On Vector Product Algebras

By Markus Rost.

This text contains some remarks on vector product algebras and the graphical techniques. It is partially contained in the diploma thesis of D. Boos and S. Maurer.

This material dates back to 1995-1996 and was compiled in July 2004. Last changes: July 12, 2004. April 11, 2024 (equation numbers on page 5 were missing).

1. Vector Product Algebras

A vector product algebra consists of a vector space V together with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on V, and a linear map $V\otimes V\to V$, $x\otimes y\mapsto x\times y$ such that

$$\langle x\times y,z\rangle=\langle x,y\times z\rangle,\qquad x\times x=0,\qquad (x\times y)\times x=\langle x,x\rangle y-\langle x,y\rangle x$$

In a vector product algebra one has $x\times(y\times x)=(x\times y)\times x$. Moreover $\langle x\times y,z\rangle$ is alternating.

V is called associative if

$$(x\times y)\times z=\langle x,z\rangle y-\langle y,z\rangle x$$

V is called commutative if the product is trivial:

$$x\times y=0$$

In this case $\langle x,x\rangle y=\langle x,y\rangle x$.

2. The Fundamental Relation in Vector Product Algebras

For a vector product algebra one introduces the following tensors $R_n\colon V^{\otimes n}\to V$.

$$R_1(x)=x$$
$$R_2(x,y)=x\times y,$$
$$R_3(x,y,z)=(x\times y)\times z-y\langle x,z\rangle+x\langle y,z\rangle$$
$$R_4(x_1,x_2,x_3,t)=R_3(x_1,x_2,x_3)\times t-\sum_{i=1}^{3}x_i\langle x_{i+1}\times x_{i+2},t\rangle+\sum_{i=1}^{3}x_{i+1}\times x_{i+2}\langle x_i,t\rangle$$

with i taken mod3.

One has the following fundamental relation for vector product algebras. It holds over any ring F and for possibly degenerate bilinear forms $\langle\ ,\ \rangle$.

(2.1) Main Lemma. $2 \cdot R_4 \equiv 0$.

Proof. Put

$$\Delta(u,v,w) = (u \times v) \times w + u \times (v \times w)$$

and check the equality

$$(*) \qquad \begin{gathered} 2((x \times y) \times z) \times t = \\ \Delta(x \times y, z, t) - \Delta(x, y, z \times t) + x \times \Delta(y,z,t) - \Delta(x, y \times z, t) + \Delta(x,y,z) \times t \end{gathered}$$

In fact, you will find the cancelling terms:

$$(x \times y) \times (z \times t), \quad x \times (y \times (z \times t)), \quad x \times ((y \times z) \times t), \quad (x \times (y \times z)) \times t$$

On the other hand, the anti-commutativity and the polarization of the second axiom give

$$\Delta(u,v,w) = 2w\langle u,v\rangle - u\langle v,w\rangle - v\langle u,w\rangle$$

Inserting this in $(*)$ leads to the claim.

3. Relations for the Dimension in a Vector Product Algebra

We next consider the norms N_n of the tensors R_n. Let

$$Q_n(x_1,\ldots,x_{n+1}) = \big\langle R_n(x_1,\ldots,x_n), x_{n+1}\big\rangle$$

and put

$$N_n = \mathrm{N}_{V^{\otimes(n+1)}}(Q_n)$$

In other words, if e_i is an orthonormal basis of V (over some algebraic closure of V), then

$$N_n = \sum_{i_1,\ldots,i_{n+1}} Q_n(e_{i_1},\ldots,e_{i_{n+1}})^2 = \sum_{i_1,\ldots,i_n} \mathrm{N}\left(R_n(e_{i_1},\ldots,e_{i_n})\right)$$

Let $d = \dim V \in F$ be the dimension of V considered as an element of the ground field.

Since $2R_4 = 0$ it follows immediately from the next Proposition that

$$4d(d-1)(d-3)(d-7) = 0$$

in any vector product algebra. Similarly for the associative and commutative case.

(2.2) Proposition.

$$
\begin{aligned}
\mathrm{N}_1 &= d\\
\mathrm{N}_2 &= d(d-1)\\
\mathrm{N}_3 &= d(d-1)(d-3)\\
\mathrm{N}_4 &= d(d-1)(d-3)(d-7)
\end{aligned}
$$

Proof. The claim for N_1 is obvious. Next note

$$
\sum_i e_i \times (v \times e_i) = \sum_i \langle e_i, e_i\rangle v - \sum_i \langle e_i, v\rangle e_i = dv - v = (d-1)v \tag{2.2a}
$$

Hence

$$
\sum_i \mathrm{N}(x \times e_i) = (d-1)\,\mathrm{N}(x)
$$

and

$$
N_2 = \sum_{i,j} \langle e_i \times e_j, e_i \times e_j\rangle = \sum_{i,j} \big\langle e_i, e_j \times (e_i \times e_j)\big\rangle = (d-1)\sum_i \langle e_i, e_i\rangle = d(d-1)
$$

