is also just the usual p-th power of endomorphisms.

1.3. **Quotients.** Confer [Jan03, I.6, I.7] for the definition of images of algebraic k-group homomorphisms and of quotients G/H for an algebraic k-group inclusion $H \subset G$. Note that in general,

$$G/H(A) \neq G(A)/H(A)$$

in contrast to kernels of group morphisms $f : G \to H$ which satisfy

$$\mathrm{Ker}(f)(A) = \ker(f(A)) \subset G(A)$$

If $N \subset G$ is a normal algebraic k-subgroup, then the quotient G/N is an algebraic k-group according to [Jan03, I.6.5(1)] as we are working over a field k. Denote the projection as $\pi : G \to G/N$. It has the universal property of a factor group. Further, by [Jan03, I.6.3] the functor

$$\pi^* : G/N-\mathrm{rep} \longrightarrow G-\mathrm{rep}$$

is fully faithful and its image consists of those G-representations on which N acts trivially. So, this subcategory is equivalent to $G/N-\mathrm{rep}$ under π^*.

Now the kernel of a morphism $f : G \to H$ of algebraic k-groups is a normal closed algebraic k-subgroup of G by [Jan03, I.2.1]. So, the quotient $G/\mathrm{Ker}(f)$ is an algebraic k-group. First we have the following Lemma.

Lemma 1.22. *Let $f : G \to H$ be a morphism of algebraic k-groups. Then f induces a closed immersion*

$$G/\mathrm{Ker}(f) \hookrightarrow H$$

which is given by the kernel of the corresponding morphism

$$f^{\#} : k[H] \to k[G]$$

of k-algebras.

Proof. The quotient $G/\operatorname{Ker}(f)$ is an algebraic k-group as $\operatorname{Ker}(f)$ is a normal subgroup. By [DG80, II§5,5.1b], the embedding

$$f : G/\operatorname{Ker}(f) \hookrightarrow H$$

is a closed immersion and $G/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$. So let $J \subset k[H]$ be the defining ideal. We get a factorization

$$k[H] \to k[H]/J \to k[G]$$

of $f^{\#}$. Now let $I = \operatorname{Ker}(f^{\#})$ and $C \subset H$ the closed subscheme defined by I. Then $J \subset I$ and we also get a factorization

$$k[H] \to k[H]/J \to k[H]/I \to k[G]$$

This provides

$$G \xrightarrow{f} C \subset \operatorname{Im}(f) \subset H$$

Hence $C = \operatorname{Im}(f)$ and $I = J$ as claimed. □

Now for the case that $G/\operatorname{Ker}(f) \cong H$, we get the following for representations.

Lemma 1.23. *Let $f : G \to H$ be a morphism of algebraic k-groups, such that it induces an isomorphism*

$$G/\operatorname{Ker}(f) \xrightarrow{\cong} H$$

Then the functor

$$f^* : H{-}\mathrm{rep} \longrightarrow G{-}\mathrm{rep}$$

maps irreducible representations to irreducible representations.

Proof. We can replace f by the projection

$$\pi : G \to G/\operatorname{Ker}(f)$$

So let V be an irreducible $G/\operatorname{Ker}(f)$-representation and $0 \neq W \subset \pi^*(V)$ a G-subrepresentation. Then $\operatorname{Ker}(f)$ acts trivially on W as it does on π^*V. That is, there is an induced $G/\operatorname{Ker}(f)$-representation on W, which we denote by W'. That is, $\pi^*W' = W$. Recall that the functor π^* is an equivalence of categories between $G/N{-}\mathrm{rep}$ and its image. As $W \subset \pi^*V$ in the image of π^*, we obtain that W' is a subrepresentation of V. By the irreducibility of V, we get $W = V$ as k-vector spaces which shows the irreducibility of π^*V. □

1.4. **The Frobenius Morphisms.** As we are working over a field k of prime characteristic p, there is the important notion of Frobenius morphisms. For this, confer [Jan03, I.9]. Note that Jantzen works with perfect fields for convenience. As we work over arbitrary fields, we have to give the general constructions.

Definition 1.24. Let G be an algebraic k-group. Set the *r-th Frobenius twist of G* as the affine k-scheme $G^{(r)}$ which is represented by the algebra $k[G] \otimes_{k,f^r} k$. Here $f^r : k \to k$ is the p^r-th power morphism.

Notation 1.25. For an $A \in k-\mathrm{Alg}$ and $r \in \mathbb{N}$ denote by $A^{(-r)}$ the k-algebra

$$k \xrightarrow{f^r} k \to A$$

and the natural k-algebra morphism

$$\begin{array}{ccc} A & \xrightarrow{\gamma_r} & A^{(-r)} \\ a & \mapsto & a^{p^r} \end{array}$$

which is nothing else than the r-th power of the Frobenius morphism. Further denote

$$A^{(r)} = A \otimes_{k,f^r} k$$

Also for a finite dimensional k-vector space denote

$$V^{(r)} := V \otimes_{k,f^r} k$$

We get for all $A \in k-\mathrm{Alg}$

$$G^{(r)}(A) = \mathrm{Hom}_k(k[G] \otimes_{k,f^r} k, A) \cong \mathrm{Hom}_k(k[G], A^{(-r)}) = G(A^{(-r)})$$

This provides a natural structure of an algebraic k-group for $G^{(r)}$ as $A \mapsto A^{(-r)}$ is functorial.

Definition 1.26. Set the *r-th Frobenius morphism* $F_G^r : G \to G^{(r)}$ to be

$$F_G^r(A) := G(\gamma_r) : G(A) \to G(A^{(-r)}) \cong G^{(r)}(A)$$

using the identification we just made.

By definition, F^r is a group homomorphism. Further, we get

$$F^r_{G^{(s)}} \circ F^s_G = F^{r+s}_G$$

Example 1.27. For a finite dimensional k-vector space V and the additive group $\mathbb{G}_a(V)$, we get a canonical isomorphism

$$\mathbb{G}_a(V)^{(r)} \cong \mathbb{G}_a(V^{(r)})$$

This follows by

$$\mathbb{G}_a(V^{(r)})(A) = \mathrm{Hom}_k(V^{(r)}, A) = \mathrm{Hom}_k(V, A^{(-r)}) \cong \mathbb{G}_a(V)^{(r)}$$

The r-th Frobenius morphism then translates to

$$F^r_{\mathbb{G}_a(V)} : \mathbb{G}_a(V) \to \mathbb{G}_a(V^{(r)})$$

which is induced by $\gamma_r : A \to A^{(-r)}$.

Example 1.28. For the general linear group $\mathrm{GL}(V)$ we get a canonical isomorphism

$$\mathrm{GL}(V)^{(r)} \cong \mathrm{GL}(V^{(r)})$$

This follows by

$$\mathrm{GL}(V^{(r)})(A) = \mathrm{GL}_A(V^{(r)} \otimes_k A) = \mathrm{GL}_{A^{(-r)}}(V \otimes_k A^{(-r)}) \cong \mathrm{GL}(V)^{(r)}(A)$$

as $V^{(r)} \otimes_k A = V \otimes_k A^{(-r)}$. The r-th Frobenius morphism translates to the canonical $\mathrm{GL}(V)$-representation $V^{(r)}$:

$$F^r_{\mathrm{GL}(V)} : \mathrm{GL}(V) \to \mathrm{GL}(V^{(r)})$$

Example 1.29. Let R be a finite dimensional k-algebra. Then we get a canonical isomorphism

$$\underline{\mathrm{Aut}}(R)^{(r)} \cong \underline{\mathrm{Aut}}(R^{(r)})$$

This follows by

$$\underline{\mathrm{Aut}}(R^{(r)})(A) = \mathrm{Aut}_A(R^{(r)} \otimes_k A) = \mathrm{Aut}_{A^{-r}}(R \otimes_k A^{(-r)}) \cong \underline{\mathrm{Aut}}(R)^{(r)}(A)$$

as $R^{(r)} \otimes_k A \cong R \otimes_k A^{(-r)}$. The r-th Frobenius morphism translates to the morphism

$$F^r_{\underline{\mathrm{Aut}}(R)} : \underline{\mathrm{Aut}}(R) \to \underline{\mathrm{Aut}}(R^{(r)})$$

which is induced by $(-) \otimes_{A,\gamma_r} A^{(-r)}$.

Remark 1.30. The morphism F^r_G corresponds to the morphism

$$\begin{array}{ccc} k[G] \otimes_{k,f^r} k & \xrightarrow{(F^r_G)^\#} & k[G] \\ a \otimes \lambda & \mapsto & a^{p^r}\lambda \end{array}$$

of Hopf-algebras: The universal element of the morphism $G(\gamma_r)$ is

$$\left(\gamma_r : k[G] \to k[G]^{(-r)}\right) \in G(k[G]^{(-r)})$$

Under the isomorphism $G(k[G]^{(-r)}) \cong G^{(r)}(k[G])$ this is mapped to the map above as claimed.

The morphism F^r_G is often called the *geometric* Frobenius morphism. It is a morphism over k. Further the r-th power of the absolute one $f^r : G \to G$ which corresponds to the p^r-th power map on $k[G]$ factors through F^r_G: The composition

$$k[G] \xrightarrow{a \mapsto a \otimes 1} k[G] \otimes_{k,f^r} k \xrightarrow{(F^r_G)^\#} k[G]$$

coincides with f^r. The first map corresponds to a morphism $G^{(r)} \to G$ which is called the *arithmetic* Frobenius morphism as it is the p^r-th power map on k.

Remark 1.31. Let G be an algebraic k-group which is *defined over* $\mathbb{F}_p$. That is, there is an algebraic $\mathbb{F}_p$-group $G_{\mathbb{F}_p}$ such that $G = (G_{\mathbb{F}_p})_k$. Then

$$k[G] = \mathbb{F}_p[G_{\mathbb{F}_p}] \otimes_{\mathbb{F}_p} k$$

and

$$k[G^{(r)}] = (\mathbb{F}_p[G_{\mathbb{F}_p}] \otimes_{\mathbb{F}_p} k) \otimes_{k,f^r} k \cong \mathbb{F}_p[G_{\mathbb{F}_p}] \otimes_{\mathbb{F}_p} k = k[G]$$

as k-algebras. This uses the fact that the diagram

$$\begin{tikzcd} \mathbb{F}_p \arrow[r, "f^r=\mathrm{id}"] \arrow[d] & \mathbb{F}_p \arrow[d] \\ k \arrow[r, "f^r"] & k \end{tikzcd}$$

commutes. That is, we get a canonical isomorphism $G^{(r)} \cong G$ as algebraic k-groups. Then the r-th Frobenius morphism can be identified with a group homomorphism

$$F_G^r : G \to G$$

which corresponds to the k-algebra map

$$\begin{array}{ccc} \mathbb{F}_p[G_{\mathbb{F}_p}] \otimes_{\mathbb{F}_p} k & \xrightarrow{(F_G^r)^\#} & \mathbb{F}_p[G_{\mathbb{F}_p}] \otimes_{\mathbb{F}_p} k \\ a \otimes \lambda & \mapsto & a^{p^r} \otimes \lambda \end{array}$$

Example 1.32. The additive group $\mathbb{G}_a$ is defined over $\mathbb{F}_p$ and the r-th Frobenius morphism $F_{\mathbb{G}_a}^r : \mathbb{G}_a \to \mathbb{G}_a$ corresponds to the k-algebra map

$$k[X] \xrightarrow{X \mapsto X^{p^r}} k[X]$$

The general linear group GL_n is defined over $\mathbb{F}_p$ and $F_{\mathrm{GL}_n}^r : \mathrm{GL}_n \to \mathrm{GL}_n$ corresponds to the k-algebra map

$$k[a_{ij}]_{1\le i,j\le n}[\det^{-1}] \xrightarrow{a_{ij} \mapsto a_{ij}^{p^r}} k[a_{ij}]_{1\le i,j\le n}[\det^{-1}]$$

Notation 1.33. Denote the kernel of the group homomorphism $F_G^r : G \to G^{(r)}$ by G_r, the *r-th Frobenius kernel of G*.

Note that for all $r, s \geq 1$, we obtain

$$(G^{(s)})_r \cong (G_r)^{(s)}$$

by the very definition of the twists and kernels.

Example 1.34. The r-th Frobenius kernel of the additive group $\mathbb{G}_a$ is given by the Hopf algebra

$$k[X]/X^{p^r}$$

Further

$$\mathbb{G}_{a,r}(A) = (A_r, +)$$

where

$$A_r = \mathrm{Ker}(A \xrightarrow{f^r} A)$$

More arbitrary, for a finite dimensional k-vector space V, the r-th Frobenius kernel of the additive group $\mathbb{G}_a(V)$ is just

$$\mathbb{G}_a(V)_r(A) = \mathrm{Hom}_k(V, A_r)$$

Further, the Hopf algebra is given by

$$S^\bullet V/\langle V^{(r)}\rangle$$

where $V^{(r)}$ is identified with the set of all $v^{p^r} \in S^\bullet V$ for $v \in V$.

By choosing a basis of V, we get the Hopf algebra

$$k[X_1, \ldots, X_n]/(X_1^{p^r}, \ldots, X_n^{p^r})$$

of $(\mathbb{G}_a)_r^n = (\mathbb{G}_{a,r})^n$. In fact, this will be the k-algebra whose automorphism group we are going to study in this thesis.

Remark 1.35. The r-th Frobenius kernel is a closed algebraic k-subgroup defined by the following ideal: Let $\epsilon_G : k[G] \to k$ be the counit which corresponds to $1 \in G$ and let $I_1 = \mathrm{Ker}(\epsilon_G)$. Then G_r is defined by the ideal of $k[G]$ generated by all f^{p^r} for $f \in I_1$. Further we see that G_r is *infinitesimal*: It is finite and $I_1 = \mathrm{Ker}(\epsilon_{G_r})$ in $k[G_r]$ is nilpotent.

Due to the factorization

$$F^s_{G^{(r)}} \circ F^r_G = F^{r+s}_G$$

we get a chain of closed k-subgroups

$$G_1 \subset G_2 \subset G_3 \subset \cdots \subset G$$

Moreover for F^r_G restricted to G_{r+s} this provides a factorization of $F^r_{G_{r+s}}$

$$F^r_{G_{r+s}} : G_{r+s} \xrightarrow{F^r_G} (G^{(r)})_s \hookrightarrow (G^{(r)})_{r+s} \cong (G_{r+s})^{(r)}$$

Further the inclusion $G_r \subset G$ induces an isomorphism

$$\mathrm{Lie}(G_r) \cong \mathrm{Lie}(G)$$

of p-Lie algebras as

$$\mathrm{Der}_k(k[G_r], k) \cong \mathrm{Der}_k(k[G], k)$$

by the Remark above. Moreover $F^r_G : G \to G^{(r)}$ induces

$$\mathrm{Lie}(F^r_G) = 0 : \mathrm{Lie}(G) \to \mathrm{Lie}(G^{(r)})$$

as

$$\mathrm{Der}_k(k[G], k) \xrightarrow{((F^r_G)^\#)^*} \mathrm{Der}_k(k[G^{(r)}], k)$$

equals 0. That is, the Lie algebra is not effected by the Frobenius morphisms. This is the reason why in prime characteristic, the canonical functor

$$G\text{−rep} \to \mathrm{Lie}(G)\text{−rep}$$

in general is not an equivalence of categories. But the functor

$$G_1\text{−rep} \to \mathrm{Lie}(G)\text{−}p\text{−rep}$$

always is according to [Jan03, I.9.6]. So whenever G equals its first Frobenius kernel, the first functor is an equivalence of categories between G−rep and the subcategory of p-Lie algebra representations of $\mathrm{Lie}(G)$.

The next Proposition and its proof is essentially [Jan03, I.9.5]. But as Jantzen works with perfect fields, we need our own general version.

Proposition 1.36. *Let G be a reduced algebraic k-group. Then the r-th Frobenius morphism $F^r_G : G \to G^{(r)}$ induces an isomorphism*

$$G/G_r \xrightarrow{\cong} G^{(r)}$$

and for all $s \geq 1$

$$G_{r+s}/G_r \cong (G^{(r)})_s$$

Proof. According to Lemma 1.22, the embedding

$$F_G^r : G/G_r \hookrightarrow G^{(r)}$$

is a closed immersion and it is given by the ideal which is the kernel of

$$k[G] \otimes_{k,f^r} k \xrightarrow{(F_G^r)^\#} k[G]$$

This morphism acts as $(F_G^r)^\#(a \otimes \lambda) = a^{p^r}\lambda$. As $k[G]$ is a reduced k-algebra, the kernel is 0. That is, we get an isomorphism

$$F_G^r : G/G_r \xrightarrow{\cong} G^{(r)}$$

induced by F^r as claimed.

Now we consider the subgroup $(G^{(r)})_s \subset G^{(r)}$. As we know that F_G^r is an epimorphism, we get that F_G^r induces an epimorphism

$$F_G^r : (F_G^r)^{-1}((G^{(r)})_s) \to (G^{(r)})_s$$

But due to the factorization $F_{G^{(r)}}^s \circ F_G^r = F_G^{r+s}$, we get

$$G_{r+s} = (F_G^r)^{-1}((G^{(r)})_s)$$

So, we get an epimorphism

$$F_G^r : G_{r+s} \to (G^{(r)})_s)$$

and hence an isomorphism

$$G_{r+s}/G_r \cong (G^{(r)})_s$$

as claimed. □

If G is defined over $\mathbb{F}_p$ and reduced, we obtain an isomorphism

$$G/G_r \xrightarrow{\cong} G$$

induced by $F_G^r : G \to G$ and

$$G_{r+s}/G_r \cong G_s$$

Notation 1.37. Let G be an algebraic k-group and V a $G^{(r)}$-representation. Then we denote by

$$V^{[r]} := (F_G^r)^*(V)$$

the *r-th Frobenius twist of V*.

If G is defined over $\mathbb{F}_p$, the r-th Frobenius twist provides an endofunctor

$$G\text{−rep} \xrightarrow{V \mapsto V^{[r]}} G\text{−rep}$$

Corollary 1.38. *Let G be a reduced algebraic k-group and V an irreducible $G^{(r)}$-representation. Then the r-th Frobenius twist $V^{[r]}$ is also irreducible.*

Proof. By the previous Proposition, the r-th Frobenius morphism $F_G^r : G \to G^{(r)}$ induces an isomorphism

$$G/G_r \xrightarrow{\cong} G^{(r)}$$

As $V^{[r]} = (F_G^r)^*(V)$, we get the claim by Lemma 1.23. □

2. Basics About the Algebraic Group $G(n,r)$

Let k be a field of prime characteristic p. Let us denote

$$U = k^n$$

Further let again $U^{(r)} = U \otimes_{k,f^r} k$ where $f^r : k \to k$ is the r-th power of the Frobenius morphism. Then we can consider the k-linear map

$$\begin{array}{rcl} U^{(r)} & \to & S^{p^r}U \\ u \otimes 1 & \mapsto & u^{p^r} \end{array}$$

where $S^{p^r}U$ is the p^r-th symmetric power of U. This is an injective map and we can introduce the k-algebra

$$R(n,r) := S^{\bullet}U/\langle U^{(r)}\rangle$$

the quotient of the symmetric algebra of U by the ideal generated by the image of the map above. If we choose a basis of U, say the canonical one, we obtain

$$R(n,r) = k[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$$

a truncated polynomial ring.

Notation 2.1. For $A \in k-\mathrm{Alg}$, we set

$$R(n,r)_A := R(n,r) \otimes_k A$$

Note that under the identification $R(n,r) = k[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$ we obtain

$$R(n,r)_A = A[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$$

We will formulate most of the results coordinate-free. But we will always explain the concrete meaning under the polynomial ring identification. Also, for convenience, we will use this identification for some proofs.

Remark 2.2. Note that the k-algebra $R(n,r)$ is $\mathbb{Z}$-graded since the symmetric algebra

$$S^{\bullet}U = \bigoplus_{i\geq 0} S^iU$$

is $\mathbb{Z}$-graded and the ideal $\langle U^{(r)}\rangle$ is homogeneous. Under the identification

$$R(n,r) = k[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$$

we get

$$(R(n,r))^i = \{P(x_1,\dots,x_n) \in R(n,r) \mid \deg(P) = i\}$$

Now we can introduce the algebraic group $G(n,r)$.

Definition 2.3. Define the algebraic group $G(n,r) = \underline{\mathrm{Aut}}_k(R(n,r))$ over k by

$$G(n,r)(A) := \mathrm{Aut}_A(R(n,r)_A)$$

the group of A-algebra automorphisms of $R(n,r)_A$ for all $A \in k-\mathrm{Alg}$.

Remark 2.4. There is a canonical closed embedding

$$G(n,r) \subset \mathrm{GL}(R(n,r))$$

since each A-algebra automorphism is also an A-module automorphism. This explains why $G(n,r)$ is an algebraic k-group.

Note that

$$(U \otimes_k A)^{(r)} = (U \otimes_k A) \otimes_{A,f^r} A = (U \otimes_{k,f^r} k) \otimes_k A = U^{(r)} \otimes_k A$$

since the Frobenius on A commutes with the one on k. This implies

$$R(n,r)_A = (S^\bullet_k(U)/\langle U^{(r)} \rangle) \otimes_k A = S^\bullet_A(U \otimes_k A)/\langle (U \otimes_k A)^{(r)} \rangle$$

In particular, an A-algebra automorphism of $(S^\bullet U/\langle U^{(p)}\rangle) \otimes_k A$ is uniquely determined by an A-linear map

$$U \otimes_k A \to (S^\bullet U/\langle U^{(p)}\rangle) \otimes_k A$$

which in turn is uniquely determined by a k-linear map

$$U \to (S^\bullet U/\langle U^{(p)}\rangle) \otimes_k A$$

That is,

$$G(n,r) \subset \underline{\mathrm{Hom}}_k(U, R(n,r))$$

where $\underline{\mathrm{Hom}}_k(U, R(n,r))$ is the set-valued functor

$$\begin{array}{rcl} k{-}\mathrm{Alg} & \to & \mathrm{Set} \\ A & \mapsto & \mathrm{Hom}_A(U \otimes_k A, R(n,r) \otimes A) = \mathrm{Hom}_k(U, R(n,r)_A) \end{array}$$

Remark 2.5. If we identify $R(n,r) = k[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r})$ this just says that an A-algebra endomorphism of

$$R(n,r)_A = A[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r})$$

is determined by the images of the variables $x_1, \ldots, x_n$, that is, by n elements $(f_1, \ldots, f_n)$ with $f_i \in R(n,r)_A$. So we will identify

$$f = (f_1, \ldots, f_n)$$

for $f \in G(n,r)$ with $f_i = f(x_i)$.

2.1. **Two Conditions.** We already noticed that

$$G(n,r) = \underline{\mathrm{Aut}}_k(R(n,r)) \subset \underline{\mathrm{Hom}}_k(U, R(n,r))$$

Our aim is to give two conditions which will determine when an element of the right hand side is contained in the left hand side.

In order to do this, we introduce the evaluation at 0.

Definition 2.6. Consider the k-algebra morphism

$$\mathrm{ev}_0 : R(n,r) \to k$$

which is induced by $0 : U \to k$. This induces a natural transformation

$$\begin{array}{rcl} \underline{\mathrm{Hom}}_k(U, R(n,r)) & \to & \mathbb{G}_a(U) = \underline{\mathrm{Hom}}_k(U,k) \\ f & \mapsto & \mathrm{ev}_0 \circ f =: f(0) \end{array}$$

called *the evaluation at* 0.

Remark 2.7. As $k \hookrightarrow R(n,r)$, we can also consider

$$f(0) \in \underline{\mathrm{Hom}}_k(U, R(n,r))$$

Remark 2.8. Under the identification $R(n,r) = k[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$ the morphism

$$\mathrm{ev}_0 : R(n,r) \to k$$

is just $\mathrm{ev}_0(P(x_1,\dots,x_n)) = P(0,\dots,0)$ for a polynomial $P \in R(n,r)$. That is, the natural transformation acts as

$$(f_1,\dots,f_n) \mapsto (f_1(0),\dots,f_n(0))$$

for polynomials $f_i \in R(n,r)_A$.

Now we can give a criterion when a k-linear map

$$f : U \to R(n,r)_A$$

induces an endomorphism

$$f : R(n,r)_A \to R(n,r)_A$$

Namely, this is equivalent to

$$f(U)^{p^r} \subset \langle (U \otimes_k A)^{(r)} \rangle$$

But as

$$(f - f(0))(U) \subset \langle U \otimes_k A \rangle$$

this is equivalent to

$$(f(0)(U))^{p^r} = 0$$

In other words,

$$f(0) \in \mathrm{Hom}_k(U, A_r) = \mathbb{G}_a(U)_r(A)$$

Remark 2.9. Under the identification $R(n,r) = k[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$ this means that n-polynomials $(f_1,\dots,f_n)$ with $f_i \in R(n,r)_A$ define an A-algebra endomorphism if and only if

$$f_i(0)^{p^r} = 0$$

for all $i = 1,\dots,n$.

It is left to determine when an endomorphism is an automorphism. For this we consider the graded k-algebra

$$\mathrm{gr}_I(R(n,r)_A) = (R(n,r)_A)/I \oplus I/I^2 \oplus I^2/I^3 \oplus \dots$$

by the ideal $I = \langle U \otimes_k A \rangle$. The direct sum is finite since $I^{n(p^r-1)+1} = 0$. Furthermore we see that

$$I^i/I^{i+1} \cong (R(n,r)_A)^i$$

and we get an isomorphism of $\mathbb{Z}$-graded A-algebras

$$\mathrm{gr}_I(R(n,r)_A) \cong R(n,r)_A$$

For an endomorphism $f : R(n,r)_A \to R(n,r)_A$ with $f(I) \subset I$, we get an induced endomorphism

$$\mathrm{gr}\, f : \mathrm{gr}_I\, R(n,r)_A \to \mathrm{gr}_I\, R(n,r)_A$$

which is in fact induced by a linear map

$$f_0 : U \otimes_k A \to U \otimes_k A$$

and gives again an endomorphism of $R(n,r)_A$ by the isomorphism. Note that $\mathrm{gr}\, f$ is invertible as a morphism of A-algebras if and only if f_0 is invertible as an A-linear map.

Remark 2.10. Under the identification

$$R(n,r)_A = A[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$$

the ideal I is just $(x_1,\dots,x_n)R(n,r)_A$. Then an endomorphism f with $f(I) \subset I$ is given by n polynomials $(f_1,\dots,f_n)$ with $f_i \in (x_1,\dots,x_n)R(n,r)_A$. We obtain

$$(\operatorname{gr} f)(x_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(0)x_j$$

which is just the degree 1 part of f_i. That is, the map $f \mapsto \operatorname{gr} f$ cuts off the higher degree terms of the defining polynomials. Furthermore the linear map $f_0 : A^n \to A^n$ is given by

$$J_f := \left(\frac{\partial f_j}{\partial x_i}(0)\right)_{ij} \in M_n(A)$$

the *Jacobian matrix.*

The linear map $f_0 \in \operatorname{End}_A(U \otimes_k A)$ in fact determines when f is an automorphism. This is made precise by the algebraic *Inverse Function Theorem.* Its proof is mainly inspired by [Eis95, Chapter 7.6].

Proposition 2.11. *Let $A \in k-\mathrm{Alg}$ and*

$$f : R(n,r)_A \to R(n,r)_A$$

an A-algebra morphism with the property that

$$f(U \otimes_k A) \subset \langle U \otimes_k A\rangle$$

Then f is an isomorphism if and only if

$$f_0 : U \otimes_k A \to U \otimes_k A$$

is an isomorphism of A-modules.

Proof. As we already noticed, f_0 is invertible if and only if $\operatorname{gr} f$ is invertible. So let f be invertible. We get $f(I) = I$ and hence $f^{-1}(I) \subset I$. This gives us $\operatorname{gr}(f^{-1})$ which is inverse to $\operatorname{gr}(f)$ as gr acts functorially.

Now let $\operatorname{gr} f$ be bijective. Let us start by showing the surjectivity of f. Let $y \in R(n,r)_A$ and i maximal such that $y \in I^i$. As $\operatorname{gr} f$ is surjective, there is an $a_1 \in I^i$ such that

$$y - f(a_1) \equiv 0 \mod I^{i+1}$$

That is, $y - f(a_1) \in I^{i+1}$. Again by the surjectivity of $\operatorname{gr} f$ there is an $a_2 \in I^{i+1}$ such that

$$y - f(a_1) - f(a_2) \equiv 0 \mod I^{i+2}$$

Now continue this process. As the filtration is finite, this process terminates and produces elements $a_1,\dots,a_N \in R(n,r)_A$ with

$$f\left(\sum_{j=1}^{N} a_j\right) = \sum_{j=1}^{N} f(a_j) = y$$

which shows the surjectivity of f.

For the injectivity of f let $0 \neq a \in R(n,r)_A$. Further let $0 \neq \mathrm{in}(a) \in (R(n,r)_A)^l = I^l/I^{l+1}$ be the homogenous part of a of lowest degree, the *initial term.* Then

$$(\mathrm{gr}\, f)(\mathrm{in}(a)) \neq 0$$

since $\mathrm{gr}\, f$ is injective. But as

$$a \equiv \mathrm{in}(a) \mod I^{l+1}$$

we get

$$f(a) \equiv (\mathrm{gr}\, f)(\mathrm{in}(a)) \mod I^{l+1}$$

which is not 0. This shows the injectivity of f and finishes the proof. □

Remark 2.12. Under the identification

$$R(n,r)_A = A[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r})$$

this means that an endomorphism f given by n polynomials $(f_1, \ldots, f_n)$ with $f_i \in (x_1, \ldots, x_n)R(n,r)_A$ is invertible if and only if its Jacobian matrix $J_f \in M_n(A)$ is invertible.

If we have an arbitrary morphism $f : R(n,r)_A \to R(n,r)_A$, then $f - f(0)$ satisfies the condition

$$(f - f(0))(I) \subset I$$

Notation 2.13. Denote for $f : R(n,r)_A \to R(n,r)_A$

$$f_0 := (f - f(0))_0$$

which extends the notation to arbitrary morphisms.

Remark 2.14. Under the identification

$$R(n,r)_A = A[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r})$$

the map f_0 for arbitrary f is still given by the Jacobian matrix J_f as this does not depend on the part $f(0)$.

As f is invertible if and only if $f - f(0)$ is, we get the following Corollary.

Corollary 2.15. *Let*

$$f : R(n,r)_A \to R(n,r)_A$$

be an endomorphism of A-algebras. Then f is invertible if and only if the A-module map

$$f_0 : U \otimes_k A \to U \otimes_k A$$

is invertible.

