

The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli

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Abstract

Methods of Harder and Narasimhan from the theory of moduli of vector bundles are applied to moduli of quiver representations. Using the Hall algebra approach to quantum groups, an analog of the Harder-Narasimhan recursion is constructed inside the quantized enveloping algebra of a Kac-Moody algebra. This leads to a canonical orthogonal system, the HN system, in this algebra. Using a resolution of the recursion, an explicit formula for the HN system is given. As an application, explicit formulas for Betti numbers of the cohomology of quiver moduli are derived, generalizing several results on the cohomology of quotients in 'linear algebra type' situations.

1 Introduction

The geometry of quiver representations is by now recognized as an area with many connections to such diverse fields as representation theory of algebras, geometric invariant theory, quantum group theory and representation theory of (Kac-Moody) Lie algebras.

The connection to methods of Geometric Invariant Theory is provided by the construction of moduli spaces of quiver representations of [Kin]. In particular, one can expect analogies to the theory of moduli of vector bundles on curves, generalizing the approach advertised in [Ra].

On the other hand, the connection to quantum group theory is given by the realization of quantized enveloping algebras of (Kac-Moody) Lie algebras of [Rin], [Gr] via the Hall algebra approach, which can be interpreted (see [Kap]) as a convolution algebra construction on parameter spaces of quiver representations.

The aim of the present paper is to develop a synthesis of both methods. We

start with a particular instance of the above mentioned analogy to vector bundle theory, namely the Harder-Narasimhan recursion [HN], which was originally used for computing Betti numbers of moduli spaces.

The first main result of this paper (Proposition 4.8) is a materialization of the Harder-Narasimhan recursion in the quantized enveloping algebra of a symmetric Kac-Moody Lie algebra. It leads to a canonical orthogonal system, the HN system, in such algebras (Theorem 4.9), which is recursively computable.

The HN system comprises a surprising amount of information on the moduli spaces of (semi-)stable quiver representations: as the second main result, just by evaluating at a character of the quantum group, we recover in Theorem 6.7 the Betti numbers of such moduli spaces completely (modulo the standard assumption (see e.g. [Kir]) that semistability and stability coincide). The proof uses the Weil conjectures, in analogy to the original approach of Harder and Narasimhan in the vector bundle situation (see also [Kir], [Gö] for similar situations).

The third main result is a resolution of the Harder-Narasimhan recursion, in the spirit of [Z], [LR] in the vector bundle case (Theorem 5.1), together with a fast algorithm for Betti number computation (Corollary 6.9). Whereas the cited works use involved explicit calculations, resp. the Langlands lemma from the theory of Eisenstein series, the present proof uses only some simple (polygonal and simplicial) combinatorics. It should be noted that our materialization of the Harder-Narasimhan recursion in a non-commutative algebra (the quantum group) is already anticipated in ([Z], p. 457), where one of the key insights for resolving the recursion is a noncommutative approach to certain polynomial expressions.

At the moment, the immediate applications of the HN system to quantum group theory are largely conjectural: one can expect explicit descriptions of PBW type bases, and applications to the general structure theory of Hall algebras in the spirit of [SV] (see section 7).

The applications of the cohomology formulas are more direct (and numerous), in that they unify, generalize, and make more explicit, several known results on cohomology of moduli of 'linear algebra type' problems, for example, the formulas for Betti numbers of sequences of subspaces in ([Kir], 16), and for families of linear maps in [Dr], [ES].

In another direction, note the usage of some of the present methods in [CBV] for a partial proof of the Kac conjectures (see [Kac]). Finally, one can hope for analogs of the present methods to be possible in the setup of Hall algebras for categories of coherent sheaves on curves in the framework of [Kap], which would probably bring the subject back to its roots in vector bundle theory.