Moreover

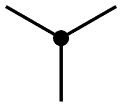
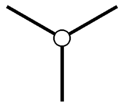
$$
\begin{aligned}
N_3 &= \sum_{i,j,k} \mathrm{N}\left(R_3(e_i, e_j, e_k)\right)\\
&= \sum_{i,j,k} \mathrm{N}\left((e_i \times e_j) \times e_k) - e_j\langle e_i, e_k\rangle + e_i\langle e_k, e_j\rangle\right)\\
&= \sum_{i,j,k} \Big[\mathrm{N}\left((e_i \times e_j) \times e_k\right) + \mathrm{N}(e_j)\langle e_i, e_k\rangle^2 + \mathrm{N}(e_i)\langle e_k, e_j\rangle^2\\
&\qquad\qquad - 2\big\langle (e_i \times e_j) \times e_k, e_j\big\rangle\langle e_i, e_k\rangle + 2\big\langle (e_i \times e_j) \times e_k, e_i\big\rangle\langle e_j, e_k\rangle\\
&\qquad\qquad\qquad - 2\langle e_i, e_k\rangle\langle e_j, e_k\rangle\langle e_k, e_i\rangle\Big]\\
&= d(d-1)^2 + d^2 + d^2 - 2N_2 - 2N_2 - 2d\\
&= d(d-1)^2 + 2d(d-1) - 4d(d-1) = d(d-1)(d-3)
\end{aligned}
$$

Finally, by re-indexing and using $\langle e_i, e_i \times e_j\rangle = \langle e_i, e_k \times e_i\rangle = 0$ one finds:

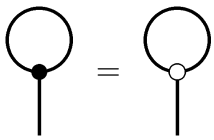
$$\begin{aligned}
N_4 = &\sum_{i,j,k,l} \mathrm{N}\left(R_4(e_i,e_j,e_k,e_l)\right)\\
= &\sum_{i,j,k,l} \mathrm{N}\big(R_3(e_i,e_j,e_k)\times e_l - e_i\langle e_j\times e_k, e_l\rangle - e_j\langle e_k\times e_i, e_l\rangle - e_k\langle e_i\times e_j, e_l\rangle\\
&\qquad + e_j\times e_k\langle e_i,e_l\rangle + e_k\times e_i\langle e_j,e_l\rangle + e_i\times e_j\langle e_k,e_l\rangle\big)\\
= &\sum_{i,j,k,l}\Big[\mathrm{N}\left(R_3(e_i,e_j,e_k)\times e_l\right) + 3\cdot\mathrm{N}(e_i)\langle e_j\times e_k, e_l\rangle^2 + 3\cdot\mathrm{N}(e_j\times e_k)\langle e_i,e_l\rangle^2\\
&\quad - 3\cdot 2\big\langle R_3(e_i,e_j,e_k)\times e_l, e_i\big\rangle\langle e_j\times e_k, e_l\rangle\\
&\qquad + 3\cdot 2\big\langle R_3(e_i,e_j,e_k)\times e_l, e_j\times e_k\big\rangle\langle e_i,e_l\rangle\\
&\quad + 3\cdot 2\langle e_i,e_j\rangle\langle e_j\times e_k, e_l\rangle\langle e_k\times e_i, e_l\rangle\\
&\qquad + 3\cdot 2\langle e_i, e_j\times e_k\rangle\langle e_j\times e_k, e_l\rangle\langle e_i,e_l\rangle\\
&\qquad\quad + 3\cdot 2\langle e_j\times e_k, e_k\times e_i\rangle\langle e_i,e_l\rangle\langle e_j,e_l\rangle\Big]\\
= &\sum_{i,j,k}(d-1)\cdot\mathrm{N}\left(R_3(e_i,e_j,e_k)\right) + 3\cdot\sum_{j,k} d\cdot\mathrm{N}(e_j\times e_k) + 3\cdot\sum_{j,k} d\cdot\mathrm{N}(e_j\times e_k)\\
&\quad + 3\cdot 2\sum_{i,j,k}\big\langle R_3(e_i,e_j,e_k)\times e_i, e_j\times e_k\big\rangle\\
&\qquad - 3\cdot 2\sum_{i,j,k}\big\langle R_3(e_i,e_j,e_k)\times(e_j\times e_k), e_i\big\rangle\\
&\quad + 3\cdot 2\sum_{i,k}\langle e_i\times e_k, e_k\times e_i\rangle\\
&\qquad - 3\cdot 2\sum_{j,k}\langle e_j\times e_k, e_j\times e_k\rangle\\
&\qquad\quad + 3\cdot 2\sum_{i,k}\langle e_i\times e_k, e_k\times e_i\rangle\\
= &\,(d-1)N_3 + 6d{\cdot}N_2 - 12N_3 - 18N_2 = (d-1-12+6)N_3\\
= &\,d(d-1)(d-3)(d-7).
\end{aligned}$$