So we obtain a diagram of natural transformations

$$\begin{tikzcd}
\underline{\mathrm{End}}_{k-\mathrm{Alg}}(R(n,r)) \arrow[rr, "f \mapsto \mathrm{gr}(f - f(0))"] \arrow[dr, "f \mapsto f_0"'] & & \underline{\mathrm{End}}_{k-\mathrm{Alg}}(R(n,r)) \\
& \underline{\mathrm{End}}_k(U) \arrow[ur, hook] &
\end{tikzcd}$$

where the transformations reflect isomorphisms.

Notation 2.16. Let $f : U \otimes_k A \to R(n,r)_A$ be A-linear. Then the composition

$$U \otimes_k A \xrightarrow{f} R(n,r)_A \xrightarrow{\pi} R(n,r)^1_A = U \otimes_k A$$

is induced by a morphism $f_0 \in \mathrm{End}_A(U \otimes_k A)$. Here π is the projection onto the first part of the $\mathbb{Z}$-grading.

Note that in the case, that f comes from $\mathrm{End}_A(R(n,r)_A)$, this notation coincides with the old one.

Finally, we reached our aim and we can conclude

Proposition 2.17. *The algebraic group $G(n,r)$ identifies with the closed subfunctor*

$$\{f \in \underline{\mathrm{Hom}}_k(U, R(n,r)) \mid f(0) \in \mathbb{G}_a(U)_r, f_0 \in \mathrm{GL}(U)\} \subset \underline{\mathrm{Hom}}_k(U, R(n,r))$$

Remark 2.18. Under the identification

$$R(n,r)_A = A[x_1, \dots, x_n]/(x_1^{p^r}, \dots, x_n^{p^r})$$

we obtain

$$G(n,r)(A) = \{f = (f_1, \dots, f_n) \mid \forall i : f_i(0)^{p^r} = 0,\ J_f \in \mathrm{GL}_n(A)\}$$

This Proposition shows in particular, that $G(n,r)$ is defined over $\mathbb{F}_p$ as all involved functors $\underline{\mathrm{Hom}}_k(U, R(n,r))$, $\mathbb{G}_a(U)$, $\mathrm{GL}(U)$ as well as the maps $f \mapsto f(0)$, $f \mapsto f_0$ are.

2.2. **Important Subgroups.** Our next aim is to introduce three crucial closed subgroups of $G(n,r)$ where we will make heavy use of Proposition 2.17.

Let us start by the observation, that we have an inclusion of algebraic k-groups

$$\mathrm{GL}(U) \subset G(n,r)$$

which is induced by

$$\underline{\mathrm{End}}_k(U) \hookrightarrow \underline{\mathrm{End}}_{k-\mathrm{Alg}}(R(n,r))$$

Definition 2.19. Define the subgroup $G^0 = G(n,r)^0$ by

$$G^0 := \mathrm{GL}(U) \subset G(n,r)$$

Remark 2.20. Under the identification

$$R(n,r)_A = A[x_1, \dots, x_n]/(x_1^{p^r}, \dots, x_n^{p^r})$$

and $\mathrm{GL}(U) = \mathrm{GL}_n$, the subgroup

$$G^0 = \mathrm{GL}_n \subset G(n,r)$$

is given as follows: For a matrix $(a_{ij})_{ij} \in \mathrm{GL}_n$ we assign the element $g = (g_1, \dots, g_n) \in G(n,r)$ by

$$g_i = \sum_{j=1}^{n} a_{ji} x_j$$

That is, the matrix acts on the generators $x_1, \dots, x_n$ as it does on the basis $e_1, \dots, e_n$ of $k^n = U$.

Now we define the subgroup G^-.

Definition 2.21. Define the subgroup $G^- = G(n,r)^-$ by the image of the group homomorphism

$$\begin{array}{rcl} \mathbb{G}_a(U)_r & \hookrightarrow & G(n,r) \\ f & \mapsto & f + \mathrm{id} \end{array}$$

Remark 2.22. The inclusion $G^- \hookrightarrow G(n,r)$ is well defined as $(f+\mathrm{id})(0) = f$ and $(f+\mathrm{id})_0 = \mathrm{id}$ for $f \in \mathbb{G}_a(U)_r$. Further

$$G^- = \{f \in G(n,r) \mid f = f(0) + \mathrm{id}\}$$

As $G^- \cong \mathbb{G}_a(U)_r$, we get

$$k[G^-] \cong k[\mathbb{G}_a(U)_r] = S^\bullet U/\langle U^{(r)}\rangle = R(n,r)$$

This will be of importance for later computations.

Remark 2.23. Under the identification

$$R(n,r) = k[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$$

the subgroup

$$G^- \subset G(n,r)$$

is given by the elements $g = (g_1,\dots,g_n) \in G(n,r)$ with

$$g_i = a_i + x_i$$

where $a_i^{p^r} = 0$. The isomorphism $G^- \cong \mathbb{G}_a(U)_r \cong (\mathbb{G}_{a,r})^n$ can explicitly be described as

$$\begin{array}{rcl} G^- & \xrightarrow{\cong} & \mathbb{G}_{a,r} \times \ldots \times \mathbb{G}_{a,r} \\ (a_1 + x_1,\dots,a_n + x_n) & \mapsto & (a_1,\dots,a_n) \end{array}$$

Also note that G^- is *unipotent*.

The next subgroup is in some sense complementary to G^0 and G^-.

Definition 2.24. Define the subgroup $G^+ = G(n,r)^+$ by

$$G^+ = \{f \in G(n,r) \mid f(0) = 0, f_0 = \mathrm{id}\}$$

Remark 2.25. Note that G^+ is the image of

$$\begin{array}{rcl} \underline{\mathrm{Hom}}_k(U, R(n,r)^{\geq 2}) & \hookrightarrow & G(n,r) \\ f & \mapsto & \mathrm{id} + f \end{array}$$

This shows that G^+ is closed under multiplication. In order to show that it is closed under taking inverses, consider $f \in G^+$, that is, $f(0) = 0$ and $f_0 = \mathrm{id}$. Now let $g = f^{-1} \in G(n,r)$ the inverse. Write $g = g(0) + g'$. That is, $g'(0) = 0$. Then by

$$f \circ (g(0) + g') = g(0) + f \circ g' = \mathrm{id}$$

with $(f \circ g')(0) = 0$ we get $g(0) = 0$. Now we have

$$\mathrm{id} = (f \circ g)_0 = f_0 \circ g_0 = g_0$$

as $f(0) = 0 = g(0)$ which finally shows that $g \in G^+$.

Further, we get

$$k[G^+] \cong S^\bullet \,\mathrm{Hom}_k(U, R(n,r)^{\geq 2})^\vee$$

which shows that G^+ is an affine space $\mathbb{A}^N$ with

$$N = \dim_k(U) \cdot \dim_k(R(n,r)^{\geq 2}) = n(np^r - n - 1) = n(n(p^r - 1) - 1)$$

Remark 2.26. Under the identification

$$R(n,r) = k[x_1, \dots, x_n]/(x_1^{p^r}, \dots, x_n^{p^r})$$

the subgroup

$$G^+ \subset G(n,r)$$

is given by the elements $g = (g_1, \dots, g_n) \in G(n,r)$ with

$$g_i = x_i + \sum_{I, \deg(I) \geq 2} a_I x^I$$

where $I \in \{0, \dots, p^r - 1\}^n$ is a multi index and the degree map is just summing up

$$\deg : \{0, \dots, p^r - 1\}^n \xrightarrow{\Sigma} \mathbb{N}$$

Affine directions can be seen as follows: Take $i \in \{1, \dots, n\}$ and a multi index I with $\deg(I) \geq 2$. Then we can define $g = (g_1 \dots, g_n) \in G(n,r)(k[a])$ by

$$g_j = \begin{cases} x_i + ax^I & j = i \\ x_j & j \neq i \end{cases}$$

This follows from the identification

$$k[G^+] = k[a_{(i,I)}]_{i \in \{1,\dots,n\}, I \in \{0,\dots,p^r-1\}^n, \deg(I) \geq 2}$$

where the index i corresponds the the basis element $e_i \in U$ and the multi index I to the basis element $x^I \in R(n,r)^{\geq 2}$.

Also note that G^+ is *unipotent*.

Now we will proof the crucial Lemma which shows that the three subgroups G^-, G^0 and G^+ are complementary by the multiplication map. Later, we will call such a structure a *(pre)triangulation of* $G(n,r)$.

Lemma 2.27. *The multiplication map*

$$m : G^+ \times G^0 \times G^- \to G(n,r)$$

is an isomorphisms of k-functors.

Proof. Let us define a map

$$\begin{array}{rcl} G(n,r) & \xrightarrow{g} & G^+ \times G^0 \times G^- \\ f & \mapsto & ((f - f(0)) \circ f_0^{-1}, f_0, f(0) + \mathrm{id}) \end{array}$$

which is inverse to m: First, we have

$$(f - f(0)) \circ f_0^{-1} \circ f_0 \circ (f(0) + \mathrm{id}) = (f - f(0)) \circ (f(0) + \mathrm{id}) = f$$

which shows that $m \circ g = \mathrm{id}$. Further for $\mathrm{id} + h \in G^+$, $g \in G^0$, and $i + \mathrm{id} \in G^-$ we obtain

$$f = (\mathrm{id} + h) \circ g \circ (i + \mathrm{id}) = i + g + hg$$

That is, $f(0) = i$, $f_0 = g$, and $(f - f(0)) \circ f_0^{-1} = \mathrm{id} + h$ which shows that $g \circ m = \mathrm{id}$. $\square$

Notation 2.28. For $f \in G(n,r)$ denote the unique preimage of m as

$$(f_+, f_0, f_-) \in G^+ \times G^0 \times G^-$$

That is, $f = f_+ f_0 f_-$.

Remark 2.29. Note that

$$f_- = f(0) + \mathrm{id}$$

Definition 2.30. Let us denote by $\overline{G}^-$ and $\overline{G}^+$ the closed subgroups of G given by

$$\overline{G}^- := \{f \in G(n,r) \mid f(0) + f_0 = f\}$$

and

$$\overline{G}^+ := \{f \in G(n,r) \mid f(0) = 0\}$$

Remark 2.31. Note that

$$\overline{G}^- = G^- \rtimes G^0 \quad \text{and} \quad \overline{G}^+ = G^+ \rtimes G^0$$

where G^0 acts by conjugation on G^-, G^+ respectively.

Further we get that G^- equals its r-th Frobenius kernel as it can be identified with $(\mathbb{G}_{a,r})^n$. Thus by use of the multiplication isomorphism

$$m : G^+ \times G^0 \times G^- \to G(n,r)$$

we can ask if the subfunctors

$$U_i = U_i(n,r) := m(G^+ \times G^0 \times G_i^-) \subset G(n,r)$$

for $1 \le i \le r$ are subgroups. In fact, they are which is proven by the next Lemma. But note first that by using Proposition 2.17, we obtain

$$U_i = \{f \in G(n,r) \mid f(0) \in \mathbb{G}_a(U)_i\}$$

as $G_i^- \cong \mathbb{G}_a(U)_i$.

Remark 2.32. Under the identification

$$R(n,r) = k[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r})$$

and $g = (g_1, \ldots, g_n)$ with $g_i \in R(n,r)$, we get $G_i^- \cong (\mathbb{G}_{a,i})^n$ and hence

$$U_i = \{g \in G(n,r) \mid g_j(0)^{p^i} = 0 \ \forall j = 1, \ldots, n\}$$

Lemma 2.33. *The subfunctors*

$$U_i \subset G(n,r)$$

are algebraic k-subgroups.

Proof. We have to show that U_i is closed under multiplication and taking inverses.

For this, we will use the following rule:

Claim. Let $h_- \in G_i^-$ and $g \in U_i$, then $(h_- g)_- \in G_i^-$.

For convenience, we will show this using the identification made above: As $G^- \cong (\mathbb{G}_{a,r})^n$ by $g \mapsto (g_1(0), \dots, g_n(0))$, this is equivalent to

$$(h_- g)_j(0)^{p^i} = 0$$

for all $j = 1 \dots, n$. So let $h_- = (a_1 + x_1, \dots, a_n + x_n)$ and $g = (g_1, \dots, g_n)$. Then

$$(h_- g)_j(0) = g_j(a_1 + x_1, \dots, a_n + x_n)(0) = g_j(a_1, \dots, a_n)$$

whose p^i-th power vanishes as $a_j^{p^i} = 0$ and $g_j(0)^{p^i} = 0$ as $g \in U_i$. This shows the claim.

So let $h = h_+ h_0 h_-, g = g_+ g_0 g_- \in U_i$. That is, $h_-, g_- \in G_i^-$. Then

$$(hg)_- = (h_- g_+ g_0)_- g_- \in G_i^-$$

by the claim above. This shows that U_i is closed under multiplication.

Now let $g = g_+ g_0 g_- \in U_i$. That is, $g_- \in G_i^-$ and also $g_-^{-1} \in G_i^-$, as $G_i^- \subset G(n,r)$ is a subgroup. Then

$$((g_+ g_0 g_-)^{-1})_- = (g_-^{-1} g_0^{-1} g_+^{-1})_- \in G_i^-$$

by the claim above. This shows that U_i is closed under taking inverses. □

2.3. **Weight Spaces.** If we turn to representations $G(n,r) \to \mathrm{GL}(V)$ of the group $G(n,r)$, we obtain a simple weight space filtration as follows: The multiplicative group $\mathbb{G}_m$ is contained in $\mathrm{GL}(U) = G^0$ by scalar operations. Thus we can associate to each G^0-representation V a $\mathbb{G}_m$-representation V by restriction. Then we take the comodule map

$$\phi : V \to V \otimes_k k[X, X^{-1}]$$

and write

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n X^n$$

with $\phi_n \in \mathrm{End}(V)$. Note that $X(\mathbb{G}_m) \cong \mathbb{Z}$ where we associate to each $n \in \mathbb{Z}$ the group homomorphism $(-)^n : \mathbb{G}_m \to \mathbb{G}_m$. This corresponds to the Hopf algebra map $X \mapsto X^n$. That is, by setting $V_n := \phi_n(V)$, we get the usual weight-space filtration

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

The equality holds since $\phi_n^2 = \phi_n$ and $\phi_i \circ \phi_j = 0$ for $i \neq j$.

Remark 2.34. Note that for a G^0-representation V the weight space filtration

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

is G^0-invariant. That is, for all $n \in \mathbb{Z}$, we get that $V_n \subset V$ is a G^0-subrepresentation. This follows by the description

$$V_n = \{v \in V \mid a(v) = a^n v \ \forall a \in \mathbb{G}_m\}$$

and the fact that all elements of $G^0 = \mathrm{GL}(U)$ commute with the ones of $\mathbb{G}_m \subset G^0$.

Example 2.35. For the canonical representation $G(n,r) \subset \mathrm{GL}(R(n,r))$, we obtain

$$R(n,r) = \bigoplus_{i \geq 0} R(n,r)_i$$

where

$$R(n,r)_i = R(n,r)^i$$

That is, the weight filtration coincides with the $\mathbb{Z}$-grading.

2.4. **The Lie Algebra.** As we already know, $G(n,r)$ is a closed subgroup of $\mathrm{GL}(R(n,r))$. According to Proposition 1.20, we get

$$\mathrm{Lie}(G(n,r)) = \mathrm{Der}_k(R(n,r)) \subset \mathrm{End}_k(R(n,r))$$

Example 2.36. For $r = 1$, we get as p-Lie algebras

$$\mathrm{Lie}(G(n,1)) = \mathrm{Der}_k(R(n,1)) \cong W(n,(1,\dots,1))$$

the Jacobson-Witt algebra, a Lie algebra of *Cartan type*. Confer [SF88, 3.5.9,4.2.1]. But note that for $r > 1$, we have

$$\mathrm{Lie}(G(n,r)) = \mathrm{Der}_k(R(n,r)) \neq W(n,(r,\dots,r))$$

even as Lie algebras. This follows as $\mathrm{Lie}(G(n,r))$ carries the structure of a p-Lie algebra (or is restrictable) but $W(n,(r,\dots,r))$ does not by [SF88, 4.2.4(2)].

Now note that for any k-vector space V a k-derivation $S^\bullet V/\langle V^{(r)}\rangle \to A$ is uniquely determined by a k-linear map $V \to A$ as we are in prime characteristic p. Denote this correspondence by

$$\begin{array}{ccc} \mathrm{Hom}_k(V,A) & \xrightarrow{\cong} & \mathrm{Der}_k(S^\bullet V/\langle V^{(r)}\rangle, A) \\ f & \mapsto & \widehat{f} \end{array}$$

As $\widehat{f+g} = \widehat{f} + \widehat{g}$, this correspondence is an isomorphism of k-vector spaces. This provides

$$\mathrm{Hom}_k(U,R(n,r)) \cong \mathrm{Der}_k(R(n,r)) = \mathrm{Lie}(G(n,r)) \subset \mathrm{End}_k(R(n,r))$$

as k-vector spaces. In particular, we get the following Lemma.

Lemma 2.37. *Under the identification*

$$R(n,r) = k[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$$

the operators

$$\delta_{(i,I)} = x^I \frac{\partial}{\partial x_i} \in \mathrm{End}_k(R(n,r))$$

with $i \in \{1,\dots,n\}$ *and* $\mathrm{I} \in \{0,\dots,p^r-1\}^n$ *provide a k-basis of*

$$\mathrm{Lie}(G(n,r)) \subset \mathrm{End}_k(R(n,r))$$

Proof. We use the isomorphism

$$L(G(n,r)) \cong \mathrm{Hom}_k(U,R(n,r))$$

from above. If we choose the k-basis $e_1 \dots, e_n$ of $k^n = U$, we get as a k-basis of $R(n,r)$ the monomials x^I. Then a k-basis of $\mathrm{Lie}(G(n,r))$ is given by the maps

$$\delta_{(i,I)}(e_j) = \begin{cases} x^I & j = i \\ 0 & j \neq i \end{cases}$$

By the isomorphism, we get that the image of this operator in $\mathrm{End}_k(R(n,r))$ is obtained by extending it as a k-derivation of $R(n,r)$. But this provides precisely

$$\delta_{(i,I)} = x^I \frac{\partial}{\partial x_i} \in \mathrm{End}_k(R(n,r))$$

which shows the claim. □

Our next aim is to study how the inclusions $G^-, G^0, G^+ \subset G(n,r)$ behave under this identification. As these three subgroups are closed, we obtain inclusions of Lie algebras $\mathrm{Lie}(G^\alpha) \subset \mathrm{Lie}(G(n,r))$ for $\alpha \in \{-,0,+\}$. In fact, we obtain the following Lemma.

Lemma 2.38. *Under the isomorphism* $\mathrm{Lie}(G(n,r)) \cong \mathrm{Hom}_k(U, R(n,r))$, *the k-vector space morphism*

$$\iota_* : \mathrm{Lie}(G^-) \oplus \mathrm{Lie}(G^0) \oplus \mathrm{Lie}(G^+) \to \mathrm{Lie}(G(n,r))$$

induced by the inclusions $\iota_\alpha : G^\alpha \hookrightarrow G(n,r)$ *for* $\alpha \in \{-,0,+\}$ *translates to the canonical isomorphism of k-vector spaces*

$$U^\vee \oplus \mathrm{End}_k(U) \oplus \mathrm{Hom}_k(U, R(n,r)^{\geq 2}) \to \mathrm{Hom}_k(U, R(n,r))$$

which is induced by $k \subset R(n,r)$, $U \subset R(n,r)$, *and* $R(n,r)^{\geq 2} \subset R(n,r)$. *Hence ι_* is also an isomorphism of k-vector spaces.*

Remark 2.39. Under the identification

$$R(n,r) = k[x_1, \dots, x_n]/(x_1^{p^r}, \dots, x_n^{p^r})$$

we computed a k-basis of $\mathrm{Lie}(G(n,r))$ in Lemma 2.37. In view of its proof, in order to get the Lemma, it suffices to prove that the inclusions ι_α induce

$$\mathrm{Lie}(G^-) \cong \bigoplus_{i=1}^{n} k\delta_{(i,(0,\dots,0))} \subset \mathrm{End}(R(n,r))$$

as well as

$$\mathrm{Lie}(G^0) \cong \bigoplus_{i,j} k\delta_{(j,\hat{\imath})} \subset \mathrm{End}(R(n,r))$$

where $\hat{\imath} \in \{0, \dots, p^r - 1\}^n$ is the multi-index with $(\hat{\imath})_k = \delta_{ik}$. Note that $\delta_{(j,\hat{\imath})}$ corresponds to the (i,j)-th elementary matrix $E_{ij} \in M_n(k) \cong \mathrm{Lie}(G^0)$. Finally we need

$$\mathrm{Lie}(G^+) \cong \bigoplus_{(i,I),\ \deg(I) \geq 2} k\delta_{(i,I)} \subset \mathrm{End}(R(n,r))$$

Proof of 2.38. We start with G^- and use the identification $G^- \cong (\mathbb{G}_{a,r})^n$. Let $\mathbb{G}_{a,r} \subset G^-$ be the i-th component. Further we know that

$$\mathrm{Lie}(\mathbb{G}_{a,r}) = \mathrm{Der}_k(k[a_i]/a_i^{p^r}, k) = k\delta_i$$

where

$$\delta_i := k[a_i]/a_i^{p^r} \xrightarrow{\frac{\partial}{\partial a_i}|_{a_i=0}} k$$

Then the induced operators $\delta_1, \ldots, \delta_n \in \mathrm{Lie}(G^-)$ are a k-basis. The image of δ_i in $\mathrm{Lie}(G(n,r)$ is computed by the $k[\mathbb{G}_{a,r}]$-comodule map

$$\begin{array}{rcl} R(n,r) & \to & R(n,r) \otimes k[a_i]/a_i^{p^r} \\ P(x_1, \ldots, x_n) & \mapsto & P(x_1, \ldots, a_i + x_i, \ldots, x_n) \end{array}$$

composed with $\frac{\partial}{\partial a_i}|_{a_i=0}$. This provides

$$\delta_i = \frac{\partial}{\partial x_i} = \delta_{(i,(0,\ldots,0))} \in \mathrm{End}(R(n,r))$$

This proves the assertion for G^-.

Now we proceed with $G^- \cong \mathrm{GL}_n$. Consider the (r,s)-th component of GL_n. The corresponding $\mathrm{Lie}(\mathrm{GL}_n)$-element is the derivation

$$D_{rs} : k[a_{ij}][\det^{-1}] \xrightarrow{\frac{\partial}{\partial a_{rs}}} k[a_{ij}][\det^{-1}] \xrightarrow{a_{ij} \mapsto \delta_{ij}} k$$

Its image in $\mathrm{End}(R(n,r))$ is computed by the $k[a_{ij}][\det^{-1}]$-comodule map

$$\begin{array}{rcl} R(n,r) & \to & R(n,r) \otimes k[a_{ij}][\det^{-1} \\ P(x_1, \ldots, x_n) & \mapsto & P(Ax_1, \ldots, Ax_n) \end{array}$$

where $A = (a_{ij})_{ij}$ is the universal matrix, composed with D_{rs}. This provides

$$D_{rs} = x_r \frac{\partial}{\partial x_s} = \delta_{(s,\hat{r})} \in \mathrm{End}(R(n,r))$$

as claimed.

Finally we take the affine space $G^+ \cong \mathbb{A}^N$. Let $i \in \{1, \ldots, n\}$ and $I \in \{0, \ldots, p^r - 1\}^n$ be a multi-index with $\deg(I) \geq 2$. Consider the (i,I)-th component of G^+ which is isomorphic to the affine line $\mathbb{A}^1$. Let $\iota_{(i,I)} : \mathbb{A}^1 \hookrightarrow G^+$ be the inclusion. Then the corresponding $\mathrm{Lie}(G^+)$-element is the derivation

$$D_{(i,I)} : k[G^+] \xrightarrow{\iota^{\#}_{(i,I)}} k[a] \xrightarrow{\frac{\partial}{\partial a}|_{a=0}} k$$

Its image in $\mathrm{End}(R(n,r))$ is computed by the map

$$\begin{array}{rcl} R(n,r) & \to & R(n,r) \otimes k[a] \\ P(x_1, \ldots, x_n) & \mapsto & P(x_1, \ldots, x_i + ax^I, \ldots, x_n) \end{array}$$

composed with $\frac{\partial}{\partial a}|_{a=0}$. This provides

$$D_{(i,I)} = x^I \frac{\partial}{\partial x_i} = \delta_{(i,I)} \in \mathrm{End}(R(n,r))$$

as claimed. □

Corollary 2.40. *Let $1 \leq i \leq r$. Then the inclusion $U_i(n,r) \subset G(n,r)$ induces an isomorphism*

$$\mathrm{Lie}(U_i(n,r)) \xrightarrow{\cong} \mathrm{Lie}(G(n,r))$$

of Lie algebras.

Proof. By definition

$$U_i \cong G^+ \times G^0 \times G_i^-$$

and the inclusion $G_i^- \subset G^-$ induces an isomorphism of Lie algebras

$$\mathrm{Lie}(G_i^-) \xrightarrow{\cong} \mathrm{Lie}(G^-)$$

So the claim follows from the previous Lemma. □

Notation 2.41. For $\delta_{(i,I)} \in \mathrm{Lie}(G^-)$, that is, $I = (0, \ldots, 0)$, we shortly denote

$$\delta_i := \delta_{(i,(0,\ldots,0))} \in \mathrm{Lie}(G^-)$$

If $\gamma : H \to \mathrm{GL}(V)$ is a representation where H is either $G(n,r)$ or one of its subgroups mentioned above, we shortly denote the images of the $\mathrm{Lie}(H)$-generators $\delta_{(i,I)}$ under the induced representation

$$\mathrm{Lie}(\gamma) : \mathrm{Lie}(H) \to \mathrm{End}(V)$$

also by $\delta_{(i,I)} \in \mathrm{End}(V)$ if no confusion is possible.

Remark 2.42. Note that for the $\mathrm{Lie}(G^-)$-basis δ_i, we obtain that

$$[\delta_i, \delta_j] = 0 \in \mathrm{End}(R(n,r))$$

for all $1 \leq i, j \leq n$ since $\delta_i = \frac{\partial}{\partial x_i} \in \mathrm{End}(R(n,r))$. That is, for each G^--representation V, we also obtain

$$[\delta_i, \delta_j] = 0 \in \mathrm{End}(V)$$

by the induced $\mathrm{Lie}(G^-)$-representation. That is, these operators commute in $\mathrm{End}(V)$.

Now we proceed with the weight space filtration of the adjoint representation.

Lemma 2.43. *For the adjoint representation*

$$\mathrm{Ad} : G(n,r) \to \mathrm{GL}(\mathrm{Lie}(G(n,r)))$$

we obtain the following weight space filtration

$$\mathrm{Lie}(G(n,r)) = \bigoplus_{i \geq -1} \mathrm{Lie}(G(n,r))_i$$

with

$$\mathrm{Lie}(G(n,r))_i = \mathrm{Hom}_k(U, R(n,r)^{i+1})$$

That is,

- $\mathrm{Lie}(G^-) = \mathrm{Lie}(G(n,r))_{-1}$
- $\mathrm{Lie}(G^0) = \mathrm{Lie}(G(n,r))_0$
- $\mathrm{Lie}(G^+) = \mathrm{Lie}(G(n,r))_{\geq 1}$

In other words

$$\mathrm{Ad}(a)(f) = a^{i-1} f$$

for all $a \in \mathbb{G}_m$ *and* $(f : U \to R(n,r)^i) \in \mathrm{Lie}(G(n,r))$.

Proof. Recall that

$$\mathrm{Lie}(G(n,r)) \cong \mathrm{Hom}_k(U, R(n,r)) \hookrightarrow \mathrm{End}_k(R(n,r))$$

by extension as k-derivations. Let $a \in \mathbb{G}_m$ and $f : U \to R(n,r)^i \in \mathrm{Lie}(G(n,r))$. First note that

$$\mathrm{Ad}(a)(f) = a \circ \hat{f} \circ a^{-1} \in \mathrm{End}(R(n,r))$$

where a acts as a^j on $R(n,r)^j$. As $f : U \to R(n,r)^i$, we obtain

$$a \circ f = a^i f \in \mathrm{Hom}_k(U, R(n,r))$$

and finally

$$a \circ f \circ a^{-1} = a^{i-1} f \in \mathrm{Hom}_k(U, R(n,r)) = \mathrm{Lie}(G(n,r))$$

which shows the claim. □

Remark 2.44. In other words, G^+ realizes the positive weight part of $\mathrm{Lie}(G(n,r))$, G^0 the zero weight part, and G^- the negative weight part. This justifies and explains our notion of G^-, G^0 and G^+.

Proposition 2.45. *Let $\gamma : G(n,r) \to \mathrm{GL}(V)$ be a $G(n,r)$-representation with induced representation $\mathrm{Lie}(\gamma) : \mathrm{Lie}(G(n,r)) \to \mathrm{End}(V)$. Then an element $f : U \to R(n,r)^i$ in $\mathrm{Lie}(G(n,r))$ acts on the weight spaces of V as*

$$f(V_k) \subset V_{k+i-1}$$

In other words

$$f \circ \phi_k = \phi_{k+i-1} \circ f$$

Proof. In order to show $f(V_k) \subset V_{k+i-1}$ we have to check the equation

$$\gamma(a)(\mathrm{Lie}(\gamma)(f)(v)) = a^{k+i-1}\, \mathrm{Lie}(\gamma)(f(v))$$

for all $v \in V_n$ and all $a \in \mathbb{G}_m$. For this, use the well known equation

$$\gamma(a) \circ \mathrm{Lie}(\gamma)(f) \circ \gamma(a)^{-1} = \mathrm{Lie}(\gamma)(\mathrm{Ad}(a)(f))$$

for all $a \in \mathbb{G}_m$. Together with $\mathrm{Ad}(a)(f) = a^{i-1} f$ by the previous Lemma we get

$$\gamma(a) \circ \mathrm{Lie}(\gamma)(f) = a^{i-1}\, \mathrm{Lie}(\gamma)(f) \circ \gamma(a)$$

Now we get the claim by applying this equation to $v \in V_k$ since $\gamma(a)(v) = a^k v$ for all $a \in \mathbb{G}_m$. □

In particular, let V be a $G(n,r)$-representation whose $\mathbb{G}_m$-weight filtration looks like

$$V = V_k \oplus \ldots \oplus V_N$$

with $N \geq k$. That is, $V = V_{\geq k}$ and $V_{>k} \subsetneq V$ or in other words V_k is the lowest non-zero weight space. Then we always know that

$$V_k \subset \bigcap_{i=1}^{n} \mathrm{Ker}(\delta_i)$$

by the previous Proposition since $\delta_i(V_k) \subset V_{k-1} = 0$.