The paper is organized as follows: in section 2, we recall the notions of (semi-) stability and of the Harder-Narasimhan filtration for categories of quiver repre-

sentations. These are applied in section 3 to the representation varieties of quivers; we construct the Harder-Narasimhan stratification of these spaces and prove some basic geometric properties. In section 4, we first recall the Hall algebra construction and its relation to quantum groups. Then we use the results of the previous sections to obtain a quantum group version of the Harder-Narasimhan recursion, yielding the above mentioned Harder-Narasimhan system. Section 5 is devoted to the resolution of the recursion. In section 6, we take a closer look at the geometry of quiver moduli over finite fields, in order to apply the Weil conjectures, and to derive the formulas for Betti numbers. Finally, section 7 contains some conjectural applications, and the above mentioned examples of quiver moduli which allow for explicit cohomology formulas.

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2 Quiver representations and the HN filtration

Let Q be a finite quiver (oriented graph) without oriented cycles. We denote by I its set of vertices, and by r_{ij} for $i, j \in I$ the number of arrows from i to j . The free abelian group $\mathbf{Z}I$ generated by I carries a bilinear form, the Euler form, defined by $\langle i, j \rangle = \delta_{ij} - r_{ij}$.

Let \mathbf{k} be an arbitrary field. We consider the well-known category $\text{mod}_{\mathbf{k}}Q$ of finite-dimensional \mathbf{k} -representations of Q (see [ARS] for general notions and facts on quiver representations). For a representation X given by \mathbf{k} -vector spaces X_i for $i \in I$, we define its dimension type $\underline{\dim}X$ as $\sum_{i \in I} \dim_{\mathbf{k}}(X_i)i \in \mathbf{N}I$. We have

$$\langle \underline{\dim}X, \underline{\dim}Y \rangle = \dim_{\mathbf{k}} \text{Hom}_{\mathbf{k}Q}(X, Y) - \dim_{\mathbf{k}} \text{Ext}_{\mathbf{k}Q}^1(X, Y)$$

for all representations $X, Y \in \text{mod}_{\mathbf{k}}Q$.

We introduce a (non-canonical) notion of stability in $\text{mod}_{\mathbf{k}}Q$. Fix once and for all a linear form $\Theta = \sum_{i \in I} \Theta_i i^*$ on $\mathbf{Z}I$, called a weight for Q . We define a slope function on $\mathbf{N}I \setminus \{0\}$ by $\mu(d) = \Theta(d) / \dim d$, where the linear form \dim on $\mathbf{Z}I$ is defined by $\dim i = 1$ for $i \in I$. Using the slope function, we can define a notion of stability for representations of Q ; for a representation $0 \neq X \in \text{mod}_{\mathbf{k}}Q$, we set $\mu(X) = \mu(\underline{\dim}X)$.

Definition 2.1 *A \mathbf{k} -representation X of Q is called semistable (resp. stable) if $\mu(U) \leq \mu(X)$ (resp. $\mu(U) < \mu(X)$) for all proper subrepresentations $0 \neq U \subset X$.*

Examples: In each of the following examples, it is easy to work out the corresponding meaning of stability; we just state the results. The examples illustrate that the quiver setup unifies several 'linear algebra type' problems. They will be used as standard examples throughout the paper.