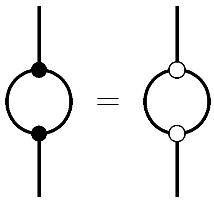
4. Graph Considerations for Vector Product Algebras

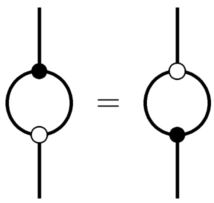
We consider 3-valent graphs with cyclically oriented vertices. We describe the orientation at a vertex by replacing it by an oriented disk. The orientation of a disk is indicated by black or white coloring:

 (positive),  (negative).

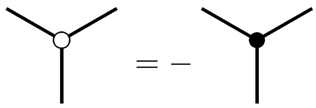
First note that rotation around the vertical symmetry axis give the following identities:

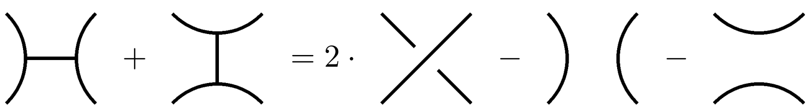
(0.1) 

(0.2) 

(0.3) 

The following rules (R1) and (R2) are the graph versions of the axioms for vector product algebras.

(R1) 

(R2) 

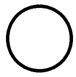
Here we use the following **convention**: If in a plane graph no orientation is indicated we understood the positive orientation (black coloring). The rule (R1) makes it possible to give this orientation to the pictures in this section.

5. Graphical proof of $d(d-1)(d-3)(d-7)=0$

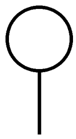
We assume now that 2 is invertible in the ground ring and show that the two rules (R1) and (R2) imply

$$d(d-1)(d-3)(d-7)=0$$

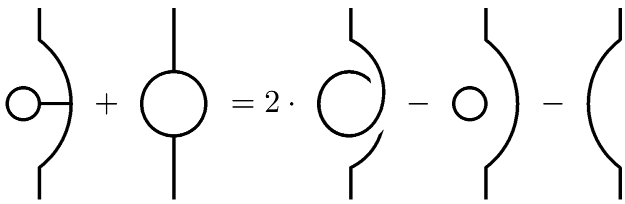
where

$d =$ 

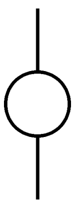
Since 2 is invertible, one has

(1.0)  $= 0$

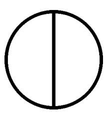
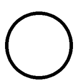
by (0.1) and (R1).
Next consider the following consequence of (R2):



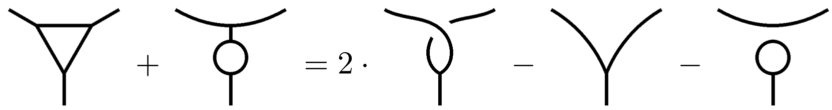
By (1.0) this gives

(1.1)  $= -(d-1)\cdot$

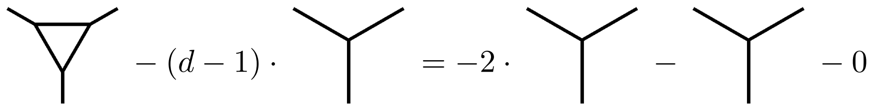
This yields immediately

(1.2)  $= -(d-1)\cdot$  $= -d(d-1)$

We next prove Springer's formula. (R2) gives

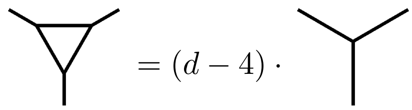


Inserting (1.0) and (1.1) shows



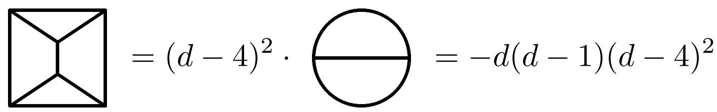
Hence

(1.3)



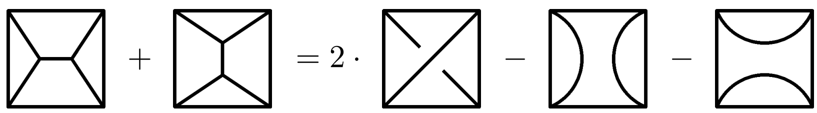
Moreover one finds

(1.4)



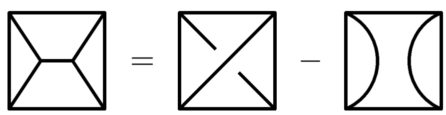
by applying (1.3) to both of the triangles and using (1.2).