3. Triangulated Groups

3.1. **Pretriangulations and Triangulated Morphisms.** We will work with algebraic k-groups H which satisfy the following definition.

Definition 3.1. An algebraic group H is called *pretriangulated* if there are three algebraic k-subgroups (H^-, H^0, H^+) of H such that the multiplication map

$$m : H^+ \times H^0 \times H^- \to H$$

is an isomorphism of k-schemes.

The three subgroups (H^+, H^0, H^-) are called a *pretriangulation* of H. Further we call H^+, H^- the *positive (negative) wing* of H. The subgroup H^0 is called the *heart* of H.

Note that the definition depends on a choice of the three subgroups. Whenever we work with a pretriangulated group H we assume a fixed choice of such three subgroups. Furthermore the three subgroups have to be closed.

Example 3.2. First of all, our group of interest

$$G = G(n,r) = \underline{\mathrm{Aut}}(R(n,r))$$

is pretriangulated by the three subgroups (G^+, G^0, G^-) according to Lemma 2.27. Confer section 2.2 for the definition of these three subgroups. Further the subgroups $U_i \subset G(n,r)$ for $1 \leq i \leq r$ are pretriangulated by (G^+, G^0, G_i^-).

Moreover, the r-th Frobenius kernel G_r of a split reductive group G is pretriangulated by the three subgroups (U_r^+, T_r, U_r^-) according to [Jan03, II.3.2].

Remark 3.3. Note that the notion of pretriangulations is symmetric. That is, if H is pretriangulated by (H^+, H^0, H^-), it is also pretriangulated by (H^-, H^0, H^+). This follows by the commutative diagram

$$\begin{tikzcd}
H^+ \times H^0 \times H^- \arrow[r, "m"] \arrow[d, "\iota \circ \tau"'] & H \arrow[d, "\iota"] \\
H^- \times H^0 \times H^+ \arrow[r, "m"] & H
\end{tikzcd}$$

where ι is the inverting map and τ is the twist of factors which are both isomorphisms of k-schemes.

Definition 3.4. Let G and H be pretriangulated. A group homomorphism $f : G \to H$ is said to be *triangulated* if it respects the pretriangulations. That is, for all $\alpha \in \{-, 0, +\}$, the restriction of f to G^α factors through H^α. In other words, there are three group homomorphisms $f^\alpha : G^\alpha \to H^\alpha$ such that

$$f = f^- \times f^0 \times f^+$$

Example 3.5. Let H be pretriangulated by (H^+, H^0, H^-). Then the r-th Frobenius twist $H^{(r)}$ is pretriangulated by $((H^+)^{(r)}, (H^0)^{(r)}, (H^-)^{(r)})$ and the r-th Frobenius morphism

$$F_H^r : H \to H^{(r)}$$

is a triangulated morphism with $F_H^r = F_{H^+}^r \times F_{H^0}^r \times F_{H^-}^r$.

Notation 3.6. For a pretriangulated group H, denote the unique preimage of $h \in H$ under m as

$$(h_+, h_0, h_-) \in H^+ \times H^0 \times H^-$$

That is, $h = h_+ h_0 h_-$.

Lemma 3.7. *Let G and H be pretriangulated and $f : G \to H$ a triangulated morphism with $f = f^- \times f^0 \times f^+$. Then*

$$\operatorname{Ker}(f) = \operatorname{Ker}(f^+) \times \operatorname{Ker}(f^0) \times \operatorname{Ker}(f^-)$$

and the canonical morphism

$$G \to G^+ / \operatorname{Ker}(f^+) \times G^0 / \operatorname{Ker}(f^0) \times G^- / \operatorname{Ker}(f^-)$$

induces an isomorphism

$$G / \operatorname{Ker}(f) \cong G^+ / \operatorname{Ker}(f^+) \times G^0 / \operatorname{Ker}(f^0) \times G^- / \operatorname{Ker}(f^-)$$

Proof. The decomposition of $\operatorname{Ker}(f)$ just follows from the decomposition of f. Now consider the morphism

$$\phi' : G \to G^+ / \operatorname{Ker}(f^+) \times G^0 / \operatorname{Ker}(f^0) \times G^- / \operatorname{Ker}(f^-)$$

which is the composition of

$$G \xrightarrow{m^{-1}} G^+ \times G^0 \times G^- \xrightarrow{\pi_+ \times \pi_0 \times \pi_-} G^+ / \operatorname{Ker}(f^+) \times G^0 / \operatorname{Ker}(f^0) \times G^- / \operatorname{Ker}(f^-)$$

where $\pi_\alpha : G^\alpha \to G^\alpha / \operatorname{Ker}(f^\alpha)$ is the projection. By the decomposition of $\operatorname{Ker}(f)$, we get: $h = h_+ h_0 h_- \in \operatorname{Ker}(f)$ if and only if $h_\alpha \in \operatorname{Ker}(f^\alpha)$ for all $\alpha \in \{-, 0, +\}$. This shows that $\operatorname{Ker}(\phi') = \operatorname{Ker}(f)$. According to Lemma 1.22, this induces a closed immersion

$$\phi : G / \operatorname{Ker}(f) \hookrightarrow G^+ / \operatorname{Ker}(f^+) \times G^0 / \operatorname{Ker}(f^0) \times G^- / \operatorname{Ker}(f^-)$$

Now denote $A^\alpha = k[H^\alpha]$, $B^\alpha = k[G^\alpha]$ and $I^\alpha = \operatorname{Ker}(f^\alpha)^\#$. Then the closed immersion ϕ is given by the kernel of

$$k[G^+ / \operatorname{Ker}(f^+)] \otimes k[G^0 / \operatorname{Ker}(f^0)] \otimes k[G^- / \operatorname{Ker}(f^-)] \xrightarrow{\phi^\#} k[G]$$

In order to get our claim, we need to prove the injectivity of this map. According to Lemma 1.22 again, we get

$$k[G^\alpha / \operatorname{Ker}(f^\alpha)] \cong A^\alpha / I^\alpha$$

for all $\alpha \in \{-, 0, +\}$. So we can consider

$$(\psi^\alpha)^\# : k[G^\alpha / \operatorname{Ker}(f^\alpha)] \cong A^\alpha / I^\alpha \xrightarrow{(f^\alpha)^\#} B^\alpha$$

and it suffices to show that

$$A^+ / I^+ \otimes A^0 / I^0 \otimes A^- / I^- \xrightarrow{(f^-)^\# \otimes (f^0)^\# \otimes (f^+)^\#} B^+ \otimes B^0 \otimes B^-$$

is injective. For that, we have to show that

$$A^+ \otimes A^0 \otimes A^- / \operatorname{Ker}((f^-)^\# \otimes (f^0)^\# \otimes (f^+)^\#) \cong A^+ / I^+ \otimes A^0 / I^0 \otimes A^- / I^-$$

This follows by the observation that the ideal

$$I := (I^+ \otimes A^0 \otimes A^-) + (A^+ \otimes I^0 \otimes A^-) + (A^+ \otimes A^0 \otimes I^-) \subset A^+ \otimes A^0 \otimes A^-$$

is contained in $\operatorname{Ker}((f^-)^\# \otimes (f^0)^\# \otimes (f^+)^\#)$ and induces an isomorphism

$$A^+ \otimes A^0 \otimes A^- / I \cong A^+ / I^+ \otimes A^0 / I^0 \otimes A^- / I^-$$

This finishes the proof. □

Remark 3.8. For a triangulated morphism $f : G \to H$, we obtain a closed immersion

$$G/\operatorname{Ker}(f) \hookrightarrow H$$

by Lemma 1.22. By the previous Lemma and its proof, we obtain a pretriangulation of $\operatorname{Ker}(f)$ and $G/\operatorname{Ker}(f)$. Then the closed immersion translates to

$$G^+/\operatorname{Ker}(f^+) \times G^0/\operatorname{Ker}(f^0) \times G^-/\operatorname{Ker}(f^-) \hookrightarrow H^+ \times H^0 \times H^-$$

which is induced by $f^+ \times f^0 \times f^-$.

Example 3.9. Let H be pretriangulated by (H^+, H^0, H^-). As we already saw, the r-th Frobenius morphism F_H^r is triangulated with $F_H^r = F_{H^+}^r \times F_{H^0}^r \times F_{H^-}^r$. Thus the Lemma provides a pretriangulation

$$H_r \cong H_r^+ \times H_r^0 \times H_r^-$$

of the r-th Frobenius kernel of H.

3.2. **Triangulations and Irreducible Representations.** Now we extend our definition of a pretriangulated group. We will need this to develop the machinery which is necessary to understand irreducible representations of triangulated groups.

Definition 3.10. An algebraic group H is called *triangulated* if there is a pretriangulation (H^-, H^0, H^+) of H satisfying the following statements:

(1) There are two semidirect products by conjugation which are also subgroups of H:

$$\overline{H}^- := H^- \rtimes H^0 \text{ and } \overline{H}^+ := H^+ \rtimes H^0$$

(2) H^- and H^+ are unipotent.
(3) H^- is finite.

The aim of this section is the following: For a triangulated group H we want to establish a one-to-one correspondence between the isomorphism classes of irreducible representations of H and those of its heart H_0. This reads similar to the standard machinery for parametrizing irreducible representations of reductive groups and their Frobenius kernels as it occurs for example in [Jan03, II.2,II.3].

Example 3.11. Let H be triangulated by (H^+, H^0, H^-). Then the pretriangulation (H_r^+, H_r^0, H_r^-) of the r-th Frobenius kernel H_r is also a triangulation.

The pretriangulation (G^+, G^0, G^-) of our group of interest

$$G = G(n, r) = \underline{\operatorname{Aut}}(R(n, r))$$

is a triangulation (confer section 2.2). Furthermore the pretriangulation (G^+, G^0, G_i^-) of $U_i \subset G(n, r)$ is a triangulation.

Moreover, the pretriangulation (U_r^+, T_r, U_r^-) of an r-th Frobenius kernel G_r of a split reductive group G is a triangulation.

Remark 3.12. Note that the notion of triangulated groups is *not* symmetric. For example, the subgroup $G^+ \subset G(n,r)$ is an affine space, so it is not finite and the pretriangulation (G^-, G^0, G^+) of $G(n,r)$ is not a triangulation.

In fact, our machinery for parametrizing irreducible representations of triangulated groups generalizes the one which applies to Frobenius kernels of reductive groups.

Remark 3.13. Note that the multiplication isomorphism

$$m : \overline{H}^+ \times H^- \to H$$

of a triangulated group H is compatible with the following actions:

- the action of H^- by right multiplication on H^- and H
- the action of $\overline{H}^+$ by left multiplication on $\overline{H}^+$ and H
- the action of H^0 on H^- by conjugation
- the action of H^0 by right multiplication on $\overline{H}^+$ and H

Hence this also holds for the corresponding isomorphism

$$k[H] \xrightarrow{m^\#} k[\overline{H}^+] \otimes_k k[H^-]$$

and the induced actions of H^-, $\overline{H}^+$, and H^0.

Definition 3.14. For a triangulated group H, define the functor

$$\mathrm{I} : H^0\text{−rep} \longrightarrow H\text{−rep}$$

as

$$\mathrm{I}(V) := \mathrm{ind}_{\overline{H}^+}^{H}(V_{\mathrm{tr}})$$

Here V_{tr} is the trivial extension of V with respect to the H^+-part of $\overline{H}^+$ and $\mathrm{ind}_{\overline{H}^+}^{H}$ is the induction functor (cf. [Jan03, I.3]) which is right adjoint to the restriction functor $\mathrm{res}_{\overline{H}^+}^{H}$.

Remark 3.15. Note that the small letter notion of "rep" refers to finite dimensional representations. It is not clear that I maps finite dimensional representations to finite dimensional representations. But this follows from the following Lemma since H^- is finite.

Here is a computation of the functor I which also shows the exactness.

Lemma 3.16. *Let H be triangulated. For any H^0-representation V we get*

$$\mathrm{res}_{\overline{H}^-}^{H} \mathrm{I}(V) \cong \mathrm{ind}_{H^0}^{\overline{H}^-} V \cong k[H^-] \otimes_k V$$

as $\overline{H}^-$-representations. Here H^0 acts on $k[H^-]$ as it does on H^- by conjugation and as given on V. The group H^- acts on $k[H^-]$ via the right regular representation and trivially on V. In general the action of an element $h \in H$ on $\mathrm{I}(V)$ is given as follows

$$h(x \otimes v) = \Psi_h(\phi_h(x) \otimes v)$$

Here $\phi_h : k[H^-] \to k[H^-]$ corresponds to the map $a \mapsto (ah)_-$ with $a \in H^-$ and

$$\Psi_h \in H^0(k[H^-]) = \mathrm{Hom}_k(k[H^0], k[H^-])$$

corresponds to the map $a \mapsto (ah)_0$ with $a \in H^-$ and thus acts on $k[H^-] \otimes V$ as V is an H^0-representation.

Proof. We use the description of [Jan03, I.3.3], but we interchange the roles of the left and right multiplication. This can be done by precomposing all morphisms involved in the description of the induction with the inverse map $\iota : H \to H$. Then we get

$$\mathrm{I}(V) = \left\{ f \in \mathrm{Mor}(H, V_a) \;\middle|\; f(hg) = hf(g)\ \forall g \in H,\ \forall h \in \overline{H}^+ \right\}$$

where V_a is the k-functor defined by $V_a(A) = A \otimes_k V$ for all k-algebras A. Recall that $\overline{H}^+$ only acts on V by its H^0-component. The H-action of $\mathrm{I}(V)$ is given by right translation. By the definition of a pretriangulation, the multiplication map

$$m : \overline{H}^+ \times H^- \to H$$

is an isomorphism. That is, a morphism $f : H \to V_a$ with the compatibility condition above is uniquely determined by its restriction to H^-. Thus we get

$$\mathrm{I}(V) \subset \mathrm{Mor}(H^-, V_a) \cong k[H^-] \otimes_k V$$

which is an equality. Now the H-action computes as:

$$(gf)(a) = f(ag) = (ag)_0 f((ag)_-)$$

for all $f : H^- \to V_a$, $a \in H^-$ and $g \in H$. This shows the claim about the general H-action. As for $h = h_0 \in H^0$, we have $(ah_0) = h_0(h_0^{-1} a h_0)$ for $a \in H^-$, we get $(ah_0)_- = h_0^{-1} a h_0$ and $(ah_0)_0 = h_0$. Thus the action of H^- and H^0 translates as claimed in the statement of the Lemma, hence

$$\mathrm{res}^H_{\overline{H}^-} \mathrm{I}(V) \cong \mathrm{ind}^{\overline{H}^-}_{H^0} V$$

by [Jan03, I.3.8(2)] as $\overline{H}^- = H^- \rtimes H^0$. □

Example 3.17. For our group of interest $G(n,r) = \underline{\mathrm{Aut}}(R(n,r))$, we obtain

$$\mathrm{I}(V) \cong k[G^-] \otimes_k V \cong R(n,r) \otimes_k V$$

as $k[G^-] \cong R(n,r)$. Using the identification

$$R(n,r) \cong k[G^-] = k[a_1, \ldots, a_n]/(a_1^{p^r}, \ldots, a_n^{p^r})$$

the action of $g = (g_i)_i \in G(n,r)$ translates to

$$g(P(a_1 \ldots, a_n) \otimes v) = \left(\frac{\partial g_j(a_1, \ldots, a_n)}{\partial a_i} \right)_{ij} (g(P) \otimes v)$$

This can be seen as follows: For $a = (a_i + x_i)_i$, we obtain

$$ag = (g_i(a_1 + x_1, \ldots, a_n + x_n))_i$$

Hence

$$(ag)_- = ((ag)_i(0) + x_i)_i = (g_i(a_1, \ldots, a_n) + x_i)_i \in G^-$$

and

$$(ag)_0 = \left(\frac{\partial g_j(a_1 + x_1, \ldots a_n + x_n)}{\partial x_i}(0) \right)_{ij} = \left(\frac{\partial g_j(a_1, \ldots, a_n)}{\partial a_i} \right)_{ij} \in G^0$$

Finally we have $g(P) = P(g_i(a_1, \ldots, a_n))$ in a_i-variables which shows the claim.

Note that for an H-representation W, the invariants W^{H^-} are H^0-invariant as $\overline{H}^- = H^- \rtimes H^0$ is a semi-direct product. Thus we can view W^{H^-} as an H^0-representation.

Lemma 3.18. *Let H be triangulated and V be a H^0-representation. Then*

$$\mathrm{I}(V)^{H^-} \cong V$$

as H^0-representations.

Proof. We use the computation

$$\mathrm{res}^H_{H^-} \mathrm{I}(V) \cong k[H^-] \otimes_k V$$

of the previous Lemma. Then we get

$$(k[H^-] \otimes_k V)^{H^-} = k[H^-]^{H^-} \otimes_k V \cong k \otimes_k V \cong V$$

by [Jan03, I 2.10(5)] which shows the claim. □

We obtain irreducible representations from I by taking socles:

Proposition 3.19. *Let H be triangulated. Then for an irreducible H^0-representation V the socle of $\mathrm{I}(V)$ is an irreducible H-representation. Furthermore,*

$$(\mathrm{soc}\,\mathrm{I}(V))^{H^-} \cong V$$

and

$$\mathrm{soc}\,\mathrm{I}(V) = HV \subset \mathrm{I}(V)$$

Proof. Let us assume that there are two distinct irreducible subrepresentations U, W of $\mathrm{I}(V)$ for V irreducible. The sum of these two is direct hence

$$U \oplus W \subset \mathrm{I}(V)$$

Now take H^--invariants. Since H^- is unipotent we have that both U^{H^-} and W^{H^-} are nonzero by [Jan03, I.2.14(8)]. Further we have the inclusion of H^0-representations

$$U^{H^-} \oplus W^{H^-} \subset (\mathrm{I}(V))^{H^-} \cong V$$

by Lemma 3.18. This contradicts the irreducibility of V as H^0-representation.

Now let

$$U := (\mathrm{soc}\,\mathrm{I}(V))^{H^-} \subset (\mathrm{I}(V))^{H^-} \cong V$$

which is an inclusion of H^0-representations. Again $U \neq 0$ as H^- is unipotent. That is, $U = V$ by the irreducibility of V. The last statement follows from the irreducibility of the socle. □

In fact, we obtain all irreducible representations as I-socles:

Proposition 3.20. *Let H be triangulated. Then for each irreducible H-representation W there is an irreducible H^0-representation V such that*

$$W \cong \mathrm{soc}\,\mathrm{I}(V)$$

Furthermore V is unique up to isomorphism.

Proof. Let W be an irreducible H-representation. According to [Jan03, I.2.14(1)] W is finite dimensional. Further the dual $W^\vee$ is nonzero. As H^+ is unipotent, we get $(W^\vee)^{H^+} \neq 0$. By [Jan03, I.2.14(2)] there is an irreducible H^0-representation V' such that

$$V' \subset (W^\vee)^{H^+} \subset W^\vee$$

Thus we obtain a nonzero map

$$V' \to W^\vee$$

which is H^0-equivariant and H^+-equivariant. Hence it is $\overline{H}^+$-equivariant. That is, by setting $V := (V')^\vee$ as the dual and applying dualization, we obtain a nonzero map

$$f : W \to V$$

which is $\overline{H}^+$-equivariant. Note that V is also an irreducible H^0-representation. By the adjoint property of the induction, there is a unique H-equivariant map

$$f' : W \to \mathrm{I}(V)$$

such that $\epsilon \circ f' = f$ where

$$\epsilon : \mathrm{I}(V) \to V$$

is the map $x \otimes v \mapsto v$, that is, the projection onto V. Hence $f' \neq 0$. By the irreducibility of W, the map f' is injective and its image is an irreducible H-representation. Thus

$$\mathrm{Im}(f') \subset \operatorname{soc} \mathrm{I}(V)$$

This is an equality since $\operatorname{soc} \mathrm{I}(V)$ is irreducible by the previous Proposition. Hence $W \cong \operatorname{soc} \mathrm{I}(V)$.

For the uniqueness consider irreducible H^0-representations V, V' with $\operatorname{soc} \mathrm{I}(V) \cong \operatorname{soc} \mathrm{I}(V')$. By the previous Proposition, the H^--invariants of these socles are V, V' respectively. But this means that V and V' have to be isomorphic. $\square$

Now we reached the aim of the section: The map

$$\begin{array}{rcl} \{\text{irred. } H^0\text{−rep}\}/_{\cong} & \to & \{\text{irred. } H\text{−rep}\}/_{\cong} \\ V & \mapsto & \operatorname{soc} \mathrm{I}(V) \end{array}$$

is a bijection: The injectivity follows from the formula

$$(\operatorname{soc} \mathrm{I}(V))^{H^-} \cong V$$

for an irreducible H_0-representation V. The surjectivity follows immediately from the previous Proposition.

4. Representations of Reductive Groups

In the last section, we studied the irreducible representations of triangulated groups H which are in one to one correspondence to those of their hearts H^0. In particular, we obtained this for our group of interest $G(n,r)$. Further the heart of $G(n,r)$ is isomorphic to the split reductive group GL_n. That is, before we proceed with the computation of the irreducible representations of $G(n,r)$ and its representation ring, we need to understand the irreducible representations and representation ring of split reductive groups.

We will work with split reductive groups G in the sense of [Jan03, II.1].

Notation 4.1. An algebraic k-group G is called *split reductive* if there is a split and connected reductive $\mathbb{Z}$-group $G_{\mathbb{Z}}$ such that $G = (G_{\mathbb{Z}})_k$.

A *maximal torus* $T \subset G$ is always assumed to arise from a split maximal torus $T_{\mathbb{Z}} \subset G_{\mathbb{Z}}$. That is, $T = T_k$.

Example 4.2. The general linear group GL_n is split reductive in this sense as its Hopf algebra is defined over $\mathbb{Z}$. A canonical maximal torus is given by the diagonal matrices $T = D_n$. Whenever we deal with GL_n, we will consider this maximal torus.

Remark 4.3. Note that as we work over a field k of characteristic $p > 0$, every split reductive group G is defined over $\mathbb{F}_p$ by $(G_{\mathbb{Z}})_{\mathbb{F}_p}$.

4.1. Irreducible Representations.

Irreducible representations of split reductive groups are parametrized by the *dominant weights* as it occurs for example in [Jan03, II.2]. In particular the things presented here are partly taken from [Jan03, II.1,II.2].

This parametrization works as follows: Let G be a split reductive group and let us choose a maximal torus T in G. Further let $X(T)$ be the character group. As $T \cong \mathbb{G}_m^r$ for an r, we get that $X(T)$ is a free abelian group.

Example 4.4. As we consider $T = D_n$ for GL_n, the character group $X(T)$ is a free abelian group generated by the projections

$$\epsilon_i : T \to \mathbb{G}_m$$

which maps a diagonal matrix to its i-th entry.

Recall that the elements of $X(T)$ induce for each G-representation V a weight space filtration

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda$$

with

$$V_\lambda = \{v \in V \mid tv = \lambda(t)v \ \forall t \in T\}$$

Remark 4.5. For GL_n, we canonically get $\mathbb{G}_m \subset \mathrm{GL}_n$ by scaler operations and also $\mathbb{G}_m \subset D_n = T$. Thus, it makes also sense to consider $\mathbb{G}_m$-weights. There is a connection between the $\mathbb{G}_m$-weight spaces and the T-weight spaces which requires the following *degree map*:

$$\deg : X(T) \to \mathbb{Z}$$

induced by

$$\deg(\epsilon_i) = 1$$

Now the $\mathbb{G}_m$-weight spaces of a GL_n-representation V can be described as

$$V_i = \bigoplus_{\lambda \in \deg^{-1}(i)} V_\lambda$$

Furthermore, denote by

$$Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

the *cocharacter group*. This is again a free abelian group.

Example 4.6. For GL_n, the cocharacter group $Y(T)$ is generated by

$$\epsilon_i' : \mathbb{G}_m \to T$$

which acts as

$$\epsilon_i'(a) = aE_{ii} + \sum_{j \neq i} E_{jj}$$

the diagonal matrix with i-th entry a and all others 1.

There is a bilinear pairing

$$\langle -, - \rangle : X(T) \times Y(T) \to \mathbb{Z}$$

given as follows: For $f \in X(T)$ and $\phi \in Y(T)$, the composition $f \circ \phi$ is an endomorphism of $\mathbb{G}_m$ and thus it corresponds to a unique integer $\langle f, \phi \rangle$ by taking its power morphism.

Example 4.7. For GL_n, we obtain that the ϵ_i's and ϵ_j''s are dual to each other:

$$\langle \epsilon_i, \epsilon_j' \rangle = \delta_{ij}$$

Now we consider the *root system* $R \subset X(T)$ which are the non-zero weights of the adjoint representation on $\mathrm{Lie}(G)$. Further we denote by R^+ the *positive roots* and by S the *simple roots* (cf. [Bou68, VI] and [Jan03, II.1]). As R is a root system, for each $\alpha \in R$, there is a *coroot* $\alpha^\vee \in R^\vee \subset Y(T)$.

Example 4.8. For GL_n, we get

$$R = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n,\ i \neq j\}$$

and

$$R_+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$$

as well as

$$S = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n\}$$

For each $\alpha = \epsilon_i - \epsilon_j \in R$ the coroot is given by

$$\alpha^\vee = \epsilon_i' - \epsilon_j'$$

Now we can introduce a partial order on $X(T)$.

Definition 4.9. Let $\lambda, \mu \in X(T)$. Then $\lambda \leq \mu$ if and only if

$$\mu - \lambda \in \sum_{\alpha \in S} \mathbb{N}\alpha = \sum_{\beta \in R_+} \mathbb{N}\beta$$

Example 4.10. For GL_n and $j > i$, we get $\epsilon_j < \epsilon_i$ as $\epsilon_i - \epsilon_j \in R^+$.

Further we can introduce the weights which parametrize the irreducible representations of a split reductive group G.

Definition 4.11. The *dominant weights* are

$$X(T)_+ := \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in S\}$$

The name *dominant* is explained by the following parametrization Theorem of irreducible G-representations which follows from the results of [Jan03, II.2].

Theorem 4.12. *Let G be a split reductive group and T a maximal torus. Then for each dominant weight $\lambda \in X(T)_+$ there is a unique irreducible G-representation $L(\lambda)$ characterized by:*

(1) *All nonzero weights μ of $L(\lambda)$ satisfy $\mu \leq \lambda$.*
(2) *The dimension of the highest weight space $L(\lambda)_\lambda$ is 1.*

Furthermore the $L(\lambda)$ for $\lambda \in X(T)_+$ form a complete list of pairwise nonisomorphic G-representations.

Of course, the *uniqueness* is meant up to isomorphism.

Our next aim is to understand the irreducible GL_n-representations more concretely and to obtain some computational rules. In order to do this, we consider the *fundamental weights*

$$\epsilon_1 + \ldots + \epsilon_i$$

for all $i = 1, \ldots, n$. Note that the fundamental weights are a $\mathbb{Z}$-basis of $X(T)$. In fact, they are dominant and they uniquely generate all dominant weights which is made precise in the following Lemma.

Lemma 4.13. *The dominant weights of GL_n are freely generated by the fundamental weights in the following way:*

$$X(T)_+ = \mathbb{N}\epsilon_1 \oplus \ldots \oplus \mathbb{N}(\epsilon_1 + \ldots + \epsilon_{n-1}) \oplus \mathbb{Z}(\epsilon_1 + \ldots + \epsilon_n)$$

Proof. First observe that for $\alpha = \epsilon_j - \epsilon_{j+1} \in S$, we get

$$\langle \epsilon_1 + \ldots + \epsilon_i, \alpha^\vee \rangle = \sum_{k=1}^{i} \left(\langle \epsilon_k, \epsilon'_j \rangle - \langle \epsilon_k, \epsilon'_{j+1} \rangle \right) = \delta_{ij}$$

which shows that

$$\epsilon_1 + \ldots + \epsilon_i \in X(T)_+$$

for all $i = 1, \ldots, n$. Now express an arbitrary $\lambda \in X(T)$ uniquely as

$$\lambda = \sum_{i=1}^{n} n_i(\epsilon_1 + \ldots + \epsilon_i)$$

with $n_i \in \mathbb{Z}$. Then $\lambda \in X(T)_+$ if and only if

$$\langle \lambda, \alpha^\vee \rangle \geq 0$$

for all $\alpha \in S$. But for $\alpha = \epsilon_i - \epsilon_{i+1}$ with $1 \leq i < n$ this condition means

$$\langle \lambda, \alpha^\vee \rangle = n_i \geq 0$$

by the computation above. That is $\lambda \in X(T)_+$ if and only if $n_i \geq 0$ for all $1 \leq i < n$ which shows the claim. $\square$

Let us denote again $U = k^n$. Then $\mathrm{GL}_n = \mathrm{GL}(U)$ acts canonically on U. This action extends to the exterior powers $\Lambda^i U$ by operating on the factors simultaneously. The role of these representations is explained in the following Lemma which is taken from [Jan03, II.2.15].

But first, we have to introduce the *Weyl group* of GL_n, namely $W = S_n$, the n-th symmetric group. It acts on $X(T)$ by permuting the ϵ_i's and on each GL_n-representation V as permutation matrices. In fact, the W-action behaves with respect to the weight spaces as

$$w(V_\lambda) = V_{w\lambda}$$

for all $w \in W$ and $\lambda \in X(T)$ (cf. [Jan03, II.1.19]).

Lemma 4.14. *For all $i = 1, \ldots, n$, we have*

$$\Lambda^i U \cong L(\epsilon_1 + \ldots + \epsilon_i)$$

Proof. The weight space filtration is given by

$$\Lambda^i U = \bigoplus_{j_1 < \ldots < j_i} k(e_{j_1} \wedge \ldots \wedge e_{j_i})$$

and the weight of $e_{j_1} \wedge \ldots \wedge e_{j_i}$ is $\sum_i \epsilon_{j_i}$. By $\epsilon_i > \epsilon_{i+1}$, we get that the highest weight space is

$$k(\epsilon_1 + \ldots + \epsilon_i)$$

By the parametrization Theorem 4.12, it is left to show that $\Lambda^i U$ is irreducible. We already know that all weight spaces are 1-dimensional. Furthermore, all occurring nonzero weights are conjugate under the Weyl group W. Let $0 \neq V \subset \Lambda^i U$ be a subrepresentation. Then

$$V_\lambda \subset (\Lambda^i U)_\lambda$$

for all weights $\lambda \in X(T)$. In particular $V_\lambda \neq 0$ for one of the $\Lambda^i U$-weights λ. But by dimension reasons, we get

$$V_\lambda = (\Lambda^i U)_\lambda$$

Now all nonzero weights μ of $\Lambda^i U$ occur as $w\lambda = \mu$ for a $w \in W$. That is, we obtain

$$V_\mu = V_{w\lambda} = wV_\lambda = w(\Lambda^i U)_\lambda = (\Lambda^i(U))_{w\lambda} = (\Lambda^i(U))_\mu$$

Hence $V = \Lambda^i U$ which shows the irreducibility of $\Lambda^i U$. $\square$

Unfortunately, in our prime characteristic p, the irreducible represention of arbitrary dominant weights are not that easy to compute. At least there is a presentation of all $L(\lambda)$ as quotients of explicit representations.