- A** If $Q = i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_n$ and $\Theta(i_k) = -k$, then the stable representations are precisely the unique indecomposables E_{kl} of dimension type $i_k + \dots + i_l$ for $k \leq l$, and the semistables are their powers (see section 7 for a possible generalization).
- B** Let Q be given by vertices $I = \{i_0, i_1, \dots, i_n\}$ and arrows $i_k \rightarrow i_0$ for $k = 1 \dots n$. Let $d = mi_0 + i_1 + \dots + i_n$, $\Theta = -i_0^*$. Then a semistable (resp. stable) representation corresponds to a tuple (v_1, \dots, v_n) of non-zero vectors in \mathbf{k}^m such that for any proper non-zero subspace $U \subset \mathbf{k}^m$, the number of vectors v_k in U is \leq (resp. $<$) $\frac{n}{m} \cdot \dim U$. Thus, we arrive at one of the principal examples of Mumford's GIT ([M], 3.).
- C** More generally, fix $r, N \in \mathbf{N}$. Let Q be given by vertices $I = \{i_0, i_{\nu,p} : \nu = 1 \dots r, p = -N \dots N\}$, arrows $i_{\nu,p} \rightarrow i_{\nu,p-1}$ for $\nu = 1 \dots r, p = N \dots 1-N$, and $i_{\nu,-N} \rightarrow i_0$ for $\nu = 1 \dots r$. Let $d = \sum_{\nu,p} d_{\nu,p} i_{\nu,p} + d_0 i_0$, where $0 = d_{\nu,N} \leq \dots \leq d_{\nu,-N} = d_0$ for $\nu = 1 \dots r$, and $\Theta = -i_0^*$. Then the semistable (resp. stable) representations correspond to d_0 -dimensional \mathbf{k} -vector spaces V together with a family of descending flags F_ν^p (where $\dim F_\nu^p = d_{\nu,p}$), fulfilling the conditions of [F].
- D** If $Q = i \xrightarrow{(n)} j$ (with n arrows pointing from i to j), $d = ai + bj$, $\Theta = i^*$, then semistable (resp. stable) representations correspond to tuples $(f_1 \dots f_n)$ of linear maps $f_k : \mathbf{k}^a \rightarrow \mathbf{k}^b$ such that for each non-zero proper subspace $U \subset \mathbf{k}^a$, the dimension of $\sum_k f_k(U)$ is \geq (resp. $>$) $\frac{b}{a} \cdot \dim U$ (see [ES], [Dr]).

The following properties of semistable (resp. stable) representations are well-known and easy to prove (see e.g. [HN], [Sh], [Ru], ...).

Lemma 2.2 *Given a short exact sequence $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$ in $\text{mod}_{\mathbf{k}} Q$, we have*

$$\mu(M) \leq \mu(X) \text{ iff } \mu(X) \leq \mu(N) \text{ iff } \mu(M) \leq \mu(N)$$

and

$$\min(\mu(M), \mu(N)) \leq \mu(X) \leq \max(\mu(M), \mu(N)).$$

If $\mu(M) = \mu(X) = \mu(N)$, then X is semistable if and only if M and N are semistable.

Denote by $\text{mod}_{\mathbf{k}}^\mu Q$ the full subcategory of $\text{mod}_{\mathbf{k}} Q$ consisting of semistable representations of Q of slope $\mu \in \mathbf{Q}$.

Lemma 2.3 *For all $\mu \in \mathbf{Q}$, the category $\text{mod}_{\mathbf{k}}^{\mu}Q$ is an abelian subcategory of $\text{mod}_{\mathbf{k}}Q$ whose simple objects are the stable representations of Q of slope μ . Moreover, we have $\text{Hom}_Q(\text{mod}_{\mathbf{k}}^{\mu}Q, \text{mod}_{\mathbf{k}}^{\nu}Q) = 0$ provided $\mu > \nu$.*

Finally, we introduce the key notion of this section.

Definition 2.4 *Let X be a representation of Q . A Harder-Narasimhan (HN) filtration of X is a filtration $0 = X_0 \subset X_1 \subset \dots \subset X_s = X$ such that the quotients X_k/X_{k-1} are semistable for $k = 1 \dots s$, and $\mu(X_1/X_0) > \mu(X_2/X_1) > \dots > \mu(X_s/X_{s-1})$.*

Proposition 2.5 *Any \mathbf{k} -representation X of Q possesses a unique Harder-Narasimhan filtration.*

Proof: We proceed by induction on the dimension of X . Let $X_1 \subset X$ be a subrepresentation of maximal slope, and of maximal dimension among the subrepresentations with this property. It is easy to see that this determines X_1 uniquely. By induction, X/X_1 possesses a unique HN filtration, which we can lift to one of X via the projection $X \rightarrow X/X_1$. Uniqueness follows inductively from the fact that the first term of a HN filtration already has to be the X_1 just constructed. \square

Definition 2.6 *An element $d \in \mathbf{N}I$ is called semistable if there exists a semistable representation of dimension type d . A tuple $d^* = (d^1, \dots, d^s)$ is called a HN type if each d^k is semistable and $\mu(d^1) > \dots > \mu(d^s)$. The sum $|d^*| = \sum_{k=1}^s d^k$ is called the weight of d^* , and $l(d^*) = s$ is called the length of d^* .*

3 Representation varieties and the HN stratification

We first recall the varieties of quiver representations of fixed dimension type.