Now comes the final step. Rule (R2) gives



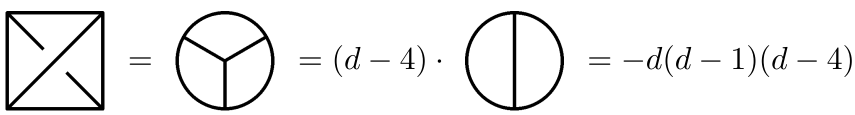
The two leftmost and the two rightmost graphs are the same. So, after dividing by 2 we have

(1.5)



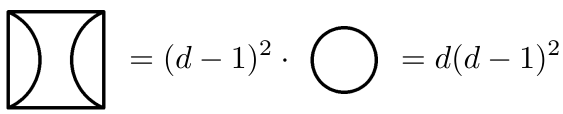
For the middle term of (1.5) one finds

(1.6)



Here the first equality follows, since both pictures are just different projections of the same graphs but with sign changes at two vertices. Rule (R1) give then equality. The other two equalities follow from (1.3) and (1.2).
To compute the rightmost graph in (1.5) one applies formula (1.1) twice and finds

(1.7)



The formulas (1.4), (1.6), and (1.7) give the desired relation. $\square$

Here are some of the translations of the graph formulas above to the algebraic formulas in [Rost, M., On the Dimension of a Composition Algebra, *Doc. Math.* **1** No. 10 (1996) 209–214]:

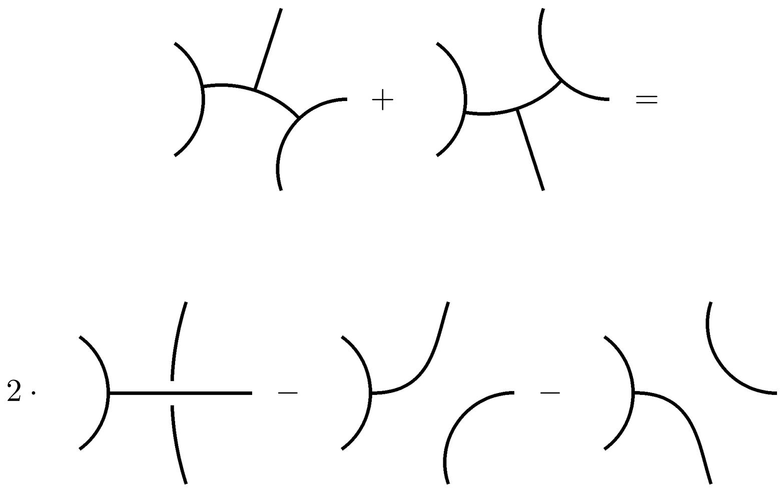
(R1)↔(2.4a)
(R2)↔(2.5a,b)
(1.1)↔(3.1)
(1.2)↔(3.2)
(1.3)↔(3.3)

Exercise. Draw the pictures which derive the relations $d(d-1)(d-3) = 0$ and $d(d-1) = 0$ for "associative" and "commutative" vector product algebras (the "toy modells" in loc. cit.)

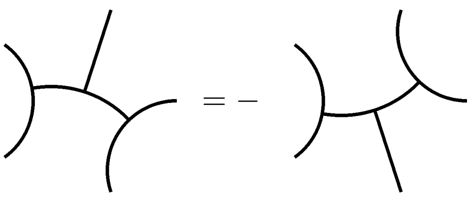
6. Graphical illustration of $2R_4 = 0$

The relation (R2) gives

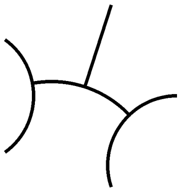
(1.8)



Hence

 $+$ sum of graphs with less vertices

The two graphs differ just by sign and a rotation of order 5. Therefore, iterating this relation 5 times yields

$2 \cdot$  $=$ sum of graphs with less vertices

This shows that the morphism

$$V^{\otimes 4} \to V, \qquad x_1 \otimes x_2 \otimes x_3 \otimes x_4 \mapsto ((x_1 \times x_2) \times x_3) \times x_4$$

can be expressed as a sum of tensors involving only 1 $\times$-product. The precise formula for this is

$$2R_4 = 0$$

Together with (1.0) and (1.1) it follows from (1.8):

(1.1) Proposition. *Any closed connected graph with at most $2k-2$ vertices can be expressed by a polynomial in d of degree $\leq k$. In particular,* $\mathrm{End}(\varnothing)$ *is generated by d.*

If I remember well one can use (1.8) directly to show at least the following qualitative result:

(1.2) Proposition. *One has $P(d) = 0$ in* $\mathrm{End}(\varnothing)$ *for some polynomial $P \in \mathbb{Z}[d]$ of degree* 4*.*

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