Notation 4.15. Denote for all $\lambda = \sum_{i=1}^n n_i(\epsilon_1 + \ldots + \epsilon_i) \in X(T)_+$, that is, $n_i \geq 0$ for $i = 1, \ldots, n-1$, the generated subrepresentation

$$W(\lambda) := \mathrm{GL}_n(v(\lambda)) \subset \mathrm{Sym}^{n_1}(U) \otimes \mathrm{Sym}^{n_2}(\Lambda^2 U) \otimes \ldots \otimes \mathrm{Sym}^{n_n}(\Lambda^n U)$$

where

$$v(\lambda) = e_1^{n_1} \otimes (e_1 \wedge e_2)^{n_2} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_n)^{n_n}$$

Note that for $n_n < 0$, we take

$$\mathrm{Sym}^{n_n}(\Lambda^n U) = (\Lambda^n U)^{\otimes n_n} := ((\Lambda^n U)^\vee)^{\otimes - n_n}$$

as $\Lambda^n U$ is 1-dimensional.

Note that for a fundamental weight $\lambda = \epsilon_1 + \ldots + \epsilon_i$, we get

$$W(\epsilon_1 + \ldots + \epsilon_i) = \Lambda^i U = L(\epsilon_1 + \ldots + \epsilon_i)$$

This rule generalizes as follows.

Lemma 4.16. *For all dominant weights $\lambda \in X(T)_+$ there is a subrepresentation $V \subset W(\lambda)$ such that*

$$W(\lambda)/V \cong L(\lambda)$$

Further, the residue class of $v(\lambda)$ is a highest weight vector of $L(\lambda)$.

Proof. The highest weight space of

$$\mathrm{Sym}^{n_1}(U) \otimes \mathrm{Sym}^{n_2}(\Lambda^2 U) \otimes \ldots \otimes \mathrm{Sym}^{n_n}(\Lambda^n U)$$

is $kv(\lambda)$. That is, it is 1-dimensional and of weight λ. Thus the highest weight space of $W(\lambda) = \mathrm{GL}_n(v(\lambda))$ is also $kv(\lambda)$. Hence $W(\lambda)$ contains $L(\lambda)$ as a simple composition factor and $\overline{v(\lambda)} \in L(\lambda)$ is a highest weight vector. Thus in a composition series

$$0 = W_0 \subset W_1 \subset \ldots \subset W_{s-1} \subset W_s = W(\lambda)$$

there is a unique $1 \leq i \leq s$ such that

$$W_i/W_{i-1} \cong L(\lambda)$$

That is, $v(\lambda) \in W_i$ but $v(\lambda) \notin W_{i-1}$. As $W(\lambda) = \mathrm{GL}_n(v(\lambda))$, we obtain $s = i$ and

$$W(\lambda)/W_{s-1} \cong L(\lambda)$$

as claimed. □

There are some more rules applying to general split reductive groups G: Recall that G is defined over $\mathbb{F}_p$. Hence the r-th Frobenius twist $G^{(r)}$ is canonically isomorphic to G as algebraic k-group. In particular, $G^{(r)}$ is also split reductive. Furthermore, a chosen maximal torus T of G induces a maximal torus $T^{(r)} \cong T$ of $G^{(r)}$. We can thus write $X(T)$ and $X(T)_+$ for both G and $G^{(r)}$. Note that the restriction of F^r_G to the torus T is the p^r-th power morphism

$$F^r_T : T \xrightarrow{(-)^{p^r}} T^{(r)}$$

Thus the induced map

$$X(T) \cong X(T^{(r)}) \xrightarrow{X(F^r_T)} X(T)$$

is multiplication by p^r. This implies, that for a $G^{(r)}$-representation V, the weights of the r-th Frobenius twist $V^{[r]}$ are the ones of V multiplied by p^r.

Our first Proposition computes r-th Frobenius twist of irreducible $G^{(r)}$-representations.

Proposition 4.17. *Let G be split reductive. Then for all dominant weights $\lambda \in X(T)_+$, we get*

$$L(\lambda)^{[r]} \cong L(p^r\lambda)$$

Proof. According to Corollary 1.38, the G-representation

$$L(\lambda)^{[r]}$$

is irreducible. By Theorem 4.12, it is left to determine its highest weight. The weights of $L(\lambda)^{[r]}$ are the ones of $L(\lambda)$ multiplied by p^r. As

$$\lambda \geq \mu \Longleftrightarrow p^r\lambda \geq p^r\mu$$

the highest weight of $L(\lambda)^{[r]}$ is $p^r\lambda$. This shows the claim. $\square$

This rule generalizes as *Steinberg's Tensor Product Theorem* which can be found as [Jan03, II.3.16,II.3.17]. We will state it in the next Proposition. First we have to introduce a new notation.

Notation 4.18. For all $r \geq 1$ set

$$X_r(T) := \{\lambda \in X(T) \mid \forall\ \alpha \in S:\ 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r\}$$

Example 4.19. For GL_n, we get

$$X_r(T) = \left\{ \lambda = \sum_{i=1}^{n} n_i(\epsilon_1 + \ldots + \epsilon_i) \in X(T) \mid \forall\ 1 \leq i \leq n-1 : 0 \leq n_i < p^r \right\}$$

Note that

$$X_1(T) \subset X_2(T) \subset \cdots \subset X_r(T) \subset \cdots \subset X(T)_+$$

Proposition 4.20. *Let G be split reductive, $\lambda \in X_r(T)$, and $\mu \in X(T)_+$. Then*

$$L(\lambda + p^r\mu) \cong L(\lambda) \otimes L(\mu)^{[r]}$$

Jantzen works with perfect fields but according to his introduction to [Jan03, II.3], the results of this section we are referring to hold for arbitrary fields.

4.2. **Irreducible Representations of Frobenius Kernels.** Our next aim is to give a parametrization of the irreducible representations of Frobenius kernels G_r of split reductive groups G. Furthermore, we want to relate them to the irreducible representations of G.

For each $\lambda \in X(T)$, in [Jan03, II.3], a G_r-representation $L_r(\lambda)$ is introduced. The following Theorem follows immediately from [Jan03, II.3.10].

Theorem 4.21. *Let G be split reductive. Then the following statements hold:*

1. *For all $\lambda \in X(T)$, the G_r-representation $L_r(\lambda)$ is irreducible.*
2. *If Λ is a set of representatives of $X(T)/p^rX(T)$, then the $L_r(\lambda)$ with $\lambda \in \Lambda$ form a complete list of pairwise nonisomorphic G_r-representations*

In order to understand these representations more concretely, there is the following Proposition which is [Jan03, II.3.15].

Proposition 4.22. *For all $\lambda \in X_r(T)$, we get*

$$\operatorname{res}^G_{G_r} L(\lambda) \cong L_r(\lambda)$$

So, for all $\lambda \in X(T)_r$, the $L_r(\lambda)$ arises from an irreducible G-representations by restriction. If there is a set of representatives Λ for $X(T)/p^r X(T)$ with $\Lambda \subset X_r(T)$, all irreducible G_r-representations arise in this way. So the question is: When does this happen? By [Jan03, II.3.15 Remark 2], such a Λ exists for groups G which are semi-simple and simply connected: In this case $X_r(T)$ itself is a set of representatives. It also holds for GL_n which requires the following notation.

Notation 4.23. For GL_n, denote

$$X'_r(T) := \left\{ \lambda = \sum_{i=1}^{n} n_i(\epsilon_1 + \ldots + \epsilon_i) \in X(T) \mid \forall\ 1 \leq i \leq n : 0 \leq n_i < p^r \right\}$$

Now $X'_r(T)$ is indeed a set of representatives for $X(T)/p^r X(T)$ and

$$X'_r(T) \subset X_r(T)$$

So all irreducible $(\mathrm{GL}_n)_r$-representations arise as restrictions of $L(\lambda)$ for $\lambda \in X'_r(T)$.

4.3. **The Representation Ring of** GL_n**.** Our next aim is to give a computation of the *representation ring of* GL_n, that is, the Grothendieck-ring of the category of finite dimensional representations. Note that there is an general Theorem computing it for split reductive groups which works as follows: Consider the group ring of the weight group $\mathbb{Z}[X(T)]$ and the *character map*

$$\begin{array}{rcl} \mathrm{Rep}(\mathrm{GL}_n) & \xrightarrow{\mathrm{ch}} & \mathbb{Z}[X(T)] \\ [V] & \mapsto & \displaystyle\sum_{\lambda \in X(T)} \dim(V_\lambda) e(\lambda) \end{array}$$

where V_λ is the λ-th weight space and $e(\lambda)$ the basis element of $\mathbb{Z}[X(T)]$ corresponding to $\lambda \in X(T)$. We already noticed that for all $w \in W$, we have

$$wV_\lambda = V_{w\lambda}$$

Thus we get

$$\mathrm{ch}(V) \in \mathbb{Z}[(X(T)]^W$$

Now there is the computation

$$\mathrm{Rep}(\mathrm{GL}_n) = \bigoplus_{\lambda \in X(T)_+} \mathbb{Z}[L(\lambda)]$$

as an abelian group by the parametrization Theorem 4.12 and Jordan-Hölder (cf. Remark 1.8). That is, the classes of the $L(\lambda)$ for $\lambda \in X(T)_+$ are a $\mathbb{Z}$-basis of $\mathrm{Rep}(\mathrm{GL}_n)$. Again by the parametrization Theorem, we obtain

$$\mathrm{ch}([L(\lambda)]) = e(\lambda) + \sum_{\mu < \lambda} \dim(L(\lambda)_\mu) e(\mu)$$

for all $\lambda \in X(T)_+$. That is, ch maps a $\mathbb{Z}$-basis of $\mathrm{Rep}(\mathrm{GL}_n)$ to a $\mathbb{Z}$-linearly independent set in $\mathbb{Z}[X(T)]$ which shows that ch is injective. By [Jan03, II.5.8] the $\mathrm{ch}([L(\lambda)])$ also generate $\mathbb{Z}[X(T)]^W$ and hence

$$\mathrm{ch} : \mathrm{Rep}(\mathrm{GL}_n) \to \mathbb{Z}[X(T)]^W$$

is an isomorphism of rings.

We want to understand this isomorphism more explicitly. For this note that we can compute $\mathbb{Z}[X(T)]^W$: As $X(T) = \mathbb{Z}^n$, we get

$$\mathbb{Z}[X(T)] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] = \mathbb{Z}[t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}]$$

the Laurent polynomial ring in n variables by the identification $e(\epsilon_j) = t_j$. Using $W = S_n$, we get

$$\mathbb{Z}[X(T)]^W = \mathbb{Z}[s_1, \ldots, s_n, s_n^{-1}]$$

with s_i the i-th elementary symmetric polynomial in the variables $t_1, \ldots, t_n$. Then we have as preimage of s_i the representation $\Lambda^i U$:

$$\mathrm{ch}(\Lambda^i U) = \sum_{j_1 < \ldots < j_i} e(\epsilon_{j_1} + \ldots + \epsilon_{j_i}) = \sum_{j_1 < \ldots < j_i} t_{j_1} \cdots t_{j_i} = s_i$$

Note that this provides another explicit proof of the surjectivity of ch for the reductive group GL_n. What we got in the end is that

$$\mathrm{Rep}(\mathrm{GL}_n) = \mathbb{Z}\left[[\Lambda^1 U], \ldots, [\Lambda^n U], [(\Lambda^n U)]^{-1}\right]$$

a free polynomial ring with the last variable inverted.

Now we are ready to prove a result which will be important for the computation of the representation ring of $G(n, r)$. Again we use the identification

$$R(n, r) = k[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r})$$

which is a $\mathrm{GL}_n = \mathrm{GL}(U)$-representation. As we are only working with $\mathrm{GL}(U)$, we denote

$$F := F^1_{\mathrm{GL}(U)} \text{ and } F^r = F^r_{\mathrm{GL}(U)}$$

We want to consider the map

$$\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)})) \xrightarrow{[R(n,r)]\cdot(-)+F^*} \mathrm{Rep}(\mathrm{GL}(U))$$

which will be crucial for the computation of the representation ring of $G(n, r)$. Our aim now is to compute the kernel by using the structure of $\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$-modules: $\mathrm{Rep}(\mathrm{GL}(U))$ is such a module by F^* and $[R(n, r)]\cdot(-)$ as well as F^* are maps of $\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$-modules hence the sum as well. We introduce the $\mathrm{Rep}(\mathrm{GL}(U^{(r)}))$-element

$$\delta_r = \sum_{i=1}^{n} (-1)^i [\Lambda^i U^{(r)}]$$

Furthermore we consider the $\mathrm{Rep}(\mathrm{GL}(U))$-element

$$\delta = \delta_0 = \sum_{i=1}^{n} (-1)^i [\Lambda^i U]$$

Our aim is to prove the following.

Proposition 4.24. *The kernel of*

$$\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)})) \xrightarrow{[R(n,r)]\cdot(-)+F^*} \mathrm{Rep}(\mathrm{GL}(U))$$

is generated by $(\delta, -(F^{r-1})^*(\delta_r))$ *as an* $\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$*-module.*

In order to prove this, we use the computation of $\mathrm{Rep}(\mathrm{GL}_n)$ by the character map as given above.

As the character map ch factors through $\mathrm{Rep}(T)$, we can compute the character of $[R(n, r)]$ by the observation that

$$R(n, r) = (k[x_1]/x_1^{p^r}) \otimes_k \ldots \otimes_k (k[x_n]/x_n^{p^r})$$

as a T-representation. Since

$$\mathrm{ch}\left(\left[k[x_i]/x_i^{p^r}\right]\right) = \sum_{j=0}^{p^r-1} e(j\epsilon_i) = \sum_{j=1}^{p^r-1} t_i^j = \frac{t_i^{p^r} - 1}{t_i - 1} \in \mathbb{Z}[t_1^{\pm 1} \ldots, t_n^{\pm 1}]$$

we obtain

$$\mathrm{ch}([R(n, r)]) = \prod_{i=1}^{n} \mathrm{ch}([k[x_i]/x_i^{p^r}]) = \prod_{i=1}^{n} \frac{t_i^{p^r} - 1}{t_i - 1}$$

Then the character of δ_r computes as

$$\mathrm{ch}(\delta_r) = \sum_{i=1}^{n} (-1)^i \, \mathrm{ch}([\Lambda^i U^{(r)}]) = \sum_{i=1}^{n} (-1)^i s_i = \prod_{i=1}^{n} (t_i - 1) \in \mathbb{Z}[s_1 \ldots, s_n, s_n^{-1}]$$

by a well known characterization of the elementary symmetric polynomials s_i.

Now, we translate the map $(F^r)^*$: We already noticed that the application of the functor $(F^r)^*$ acts on the weight filtration by multiplication by p^r of the occurring weights. That is, we obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(\mathrm{GL}(U^{(r)})) \arrow[r, "\mathrm{ch}"] \arrow[d, "(F^r)^*"'] & \mathbb{Z}[t_1^{\pm 1} \ldots, t_n^{\pm 1}] \arrow[d, "(\psi^p)^r"] \\
\mathrm{Rep}(\mathrm{GL}(U)) \arrow[r, "\mathrm{ch}"] & \mathbb{Z}[t_1^{\pm 1} \ldots, t_n^{\pm 1}]
\end{tikzcd}$$

where ψ^p is the p-th Adams operation which is defined by $\psi^p(t_i) = t_i^p$. Thus $(F^r)^*$ is injective. This also shows that

$$(F^r)^* = (\psi^p)^r : \mathrm{Rep}(\mathrm{GL}_n) \to \mathrm{Rep}(\mathrm{GL}_n)$$

is the r-th power of the p-th Adams operation on $\mathrm{Rep}(\mathrm{GL}_n)$ under the isomorphism $\mathrm{GL}(U^{(r)}) \cong \mathrm{GL}_n$.

As

$$\psi^p((\psi^p)^{r-1}(\mathrm{ch}\,\delta)) = \psi^{p^r}\left(\prod_{i=1}^{n}(t_i - 1)\right) = \prod_{i=1}^{n}(t_i^{p^r} - 1) = \mathrm{ch}([R(n, r)])\,\mathrm{ch}(\delta)$$

we get

$$(\delta, -(F^{r-1})^*(\delta_r)) \in \mathrm{Ker}([R(n, r)] \cdot (-) + F^*)$$

We also saw, that the map F^* is injective. Further the multiplication with $[R(n, r)]$ is also injective. That is, the claim of the Proposition

$$\mathrm{Ker}([R(n, r)] \cdot (-) + F^*) = (\delta, -(F^{r-1})^*(\delta_r))\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$$

is equivalent to the equality

$$\begin{aligned} & F^*(\mathrm{Rep}(\mathrm{GL}(U^{(1)})) \cap [R(n,r)]\mathrm{Rep}(\mathrm{GL}(U)) \\ = \; & [R(n,r)]\delta F^*(\mathrm{Rep}(\mathrm{GL}(U^{(1)}))) \\ \subset \; & \mathrm{Rep}(\mathrm{GL}(U)) \end{aligned}$$

By $(\delta, -(F^{r-1})^*(\delta_r)) \in \mathrm{Ker}([R(n,r)]\cdot(-) + F^*)$, we know that the right hand side is contained in the left hand side. The equality for $r=1$ is proved in the next Lemma.

Lemma 4.25. *The equality*

$$\begin{aligned} & F^*(\mathrm{Rep}(\mathrm{GL}(U^{(1)})) \cap [R(n,1)]\mathrm{Rep}(\mathrm{GL}(U)) \\ = \; & [R(n,1)]\delta F^*(\mathrm{Rep}(\mathrm{GL}(U^{(1)}))) \end{aligned}$$

holds.

Proof. As

$$\mathrm{ch} : \mathrm{Rep}(\mathrm{GL}(U)) \to \mathbb{Z}[t_1^{\pm 1} \ldots, t_n^{\pm 1}]$$

is an injective ring homomorphism, it suffices to prove

$$\mathbb{Z}[t_1^{\pm p} \ldots, t_n^{\pm p}] \cap U_1 \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] = (U_1\delta)\mathbb{Z}[t_1^{\pm p} \ldots, t_n^{\pm p}]$$

with

$$U_1 = \prod_{i=1}^n \frac{t_i^p - 1}{t_i - 1} \text{ and } \delta = \prod_{i=1}^n (t_i - 1)$$

since

$$\psi^p(\mathbb{Z}[t_1^{\pm 1} \ldots, t_n^{\pm 1}]) = \mathbb{Z}[t_1^{\pm p} \ldots, t_n^{\pm p}]$$

and $\mathrm{ch} \circ F^* = \psi^p \circ \mathrm{ch}$.

Now let

$$P_i := \left\langle \frac{t_i^p - 1}{t_i - 1} \right\rangle \subset \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

be generated ideals. These are prime ideals of height 1 since the elements $\frac{t_i^p-1}{t_i-1}$ are irreducible and $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is factorial according to the following Lemma. Further we consider the ideals

$$Q_i := \langle t_i^p - 1 \rangle \subset \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$$

which are also prime ideals of height 1. Our claim is that

$$P_i \cap \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}] = Q_i$$

as prime ideals of $\mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$. Note that we have

$$Q_i \subset P_i \cap \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$$

That is, the claim follows, if the height of the prime ideal $P_i \cap \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$ is also 1. But this follows from the fact that P_i is of height 1 and the "going-up" Theorem [Mat86, Theorem 9.4] since $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is integral over $\mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$.

Now observe that

$$\langle U_1 \rangle = P_1 \cdot \ldots \cdot P_n = P_1 \cap \ldots \cap P_n \subset \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

and

$$\langle U_1\delta \rangle = Q_1 \cdot \ldots \cdot Q_n = Q_1 \cap \ldots \cap Q_n \subset \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$$

Together with our claim, this provides

$$\langle U_1\rangle \cap \mathbb{Z}[t_1^{\pm p},\dots,t_n^{\pm p}] = \langle U_1\delta\rangle \subset \mathbb{Z}[t_1^{\pm p},\dots,t_n^{\pm p}]$$

which finishes the proof. □

Now we are ready to prove Proposition 4.24.

Proof of 4.24. By the previous Lemma, we know the claim for $r=1$. So let $r \geq 2$ and denote

$$U_s = \mathrm{ch}[R(n,s)] = \prod_{i=1}^{n} \frac{t_i^{p^s}-1}{t_i-1}$$

We get

$$U_1\psi^p(U_{r-1}) = U_r$$

and

$$(\psi^p)^{r-1}(\delta) = \delta U_{r-1}$$

Hence we get a factorization of $\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$-module maps

$$\begin{tikzcd}
\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)})) \arrow[r, "{[R(n,r)]\cdot(-)+F^*}"] \arrow[d, "{[F^*(R(n,r-1)^{(1)})]\cdot(-)\oplus\mathrm{id}}"'] & \mathrm{Rep}(\mathrm{GL}(U)) \\
\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)})) \arrow[ur, "{[R(n,1)]\cdot(-)+F^*}"'] &
\end{tikzcd}$$

By the case $r=1$ the kernel of the map $[R(n,1)]\cdot(-)\oplus F^*$ consists of the elements of type

$$(\delta\psi^p(a), -\delta a)$$

for $a \in \mathrm{Rep}(\mathrm{GL}(U)^{(1)}$. Furthermore the map $F^*(R(n,r-1)^{(1)})\cdot(-)\oplus\mathrm{id}$ is injective. Also, no prime factor t_i-1 of δ divides $\psi^p(U_{r-1})$ in the factorial ring $\mathbb{Z}[t_1^{\pm p},\dots,t_n^{\pm p}]$. Hence, the images of the elements of the kernel of the map $[R(n,r)]\cdot(-)+F^*$ are the elements of the form

$$(\delta\psi^p(U_{r-1})\psi^p(a), -\delta U_{r-1}a) = (\psi^p(U_{r-1})\cdot(-)\oplus\mathrm{id})(\delta\psi^p(a), -(\psi^p)^{r-1}(\delta)a)$$

for all $a \in \mathrm{Rep}(\mathrm{GL}(U)^{(1)})$. Whence the claim. □

Lemma 4.26. *The ring*

$$\mathbb{Z}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$$

is factorial (UFD).

Proof. It is well known that

$$\mathbb{Z}[t_1,\dots,t_n]$$

is factorial. By Hilbert's Basis Theorem [Mat86, Theorem 3.3], it is also noetherian. Hence also the localization

$$\mathbb{Z}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$$

is noetherian. Now according to [Mat86, Theorem 20.1] an integral noetherian ring is factorial if and only if each prime ideal of height 1 is principal. So let P be a prime ideal of the localization of height 1. But prime ideals of a localization with respect to S correspond to prime ideals of the original ring which have empty intersection with S. That is, P corresponds to a prime ideal Q of $\mathbb{Z}[t_1, \ldots, t_n]$ which is again of height 1. By the Theorem mentioned above, $Q = (q)$ is principal with $q \notin S$ irreducible and hence prime. But since we localize with respect to prime elements $t_1, \ldots, t_n$, the element q is also prime in the localization. Hence $(q) \subset P$ is an equality since the height of P is 1. This shows that P is principal and finishes the proof. $\square$

5. r-Triangulated Groups with Reductive Hearts

5.1. **r-Triangulations.** First of all, we extend our definition of a triangulated group.

Definition 5.1. An algebraic group H is called an *r-triangulated* group if there is a triangulation (H^+, H^0, H^-) such that the negative wing H^- coincides with its r-th Frobenius kernel H_r^-. A triangulation (H^+, H^0, H^-) of H satisfying this additional condition $H^- = H_r^-$ is called an *r-triangulation* of H.

Remark 5.2. If H is r-triangulated, then it is also $r+1$-triangulated as $H_r^- \subset H_{r+1}^- \subset H^-$.

Further let H be r-triangulated by (H^+, H^0, H^-). Then the r-th Frobenius kernel H_r is r-triangulated by (H_r^+, H_r^0, H^-).

Example 5.3. The group $G = G(n,r) = \underline{\mathrm{Aut}}(R(n,r))$ is r-triangulated by the triangulation (G^+, G^0, G^-) since $G^- \cong \mathbb{G}_a(U)_r$. Moreover, for $1 \leq i \leq r$, the subgroup $U_i \subset G(n,r)$ is i-triangulated by (G^+, G^0, G_i^-).

Further, each r-th Frobenius kernel G_r of a reductive group G is r-triangulated by the three subgroups (U_r, T_r, U_r^+).

The final aim of this section is to study r-triangulated groups H whose heart H^0 is a split reductive group and whose positive wing H^+ is reduced. Our main example is again $G(n,r) = \underline{\mathrm{Aut}}(R(n,r))$ since $G_0 \cong \mathrm{GL}_n$ and $G_+ \cong \mathbb{A}^N$ is an affine space. Then we will relate the irreducible representations of such an r-triangulated group H with the ones of its r-th Frobenius kernel H_r. This will work in a similar fashion as it does for split reductive groups which we discussed in the previous section and we will make heavy use of this machinery.

Now we will introduce one main advantage of an r-triangulation. Namely, we can define a group homomorphism L_r between H and the r-th Frobenius twist of its heart $(H^0)^{(r)}$.

Notation 5.4. Let H be r-triangulated. Define the morphism

$$L_r : H \to (H^0)^{(r)}$$

as the composition

$$H \xrightarrow{\pi_0} H^0 \xrightarrow{F^r_{H^0}} (H^0)^{(r)}$$

where π_0 is the projection. That is,

$$L_r(h) = F^r_{H^0}(h_0)$$

Lemma 5.5. *Let H be an r-triangulated group. Then the following statements hold.*

1. *The morphism $L_r : H \to (H^0)^{(r)}$ is a group homomorphism.*
2. *The composition $H^0 \xrightarrow{\iota_0} H \xrightarrow{L_r} (H^0)^{(r)}$ coincides with the r-th Frobenius $F^r_{H^0}$ on H^0 where ι_0 is the inclusion.*
3. *If H^0 is reduced, L_r induces an isomorphism $H/\operatorname{Ker}(L_r) \cong (H^0)^{(r)}$.*

Proof. As the r-th Frobenius on H is triangulated, we get a commutative diagram

$$\begin{tikzcd} H \arrow[r, "\pi_0"] \arrow[d, "F_H^r"'] & H^0 \arrow[d, "F_{H^0}^r"] \\ H^{(r)} \arrow[r, "\pi_0"] & (H^0)^{(r)} \end{tikzcd}$$

That is, $L_r = \pi_0 \circ F_H^r$. Further F_H^r is trivial on $H^- = H_r^-$. Now let $h = h_+h_0h_-, g = g_+g_0g_-$ be arbitrary elements of H. Then we get

$$\begin{aligned} L_r(hg) &= \pi_0(F^r(h_+)F^r(h_0)F^r(h_-)F^r(g_+)F^r(g_0)F^r(g_-)) \\ &= \pi_0(F^r(h_+)F^r(h_0)F^r(g_+)F^r(g_0)) \\ &= \pi_0(F^r(h_+)F^r(h_0g_+)F^r(g_0)) \\ &= \pi_0(F^r(h_+)F^r(h_0g_+h_0^{-1})F^r(h_0)F^r(g_0)) \\ &= F^r(h_0)F^r(g_0) \\ &= L_r(h)L_r(g) \end{aligned}$$

since $h_0g_+h_0^{-1} \in H^+$ which follows as H^0 and H^+ are semidirect.

The second statement follows from the very definition of L_r and the fact the composition

$$H^0 \xrightarrow{\iota_0} H \xrightarrow{\pi_0} H^0$$

is the identity.

For the third statement we like to consider $(H^0)^{(r)}$ as a pretriangulated group with zero wings. Then the group homomorphism L_r is triangulated as its restriction to H^+ and H^- is trivial and the restriction to H^0 coincides with $F_{H^0}^r$ by the second statement. That is, by Remark 3.8, the closed immersion

$$H/\operatorname{Ker}(L_r) \hookrightarrow (H^0)^{(r)}$$

translates as

$$H^0/H_r^0 = H^+/\operatorname{Ker}(L_r^+) \times H^0/\operatorname{Ker}(L_r^0) \times H^-/\operatorname{Ker}(L_r^-) \hookrightarrow (H^0)^{(r)}$$

which is induced by $F_{H^0}^r$. By Proposition 1.36 this is an isomorphism for H^0 reduced which shows the claim. □

Remark 5.6. Let H be r-triangulated such that the positive wing also coincides with its r-th Frobenius kernel, that is $H^+ = H_r^+$. Then the composition

$$H \xrightarrow{L_r} (H^0)^{(r)} \xrightarrow{\iota_0} H^{(r)}$$

coincides with the r-th Frobenius F_H^r on H: For $h = h_+h_0h_- \in H$ we get

$$F^r(h) = F^r(h_+)F^r(h_0)F^r(h_-) = F^r(h_0) = L_r(h)$$

Now we can consider the pullback functor

$$L_r^* : (H^0)^{(r)}\text{−rep} \longrightarrow H\text{−rep}$$

for r-triangulated groups H. This is well-defined as L_r is a group homomorphism by the previous Lemma. The functor I from section 3 and the functor L_r^* are related as follows.

Lemma 5.7. *Let H be r-triangulated, V an H^0-representation and W an $(H^0)^{(r)}$-representation. Then*

$$\mathrm{I}(V \otimes_k W^{[r]}) \cong \mathrm{I}(V) \otimes_k L_r^*(W)$$

as H-representations.

Proof. Recall that I is the composition

$$H^0\text{−rep} \xrightarrow{(-)_{\mathrm{tr}}} \overline{H}^+\text{−rep} \xrightarrow{\mathrm{ind}_{\overline{H}^+}^H} H\text{−rep}$$

Also the restriction $\mathrm{res} = \mathrm{res}_{H^0}^H$ can be written as the composition

$$H\text{−rep} \xrightarrow{\mathrm{res}_+} \overline{H}^+\text{−rep} \xrightarrow{\mathrm{res}_0} H^0\text{−rep}$$

Further $(F_{H^0}^r)^* = \mathrm{res} \circ L_r^*$ by Lemma 5.5. That is our claim reads as

$$\mathrm{ind}_{\overline{H}^+}^H(V_{\mathrm{tr}} \otimes_k \mathrm{res}\, L_r^*(W)_{\mathrm{tr}}) \cong \mathrm{ind}_{\overline{H}^+}^H(V_{\mathrm{tr}}) \otimes_k L_r^*(W)$$

By the Tensor Identity [Jan03, I.3.6], we obtain

$$\mathrm{ind}_{\overline{H}^+}^H(V_{\mathrm{tr}}) \otimes_k L_r^*(W) \cong \mathrm{ind}_{\overline{H}^+}^H(V_{\mathrm{tr}} \otimes_k \mathrm{res}_+ L_r^*(W))$$

That is, the claim follows if

$$\mathrm{res}\, L_r^*(W)_{\mathrm{tr}} = \mathrm{res}_0(\mathrm{res}_+ L_r^*(W))_{\mathrm{tr}} = \mathrm{res}_+ L_r^*(W)$$

But this holds as $L_r|_{H^+}$ is trivial and we are done. $\square$

5.2. **Reductive Hearts.** Now let H be r-triangulated such that H^0 is split reductive. Let $T \subset H^0$ be a maximal torus and $X(T)_+$ the set of dominant weights as introduced in section 4.1.

Notation 5.8. For $\lambda \in X(T)_+$ denote

$$L(\lambda, H) := \mathrm{soc}\, \mathrm{I}(L(\lambda))$$

where $L(\lambda)$ is the irreducible H^0-representation corresponding to λ (cf. Theorem 4.12).

According to the results of section 3 and Theorem 4.12, we obtain the following parametrization.

Theorem 5.9. *Let H be triangulated such that H^0 is split reductive. Then the representations $L(\lambda, H)$ with $\lambda \in X(T)_+$ form a complete list of pairwise non-isomorphic irreducible H-representations.*

Our first aim is to establish a mod p^r-periodicity with respect to the dominant weights. As a first result, we get that the functor L_r^* provides all irreducible representations whose weight is divisible by p^r.

Proposition 5.10. *Let H be r-triangulated, H^0 split reductive, and $\lambda \in X(T)_+$. Then*

$$L(p^r\lambda, H) \cong L_r^*(L(\lambda))$$

Proof. By Lemma 5.5 (3) and Lemma 1.23, we get that $L_r^*(L(\lambda))$ is an irreducible H-representation. Further its restriction to H^0 is the r-th Frobenius twist $L(\lambda)^{[r]}$. Now by Proposition 4.17, we obtain

$$L(p^r\lambda) = L(\lambda)^{[r]}$$

Further, the associated H^0-representation of the irreducible H-representation $L_r^*(L(\lambda))$ is obtained by taking G^--invariants. Thus we get an inclusion

$$(L_r^*(L(\lambda)))^{G^-} \subset L(p^r\lambda)$$

of irreducible H^0-representations which has to be an equality. This shows the claim. □

Now we are ready to prove the mod p^r-periodicity which reads as Steinberg's Tensor Product Theorem 4.20. Recall that

$$X_r(T) = \{\lambda \in X(T) \mid \forall\ \alpha \in S:\ 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r\}$$

Proposition 5.11. *Let H be r-triangulated, H^0 split reductive, $\lambda \in X_r(T)$ and $\mu \in X(T)_+$. Then*

$$L(\lambda + p^r\mu, H) \cong L(\lambda, H) \otimes_k L_r^*(L(\mu)) = L(\lambda, H) \otimes_k L(p^r\mu, H)$$

Proof. By Steinberg's Tensor Product Theorem 4.20, we obtain

$$L(\lambda + p^r\mu) \cong L(\lambda) \otimes_k L(\mu)^{[r]}$$

for H^0-representations. Together with Lemma 5.7, we get

$$\mathrm{I}(L(\lambda + p^r\mu)) \cong \mathrm{I}(L(\lambda)) \otimes_k L_r^*(L(\mu))$$

Further

$$L(\lambda + p^r\mu, H) = HL(\lambda + p^r\mu) \subset \mathrm{I}(L(\lambda + p^r\mu))$$

by Proposition 3.19. Thus

$$L(\lambda + p^r\mu, H) \cong (HL(\lambda)) \otimes_k L_r^*(L(\mu)) = L(\lambda, H) \otimes_k L_r^*(L(\mu))$$

The second equality follows from the previous Proposition. □

Now we will relate the irreducible representations of H for the weights in $X_r(T)$ with those of the r-th Frobenius kernel H_r.

Let (H^+, H^0, H^-) be a triangulation of H which lead to a functor

$$\mathrm{I} : H^0\text{−rep} \longrightarrow H\text{−rep}$$

As the r-th Frobenius kernel of H is triangulated by (H_r^+, H_r^0, H_r^-), we also get a functor

$$\mathrm{I}_r : H_r^0\text{−rep} \longrightarrow H_r\text{−rep}$$

For r-triangulated groups, these two functors are related as follows.

Lemma 5.12. *Let H be r-triangulated. Then for each H^0-representation V, we obtain*

$$\mathrm{res}^H_{H_r} \mathrm{I}(V) = \mathrm{I}_r \,\mathrm{res}^{H^0}_{H_r^0}(V)$$

Proof. This follows immediately from Lemma 3.16 as the negative wings of H and H_r coincide. □

Now let H be triangulated such that H^0 is again split reductive and T a maximal torus of H^0.

Notation 5.13. For $\lambda \in X(T)_+$ denote

$$L(\lambda, H_r) := \operatorname{soc} \mathrm{I}_r(L_r(\lambda))$$

(cf. section 4.2).

According to the results of section 3 and Theorem 4.21, we get all irreducible H_r-representations in this way.

Theorem 5.14. *Let H be triangulated such that H^0 is split reductive. Further let $\Lambda \subset X(T)_+$ be a set of representatives for $X(T)/p^r X(T)$. Then the representations $L(\lambda, H_r)$ with $\lambda \in \Lambda$ form a complete list of pairwise non-isomorphic irreducible H_r-representations.*

The next Proposition shows that for r-triangulated groups and $\lambda \in X_r(T)$ the corresponding irreducible H-representation restricts to the corresponding irreducible H_r-representation if we additionally assume that H^+ is reduced.

Proposition 5.15. *Let H be r-triangulated such that H^0 is split reductive and H^+ is reduced. Then for all $\lambda \in X_r(T)$ we get*

$$L(\lambda, H_r) = \operatorname{res}^H_{H_r} L(\lambda, H)$$

Proof. We have to show that

$$\operatorname{soc} \mathrm{I}_r(L_r(\lambda)) = \operatorname{res}^H_{H_r} \operatorname{soc} \mathrm{I}(L(\lambda))$$

According to Proposition 4.22, we have

$$\operatorname{res}^{H^0}_{H^0_r} L(\lambda) = L_r(\lambda)$$

for $\lambda \in X_r(T)$. Together with the previous Lemma we get

$$\operatorname{soc} \mathrm{I}_r(L_r(\lambda)) = \operatorname{soc}_{H_r} \mathrm{I}(L(\lambda))$$

Further by Proposition 3.19 applied to both H and H_r, we get

$$\operatorname{soc} \mathrm{I}_r(L_r(\lambda)) = H_r L(\lambda) \subset \mathrm{I}(L(\lambda))$$

and

$$\operatorname{soc} \mathrm{I}(L(\lambda)) = H L(\lambda)$$

Hence

$$\operatorname{soc} \mathrm{I}_r(L_r(\lambda)) = \operatorname{soc}_{H_r} \mathrm{I}(L(\lambda)) \subset \operatorname{soc} \mathrm{I}(L(\lambda)) = L(\lambda, H)$$

as H_r-representations. Thus it suffices to check that this subspace is H-stable as $L(\lambda, H)$ is an irreducible H-representation. As $H \cong \overline{H}^+ \times H^-$ by multiplication, it suffices to show this for $\overline{H}^+$ and H^- separately. As H is r-triangulated, we have $H^- = H^-_r \subset H_r$ and get it for H^-. Now we can assume that k is algebraically closed as the whole situation base changes. According to [Jan03, I.2.8 Remark], it suffices to check that this subspace is $\overline{H}^+(k) = H^+(k) \times H^0(k)$-stable as H^+ and H^0 are reduced. But [Jan03, I.6.15(1)] tells us that $\operatorname{soc}_{H_r} \mathrm{I}(L(\lambda))$ is $H(k)$-stable. Finally

$$H(k) = \overline{H}^+(k)$$

as $H^- = H^-_r$ by assumption which finishes the proof. $\square$

Now we summarize the practical use of this theory: Let H be r-triangulated with H^0 split reductive and H^+ reduced. Furthermore suppose that there is a set of representatives Λ for $X(T)/p^r X(T)$ such that

$$\Lambda \subset X_r(T)$$

This holds for example for H^0 semi-simple and simply connected and for $H^0 = \mathrm{GL}_n$ (cf. section 4.2). Note that for all $\lambda \in X(T)_+$ there is $\lambda' \in \Lambda$ and $\mu \in X(T)_+$ such that

$$\lambda = \lambda' + p^r \mu$$

According to the mod p^r-periodicity Proposition 5.11 and the previous Proposition, the computation of the irreducible H-representations can be reduced to the computation of the irreducible H_r-representations.

In the case of $r = 1$ we are reduced to the first Frobenius kernel. But the representation theory of H_1 is the same as the one of the p-Lie algebra $\mathrm{Lie}(H)$. Thus in the case of a 1-triangulated group we are reduced to the computation of the irreducible $\mathrm{Lie}(H)$-representations.

Example 5.16. As the group $G(n, 1)$ is 1-triangulated, we are reduced to the computation of the irreducible p-$\mathrm{Lie}(G(n, 1))$-representations (or simple restricted $\mathrm{Lie}(G(n, 1))$-modules). But

$$\mathrm{Lie}(G(n, 1)) = W(n, (1, \ldots, 1))$$

the Jacobson-Witt algebra. That is, we will provide a parametrization of the irreducible p-$W(n, (1, \ldots, 1))$-representations. We will see later that this coincides with the one described in [Hol01, 2.2 Proposition] and [Nak92, II] which goes back to [She88].

Moreover, for arbitrary $r \geq 1$, the subgroup $U_1 \subset G(n, r)$ is 1-triangulated and its Lie algebra coincides with

$$\mathrm{Lie}(G(n, r)) = \mathrm{Der}_k(R(n, r))$$

Thus, the description of the irreducible U_1-representations is reduced to the one of $\mathrm{Lie}(G(n, r))$. Unfortunately, this will not directly provide the one for irreducible $G(n, r)$-representations. But in the next section, we will provide a transfer morphism

$$t_{r,1} : U_1(n, r) \to G(n, 1)$$

with the following property: The p-Lie algebra morphism

$$\mathrm{Lie}(t_{r,1}) : \mathrm{Lie}(G(n, r)) \to \mathrm{Lie}(G(n, 1))$$

induces a bijection on isomorphism classes of irreducible p-Lie algebra representations by pullback (confer Remark 6.9). That is, we will obtain the description for $\mathrm{Lie}(G(n, r))$ by the one of $\mathrm{Lie}(G(n, 1))$.

6. Transfer Homomorphisms

In order to prepare the description of the irreducible $G(n,r)$-representations, we will introduce several *transfer homomorphisms*. They will be between the $G(n,r)$, $U_i(n,r)$, and $G(n,r)^0 \cong \mathrm{GL}_n$ and their Frobenius twists respectively.

6.1. **First Type.** The first type of transfer homomorphisms are

$$\begin{array}{ccc} G(n,r) & \xrightarrow{L_r} & (G^0)^{(r)} \\ f & \mapsto & F^r_{G^0}(f_0) \end{array}$$

for $r \geq 1$. Note that we already discussed these morphisms in the general setting of r-triangulated groups in the previous section. In particular the L_r are group homomorphisms according to Lemma 5.5. This Lemma also provides that they induce isomorphisms

$$G(n,r)/\operatorname{Ker}(L_r) \cong (G^0)^{(r)}$$

Furthermore, we already saw in Proposition 5.10, that for the induced functor

$$L_r^* : (G^0)^{(r)} \longrightarrow G(n,r)\text{−rep}$$

we obtain

$$L_r^* L(\lambda) \cong L(p^r\lambda, G(n,r))$$

for all $\lambda \in X(T)_+$.

Remark 6.1. As $G^0 = \mathrm{GL}(U)$, we obtain $(G^0)^{(r)} \cong \mathrm{GL}(U^{(r)})$ and we can view L_r as a $G(n,r)$-representation

$$L_r : G(n,r) \to \mathrm{GL}(U^{(r)})$$

Remark 6.2. Under the identification

$$R(n,r) = k[x_1,\ldots,x_n]/(x_1^{p^r},\ldots,x_n^{p^r})$$

and $(G^0)^{(r)} \cong G^0 = \mathrm{GL}_n$ we obtain: For $f = (f_1,\ldots,f_n) \in G(n,r)$

$$L_r(f) = F^r_{\mathrm{GL}_n}(J_f) = \left(\left(\frac{\partial f_j}{\partial x_i}(0)\right)^{p^r}\right)_{ij}$$

6.2. **Second Type.** The next type of transfer homomorphisms are

$$T_r : G(n,r) \to G(n,r-1)^{(1)}$$

for $r \geq 2$ which can be defined by the following observation: Let $S(n,r)$ be the subalgebra of

$$R(n,r) = S^\bullet U/\langle U^{(r)}\rangle$$

generated by $U^{(1)}$. That is, by the image of

$$\begin{array}{ccc} U^{(1)} & \hookrightarrow & S^\bullet U/\langle U^{(r)}\rangle \\ u\otimes\lambda & \mapsto & u^p\lambda \end{array}$$

Then

$$R(n,r-1)^{(1)} \cong S(n,r)$$

as k-algebras as

$$R(n,r-1)^{(1)} = \left(S^\bullet U/\langle U^{(r-1)}\rangle\right)^{(1)} \cong S^\bullet U^{(1)}/\langle (U^{(1)})^{(r-1)}\rangle \cong S(n,r)$$

Further each $R(n,r)$-automorphism stabilizes $S(n,r)$ and

$$\underline{\mathrm{Aut}}(R(n,r-1)^{(1)}) \xrightarrow{\cong} G(n,r-1)^{(1)}$$

This induces T_r by restricting an $R(n,r)$-automorphism to $S(n,r)$.

Remark 6.3. Under the identification

$$R(n,r) = k[x_1,\dots,x_n]/(x_1^{p^r},\dots,x_n^{p^r})$$

the subalgebra $S(n,r)$ is generated by $x_1^p,\dots,x_n^p$. So, for $f=(f_1,\dots,f_n)\in G(n,r)$ we get

$$T_r(f) = (f_1^{(1)},\dots,f_n^{(1)}) \in G(n,r-1) \cong G(n,r-1)^{(1)}$$

Here for a polynomial $P=\sum \lambda_I x^I$, we denote the polynomial $P^{(1)} := \sum \lambda_I^p x^I$. Note that by taking the residue classes mod $x_i^{p^{r-1}}$, the coefficients λ_I vanish for the $I=(i_1,\dots,i_n)$ with at least one $i_j \geq p^{r-1}$.

Note that T_r is triangulated: The restriction to the negative wings is just the first Frobenius morphism

$$F^1_{\mathbb{G}_a(U)_r} : \mathbb{G}_a(U)_r \to (\mathbb{G}_a(U)_{r-1})^{(1)}$$

where we identify $G(n,j)^- = \mathbb{G}_a(U)_j$. The restriction to the hearts is also the first Frobenius morphism

$$F^1_{G^0} : G^0 \to (G^0)^{(1)}$$

Hence, it also respects the positive wings as $f(0)=0$ implies $0=F^1(f(0)) = (T_r(f))(0)$ and $f_0=\mathrm{id}$ implies $\mathrm{id}=F^1(f_0)=(T_r(f))_0$.

Lemma 6.4. *For all $r\geq 2$, The morphism $T_r : G(n,r)\to G(n,r-1)^{(1)}$ induces an isomorphism*

$$G(n,r)/\operatorname{Ker}(T_r) \xrightarrow{\cong} G(n,r-1)^{(1)}$$

Proof. As T_r is triangulated, the closed immersion

$$G(n,r)/\operatorname{Ker}(T_r) \hookrightarrow G(n,r-1)^{(1)}$$

translates to

$$\begin{aligned} & G(n,r)^+/\operatorname{Ker}(T_r^+) \times G^0/\operatorname{Ker}(T_r^0) \times G^-/\operatorname{Ker}(T_r^-) \\ \hookrightarrow \quad & (G(n,r-1)^+)^{(1)} \times (G^0)^{(1)} \times ((G^-)^{(1)})_{r-1} \end{aligned}$$

according to Remark 3.8. We want to show that this is an isomorphism. As T_r^- is the first Frobenius morphism and $G^-\cong \mathbb{G}_a(U)_r$, T_r induces an isomorphism on the negative wings by Proposition 1.36. Furthermore $T_r^0 = F^1_{G_0}$. So by the same Proposition again, T_r induces an isomorphism on the hearts. So it is left to show that the closed immersion

$$T_r^+ : G(n,r)^+/\operatorname{Ker}(T_r^+) \hookrightarrow (G(n,r-1)^+)^{(1)}$$

is an isomorphism. According to Lemma 1.22, this is given by the kernel of the morphism

$$(T_r^+)^\# : (k[G(n,r-1)^+)^{(1)}] \to k[G(n,r)^+]$$

So we have to show that this kernel is 0. This morphism is just

$$S^\bullet(\mathrm{Hom}_k(U, R(n,r-1)^{\geq 2})^\vee)^{(1)} \to S^\bullet \mathrm{Hom}_k(U, R(n,r)^{\geq 2})^\vee$$

which is induced by

$$\begin{array}{rcl} (\mathrm{Hom}_k(U, R(n,r-1)^{\geq 2})^\vee)^{(1)} & \xrightarrow{((\pi_*)^\vee)^{(1)}} & (\mathrm{Hom}_k(U, R(n,r)^{\geq 2})^\vee)^{(1)} \\ & \xrightarrow{f\otimes\lambda \mapsto f^p\lambda} & S^\bullet \mathrm{Hom}_k(U, R(n,r)^{\geq 2})^\vee \end{array}$$

where $\pi : R(n,r)^{\geq 2} \to R(n,r-1)^{\geq 2}$ is the projection. As both maps are injective, we get that $(T_r^+)^\#$ is injective. Whence the claim. $\square$

Now we can consider the induced functor

$$T_r^* : G(n,r-1)^{(1)}\text{–rep} \longrightarrow G(n,r)\text{–rep}$$

This has the following very useful property.

Corollary 6.5. *Let $r \geq 2$ and $\lambda \in X(T)_+$. Then we get*

$$T_r^* L(\lambda, G(n,r-1)^{(1)}) \cong L(p\lambda, G(n,r))$$

Proof. Recall that by Theorem 5.9 the $L(\lambda, G(n,r-1)^{(1)})$ and $L(\lambda, G(n,r))$ for $\lambda \in X(T)_+$, are a complete list of pairwise non-isomorphic irreducible $G(n,r-1)^{(1)}$-representations and $G(n,r)$-representations respectively.

Now by the previous Lemma and Lemma 1.23, we get

$$T_r^* L(\lambda, G(n,r-1)^{(1)}) \cong L(\mu, G(n,r))$$

for a $\mu \in X(T)_+$. By using Proposition 3.19 and the fact that T_r is triangulated with $T_r^0 = F^1_{G^0}$ and $T_r^- = F^1_{G^-}$, we obtain

$$\begin{array}{rcl} L(\mu) & = & L(\mu, G(n,r))^{G(n,r)^-} \\ & = & (T_r^* L(\lambda, G(n,r-1)^{(1)}))^{G(n,r)^-} \\ & \supset & (F^1_{G^0})^* \left(L(\lambda, G(n,r-1)^{(1)})^{(G(n,r-1)^{(1)})^-} \right) \\ & = & (F^1_{G^0})^* L(\lambda) \\ & = & L(\lambda)^{[1]} \end{array}$$

as G^0-representations. Finally we get

$$L(\mu) \supset L(\lambda)^{[1]} = L(p\lambda)$$

by Proposition 4.17. This is an inclusion of irreducible G^0-representations, hence an equality. Whence $\mu = p\lambda$ as claimed. $\square$

6.3. **Third Type.** The last type of transfer homomorphisms are

$$t_{r,i} : U_i(n,r) \to G(n,i)$$

for all $1 \le i \le r$. Recall that

$$U_i = \{f \in G(n,r) \mid f(0) \in \mathbb{G}_a(U)_i\}$$

We use the description of Proposition 2.17. So let $f \in \underline{\mathrm{Hom}}_k(U, R(n,r))$ such that $f(0) \in \mathbb{G}_a(U)_i$ and $f_0 \in \mathrm{GL}(U)$. Then we can compose f with the projection map

$$\pi : R(n,r) \to R(n,i)$$

As $(\pi \circ f)(0) = f(0) \in \mathbb{G}_a(U)_i$ and $(\pi \circ f)_0 = f_0$, we obtain an element in $G(n,i)$. So set

$$t_{r,i}(f) = \pi \circ f$$

Remark 6.6. Under the identification $R(n,r) = k[x_1, \dots, x_n]/(x_1^{p^r}, \dots, x_n^{p^r})$ this just reads as follows: For $f = (f_1, \dots, f_n) \in U_i$ with $f_j \in R(n,r)$ we just consider the residue classes of the polynomials f_j in $R(n,i)$ so

$$t_{r,i}(f_1, \dots, f_n) = (f_1, \dots, f_n) \in G(n,i)$$

This is well defined as $f_j(0)^{p^i} = 0$ for all $j = 1, \dots, n$.

Note that $t_{r,i}$ is also triangulated: The restriction to G_i^- and G^0 is just the identity, so it also respects the positive wings.

Lemma 6.7. *For all $1 \le i \le r$, the morphism $t_{r,i} : U_i(n,r) \to G(n,i)$ induces an isomorphism*

$$U_i(n,r)/\operatorname{Ker}(t_{r,i}) \xrightarrow{\cong} G(n,i)$$

Proof. As $t_{r,i}$ is triangulated, the closed immersion

$$U_i(n,r)/\operatorname{Ker}(t_{r,i}) \hookrightarrow G(n,i)$$

translates as

$$G(n,r)^+/\operatorname{Ker}(t_{r,i}^+) \times G^0/\operatorname{Ker}(t_{r,i}^0) \times G_i^-/\operatorname{Ker}(t_{r,i}^-) \hookrightarrow G(n,i)^+ \times G^0 \times G_i^-$$

according to Remark 3.8. We want to show that this is an isomorphism. As $t_{r,i}^0$ and $t_{r,i}^-$ are the identity, it suffices to show that the closed immersion

$$G(n,r)^+/\operatorname{Ker}(t_{r,i}^+) \hookrightarrow G(n,i)^+$$

is an isomorphism. According to Lemma 1.22, the defining ideal is the kernel of

$$(t_{r,i}^+)^\# : k[G(n,i)^+] \to k[G(n,r)^+]$$

But this morphism is

$$S^\bullet \operatorname{Hom}_k(U, R(n,i)^{\ge 2})^\vee \xrightarrow{S^\bullet(\pi^*)} S^\bullet \operatorname{Hom}_k(U, R(n,r)^{\ge 2})^\vee$$

which is injective as the projection $\pi : R(n,r)^{\ge 2} \to R(n,i)^{\ge 2}$ is surjective. So the defining ideal is 0. Whence the claim. □

Now we can consider the induced functor

$$t_{r,i}^* : G(n,i)\text{−rep} \longrightarrow U_i(n,r)\text{−rep}$$

It has the following very useful property.

Corollary 6.8. *Let $1 \geq i \geq r$, and $\lambda \in X(T)_+$. Then we get*

$$t_{r,i}^* L(\lambda, G(n,i)) \cong L(\lambda, U_i(n,r))$$

In particular, $t_{r,i}^*$ induces a bijection on isomorphism classes of irreducible representations.

Proof. Again by Theorem 5.9, the $L(\lambda, G(n,i))$ and $L(\lambda, U_i(n,r))$ with $\lambda \in X(T)_+$ are a complete list of pairwise non-isomorphic irreducible $G(n,i)$-representations and $U_i(n,r)$-representations respectively.

Now by the previous lemma and Lemma 1.23, we get

$$t_{r,i}^* L(\lambda, G(n,i)) \cong L(\mu, U_i(n,r))$$

for a $\mu \in X(T)_+$. Recall that $G(n,i)^- = U_i^-$ and that $t_{r,i}$ is triangulated with $t_{r,i}^0 = \mathrm{id}$. Then by using Proposition 3.19, we obtain

$$L(\mu) = L(\mu, U_i(n,r))^{U_i^-} = L(\lambda, G(n,i))^{G(n,i)^-} = L(\lambda)$$

as G^0-representations. This shows the claim. □

Remark 6.9. For $i = 1$ the morphism

$$t_{r,1} : U_1(n,r) \to G(n,1)$$

is a triangulated morphism between 1-triangulated groups. Consider the induced morphism

$$t_{r,1} : U_1(n,r)_1 \to G(n,1)_1$$

on first Frobenius kernels. Then Proposition 5.15 implies that for all $\lambda \in X_1(T)$

$$t_{r,1}^* L(\lambda, G(n,1)_1) = L(\lambda, U_1(n,r)_1)$$

That is, we get all irreducible $U_1(n,r)_1$-representations in this way by Theorem 5.14 as $X_1'(T) \subset X_1(T)$ (cf. section 4.2). Furthermore the representation theory of $U_1(n,r)_1$ is equivalent to the one of $\mathrm{Lie}(U_1(n,r)) \cong \mathrm{Lie}(G(n,r))$ as well as for $G(n,1)_1$ and $\mathrm{Lie}(G(n,1)) = W(n,(1,\ldots,1))$. That is, the induced functor

$$\mathrm{Lie}(t_{r,1})^* : \mathrm{Lie}(G(n,1)) - p - \mathrm{rep} \longrightarrow \mathrm{Lie}(G(n,r)) - p - \mathrm{rep}$$

yields a bijection on isomorphism classes of irreducible representations.

Finally, we want to understand how the map

$$\mathrm{Lie}(t_{r,1}) : \mathrm{Der}_k(R(n,r)) \to \mathrm{Der}_k(R(n,1))$$

explicitly looks like. For this recall that

$$\mathrm{Hom}_k(U, R(n,j)) \xrightarrow{\cong} \mathrm{Der}_k(R(n,j))$$

by extension as derivations. Then the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Der}_k(R(n,r)) \arrow[r, "\mathrm{Lie}(t_{r,1})"] & \mathrm{Der}_k(R(n,1)) \\
\mathrm{Hom}_k(U, R(n,r)) \arrow[u, "\cong"] \arrow[r, "\pi_*"] & \mathrm{Hom}_k(U, R(n,1)) \arrow[u, "\cong"']
\end{tikzcd}$$

where $\pi : R(n,r) \to R(n,1)$ is the projection map. That is, under the identification $R(n,j) = k[x_1,\ldots,x_n]/(x_1^{p^j},\ldots,x_n^{p^j})$, a basis element

$$\delta_{(i,I)} = x^I \frac{\partial}{\partial x_i} \in \mathrm{Der}_k(R(n,r))$$

is mapped to the element

$$\overline{x^I} \frac{\partial}{\partial x_i} \in \mathrm{Der}_k(R(n,1))$$

by taking the residue class of the monomial x^I modulo $(x_1^p,\ldots,x_n^p)$.

6.4. **Relations.** Finally, we discuss how the maps L_r, T_r, $t_{r,i}$ are related. The first two relations are rather obvious.

Lemma 6.10. *For all $r \geq 2$, the diagram*

$$\begin{tikzcd} G(n,r) \arrow[r, "T_r"] \arrow[dr, "L_r"'] & G(n,r-1)^{(1)} \arrow[d, "L_{r-1}"] \\ & (G^0)^{(r)} \end{tikzcd}$$

commutes.

Proof. We consider again $(G^0)^{(r)}$ pretriangulated with zero wings. Then $L_r = 0 \times F^r_{G^0} \times 0$ where we understand $0 : H \to 0$ as the trivial morphism for any algebraic group H. Then we get

$$L_{r-1} \circ T_r = (0 \times F^{r-1}_{(G^0)^{(1)}} \times 0) \circ (T_r^+ \times F^1_{G^0} \times F^1_{G^-}) = 0 \times F^r_{G^0} \times 0 = L_r$$

which shows the claim. □

Lemma 6.11. *For all $1 \leq i \leq r$, the diagram*

$$\begin{tikzcd} U_i(n,r) \arrow[r, "t_{r,i}"] \arrow[dr, "L_i"'] & G(n,i) \arrow[d, "L_i"] \\ & (G^0)^{(i)} \end{tikzcd}$$

commutes.

Proof. Again, we consider $(G^0)^{(i)}$ pretriangulated with zero wings and $L_i = 0 \times F^i_{G^0} \times 0$. This provides

$$L_i \circ t_{r,i} = (0 \times F^i_{G^0} \times 0) \circ (t^+_{r,i} \times \mathrm{id}_{G^0} \times \mathrm{id}_{G^-}) = 0 \times F^i_{G^0} \times 0 = L_i$$

as claimed. □

We again denote $G^- = G(n,r)^-$ and $U_i = U_i(n,r) \subset G(n,r)$. Recall that $U_i^- = G_i^-$. Our next aim is to study the induction functor $\mathrm{ind}_{U_i}^{G(n,r)}$ and its relation to the induced functors of the three morphism types.

Lemma 6.12. *For all $1 \leq i \leq r$, we get for the induction functor*

$$\mathrm{res}^{G(n,r)}_{G^-} \circ \mathrm{ind}^{G(n,r)}_{U_i} = \mathrm{ind}^{G^-}_{G_i^-} \circ \mathrm{res}^{U_i}_{G_i^-}$$

Furthermore $\mathrm{ind}^{G(n,r)}_{U_i}$ is exact.

Proof. We use the morphism description of the induction $\operatorname{ind}_{U_i}^{G(n,r)}$. Then we obtain for all U_i-representations V that

$$\begin{aligned}\operatorname{ind}_{U_i}^{G(n,r)} V &= \{f \in \operatorname{Mor}(G(n,r), V_a) \mid f(ug) = uf(g)\ \forall u \in U_i\} \\ &= \{f \in \operatorname{Mor}(G^-, V_a) \mid f(cg) = cf(g)\ \forall c \in G_i^-\}\end{aligned}$$

by using the decomposition $G(n,r) = \overline{G(n,r)}^+ \times G^-$ and $U_i = \overline{G(n,r)}^+ \times G_i^-$. The restriction of this to G^- coincides with

$$\operatorname{ind}_{G_i^-}^{G^-} \operatorname{res}_{G_i^-}^{U_i} V$$

as the G^--action is given by right translation. This shows the first claim. Now $\operatorname{ind}_{G_i^-}^{G^-}$ is exact by [Jan03, I.5.13] as

$$F_{G^-}^i : G^- \to (G^-)_{r-i}^{(i)} \subset (G^-)^{(i)}$$

induces an isomorphism

$$G^-/G_i^- \cong (G^-)_{r-i}^{(i)}$$

by Proposition 1.36. This implies the exactness of $\operatorname{ind}_{U_i}^{G(n,r)}$. □

We start with the relation of $\operatorname{ind}_{U_i}^{G(n,r)}$ to the I-functors from section 3. Recall that $G(n,r)$ is r-triangulated and U_i is i-triangulated. So let us denote the I-functor for a j-triangulated group as I_j.

Lemma 6.13. *For all $1 \le i \le r$, both triangles of the diagram*

$$\begin{tikzcd} & G(n,i)\text{-rep} \arrow[dr, "t_{r,i}^*"] & \\ G^0\text{-rep} \arrow[ur, "\mathrm{I}_i"] \arrow[rr, "\mathrm{I}_i"] \arrow[dr, "\mathrm{I}_r"'] & & U_i(n,r)\text{-rep} \arrow[dl, "\operatorname{ind}_{U_i}^{G(n,r)}"] \\ & G(n,r)\text{-rep} & \end{tikzcd}$$

commute.

Proof. The commutativity of the upper triangle follows immediately from the definition of $t_{r,i}$ and Lemma 3.16 as the negative wings of $G(n,i)$ and U_i coincide.

For the commutativity of the lower triangle observe that $\overline{G(n,r)}^+ = \overline{U_i}^+$. Then it follows from the commutativity of the diagram

$$\begin{tikzcd} & & G(n,r)\text{-rep} \\ G^0\text{-rep} \arrow[urr, bend left, "\mathrm{I}_r"] \arrow[drr, bend right, "\mathrm{I}_i"'] \arrow[r, "(-)_{\mathrm{tr}}"] & \overline{G(n,r)}^+\text{-rep} \arrow[ur, "\operatorname{ind}"] \arrow[dr, "\operatorname{ind}"'] & \\ & & U_i(n,r)\text{-rep} \arrow[uu, "\operatorname{ind}_{U_i}^{G(n,r)}"'] \end{tikzcd}$$

which involves the transitivity of induction [Jan03, I.3.5]. □

Finally there is a more complicated relation of the induction $\operatorname{ind}_{U_i}^{G(n,r)}$ to the functors L_i^*, T_j^*, and I:

Lemma 6.14. *For all $1 \leq i \leq r$, the triangle and the square of the following diagram commute up to functor isomorphism*

$$\begin{tikzcd}
 & G(n,i)\text{-rep} \arrow[dr, "t_{r,i}^*"] & \\
(G^0)^{(i)}\text{-rep} \arrow[ur, "L_i^*"] \arrow[rr, "L_i^*"] \arrow[d, "\mathrm{I}_{r-i}"'] & & U_i(n,r)\text{-rep} \arrow[d, "\operatorname{ind}_{U_i}^{G(n,r)}"] \\
G(n,r-i)^{(i)}\text{-rep} \arrow[rr, "(T^i)^*"] & & G(n,r)\text{-rep}
\end{tikzcd}$$

Here $T^i : G(n,r) \to G(n,r-i)^{(i)}$ is the composition

$$T^i := T_{r-(i-1)}^{(i-1)} \circ \cdots \circ T_r$$

Proof. The commutativity of the triangle follows from $L_i \circ t_{r,i} = L_i$. For the commutativity of the square, note that the morphism

$$T^i = T_{r-(i-1)}^{(i-1)} \circ \cdots \circ T_r$$

is triangulated as all $T_{r-j}^{(j)}$ are triangulated. More precisely

$$(T^i)^- = F_{G^-}^i : G^- \to (G^-)_{r-i}^{(i)}$$

and

$$(T^i)^0 = F_{G^0}^i : G^0 \to (G^0)^{(i)}$$

We use the morphism description of the functor I (cf. Lemma 3.16 and its proof). Let V be a $(G^0)^{(i)}$-representation. On one hand

$$(T^i)^* \, \mathrm{I}_{r-i}(V) = \operatorname{Mor}((G^-)_{r-i}^{(i)}, V_a)$$

as $(G(n,r-i)^{(i)})^- = (G^-)_{r-i}^{(i)}$. The $G(n,r)$-action is given as follows: For $g \in G(n,r)$ and $f : (G^-)_{r-i}^{(i)} \to V_a$, we get

$$(gf)(b) = (bT^i(g))_0 f((bT^i(g))_-)$$

On the other hand, G_i^- operates trivially on L_i^*V which implies

$$\begin{aligned}
\operatorname{ind}_{U_i}^{G(n,r)} L_i^*V &= \{f \in \operatorname{Mor}(G^-, L_i^*V_a) \mid f(cg) = cf(g) \; \forall c \in G_i^-\} \\
&= \operatorname{Mor}(G^-/G_i^-, L_i^*V_a)
\end{aligned}$$

(cf. the proof of Lemma 6.12). The $G(n,r)$-action is given as follows: For $g \in G(n,r)$, $f : G^- \to L_i^*V_a$ with $f(cg) = cf(g)$ for all $c \in G_i^-$, and $a \in G^-$ we get

$$(gf)(a) = f(ag) = F_{G^0}^i((ag)_0) f((ag)_-)$$

as $L_i = F_{G^0}^i \circ \pi_0$. As the i-th Frobenius $F_{G^-}^i : G^- \to (G^-)_{r-i}^{(i)}$ induces an isomorphism $G^-/G_i^- \cong (G^-)_{r-i}^{(i)}$, it induces a natural linear isomorphism

$$(T^i)^* \, \mathrm{I}_{r-i}(V) = \operatorname{Mor}((G^-)_{r-i}^{(i)}, V_a) \xrightarrow{(F_{G^-}^i)^*} \operatorname{Mor}_{G_i^-}(G^-, L_i^*V) = \operatorname{ind}_{U_i}^{G(n,r)} L_i^*V$$

For the $G(n,r)$-equivariance let $g \in G(n,r)$, $f : (G^-)^{(i)}_{r-i} \to V_a$, and $a \in G^-$. Then we can do the following computation by using the fact that T^i is triangulated with $(T^i)^- = F^i_{G^-}$ and $(T^i)^0 = F^i_{G^0}$.

$$\begin{aligned}(gf)(F^i_{G^-}(a)) &= (F^i_{G^-}(a)T^i(g))_0 f((F^i_{G^-}(a)T^i(g))_-)\\ &= (T^i(ag))_0 f((T^i(ag))_-)\\ &= F^i_{G^0}((ag)_0) f(F^i_{G^-}((ag)_-))\\ &= (g(f \circ F^i_{G^-}))(a)\end{aligned}$$

This shows the equivariance of $(F^i_{G^-})^*$:

$$(F^i_{G^-})^*(gf) = g(F^i_{G^-})^*(f)$$

Whence the commutativity of the square. $\square$

7. Differentials and Cartier's Theorem

The aim of this section is to introduce some concrete $G(n,r)$-representations which will play a crucial role in the computation of the irreducible $G(n,r)$-representations and the representation ring of $G(n,r)$.

Again, identify

$$R(n,r) = k[x_1, \dots, x_n]/(x_1^{p^r}, \dots, x_n^{p^r})$$

Recall that we have a canonical representation $G(n,r) \subset \mathrm{GL}(R(n,r))$.

Notation 7.1. For convenience, we will describe the concrete representations $G(n,r) \to \mathrm{GL}(V)$ only for k-rational points, that is, we will construct $G(n,r)(k) \to \mathrm{GL}_k(V)$ rather than $G(n,r)(A) \to \mathrm{GL}_A(V \otimes_k A)$ for all commutative k-algebras A. It will be pointed out why this suffices.

Definition 7.2. Let

$$\Omega_r := \Omega_{R(n,r)/k} = \bigoplus_{i=1}^{n} R(n,r) dx_i$$

be the *module of Kähler-differentials*. Then we obtain a representation

$$G(n,r) \to \mathrm{GL}(\Omega_r)$$

as follows: Let $g \in G(n,r)(k) = \mathrm{Aut}(R(n,r))$, that is, $g : R(n,r) \to R(n,r)$ is a k-algebra automorphism. Then consider the diagram

$$\begin{array}{ccc} R(n,r) & \xrightarrow{g} & R(n,r) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ \Omega_r & \xrightarrow{\partial g} & \Omega'_r \end{array}$$

where Ω'_r is the $R(n,r)$-module Ω_r twisted by g. Then $d \circ g$ is a differential and by the universal property of Ω_r we get ∂g which just reads as

$$\partial g(P(x_1, \dots, x_n)dx_i) = P(g_1, \dots, g_n)dg_i = P(g_1, \dots, g_n) \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} dx_j$$

with $g = (g_i)_i$ for $g_i \in R(n,r)$. This gives a representation Ω_r since for all commutative k-algebras A, we have $\Omega_{R(n,r)/k} \otimes_k A \cong \Omega_{R(n,r)_A/A}$.

Let

$$\begin{array}{ccc} R(n,r) & \xrightarrow{f^r} & R(n,r) \\ P(x_1, \dots, x_n) & \mapsto & P(x_1, \dots, x_n)^{p^r} \end{array}$$

be the r-th power of the Frobenius-morphism. It factors through

$$\begin{array}{ccc} R(n,r) & \xrightarrow{f^r} & k \\ P(x_1, \dots, x_n) & \mapsto & P(0, \dots, 0)^{p^r} \end{array}$$

Now take

$$\Omega_r \otimes_{R(n,r), f^r} k$$

which is an n-dimensional k-vector space with basis $dx_i \otimes 1$ for $i = 1, \ldots, n$. The $G(n,r)$-action on Ω_r induces a $G(n,r)$-action on $\Omega_r \otimes_{R(n,r),f} k$ by

$$g(dx_i \otimes 1) = dg_i \otimes 1 = \sum_{j=1}^{n} \left(\frac{\partial g_i}{\partial x_j}(0) \right)^p (dx_j \otimes 1)$$

for all $g = (g_i)_i \in G(n,r)$.

Remark 7.3. Note that with $U = k^n$ again, we get

$$\Omega_r \otimes_{R(n,r),f^r} k = (R(n,r) \otimes_k U) \otimes_{R(n,r),f^r} k \cong U \otimes_{k,f^r} k = U^{(r)}$$

as G^0-representations. The corresponding group homomorphism

$$G(n,r) \to \mathrm{GL}(U^{(r)})$$

coincides with the group homomorphism L_r we already discussed in the previous section (cf. Remark 6.1).

Notation 7.4. For all $i = 1, \ldots, n$ denote by Ω_r^i the i-th higher differentials. That is,

$$\Omega_r^i := \Lambda^i_{R(n,r)} \Omega_r = \bigoplus_{j_1 < \ldots < j_i} R(n,r) dx_{j_1} \wedge \ldots \wedge dx_{j_i}$$

where the i-th exterior power is taken over the ring $R(n,r)$. The $G(n,r)$-action of Ω_r extends canonically to the higher differentials by acting on all factors simultaneously.

Remark 7.5. Note that

$$\Omega_r^i \cong R(n,r) \otimes_k \Lambda^i U \cong \mathrm{I}_r(\Lambda^i U)$$

as $G(n,r)$-representations by Example 3.17. Again, we denote by I_j the I-functor for j-triangulated groups. Furthermore, for any $(G^0)^{(r)}$-representation V, we obtain

$$\mathrm{I}_r(\Lambda^i U \otimes V^{[r]}) \cong \mathrm{I}_r(\Lambda^i U) \otimes_k L_r^* V = \Omega_r^i \otimes_k L_r^* V$$

by Lemma 5.7. Moreover, for all $1 \le j \le r$, the functor

$$\mathrm{ind}_{U_j}^{G(n,r)} \circ t_{r,j}^* : G(n,j)\text{−rep} \longrightarrow G(n,r)\text{−rep}$$

maps i-th differentials to i-th differentials:

$$\begin{aligned} \mathrm{ind}_{U_j}^{G(n,r)}(t_{r,j}^*(\Omega_j^i)) &\cong \mathrm{ind}_{U_j}^{G(n,r)}(t_{r,j}^*(\mathrm{I}_j(\Lambda^i U))) \\ &\cong \mathrm{I}_r(\Lambda^i U) \\ &\cong \Omega_r^i \end{aligned}$$

by Lemma 6.13.

The higher differentials Ω_r^i are connected by the *deRham-complex*

$$0 \to R(n,r) \xrightarrow{d_1} \Omega_r^1 \xrightarrow{d_2} \cdots \xrightarrow{d_n} \Omega_r^n \to 0$$

The differentials are defined by

$$d_i(f(dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}})) := df \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}}$$

for all $j_1 < \ldots < j_{i-1}$ and $f \in R(n,r)$. In fact, this is a complex of $G(n,r)$-representations by the following Lemma.

Lemma 7.6. *For all $r \geq 1$ and $i = 1, \ldots, n$, the differential map*
$$d_i : \Omega_r^{i-1} \to \Omega_r^i$$
is a morphism of $G(n,r)$-representations.

Proof. Let us proof this by induction on i. For $i = 1$, the map
$$d_1 : R(n,r) \to \Omega_r$$
is $G(n,r)$-equivariant by the very definition of the representation Ω_r. So let $i \geq 1$ and consider d_{i+1}. Let $g = (g_i)_i \in G(n,r)$. Then
$$\begin{aligned} & d_{i+1}(g(f)dg_{j_1} \wedge \ldots \wedge dg_{j_i}) \\ = \quad & d_i(g(f)dg_{j_1} \wedge \ldots \wedge dg_{j_{i-1}}) \wedge dg_{j_i} + g(f)dg_{j_1} \wedge \ldots \wedge dg_{j_{i-1}} \wedge d_2(dg_{j_i}) \end{aligned}$$
as $d : R(n,r) \to \Omega_r$ is a derivation. The second summand vanishes since $d_2 \circ d_1 = 0$. Hence we get by induction hypothesis
$$\begin{aligned} d_{i+1}(g(f)dg_{j_1} \wedge \ldots \wedge dg_{j_i}) &= d_i(g(f)dg_{j_1} \wedge \ldots \wedge dg_{j_{i-1}}) \wedge dg_{j_i} \\ &= g(d_i(fdx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}})) \wedge dg_{j_i} \\ &= g(d_{i+1}(fdx_{j_1} \wedge \ldots \wedge dx_{j_i})) \end{aligned}$$
This shows the $G(n,r)$-equivariance of d_{i+1} and hence the claim. $\square$

Remark 7.7. For all $1 \leq j \leq r$, one can check that the functor
$$\mathrm{ind}_{U_j}^{G(n,r)} \circ t_{r,j}^* : G(n,j)\text{−rep} \longrightarrow G(n,r)\text{−rep}$$
maps differential maps to differential maps:
$$\mathrm{ind}_{U_j}^{G(n,r)}(t_{r,j}^*(d_i : \Omega_j^{i-1} \to \Omega_j^i)) = (d_i : \Omega_r^{i-1} \to \Omega_r^i)$$
As $\mathrm{ind}_{U_j}^{G(n,r)}$ is exact by Lemma 6.12, we also get
$$\mathrm{ind}_{U_j}^{G(n,r)}(t_{r,j}^*(H^i(\Omega_j^\bullet))) = H^i(\Omega_r^\bullet)$$
That is, the cohomology of $\Omega_r^\bullet$ can be computed by the one of $\Omega_j^\bullet$ for all $1 \leq j \leq r$.

We will start by computing the cohomology of $\Omega_1^\bullet$. By the previous Remark, this will also provide a computation of the cohomology of $\Omega_r^\bullet$ for any $r \geq 1$. The following Lemma provides a computation of the complex $\Omega_1^\bullet$ in terms of 1-deRham-complexes.

Lemma 7.8. *Let $K^\bullet(n) := \Omega^\bullet_{R(n,1),k}$ the n-th deRham-complex for $r = 1$. Then*
$$K^\bullet(n) \cong K^\bullet(1)^{\otimes_k n}$$
as complexes of k-vector spaces.

Proof. Let us proof the claim by induction on n. The case $n = 1$ is clear, so let $n \geq 2$. Then by induction hypothesis we obtain
$$K^\bullet(1)^{\otimes_k n} \cong K^\bullet(n-1) \otimes_k K^\bullet(1)$$
By the definition of tensor products of complexes, we obtain
$$\left(K^\bullet(1)^{\otimes_k n}\right)^i \cong \bigoplus_{l+s=i} K^l(n-1) \otimes_k K^s(1)$$

with differentials

$$d(a_l \otimes b_s) = d(a_l) \otimes b_s + (-1)^l a_l \otimes d(b_s)$$

for $a_l \in K^l(n-1)$ and $b_s \in K^s(1)$. Let us denote $C = k[x_n]/x_n^p$ and $K^\bullet(1) = \Omega^\bullet_{C,k}$. Then using

$$K^s(1) = \begin{cases} C & s = 0 \\ Cdx_n & s = 1 \\ 0 & s \geq 2 \end{cases}$$

we see that

$$\left(K^\bullet(1)^{\otimes_k n}\right)^i = K^i(n-1) \otimes_k C \oplus K^{i-1}(n-1) \otimes_k Cdx_n$$

Now note that

$$\begin{array}{rcl} R(n,1) & \to & R(n-1,1) \otimes_k C \\ x_1^{s_1} \cdots x_n^{s_n} & \mapsto & x_1^{s_1} \cdots x_{n-1}^{s_{n-1}} \otimes x_n^{s_n} \end{array}$$

is an isomorphism. So we obtain

$$K^i(n-1) \otimes_k C \cong \bigoplus_{j_1 < \ldots < j_i \leq n-1} R(n,1) dx_{j_1} \wedge \ldots \wedge dx_{j_i}$$

and

$$K^{i-1}(n-1) \otimes_k Cdx_n \cong \bigoplus_{j_1 < \ldots < j_{i-1} \leq n-1} R(n,1) dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}} \wedge dx_n$$

That is,

$$\left(K^\bullet(1)^{\otimes_k n}\right)^i \cong K^i(n)$$

as k-vector spaces. Recall that we can write the differential on $K^i(n)$ as

$$d(f(x_1, \ldots, x_n) dx_{j_1} \wedge \ldots \wedge dx_{j_i}) = \sum_{k=1}^n \frac{\partial}{\partial x_k} f dx_k \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_i}$$

On $K^{i-1}(n-1) \otimes_k Cdx_n$, the differential translates to

$$d(a \otimes f(x_n) dx_n) = d(a) \otimes f(x_n) dx_n$$

since $d(Cdx_n) = 0$ which corresponds to the one on $K^i(n)$. Finally for $K^i(n-1) \otimes_k C$, we obtain

$$d(a \otimes f(x_n)) = d(a) \otimes f(x_n) + (-1)^i a \otimes \frac{\partial}{\partial x_n} f(x_n) dx_n$$

which also corresponds to the usual differential on $K^i(n)$ since

$$dx_n \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_i} = (-1)^i dx_{j_1} \wedge \ldots \wedge dx_{j_i} \wedge dx_n$$

for $j_1 < \ldots < j_i < n$. This finishes the proof. $\square$

The following Theorem is due to Cartier and computes the cohomology of the deRham-complex in the case $r = 1$. Its proof is directly taken from the one given in [Kat70, Theorem 7.2] where it is stated in its algebraic geometric version. Our version is the representation theoretic analogue.

Theorem 7.9 (Cartier)**.** *There is a unique collection of isomorphisms of $G(n,1)$-representations*

$$C^{-1} : L_1^* \Lambda^i U^{(1)} \to H^i(\Omega_1^\bullet)$$

which satisfies

(1) $C^{-1}(1) = 1$
(2) $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$
(3) $C^{-1}(df \otimes 1) = [f^{p-1}df] \in H^1(\Omega_1^\bullet)$

Proof. Recall that

$$L_1^* U^{(1)} = \Omega_1 \otimes_{R(n,1),f} k$$

and observe

$$L_1^* \Lambda^i U^{(1)} = \Lambda^i L_1^*(U^{(1)})$$

So let us start by defining

$$\begin{array}{rcl} R(n,1) \times k & \stackrel{\delta}{\to} & H^1(\Omega_1^\bullet) \\ (f,s) & \mapsto & [sf^{p-1}df] \end{array}$$

Note that $sf^{p-1}df$ is an element of $\mathrm{Ker}(d_2)$ since $df \wedge df = 0$. So δ is well-defined as a map of sets. In order to get that δ induces our desired map C^{-1} we have to check that δ is k-bilinear and a derivation. For the biadditivity, it suffices to check

$$\delta(f+g,s) = \delta(f,s) + \delta(g,s)$$

So let us consider the difference:

$$\begin{array}{rl} & \delta(f+g,s) - \delta(f,s) - \delta(g,s) \\ = & s((f+g)^{p-1}(df+dg) - f^{p-1}df - g^{p-1}dg) \\ = & d\left(s\left(\dfrac{(f+g)^p - f^p - g^p}{p}\right)\right) \end{array}$$

Note that the last expression makes sense since every $\binom{p}{i}$ with $1 \le i \le p-1$ appearing in the binomial expansion is divisible by p. So the difference lies in $\mathrm{Im}(d_1)$ and hence vanishes in $H^1(\Omega^\bullet) = \mathrm{Ker}(d_2)/\mathrm{Im}(d_1)$.

To complete the bilinearity, observe that for all $f \in R(n,1)$ and $s, s' \in k$ we have

$$\delta(fs, s') = s'(fs)^{p-1}dfs = s's^p(f^{p-1}df) = \delta(f, s^p s')$$

In order to get the property of a derivation, we have to check

$$\delta(fg,s) = g^p \delta(f,s) + f^p \delta(g,s)$$

This follows from

$$\delta(fg,s) = (fg)^{p-1}d(fg) = f(fg)^{p-1}dg + g(fg)^{p-1}df = g^p\delta(f,s) + f^p\delta(g,s)$$

That is, we get our k-linear map C^{-1} for $i = 1$. By the demanded property (2), it uniquely extends to larger i.

The next step is to check, that C^{-1} is in fact a map of $G(n,1)$-representations. Again by property (2), it suffices to check this for $i = 1$ which is obvious since for all $g \in G(n,1)$ we have

$$C^{-1}(g(df \otimes 1)) = C^{-1}(dg(f) \otimes 1) = g(f)^{p-1}dg(f) = g(f^{p-1}df) = gC^{-1}(df \otimes 1)$$

The last step is to show that for all i the morphism C^{-1} is an isomorphism. Define complexes

$$K^\bullet(n) := \Omega^\bullet_{R(n,1),k}$$

for all $n \in \mathbb{N}$. Then

$$K^\bullet(n) \cong K^\bullet(1)^{\otimes_k n}$$

as complexes of k-vector spaces by the previous Lemma. Then we get by the Künneth formula [Wei94, Theorem 3.6.3]

$$H^i(K^\bullet(n)) \cong \bigoplus_{j_1+\ldots+j_n=i} \big(H^{j_1}(K^\bullet(1)) \otimes_k \ldots \otimes_k H^{j_n}(K^\bullet(1))\big)$$

Furthermore, we have

$$H^j(K^\bullet(1)) = \begin{cases} k & j=0 \\ k[x_1^{p-1}dx_1] & j=1 \\ 0 & j \geq 2 \end{cases}$$

since we have

$$\mathrm{Im}(d : R(1,1) \to \Omega_{R(1,1),k}) = \bigoplus_{i=0}^{p-2} x_1^i dx_1$$

This provides

- $H^0(K^\bullet(n)) = k$
- $H^1(K^\bullet(n)) = \bigoplus_{i=1}^n k[x_i^{p-1}dx_i]$
- $H^i(K^\bullet(n)) = \Lambda^i H^1(K^\bullet(n))$

which shows that C^{-1} is an isomorphism for all $i = 1, \ldots, n$. □

In fact, the Theorem provides an explicit k-basis for the cohomology spaces of the deRham-complex:

Corollary 7.10. *A k-basis of $H^i(\Omega_1^\bullet)$ is given by the classes of*

$$x_{n_1}^{p-1} \cdots x_{n_i}^{p-1} dx_{n_1} \wedge \ldots \wedge dx_{n_i}$$

for all $n_1 < \ldots < n_i$.

Proof. We use the isomorphism

$$C^{-1} : \Omega_1 \otimes_{R(n,1),f} k \to H^i(\Omega_1^\bullet)$$

of Cartier's Theorem. By the second and the third property of C^{-1}, it acts as

$$(dx_{n_1} \otimes 1) \wedge \ldots \wedge (dx_{n_i} \otimes 1) \mapsto [x_{n_1}^{p-1} \cdots x_{n_i}^{p-1} dx_{n_1} \wedge \ldots \wedge dx_{n_i}]$$

on the canonical k-basis of $\Omega_1 \otimes_{R(n,1),f} k$ which shows the claim. □

In fact, a general result computing the deRham-cohomology for all $r \geq 1$ can be deduced from Cartier's Theorem by the use of the transfer morphisms and their properties of the previous section. Recall the transfer homomorphism of second type

$$T_r : G(n,r) \to G(n,r-1)^{(1)}$$

Corollary 7.11. *For all $r \geq 2$ we get an isomorphism*

$$H^i(\Omega_r^\bullet) \cong T_r^*((\Omega_{r-1}^i)^{(1)})$$

of $G(n,r)$-representations for all $1 \leq i \leq n$.

Proof. According to Remark 7.7, Cartier's Theorem, and Lemma 6.14, we obtain

$$\begin{aligned} H^i(\Omega_r^\bullet) &\cong \operatorname{ind}_{U_1}^{G(n,r)}(t_{r,1}^*(H^i(\Omega_1^\bullet))) \\ &\cong \operatorname{ind}_{U_1}^{G(n,r)}(t_{r,1}^*(L_1^*\Lambda^i U^{(1)})) \\ &\cong T_r^*(\mathrm{I}_{r-1}(\Lambda^i U^{(1)}) \\ &\cong T_r^*((\Omega_{r-1}^i)^{(1)}) \end{aligned}$$

Whence the claim. □

Our next aim is to define more general deRham-complexes and compute their cohomology in the same fashion as for $\Omega_r^\bullet$: By Remark 7.5 and Remark 7.7, for each $(G^0)^{(1)}$-representation V, we obtain a complex

$$\Omega_1^\bullet \otimes_k L_1^* V \cong \mathrm{I}_1(\Lambda^\bullet U \otimes V^{[1]})$$

of $G(n,1)$-representations. By Cartier's Theorem, its cohomology computes as

$$\begin{aligned} H^i(\mathrm{I}_1(\Lambda^\bullet U \otimes V^{[1]})) &\cong H^i(\Omega_1^\bullet) \otimes_k L_1^* V \\ &\cong L_1^* \Lambda^i U^{(1)} \otimes_k L_1^* V \\ &\cong L_1^*(\Lambda^i U^{(1)} \otimes V) \end{aligned}$$

Consider again the functor

$$\operatorname{ind}_{U_1}^{G(n,r)} \circ t_{r,1}^* : G(n,1)\mathrm{-rep} \longrightarrow G(n,r)\mathrm{-rep}$$

By Lemma 6.13, we get a complex

$$\operatorname{ind}_{U_1}^{G(n,r)}(t_{r,1}^*(\mathrm{I}_1(\Lambda^\bullet U \otimes V^{[1]}))) = \mathrm{I}_r(\Lambda^\bullet U \otimes V^{[1]})$$

of $G(n,r)$-representations. Furthermore $\operatorname{ind}_{U_1}^{G(n,r)}$ is exact by Lemma 6.12. According to Lemma 6.14, for $r \geq 2$ we obtain

$$\begin{aligned} H^i(\mathrm{I}_r(\Lambda^\bullet U \otimes V^{[1]})) &\cong \operatorname{ind}_{U_1}^{G(n,r)}(t_{r,1}^*(H^i(\mathrm{I}_1(\Lambda^\bullet U \otimes V^{[1]})))) \\ &\cong \operatorname{ind}_{U_1}^{G(n,r)}(t_{r,1}^*(L_1^*(\Lambda^i U^{(1)} \otimes_k V))) \\ &\cong T_r^*(\mathrm{I}_{r-1}(\Lambda^i U^{(1)} \otimes_k V)) \end{aligned}$$

Finally note that under the isomorphism of G^0-representations

$$\mathrm{I}_r(\Lambda^i U \otimes V^{[1]}) \cong \Omega_r^i \otimes_k V^{[1]}$$

the differentials of the complex $\mathrm{I}_r(\Lambda^\bullet U \otimes V^{[1]})$ read as

$$\Omega_r^{i-1} \otimes_k V^{[1]} \xrightarrow{d_i \otimes \mathrm{id}_V} \Omega_r^i \otimes_k V^{[1]}$$

For the cohomology one obtains

$$H^i(\Omega_r^\bullet \otimes_k V^{[1]}) \cong T_r^*((\Omega_{r-1}^i)^{(1)} \otimes_k V)$$

8. Irreducible $G(n,r)$-Representations

The aim of this section is to give a computation of

$$L(\lambda, G(n,r)) = \operatorname{soc} \mathrm{I}(L(\lambda)) = G(n,r)L(\lambda) \subset \mathrm{I}(L(\lambda)$$

for all $\lambda \in X(T)_+$ (cf. section 5). These are a complete list of pairwise non-isomorphic $G(n,r)$-representations by Theorem 5.9. According to the results of section 5, there is a mod p^r-periodicity for the dominant weights and one can restrict to the $L(\lambda, G(n,r)_r)$ with $\lambda \in X'_r(T)$. Actually, this will be our way to cover the case $r = 1$ where we have a mod p-periodicity. But for the case $r \geq 2$, we will also establish a mod p-periodicity by using a recursive description with respect to the group $G(n, r-1)^{(1)}$.

The mod p-periodicity requires the following notation. First recall that

$$X'_r(T) := \left\{ \lambda = \sum_{i=1}^{n} n_i(\epsilon_1 + \ldots + \epsilon_i) \in X(T) \mid \forall\ 1 \leq i \leq n : 0 \leq n_i < p^r \right\}$$

Notation 8.1. As $X'_1(T)$ is a set of representatives for $X(T)/pX(T)$, we get a decomposition

$$\lambda = r(\lambda) + ps(\lambda)$$

for all $\lambda \in X(T)_+$ with $r(\lambda) \in X'_1(T)$. We call $r(\lambda)$ the mod *p-reduction* of λ.

Remark 8.2. Note that for $\lambda \in X(T)_+$, also $s(\lambda) \in X(T)_+$. Furthermore, for

$$\lambda = \sum_{i=1}^{n} n_i(\epsilon_1 + \ldots + \epsilon_i) \in X(T)_+$$

the $r(\lambda)$ and $s(\lambda)$ read as

$$r(\lambda) = \sum_{i=1}^{n} r_i(\epsilon_1 + \ldots + \epsilon_i) \in X(T)_+$$

and

$$s(\lambda) = \sum_{i=1}^{n} s_i(\epsilon_1 + \ldots + \epsilon_i) \in X(T)_+$$

where

$$n_i = s_i p + r_i$$

with $0 \leq r_i \leq p-1$.

Let us start with the dominant weights λ with $r(\lambda) = 0$. That is, they are divisible by p.

Proposition 8.3. *Let $\lambda \in X(T)_+$ with $r(\lambda) = 0$. Then for $r = 1$ we obtain*

$$L(\lambda, G(n,1)) \cong L_1^* L(s(\lambda))$$

and for $r \geq 2$ we get

$$L(\lambda, G(n,r)) \cong T_r^* L(s(\lambda), G(n, r-1)^{(1)})$$

Proof. As $\lambda = ps(\lambda)$, the claim for $r = 1$ is just Proposition 5.10 and the claim for $r \geq 2$ is Corollary 6.5. $\square$

8.1. **Lie Algebra Action on** $\mathrm{I}(V)$**.** Before we proceed with the case $r(\lambda) \neq 0$, we need to understand the Lie algebra action of $\mathrm{Lie}(G(n,r))$ on $\mathrm{I}(V)$, since we will use these operators to compute the generated subrepresentations

$$G(n,r)V \subset \mathrm{I}(V)$$

for $V = L(\lambda)$ with $\lambda \in X(T)_+$. According to Lemma 2.38, we can compute this action by restricting to the three subgroups G^-, G^0, and G^+. Since $\mathrm{I}(V) = R(n,r) \otimes_k V$ as G^0-representations, we know that for $f \in \mathrm{End}(U) = \mathrm{Lie}(G^0)$, we get

$$f(P \otimes v) = f(P) \otimes v + P \otimes f(v)$$

for all $P \in R(n,r)$ and $v \in V$. The next Lemma treats the subgroup G^-.

Lemma 8.4. *The* $\mathrm{Lie}(G^-)$*-basis element* δ_i *acts on* $\mathrm{I}(V) = R(n,r) \otimes_k V$ *as*

$$\frac{\partial}{\partial x_i} \otimes \mathrm{id}_V : R(n,r) \otimes_k V \to R(n,r) \otimes_k V$$

Proof. For the computation of the action of δ_i for $i = 1, \ldots, n$ we take the i-th component of $G^- \cong (\mathbb{G}_a^1)^n$ and restrict the $k[G(n,r)]$-comodule map of $\mathrm{I}(V)$ to this. This provides the $k[\mathbb{G}_a^1] = k[a]/a^p$-comodule map

$$\begin{array}{rcl} R(n,r) \otimes_k V & \to & R(n,r) \otimes_k V \otimes k[a]/a^p \\ P(x_1, \ldots, x_n) \otimes v & \mapsto & P(x_1, \ldots, x_{i-1}, a + x_i, x_{i+1}, \ldots, x_n) \otimes v \end{array}$$

by Example 3.17. We obtain the δ_i-action by composing this map with $\frac{\partial}{\partial a}\big|_{a=0}$ which provides the claimed one by using the chain rule for the a-derivation. □

Now it is left to compute the action of $\mathrm{Lie}(G^+)$. Recall that we computed in Corollary 2.37 as a canonical basis of $\mathrm{Lie}(G(n,r))$ the operators

$$\delta_{(i,I)} = x^I \frac{\partial}{\partial x_i} \in \mathrm{End}(R(n,r))$$

Lemma 8.5. *Let* $\delta_{(i,I)} \in \mathrm{Lie}(G(n,r))$ *be a canonical basis element and* V *a* G^0*-representation. Then the induced action on* $\mathrm{I}(V)$ *reads as*

$$\delta_{(i,I)}(P \otimes v) = \left(x^I \frac{\partial}{\partial x_i} P \right) \otimes v + \sum_{j=1}^{n} P \frac{\partial}{\partial x_j} x^I \otimes E_{ji}(v)$$

for all $P \in R(n,r)$ *and* $v \in V$.

Proof. By the previous Lemma, the formula holds for $x^I = 0$ as in this case

$$\delta_{(i,I)} = \delta_i \in \mathrm{Lie}(G^-)$$

Also, it holds for $x^I = x_j$ since in this case

$$\delta_{(i,I)} = E_{ji} \in M_n(k) = \mathrm{Lie}(G^0)$$

That is, it is left to prove it for $\delta_{(i,I)} \in \mathrm{Lie}(G^+)$. Recall that this basis element corresponds to the affine direction $g = (g_1, \ldots, g_n)$ of G^+ with

$$g_j = \begin{cases} x_i + a x^I & j = i \\ x_j & j \neq i \end{cases}$$

Thus, according to Example 3.17, the action computes as the composition of

$$R(n,r) \otimes_k V \xrightarrow{\Delta_{R(n,r)}} R(n,r) \otimes_k V \otimes_k k[a] \xrightarrow{B} R(n,r) \otimes_k V \otimes_k k[a]$$

with $\left(\frac{\partial}{\partial a}\right)\Big|_{a=0}$ where

$$B = \left(\frac{\partial g_k}{\partial x_s}\right)_{sk} \in M_n(R(n,r) \otimes_k k[a])$$

Hence,

$$\begin{aligned}
\delta_{(i,I)}(P \otimes v) &= \frac{\partial}{\partial a}\left(B(g(P) \otimes v)\right)\Big|_{a=0} \\
&= \frac{\partial}{\partial a}\left(g(P)B(1 \otimes v)\right)\Big|_{a=0} \\
&= \left(\frac{\partial}{\partial a}(g(P))B(1 \otimes v)\right)\Big|_{a=0} + \left(g(P)\frac{\partial}{\partial a}\left(B(1 \otimes v)\right)\right)\Big|_{a=0} \\
&= \delta_{(i,I)}(P) \otimes v + P\left(\frac{\partial}{\partial a}\left(B(1 \otimes v)\right)\Big|_{a=0}\right)
\end{aligned}$$

since $g(P)|_{a=0} = P$ and $B|_{a=0} = E$ the unitary matrix. Now by Corollary 2.37, we obtain

$$\delta_{(i,I)}(P) = x^I \frac{\partial}{\partial x_i} P$$

That is, in order to get our desired formula, it is left to show that

$$\frac{\partial}{\partial a}\left(B(1 \otimes v)\right)\Big|_{a=0} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} x^I \otimes E_{ji}(v)$$

We compute the left hand side as follows: The matrix B corresponds to a unique k-algebra homomorphism

$$k[a_{sk}]_{sk} \xrightarrow{f} R(n,r) \otimes k[a]$$

Then the action of this matrix on $1 \otimes v$ is computed by the image of v under the map

$$V \xrightarrow{\Delta_V} V \otimes_k k[a_{sk}]_{sk} \xrightarrow{\mathrm{id}_V \otimes f} V \otimes_k R(n,r) \otimes_k k[a]$$

The right hand side is computed by the image of v under the map

$$V \xrightarrow{\sum_{l,j} E_{lj}} \bigoplus_{l,j} V \xrightarrow{g_{\partial B}} V \otimes_k R(n,r)$$

Here the map $g_{\partial B}$ is defined as follows: The matrix $\partial B \in M_n(R(n,r))$ is derived from B by applying $\frac{\partial}{\partial a}\big|_{a=0}$ to each entry of B. Then this matrix induces the map $g_{\partial B}$ by

$$\bigoplus_{l,j} V \xrightarrow{(v_{lj})_{lj} \mapsto (v_{lj} \otimes (\partial B)_{lj})_{lj}} \bigoplus_{l,j} V \otimes_k R(n,r) \xrightarrow{\sum} V \otimes_k R(n,r)$$

We already know that the action of E_{lj} on V is computed by

$$\begin{tikzcd} V \arrow[r, "\Delta_V"] \arrow[d, "E_{lj}"'] & V \otimes_k k[a_{sk}]_{sk} \arrow[dl, "\frac{\partial}{\partial a_{lj}}\big|_{a_{sk}=\delta_{sk}}"] \\ V & \end{tikzcd}$$

where δ_{sk} is the Kronecker-δ. That is, in order to get our claimed equality above, we have to show that the diagram

$$\begin{tikzcd} k[a_{sk}]_{sk} \arrow[r, "f"] \arrow[d, "\sum_{l,j} \frac{\partial}{\partial a_{l,j}}\big|_{a_{sk}=\delta_{sk}}"'] & R(n,r) \otimes_k k[a] \arrow[d, "\frac{\partial}{\partial a}\big|_{a=0}"] \\ \bigoplus_{l,j} k \arrow[r, "g_{\partial B}"] & R(n,r) \end{tikzcd}$$

commutes. For this, note that the composition

$$k[a_{sk}]_{sk} \xrightarrow{f} R(n,r) \otimes_k k[a] \xrightarrow{\frac{\partial}{\partial a}\big|_{a=0}} R(n,r)$$

is a k-derivation. Here the $k[a_{sk}]_{sk}$-module structure on $R(n,r)$ is induced by the composition

$$k[a_{sk}]_{sk} \xrightarrow{f} R(n,r) \otimes_k k[a] \xrightarrow{a=0} R(n,r)$$

which corresponds to the matrix $B|_{a=0} = E$. That is, this module structure is the same as the one given by

$$k[a_{sk}]_{sk} \xrightarrow{\epsilon_{\mathrm{GL}_n}} k \hookrightarrow R(n,r)$$

As a k-derivation, we get a unique factorization

$$\begin{tikzcd} k[a_{sk}]_{sk} \arrow[r, "f"] \arrow[d, "d"'] & R(n,r) \otimes_k k[a] \arrow[d, "\frac{\partial}{\partial a}\big|_{a=0}"] \\ \bigoplus_{l,j} k[a_{sk}]_{sk} da_{lj} \arrow[r, "g"] & R(n,r) \end{tikzcd}$$

by a $k[a_{sk}]_{sk}$-module map g since $\Omega_{k[a_{sk}]_{sk},k} = \bigoplus_{l,j} k[a_{sk}]_{sk} da_{lj}$. Now it is left to show that for all pairs l, j the diagram

$$\begin{tikzcd} k[a_{sk}]_{sk} da_{lj} \arrow[r, "g_{lj}"] \arrow[d, "\epsilon_{\mathrm{GL}_n}"'] & R(n,r) \\ k \arrow[ur, "\cdot(\partial B)_{lj}"'] & \end{tikzcd}$$

commutes. This follows from the fact that

$$g(da_{lj}) = \frac{\partial}{\partial a}(f(a_{lj}))\big|_{a=0} = (\partial B)_{lj}$$

and that the $k[a_{sk}]_{sk}$-module structure on $R(n,r)$ can be described by

$$k[a_{sk}]_{sk} \xrightarrow{\epsilon_{\mathrm{GL}_n}} k \hookrightarrow R(n,r)$$

This finishes the proof. $\square$

Remark 8.6. Recall that $\mathrm{Lie}(G(n,1)) \cong W(n,(1,\ldots,1))$ is the Jacobson-Witt algebra. Then we get as a corollary that

$$\mathrm{I}(V)|_{\mathrm{Lie}(G(n,1))} \cong \Delta_{V|_{\mathrm{Lie}(G^0)}}$$

as $\mathrm{Lie}(G(n,1)$-representations where the right hand side is defined in [Nak92, II.2]. That is, [Nak92, Proposition 2.2.4] implies that

$$\mathrm{I}(V)|_{\mathrm{Lie}(G(n,1))} \cong \mathrm{ind}_{\mathrm{Lie}(\overline{G(n,1)}^+)}^{\mathrm{Lie}(G(n,1))}(V|_{\mathrm{Lie}(G^0)})$$

since $\mathrm{Lie}(\overline{G(n,1)}^+) = B^+$ in Nakano's notation. Be aware of the fact that the "induction" ind in the theory of Lie-algebra representations is left adjoint to the restriction functor. That is, it corresponds to coinduction of representations of finite algebraic groups as it occurs in [Jan03, I.8.14].

8.2. **Fundamental Weights.** So let us proceed with the $\lambda \in X(T)_+$ with $r(\lambda) = \epsilon_1 + \ldots + \epsilon_i$ is a fundamental weight. By Lemma 4.14 and Proposition 4.20, we know that

$$L(\lambda) \cong L(\epsilon_1 + \ldots + \epsilon_i) \otimes_k L(s(\lambda)^{[1]} \cong \Lambda^i U \otimes_k L(s(\lambda))^{[1]}$$

with $U = k^n$. By Remark 7.5, we know that

$$\mathrm{I}_1(L(\lambda)) \cong \Omega_1^i \otimes_k L_1^* L(s(\lambda))$$

which is the i-th space of the deRham-complex of $G(n,1)$-representations. At the end of section 7, we also introduced a complex

$$\mathrm{I}_r(\Lambda^\bullet U \otimes L(s(\lambda))^{[1]})$$

of $G(n,r)$-representations. That is,

$$\mathrm{I}_r(L(\lambda)) \cong \Omega_r^i \otimes_k L(s(\lambda))^{[1]}$$

is the i-th space of that complex. Recall that the differentials read as

$$\Omega_r^{i-1} \otimes L(s(\lambda))^{[1]} \xrightarrow{d_i \otimes \mathrm{id}} \Omega_r^i \otimes_k L(s(\lambda))^{[1]}$$

This complex allows us to compute the socles of the $\mathrm{I}_r(\Lambda^i U \otimes_k L(s(\lambda))^{[1]})$ which are in fact the images of the differentials:

Proposition 8.7. *Let $\lambda \in X(T)_+$ with $r(\lambda) = \epsilon_1 + \ldots + \epsilon_i$, then*

$$L(\lambda, G(n,r)) \cong \mathrm{soc}(\Omega_r^i \otimes_k L(s(\lambda))^{[1]}) = \mathrm{Im}(d_i \otimes \mathrm{id})$$

where $d_i : \Omega_r^{i-1} \to \Omega_r^i$ is the deRham-differential.

Proof. We will use $\mathrm{Lie}(G(n,r))$-operators to prove the claim. Note that under the isomorphism

$$\mathrm{I}_r(\Lambda^i U \otimes L(s(\lambda))^{[1]}) \cong \Omega_r^i \otimes_k L(s(\lambda))^{[1]}$$

of G^0-representation and in view of Lemma 8.5, an element $f \in \mathrm{Lie}(G(n,r))$ acts as $f_{|\Omega_r^i} \otimes \mathrm{id}$ since $\mathrm{Lie}(G^0)$ acts trivially on $L(s(\lambda))^{[1]}$.

According to Proposition 3.19, the socle is generated by the G^--invariants as a $G(n,r)$-representation. A generating system of these invariants is given by

$$(dx_{j_1} \wedge \ldots \wedge dx_{j_i}) \otimes v$$

for all $j_1 < \ldots < j_i$ and $v \in L(s(\lambda))^{[1]}$. Now the inclusion

$$\mathrm{soc}(\Omega_r^i \otimes L(s(\lambda))^{[1]}) \subset \mathrm{Im}(d_i \otimes \mathrm{id})$$

follows from the fact that the generators lie in the image of $d_i \otimes \mathrm{id}$:

$$(dx_{j_1} \wedge \ldots \wedge dx_{j_i}) \otimes v = (d_i \otimes \mathrm{id})((x_{j_1}(dx_{j_2} \wedge \ldots \wedge dx_{j_i})) \otimes v) \in \mathrm{Im}(d_i)$$

For the inclusion $\mathrm{Im}(d_i \otimes \mathrm{id}) \subset \mathrm{soc}(\Omega_r^i \otimes L(s(\lambda))^{[1]})$ note that $\mathrm{Im}(d_i \otimes \mathrm{id})$ is as a k-vector space generated by

$$(dx^I \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}}) \otimes v$$

for each monomial $x^I = x_1^{n_1} \cdots x_n^{n_n} \in R(n,r)$ and $j_1 < \ldots < j_{i-1}$. As $i-1 < n$, there is an index $l \notin \{j_1, \ldots, j_{i-1}\}$. Then we get that the Lie algebra operator $\delta_{(l,I)} \in \mathrm{Lie}(G(n,r))$ acts as

$$\delta_{(l,x^I)}((dx_l \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}}) \otimes v) = (dx^I \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}}) \otimes v$$

according to Lemma 8.5. This provides all image elements from the generators. □

8.3. **The Final Case.** In the remaining case where the mod p-reduction $r(\lambda)$ of a dominant weight λ is neither 0 nor a fundamental weight, we will in fact get that the socle of $\mathrm{I}(L(\lambda))$ is everything if we assume $\mathrm{char}(k) \neq 2$. In order to prove this, we need the following Lemma.

Lemma 8.8. *Let L be an irreducible G^0-representation. If for any $v \in L$ with $v \neq 0$ we have*

$$R(n,r) \otimes_k kv \subset G(n,r)L = \mathrm{soc}\,\mathrm{I}_r(L)$$

then we obtain

$$\mathrm{soc}\,\mathrm{I}_r(L) = \mathrm{I}_r(L)$$

Proof. First of all, we know that $G^0 v = L$ by the irreducibility of L and $v \neq 0$. As we have

$$\mathrm{I}(L) = R(n,r) \otimes_k L = R(n,r) \otimes_k (G^0 v)$$

as G_0-representations, the claim follows from

$$R(n,r) \otimes_k (G^0 v) = G^0(R(n,r) \otimes_k kv)$$ □

Now we are ready to compute the last case which requires lots of computations. In fact, this is the only case where we have to assume that $\mathrm{char}(k) \neq 2$. Note that for the case $r = 1$ similar computations are given in [Nak92, II.§3]. In fact, the claim for $r = 1$ follows from [Nak92, II.§3] as indicated at the end of section 5. But we also need to treat the case $r \geq 2$ which is not covered by the computations given there.

Proposition 8.9. *Assume that* $\mathrm{char}(k) \neq 2$. *Let $\lambda \in X(T)_+$ a dominant weight with $r(\lambda) \neq 0$ and $r(\lambda) \neq \epsilon_1 + \ldots + \epsilon_i$ for all $i = 1, \ldots, n$. Then*

$$L(\lambda, G(n,r)) = \mathrm{I}_r(L(\lambda))$$

Proof. Again we will use $\mathrm{Lie}(G(n, r))$-operators to prove the claim. Write $\lambda = r(\lambda) + ps(\lambda)$. Then

$$L(\lambda) \cong L(r(\lambda)) \otimes_k L(s(\lambda))^{[1]}$$

by Proposition 4.20. Note that under the isomorphism

$$\mathrm{I}_r(L(\lambda)) \cong I_r(L(r(\lambda))) \otimes_k L(s(\lambda))^{[1]}$$

of G^0-representations and in view of Lemma 8.5, an element $f \in \mathrm{Lie}(G(n, r))$ acts as $f_{|I_r(L(r(\lambda)))} \otimes \mathrm{id}$ since $\mathrm{Lie}(G^0)$ acts trivially on $L(s(\lambda))^{[1]}$. Furthermore, according to the previous Lemma, it suffices to show

$$R(n, r) \otimes_k kw \subset G(n, r)L(\lambda) \subset \mathrm{I}(L(\lambda))$$

for a $w \in L(\lambda)$ with $w \neq 0$. As we use $\mathrm{Lie}(G(n, r))$-operators to show this, it thus suffices to show

$$R(n, r) \otimes kw \subset G(n, r)L(r(\lambda)) \subset \mathrm{I}_r(L(r(\lambda)))$$

for a $w \in L(r(\lambda))$ with $w \neq 0$. That is, we can assume

$$\lambda = r(\lambda)$$

By Lemma 4.16, we know that we have a presentation

$$L(\lambda) \cong W(\lambda)/V$$

hence

$$\mathrm{I}_r(L(\lambda)) = \mathrm{I}_r(W(\lambda))/\,\mathrm{I}_r(V)$$

as I_r is exact.

Recall that

$$W(\lambda) = G^0(v(\lambda)) \subset \mathrm{Sym}^{n_1}(U) \otimes_k \mathrm{Sym}^{n_2}(\Lambda^2 U) \otimes_k \ldots \otimes \mathrm{Sym}^{n_n}(\Lambda^n U)$$

with

$$v = v(\lambda) = e_1^{n_1} \otimes (e_1 \wedge e_2)^{n_2} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_n)^{n_n}$$

Let us choose

$$w := \overline{v} \neq 0 \in L(\lambda)$$

Then

$$G(n, r)(L(\lambda)) = G(n, r)(w) = (G(n, r)(v))/\,\mathrm{I}_r(V)$$

That is, it suffices to show

$$R(n, r) \otimes_k kv \subset G(n, r)(v) \subset \mathrm{I}_r(W(\lambda))$$

because this implies

$$R(n, r) \otimes_k kw = (R(n, r) \otimes_k kv)/\,\mathrm{I}_r(V) \subset G(n, r)(v)/\,\mathrm{I}_r(V) = G(n, r)(L(\lambda))$$

According to Lemma 3.16, we obtain

$$\begin{aligned} & \mathrm{I}_r(\mathrm{Sym}^{n_1}(U) \otimes_k \mathrm{Sym}^{n_2}(\Lambda^2 U) \otimes_k \ldots \otimes_k \mathrm{Sym}^{n_n}(\Lambda^n U)) \\ = \;& R(n, r) \otimes_k \mathrm{Sym}^{n_1}(U) \otimes_k \mathrm{Sym}^{n_2}(\Lambda^2 U) \otimes_k \ldots \otimes_k \mathrm{Sym}^{n_n}(\Lambda^n U) \\ \cong \;& \mathrm{Sym}^{n_1}_{R(n,r)}(\Omega^1_r) \otimes_{R(n,r)} \mathrm{Sym}^{n_2}_{R(n,r)}(\Omega^2_r) \otimes_{R(n,r)} \ldots \otimes_{R(n,r)} \mathrm{Sym}^{n_n}_{R(n,r)}(\Omega^n_r) \end{aligned}$$

as $G(n, r)$-representations. The element v corresponds to

$$v = (dx_1)^{n_1} \otimes (dx_1 \wedge dx_2)^{n_2} \otimes \ldots \otimes (dx_1 \wedge \ldots \wedge dx_n)^{n_n}$$

Let us shortly denote

$$dx_{j_1,\ldots,j_i} := dx_{j_1} \wedge \ldots \wedge dx_{j_i}$$

Then we get

$$G(n,r)(v) = G(n,r)((dx_1)^{n_1} \otimes (dx_{1,2})^{n_2} \otimes \ldots \otimes (dx_{1,\ldots,n})^{n_n})$$

We denote the Lie algebra operators $\delta_{(i,I)} \in \mathrm{Lie}(G(n,r))$ by

$$\delta_{(i,x^I)} := \delta_{(i,I)}$$

in our computations.

As $\lambda = r(\lambda)$, we have $0 \leq n_i \leq p-1$ for all $i = 0, \ldots, n$. First note the following.

Claim 1. Let $1 \leq s_j \leq r$. If

$$x_1^{p^{s_1}-1} \cdots x_n^{p^{s_n}-1} v \in G(n,r)(v)$$

then

$$x^J v \in G(n,r)(v)$$

for all $J = (j_1, \ldots, j_n)$ with $p^{s_k - 1} \leq j_k < p^{s_k}$ for all $1 \leq k \leq n$.

This claim follows by the gradual application of the operators

$$\delta_i = \frac{\partial}{\partial x_i} \otimes \mathrm{id} : R(n,r) \otimes W(\lambda) \to R(n,r) \otimes W(\lambda)$$

Hence, it suffices to proof

$$x_1^{p^{s_1}-1} \cdots x_n^{p^{s_n}-1} v \in G(n,r)(v)$$

for all choices $1 \leq s_j \leq r$.

By assumption we have $\lambda \neq 0$. That is there is a highest index k such that $n_k \neq 0$. First note the following computational rule.

Claim 2. Let $J = (j_1, \ldots, j_n)$ such that $j_k = p^s$. Then for all $j \neq k$

$$\delta_{(k,x_j)}(x^J v) = n_k x^J (dx_1)^{n_1} \otimes \ldots \otimes (dx_{1,\ldots,k-1})^{n_{k-1}} \otimes dx_{1,\ldots,k-1,j} (dx_{1,\ldots,k})^{n_k - 1}$$

This claim just follows by the computation

$$\begin{aligned} & \delta_{(k,x_j)}(x^J v) \\ = \; & x_j \frac{\partial}{\partial x_k} x^J v \\ & + n_k x^J (dx_1)^{n_1} \otimes \ldots \otimes (dx_{1,\ldots,k-1})^{n_{k-1}} \otimes dx_{1,\ldots,k-1,j} (dx_{1,\ldots,k})^{n_k-1} \\ = \; & n_k x^J (dx_1)^{n_1} \otimes \ldots \otimes (dx_{1,\ldots,k-1})^{n_{k-1}} \otimes dx_{1,\ldots,k-1,j} (dx_{1,\ldots,k})^{n_k-1} \end{aligned}$$

as $\frac{\partial}{\partial x_k} x^J = 0$ by $j_k = p^s$.

Case 1 ($n_k \geq 2$)**.** Let us assume that $n_k \geq 2$. We will make an inductive argument after s_k downwards. Then take $I = (p^{s_1} - 1, \ldots, p^{s_n} - 1)$ and we get

$$\begin{aligned} & \delta_{(k,x^I)}(n_k^{-1} v) \\ = \; & \delta_{(k,x^I)}(n_k^{-1} (dx_1)^{n_1} \otimes \ldots \otimes (dx_{1,\ldots,k})^{n_k}) \\ = \; & \sum_{j \geq k} \frac{\partial}{\partial x_j} x^I (dx_1)^{n_1} \otimes \ldots \otimes (dx_{1,\ldots,k-1})^{n_{k-1}} \otimes dx_{1,\ldots,k-1,j} (dx_{1,\ldots,k})^{n_k-1} \\ \in \; & G(n,r)(v) \end{aligned}$$

since $dx_{1,\dots,k-1,j} = 0$ for $0 \leq j \leq k-1$. Now we apply the operator $\delta_{(k,x_k^2)}$ to this sum. For the summand with $j = k$, we obtain

$$\begin{aligned}
&\delta_{(k,x_k^2)}\left(\frac{\partial}{\partial x_k}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\dots,k})^{n_k}\right)\\
= \; & x_k^2\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\dots,k})^{n_k}\\
& +2n_kx_k\frac{\partial}{\partial x_k}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\dots,k})^{n_k}\\
= \; & (p^{s_k}-1)(p^{s_k}-2+2n_k)x^Iv\\
= \; & 2(n_k-1)x^Iv
\end{aligned}$$

If $s_k = r$, we have $x_k^{p^{s_k}} = 0$ and we obtain

$$x_k\frac{\partial}{\partial x_j}x^I = 0 = x_k^2\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}x^I$$

for $j > k$. In the case that $s_k < r$, we know that

$$x_k\frac{\partial}{\partial x_j}x^Iv \in G(n,r)(v) \text{ and } x_k^2\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}x^Iv \in G(n,r)(v)$$

by induction hypothesis and Claim 1. Hence we get for the summands with $j > k$:

$$\begin{aligned}
&\delta_{(k,x_k^2)}\left(\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes dx_{1\dots,k-1,j}(dx_{1,\dots,k})^{n_k-1}\right)\\
= \; & x_k^2\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes dx_{1\dots,k-1,j}(dx_{1,\dots,k})^{n_k-1}\\
& +(n_k-1)2x_k\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes dx_{1\dots,k-1,j}(dx_{1,\dots,k})^{n_k-1}\\
= \; & \delta_{(k,x_j)}\left(n_k^{-1}x_k^2\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}x^Iv\right)+\delta_{(k,x_j)}\left(n_k^{-1}(n_k-1)2x_k\frac{\partial}{\partial x_j}x^Iv\right)\\
\in \; & G(n,r)(v)
\end{aligned}$$

by Claim 2. That is, we get

$$2(n_k-1)x^Iv \in G(n,r)(v)$$

which is nonzero as $2 \leq n_k \leq p-1$ and $\operatorname{char}(k) = p \neq 2$. Hence

$$x_1^{p^{s_1}-1}\cdots x_n^{p^{s_k}-1}v = x^Iv \in G(n,r)(v)$$

for all choices $1 \leq s_j \leq r$ which finishes the proof for the case $n_k \geq 2$.

Case 2 $(n_k = 1)$**.** In the second case, we assume that $n_k = 1$. As $r(\lambda) = \lambda$ is not a fundamental weight by assumption, there is a highest index $i < k$ with $n_i \neq 0$. We will make an inductive argument after the sum $s_i + s_k$ downwards.

We need two additional claims.

Claim 3. Let $J = (j_1, \dots, j_n)$ such that $j_i = p^s$. Then

$$\delta_{(i,x_k)}(x^Jv) = n_ix^J(dx_1)^{n_1}\otimes\ldots\otimes dx_{1,\dots,i-1,k}(dx_{1,\dots,i})^{n_i-1}\otimes dx_{1,\dots,k}$$

This claim follows by a similar computation as for Claim 2 and the fact that $dx_{1\dots,k,\dots,k} = 0$.

Claim 4. Let $J = (j_1, \dots, j_n)$ such that $j_i = p^s$ and $j > k$. Then

$$\begin{aligned} & \delta_{(i,x_k)}\left(\delta_{(k,x_j)}(x^J v) - x_j \frac{\partial}{\partial x_k} x^J v\right) \\ = \; & n_i x^J (dx_1)^{n_1} \otimes \dots \otimes dx_{1,\dots,i-1,k}(dx_{1,\dots,i})^{n_i-1} \otimes dx_{1,\dots,k-1,j} \\ & + x^J (dx_1)^{n_1} \otimes \dots \otimes (dx_{1,\dots,i})^{n_i} \otimes dx_{1,\dots,k,\dots,k-1,j} \end{aligned}$$

where the k in $dx_{1,\dots,k,\dots,k-1,j}$ appears at the i-th position.

This claim follows by a similar computation as for Claim 2 by using

$$\begin{aligned} & \delta_{(k,x_j)}(x^J v) - x_j \frac{\partial}{\partial x_k} x^J v \\ = \; & x^J (dx_1)^{n_1} \otimes \dots \otimes (dx_{1,\dots,i})^{n_i} \otimes dx_{1,\dots,k-1,j} \end{aligned}$$

Now take again $I = (p^{s_1} - 1, \dots, p^{s_n} - 1)$. Then we get

$$\delta_{(k,x^I)}(v) = \sum_{j \geq k} \frac{\partial}{\partial x_j} x^I (dx_1)^{n_1} \otimes \dots \otimes (dx_{1,\dots,i})^{n_i} \otimes dx_{1,\dots,k-1,j} \in G(n,r)(v)$$

Now we apply the operator $\delta_{(i,x_k)}$ to this sum. For the summand $j = k$, we obtain

$$\begin{aligned} & \delta_{(i,x_k)}\left(\frac{\partial}{\partial x_k} x^I (dx_1)^{n_1} \otimes \dots \otimes (dx_{1,\dots,i})^{n_i} \otimes dx_{1,\dots,k}\right) \\ = \; & x_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} x^I (dx_1)^{n_1} \otimes \dots \otimes (dx_{1,\dots,i})^{n_i} \otimes dx_{1,\dots,k} \\ & + n_i \frac{\partial}{\partial x_k} x^I (dx_1)^{n_1} \otimes \dots \otimes dx_{1,\dots,i-1,k}(dx_{1,\dots,i})^{n_i-1} \otimes dx_{1,\dots,k} \\ = \; & (p^{s_k} - 1) \frac{\partial}{\partial x_i} x^I (dx_1)^{n_1} \otimes \dots \otimes (dx_{1,\dots,i})^{n_i} \otimes dx_{1,\dots,k} \\ & + n_i \frac{\partial}{\partial x_k} x^I (dx_1)^{n_1} \otimes \dots \otimes dx_{1,\dots,i-1,k}(dx_{1,\dots,i})^{n_i-1} \otimes dx_{1,\dots,k} \end{aligned}$$

as $dx_{1,\dots,k,\dots,k} = 0$. If $s_k = r$, we have $x_k^{p^{s_k}} = 0$, and we obtain

$$x_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} x^I = 0$$

for $j > k$. If $s_k < r$, we know that

$$x_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} x^I v \in G(n,r)(v)$$

by induction hypothesis and Claim 1. Hence we get for the summands $j > k$

$$\begin{aligned}
&\delta_{(i,x_k)}\left(\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k-1,j}\right)\\
=\;& x_k\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k-1,j}\\
&+n_i\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes dx_{1,\ldots,i-1,k}(dx_{1,\ldots,i})^{n_i-1}\otimes dx_{1,\ldots,k-1,j}\\
&+\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k,\ldots,k-1,j}
\end{aligned}$$

where the k in $dx_{1,\ldots,k,\ldots,k-1,j}$ appears at the i-th position and we know

$$\begin{aligned}
& x_k\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k-1,j}\\
=\;& \delta_{(k,x_j)}\left(x_k\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}x^I v\right)\in G(n,r)(v)
\end{aligned}$$

by Claim 2.

Now we apply the operator $\delta_{(i,x_i^2)}$ to this. As $i < k$, for $s_i = r$, we obtain $x_i^{p^{s_i}} = 0$ and hence

$$x_i\frac{\partial}{\partial x_j}x^I = 0 = x_i^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}x^I$$

for all $j \geq k$. In the case that $s_i < r$, we know that

$$x_i\frac{\partial}{\partial x_j}x^I v \in G(n,r)(v) \text{ and } x_i^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}x^I v \in G(n,r)(v)$$

for all $j \geq k$ by induction hypothesis and Claim 1. By the same argument, we get

$$x_j\frac{\partial}{\partial x_k}\left(x_i\frac{\partial}{\partial x_j}x^I\right)v \in G(n,r)(v)$$

and

$$x_j\frac{\partial}{\partial x_k}\left(x_i^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}x^I\right)v \in G(n,r)(v)$$

for $j > k$. That is, the images of the $j > k$ summands lie in $G(n,r)(v)$ since

$$\begin{aligned}
&\delta_{(i,x_i^2)}\left(\frac{\partial}{\partial x_j}x^I n_i(dx_1)^{n_1}\otimes\ldots\otimes dx_{1,\ldots,i-1,k}(dx_{1,\ldots,i})^{n_i-1}\otimes dx_{1,\ldots,k-1,j}\right)\\
&+\delta_{(i,x_i^2)}\left(\frac{\partial}{\partial x_j}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k,\ldots,k-1,j}\right)\\
=\;&\delta_{(i,x_k)}\left(\delta_{(k,x_j)}\left(x_i^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}x^I v\right)-x_j\frac{\partial}{\partial x_k}\left(x_i^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}x^I\right)v\right)\\
&+\delta_{(i,x_k)}\left(\delta_{(k,x_j)}\left(n_i 2x_i\frac{\partial}{\partial x_j}x^I v\right)-x_j\frac{\partial}{\partial x_k}\left(n_i 2x_i\frac{\partial}{\partial x_j}x^I\right)v\right)\\
\in\;& G(n,r)(v)
\end{aligned}$$

by Claim 4.

Similarly, for the summand $j=k$, its second summand computes as

$$\begin{aligned}
& \delta_{(i,x_i^2)}\left(n_i\frac{\partial}{\partial x_k}x^I(dx_1)^{n_1}\otimes\ldots\otimes dx_{1,\ldots,i-1,k}(dx_{1,\ldots,i})^{n_i-1}\otimes dx_{1,\ldots,k}\right)\\
= \ & n_i x_i^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}x^I(dx_1)^{n_1}\otimes\ldots\otimes dx_{1,\ldots,i-1,k}(dx_{1,\ldots,i})^{n_i-1}\otimes dx_{1,\ldots,k}\\
+ \ & n_i^2 2x_i\frac{\partial}{\partial x_k}x^I(dx_1)^{n_1}\otimes\ldots\otimes dx_{1,\ldots,i-1,k}(dx_{1,\ldots,i})^{n_i-1}\otimes dx_{1,\ldots,k}\\
= \ & \delta_{(i,x_k)}\left(x_i^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}x^Iv\right)+\delta_{(i,x_k)}\left(n_i 2x_i\frac{\partial}{\partial x_k}x^Iv\right)\\
\in \ & G(n,r)(v)
\end{aligned}$$

by Claim 3. Its first summand computes as

$$\begin{aligned}
& \delta_{(i,x_i^2)}\left((p^{s_k}-1)\frac{\partial}{\partial x_i}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k}\right)\\
= \ & (p^{s_k}-1)x_i^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k}\\
& +(p^{s_k}-1)2n_i x_i\frac{\partial}{\partial x_i}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k}\\
& +(p^{s_k}-1)2x_i\frac{\partial}{\partial x_i}x^I(dx_1)^{n_1}\otimes\ldots\otimes(dx_{1,\ldots,i})^{n_i}\otimes dx_{1,\ldots,k}\\
= \ & (p^{s_k}-1)(p^{s_i}-1)(p^{s_i}-2+2n_i+2)x^Iv\\
= \ & 2n_i x^Iv
\end{aligned}$$

That is, we get

$$2n_i x^I v\in G(n,r)(v)$$

which is nonzero as $\mathrm{char}(k)=p\neq 2$ and $1\leq n_i\leq p-1$. Hence

$$x_1^{p^{s_1}-1}\cdots x_n^{p^{s_n}-1}v=x^Iv\in G(n,r)(v)$$

which finishes the proof for $n_k=1$.

Finally, we proved the Proposition. □

9. The Representation Ring of $G(n,r)$

We are now going to give a computation of the *representation ring of* $G(n,r)$. That is, the Grothendieck ring of finite dimensional representations. We want to identify the subgroup G^0 with $\mathrm{GL}(U)$ where $U = k^n$. We already introduced the functor

$$\mathrm{I}_r : \mathrm{GL}(U)\text{−rep} \longrightarrow G(n,r)\text{−rep}$$

which lead to a parametrization and computation of all irreducible $G(n,r)$-representations. Recall the restriction functor

$$\mathrm{res} : G(n,r)\text{−rep} \longrightarrow \mathrm{GL}(U)\text{−rep}$$

and

$$L_r^* : \mathrm{GL}(U^{(r)})\text{−rep} \longrightarrow G(n,r)\text{−rep}$$

induced by the representation

$$L_r : G(n,r) \to \mathrm{GL}(U^{(r)})$$

Recall that

$$\mathrm{res} \circ L_r^* = (F^r)^*$$

where $F^r = F^r_{\mathrm{GL}(U)} : \mathrm{GL}(U) \to \mathrm{GL}(U^{(r)})$ is the r-th Frobenius and that

$$\mathrm{res} \circ \mathrm{I}_r = R(n,r) \otimes_k (-)$$

All three functors $\mathrm{I}_r, \mathrm{res}, L_r^*$ are exact. That is, we can consider the induced abelian group homomorphisms on the representation rings which provides the commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(\mathrm{GL}(U)) \arrow[dr, "\mathrm{I}_1"] \arrow[drr, bend left, "{[R(n,1)]\cdot(-)}"] & & \\
& \mathrm{Rep}(G(n,1)) \arrow[r, "\mathrm{res}"] & \mathrm{Rep}(\mathrm{GL}(U)) \\
\mathrm{Rep}(\mathrm{GL}(U^{(1)})) \arrow[ur, "L_1^*"] \arrow[urr, bend right, "(F^1)^*"'] & &
\end{tikzcd}$$

for $r = 1$.

For $r \geq 2$, we can also consider the functor

$$T_r^* : G(n,r-1)^{(1)}\text{−rep} \longrightarrow G(n,r)\text{−rep}$$

induced by the triangulated group homomorphism

$$T_r : G(n,r) \to G(n,r-1)^{(1)}$$

Recall that

$$T_r^* \circ L_{r-1}^* = L_r^*$$

and

$$\mathrm{res}^{G(n,r)}_{\mathrm{GL}(U)} \circ T_r^* = (F^1)^* \circ \mathrm{res}^{G(n,r-1)^{(1)}}_{\mathrm{GL}(U^{(1)})}$$

As T_r^* is also exact, we can consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Rep}(\mathrm{GL}(U)) \arrow[dr, "\mathrm{I}_r"] \arrow[drr, bend left, "{[R(n,r)]\cdot(-)}"] & & \\
& \operatorname{Rep}(G(n,r)) \arrow[r, "\mathrm{res}"] & \operatorname{Rep}(\mathrm{GL}(U)) \\
\operatorname{Rep}(G(n,r-1)^{(1)}) \arrow[ur, "T_r^*"] \arrow[r, "\mathrm{res}"] & \operatorname{Rep}(\mathrm{GL}(U^{(1)})) \arrow[ur, bend right, "(F^1)^*"'] &
\end{tikzcd}$$

for $r \geq 2$. First note, that res is injective:

Lemma 9.1. *The map*

$$\mathrm{res} : \operatorname{Rep}(G(n,r)) \to \operatorname{Rep}(\mathrm{GL}(U))$$

is injective.

Proof. Due to Theorem 5.9 and Jordan-Hölder (cf. Remark 1.8) we know that $\operatorname{Rep}(G(n,r))$ is a free abelian group with $\mathbb{Z}$-basis $[L(\lambda, G(n,r))]$ for $\lambda \in X(T)_+$. Furthermore, we know by Proposition 3.19 that $L(\lambda)$ generates $L(\lambda, G(n,r))$. Moreover, it is the lowest $\mathbb{G}_m$-weight space of weight $s = \deg(\lambda)$. As the $\mathbb{G}_m$-weight space filtration of $L(\lambda, G(n,r))$ is $\mathrm{GL}(U)$-invariant, we obtain

$$[\mathrm{res}(L(\lambda, G(n,r)))] = [L(\lambda)] + \sum_{\substack{\mu \in X(T)_+ \\ \deg(\mu) > s}} m_\mu [L(\mu)] \in \operatorname{Rep}(\mathrm{GL}(U))$$

with $m_\mu \in \mathbb{Z}$ the multiplicity of $L(\mu)$ in $\mathrm{res}\, L(\lambda, G(n,r))$. Since $[L(\lambda)]$ with $\lambda \in X(T)_+$ form a $\mathbb{Z}$-basis of $\operatorname{Rep}(\mathrm{GL}(U))$, we see that res maps a basis of $\operatorname{Rep}(G(n,r))$ to a linearly independent set. This shows the injectivity. $\square$

Now recall that we can consider the maps $[R(n,r)]\cdot(-)$ and $(F^r)^*$ as maps of $\operatorname{Rep}(\mathrm{GL}(U^{(r)}))$-modules where the structure on $\operatorname{Rep}(\mathrm{GL}(U))$ is given by $(F^r)^*$. Furthermore we have two $\operatorname{Rep}(\mathrm{GL}(U^{(r)}))$-algebras:

$$L_r^* : \operatorname{Rep}(\mathrm{GL}(U^{(r)})) \to \operatorname{Rep}(G(n,r))$$

and

$$L_{r-1}^* : \operatorname{Rep}(\mathrm{GL}(U^{(r)})) \to \operatorname{Rep}(G(n,r-1)^{(1)})$$

All morphisms considered above are maps of $\operatorname{Rep}(\mathrm{GL}(U^{(r)}))$-modules.

The diagram above provides a factorization

$$\begin{tikzcd}
\operatorname{Rep}(\mathrm{GL}(U)) \oplus \operatorname{Rep}(\mathrm{GL}(U^{(1)})) \arrow[rr, "{[R(n,1)]\cdot(-)+(F^1)^*}"] \arrow[dr, "\mathrm{I}_1 + L_1^*"'] & & \operatorname{Rep}(\mathrm{GL}(U)) \\
& \operatorname{Rep}(G(n,1)) \arrow[ur, "\mathrm{res}"'] &
\end{tikzcd}$$

for $r = 1$. In fact, we can give a computation of the kernel of $\mathrm{I}_1 + L_1^*$: By the injectivity of res, we get

$$\begin{aligned}
\operatorname{Ker}(\mathrm{I}_1 + L_1^*) &= \operatorname{Ker}(\mathrm{res} \circ (\mathrm{I}_1 + L_1^*)) \\
&= \operatorname{Ker}([R(n,1)] \cdot (-) + (F^1)^*) \\
&= (\delta, -\delta_1)\operatorname{Rep}(\mathrm{GL}(U^{(1)}))
\end{aligned}$$

according to Proposition 4.24. This gives an injective map

$$\mathrm{I}_1 + L_1^* : \mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)}))/(\delta, -\delta_1) \hookrightarrow \mathrm{Rep}(G(n,1))$$

For $\mathrm{char}(k) \neq 2$, this map is in fact also surjective which is stated by the next Theorem which finishes the computation of $\mathrm{Rep}(G(n,1))$.

For $r \geq 2$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)})) \arrow[rr, bend left, "{[R(n,r)]\cdot(-)+(F^1)^*}"] & & \\
\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(G(n,r-1)^{(1)}) \arrow[u, "\mathrm{id}\oplus\mathrm{res}"] \arrow[rr, "{[R(n,r)]\cdot(-)+(F^1)^*\circ\mathrm{res}}"] \arrow[dr, "\mathrm{I}_r + T_r^*"'] & & \mathrm{Rep}(\mathrm{GL}(U)) \\
& \mathrm{Rep}(G(n,r)) \arrow[ur, "\mathrm{res}"'] &
\end{tikzcd}$$

Note that

$$((F^1)^* \circ \mathrm{res})(L_{r-1}^*(\delta_r)) = (F^r)^*(\delta_r) = [R(n,r)] \cdot \delta$$

by Proposition 4.24. That is, the map $\mathrm{I}_r + T_r^*$ factors through

$$\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(G(n,r-1)^{(1)})/(\delta, -L_{r-1}^*(\delta_r)) \xrightarrow{\mathrm{I}_r + T_r^*} \mathrm{Rep}(G(n,r))$$

This is also a surjective map which is stated by the next Theorem.

Theorem 9.2. *Assume that* $\mathrm{char}(k) \neq 2$. *Then the map*

$$\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)}))/(\delta, -\delta_1) \xrightarrow{\mathrm{I}_1 + L_1^*} \mathrm{Rep}(G(n,1))$$

is an isomorphism of $\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$*-modules.*

For $r \geq 2$, *the map*

$$\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(G(n,r-1)^{(1)})/(\delta, -L_{r-1}^*(\delta_r)) \xrightarrow{\mathrm{I}_r + T_r^*} \mathrm{Rep}(G(n,r))$$

is a surjection of $\mathrm{Rep}(\mathrm{GL}(U^{(r)}))$*-modules.*

Proof. As we already saw the injectivity for $r = 1$, it is left to show the surjectivity. Recall that $\mathrm{Rep}(G(n,r))$ is a free abelian group with $\mathbb{Z}$-basis $[L(\lambda, G(n,r))]$ with $\lambda \in X(T)_+$ according to Theorem 5.9 and Jordan-Hölder (cf. Remark 1.8). That is, it suffices to show that these elements lie in the image of $\mathrm{I}_1 + L_1^*$, $\mathrm{I}_r + T_r^*$ respectively.

Let us start with the case that the mod p-reduction $r(\lambda)$ of λ vanishes. Then we get by Proposition 8.3

$$L(\lambda, G(n,1)) = L_1^* L(s(\lambda))$$

and for $r \geq 2$

$$L(\lambda, G(n,r)) = T_r^* L(s(\lambda), G(n,r-1)^{(1)})$$

Hence the class of $L(\lambda, G(n,1))$ lies in the image of $\mathrm{I}_1 + L_1^*$ and for $r \geq 2$, the class of $L(\lambda, G(n,r))$ lies in the image of $\mathrm{I}_r + T_r^*$ respectively.

Now we proceed with the case that

$$r(\lambda) = \epsilon_1 + \ldots + \epsilon_i \in X(T)_+$$

is a fundamental weight. We know by Proposition 8.7 that

$$L(\lambda, G(n,r)) = \mathrm{Im}(d_i \otimes \mathrm{id} : \Omega_r^{i-1} \otimes_k L(s(\lambda))^{[1]} \to \Omega_r^i \otimes_k L(s(\lambda))^{[1]})$$

the images of the deRham-differentials. As computed at the end of section 7, we know that

$$\begin{aligned}\mathrm{Ker}(d_{i+1}\otimes \mathrm{id})/\,\mathrm{Im}(d_i\otimes \mathrm{id}) &= H^i\,\mathrm{I}_1(\Lambda^\bullet U\otimes_k L(s(\lambda))^{[1]})\\ &\cong L_1^*(\Lambda^i U^{(1)}\otimes_k L(s(\lambda)))\end{aligned}$$

for $r=1$ and

$$\begin{aligned}\mathrm{Ker}(d_{i+1}\otimes \mathrm{id})/\,\mathrm{Im}(d_i\otimes \mathrm{id}) &= H^i\,\mathrm{I}_r(\Lambda^\bullet U\otimes_k L(s(\lambda))^{[1]})\\ &\cong T_r^*\,\mathrm{I}_{r-1}(\Lambda^i U^{(1)}\otimes_k L(s(\lambda)))\end{aligned}$$

for $r\geq 2$. That is,

$$[\mathrm{Ker}(d_{i+1}\otimes \mathrm{id})]-[\mathrm{Im}(d_i\otimes \mathrm{id})]=[L_1^*(\Lambda^i U^{(1)}\otimes L(s(\lambda))]\in \mathrm{Rep}(G(n,1))$$

for $r=1$ and

$$[\mathrm{Ker}(d_{i+1}\otimes \mathrm{id})]-[\mathrm{Im}(d_i\otimes \mathrm{id})]=[T_r^*(\mathrm{I}_{r-1}(\Lambda^i U^{(1)}\otimes_k L(s(\lambda)))]\in \mathrm{Rep}(G(n,r))$$

for $r\geq 2$. Recall that $\Omega_r^i\otimes_k L(s(\lambda))^{[1]}=\mathrm{I}_r(\Lambda^i U\otimes L(s(\lambda))^{[1]})$. Then for $i=n$, this means

$$\begin{aligned}&[\mathrm{Im}(d_n\otimes \mathrm{id})]\\ =\;& \mathrm{I}_1[(\Lambda^n U\otimes L(s(\lambda))^{[1]})]+L_1^*[\Lambda^n U^{(1)}\otimes L(s(\lambda))]\in \mathrm{Im}(\mathrm{I}_r+L_1^*)\end{aligned}$$

for $r=1$ and

$$\begin{aligned}&[\mathrm{Im}(d_n\otimes \mathrm{id})]\\ =\;& \mathrm{I}_r[(\Lambda^n U\otimes L(s(\lambda))^{[1]})]+T_r^*[\mathrm{I}_{r-1}(\Lambda^i U^{(1)}\otimes_k L(s(\lambda)))]\in \mathrm{Im}(\mathrm{I}_r+T_r^*)\end{aligned}$$

for $r\geq 2$. Note that we also have

$$[\mathrm{I}_r(\Lambda^i U\otimes L(s(\lambda))^{[1]})]-[\mathrm{Ker}(d_{i+1}\otimes \mathrm{id})]=[\mathrm{Im}(d_{i+1}\otimes \mathrm{id})]\in \mathrm{Rep}(G(n,r))$$

By combining the two formulas, we obtain

$$\begin{aligned}&\mathrm{I}_1[(\Lambda^i U\otimes L(s(\lambda))^{[1]})]\\ =\;& [\mathrm{Im}(d_i\otimes \mathrm{id})]+[\mathrm{Im}(d_{i+1}\otimes \mathrm{id})]+L_1^*[\Lambda^i U^{(1)}\otimes L(s(\lambda))]\end{aligned}$$

in $\mathrm{Rep}(G(n,1))$ for $r=1$ and

$$\begin{aligned}&\mathrm{I}_r[(\Lambda^i U\otimes L(s(\lambda))^{[1]})]\\ =\;& [\mathrm{Im}(d_i\otimes \mathrm{id})]+[\mathrm{Im}(d_{i+1}\otimes \mathrm{id})]+T_r^*[\mathrm{I}_{r-1}(\Lambda^i U^{(1)}\otimes_k L(s(\lambda)))]\end{aligned}$$

in $\mathrm{Rep}(G(n,r))$ for $r\geq 2$. That is, if $[\mathrm{Im}(d_{i+1}\otimes \mathrm{id})]\in \mathrm{Im}(\mathrm{I}_1+L_1^*)$, then also $[\mathrm{Im}(d_i\otimes \mathrm{id})]\in \mathrm{Im}(\mathrm{I}_1+L_1^*)$ for $r=1$ and for $r\geq 2$, if $[\mathrm{Im}(d_{i+1}\otimes \mathrm{id})]\in \mathrm{Im}(\mathrm{I}_r+T_r^*)$ then also $[\mathrm{Im}(d_i\otimes \mathrm{id})]\in \mathrm{Im}(\mathrm{I}_r+T_r^*)$ respectively. But we know this already for $i=n$, so we get it for all $i=1,\dots,n$.

Finally for the case that the mod p-reduction $r(\lambda)$ of λ is neither 0 nor a fundamental weight, we get

$$L(\lambda,G(n,r))=\mathrm{I}_r(L(\lambda))$$

by Proposition 8.9 as $\mathrm{char}(k)\neq 2$. Hence the class of $L(\lambda,G(n,1))$ lies in the image of $\mathrm{I}_1+L_1^*$ and for $r\geq 2$, the class of $L(\lambda,G(n,r))$ lies in the image of $\mathrm{I}_r+T_r^*$ respectively. This finishes the proof. $\square$

Notation 9.3. For $1\leq i\leq r-1$ denote the composition

$$T^i:=T_{r-(i-1)}^{(i-1)}\circ\cdots\circ T_r:G(n,r)\to G(n,r-i)^{(i)}$$

Remark 9.4. Consider the morphisms

$$\mathrm{Rep}(\mathrm{GL}(U)) \xrightarrow{\mathrm{I}_r} \mathrm{Rep}(G(n,r))$$

and

$$\mathrm{Rep}(\mathrm{GL}(U^{(i)})) \xrightarrow{\mathrm{I}_{r-i}} \mathrm{Rep}(G(n,r-i)^{(i)}) \xrightarrow{(T^i)^*} \mathrm{Rep}(G(n,r))$$

for $1 \leq i \leq r-1$ and

$$\mathrm{Rep}(\mathrm{GL}(U^{(r)})) \xrightarrow{L_r^*} \mathrm{Rep}(G(n,r))$$

Then the Theorem implies that the sum of these induces a surjection

$$\bigoplus_{i=0}^{r} \mathrm{Rep}(\mathrm{GL}(U^{(i)})) \to \mathrm{Rep}(G(n,r))$$

That is, one needs $r+1$ copies of $\mathrm{Rep}(\mathrm{GL}(U)) \cong \mathrm{Rep}(\mathrm{GL}(U^{(i)}))$. Further, the theorem implies that the element

$$(\delta, 0, \ldots, 0, -\delta_r)$$

lies in the kernel.

If we compose these maps with the restriction

$$\mathrm{res} : \mathrm{Rep}(G(n,r)) \to \mathrm{Rep}(\mathrm{GL}(U))$$

we obtain

$$\mathrm{Rep}(\mathrm{GL}(U)) \xrightarrow{\mathrm{I}_r} \mathrm{Rep}(G(n,r)) \xrightarrow{\mathrm{res}} \mathrm{Rep}(\mathrm{GL}(U))$$

which acts as $x \mapsto [R(n,r)]x$, for $1 \leq i \leq r-1$ we get

$$\mathrm{Rep}(\mathrm{GL}(U^{(i)})) \xrightarrow{\mathrm{I}_{r-i}} \mathrm{Rep}(G(n,r-i)^{(i)}) \xrightarrow{(T^i)^*} \mathrm{Rep}(G(n,r)) \xrightarrow{\mathrm{res}} \mathrm{Rep}(\mathrm{GL}(U))$$

which acts as $x \mapsto (\psi^p)^i([R(n,r-i)^{(i)}](\psi^p)^i(x)$, and

$$\mathrm{Rep}(\mathrm{GL}(U^{(r)})) \xrightarrow{L_r^*} \mathrm{Rep}(G(n,r)) \xrightarrow{\mathrm{res}} \mathrm{Rep}(\mathrm{GL}(U))$$

which acts as $x \mapsto (\psi^p)^r(x)$. Thus the image of

$$\mathrm{res} : \mathrm{Rep}(G(n,r)) \to \mathrm{Rep}(\mathrm{GL}(U))$$

consists precisely of the elements of the form

$$[R(n,r)]x_0 + \sum_{i=1}^{r-1} (\psi^p)^i([R(n,r-i)^{(i)}])(\psi^p)^i(x_i) + (\psi^p)^r(x_r)$$

with $x_i \in \mathrm{Rep}(\mathrm{GL}(U^{(i)})$ and we have the relation

$$[R(n,r)]\delta = (\psi^p)^r(\delta)$$

For $r \geq 2$, the kernel of the surjective map

$$\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(G(n,r-1)^{(1)}) \xrightarrow{\mathrm{I}_r + T_r^*} \mathrm{Rep}(G(n,r))$$

can be described by its image under the injective map

$$\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(G(n,r-1)^{(1)}) \xrightarrow{\mathrm{id} \oplus \mathrm{res}} \mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)}))$$

as follows.

Corollary 9.5. *For $r \geq 2$, the image*
$$(\mathrm{id} \oplus \mathrm{res})(\mathrm{Ker}(\mathrm{I}_r + T_r^*)) \subset \mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)}))$$
coincides with the kernel of
$$\mathrm{Rep}(\mathrm{GL}(U)) \oplus \mathrm{Rep}(\mathrm{GL}(U^{(1)})) \xrightarrow{[R(n,r)]\cdot(-)+F^*} \mathrm{Rep}(\mathrm{GL}(U))$$
which is generated by $(\delta, -(F^{r-1})^*(\delta_r))$ as an $\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$-module.

Proof. According to Proposition 4.24, the kernel of $[R(n,r)]\cdot(-) + F^*$ is generated by $(\delta, -(F^{r-1})^*(\delta_r))$ as an $\mathrm{Rep}(\mathrm{GL}(U^{(1)})$-module. That is, the elements of the kernel are those of the form
$$(\delta\psi^p(a), -(\psi^p)^{r-1}(\delta)a)$$
for all $a \in \mathrm{Rep}(\mathrm{GL}(U)^{(1)})$. Furthermore, we know that
$$(\psi^p)^{r-1}(\delta)a = [R(n, r-1)^{(1)}]\delta a$$
which lies in the image of
$$\mathrm{res} : \mathrm{Rep}(G(n, r-1)^{(1)}) \hookrightarrow \mathrm{Rep}(\mathrm{GL}(U^{(1)}))$$
according to the last Remark. That is,
$$\mathrm{Ker}([R(n,r)]\cdot(-) + F^*) \subset \mathrm{Im}(\mathrm{id} \oplus \mathrm{res})$$
But also
$$(\mathrm{id} \oplus \mathrm{res})(\mathrm{Ker}(\mathrm{I}_r + T_r^*)) = \mathrm{Ker}([R(n,r)]\cdot(-) + F^*) \cap \mathrm{Im}(\mathrm{id} \oplus \mathrm{res})$$
which shows the claim. □

Remark 9.6. In the Theorem, for $r \geq 2$, we only consider the $\mathrm{Rep}(\mathrm{GL}(U^{(r)}))$-module generated by $(\delta, -(F^{r-1})^*(\delta_r))$. Under the injective map $\mathrm{id} \oplus \mathrm{res}$ these elements are those of the form
$$(\delta\psi^p(a), (\psi^p)^{r-1}(\delta)(a))$$
for all $a = (\psi^p)^{r-1}(b)$ with $b \in \mathrm{Rep}(\mathrm{GL}(U^{(r)}))$. That is, the occurring $\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$-coefficients are those of the image of
$$(F^{r-1})^* : \mathrm{Rep}(\mathrm{GL}(U)^{(r)}) \to \mathrm{Rep}(\mathrm{GL}(U^{(1)}))$$
Hence, for $r \geq 2$ the induced map of the Theorem is *not* injective as $\delta \cdot \psi^p$ is injective and $(F^{r-1})^*$ is not surjective.

Unfortunately, it does not seem to be possible to introduce the structure of an $\mathrm{Rep}(\mathrm{GL}(U^{(1)}))$-module on $\mathrm{Rep}(G(n, r-1)^{(1)})$, so in order to compute the kernel of $\mathrm{I}_r + T_r^*$ one needs to apply $\mathrm{id} \oplus \mathrm{res}$ as in the Corollary.

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