Definition 3.1 *For $d = \sum_{i \in I} d_i i \in \mathbf{N}I$, define $R_d = \bigoplus_{\alpha: i \rightarrow j} \text{Hom}_{\mathbf{k}}(\mathbf{k}^{d_i}, \mathbf{k}^{d_j})$ and $G_d = \prod_{i \in I} \text{GL}_{d_i}(\mathbf{k}) \subset E_d = \bigoplus_{i \in I} \text{End}(\mathbf{k}^{d_i})$. The group G_d acts on R_d via*

$$(g_i)_i \cdot (X_{\alpha})_{\alpha} = (g_j X_{\alpha} g_i^{-1})_{\alpha: i \rightarrow j}.$$

R_d is an affine \mathbf{k} -variety parametrizing the \mathbf{k} -representations of Q of dimension type d . The G_d -orbits \mathcal{O}_X in R_d correspond bijectively to the isomorphism classes $[X]$ of representations $X \in \text{mod}_{\mathbf{k}}Q$ of dimension type d . The unique closed orbit of G_d in R_d corresponds to the unique semisimple representation of dimension type d , so the invariant ring $\mathbf{k}[R_d]^{G_d}$ reduces to the scalars.

The main use of the notion of (semi-)stability lies in the following result.

Theorem 3.2 [Kin] *Let \mathbf{k} be algebraically closed. For each $d \in \mathbf{NI}$, the subset $R_d^{ss} \subset R_d$ of semistable representations is an open subvariety. It admits a categorical (GIT) quotient $\mathcal{M}_d^{ss} = R_d^{ss}/G_d$, which is a projective variety. The quotient \mathcal{M}_d^{ss} contains a smooth open subvariety \mathcal{M}_d^s , which is a geometric quotient by G_d of $R_d^s \subset R_d^{ss}$, the subset of stable representations.*

Remark: In fact, the notions of (semi-)stability for representations translate into the corresponding notions of [M] for points in R_d , with respect to the trivial line bundle on R_d with G_d -action twisted by a suitable character χ of G_d . The moduli space can be defined as

$$\mathcal{M}_d^{ss} = \mathbf{Proj}(\oplus_{n \in \mathbf{N}} \mathbf{k}[R_d]^{G_d \cdot \chi^n}),$$

where $\mathbf{k}[R_d]^{G_d \cdot \chi^n}$ consists of semi-invariant polynomial functions $f : R_d \rightarrow \mathbf{k}$ of weight χ^n .

Examples:

1. In example B of section 2, we recover the quotient $(\mathbf{P}_{m-1})_{\text{stable}}^n / \text{PGL}_m$ of $([M], 3)$.
2. The situation of example D of section 2 is related to moduli of bundles on \mathbf{P}^2 , see [Dr].
3. As a particular case of example D, the moduli space \mathcal{M}_{mi+mj}^{ss} compactifies the affine variety given by $\mathbf{k}[M_m(\mathbf{k})^{n-1}]^{\text{GL}_m}$, the ring of invariants of $n-1$ -tuples of $m \times m$ -matrices under simultaneous conjugation. In fact, if the first map f_1 is invertible, then its stabilizer $G \simeq \text{GL}_m$ under the $\text{GL}_m \times \text{GL}_m$ -action acts by simultaneous conjugation on f_2, \dots, f_n .

Next, we introduce the Harder-Narasimhan stratification. Note that the term stratification is used in a weak sense, meaning a finite decomposition of a variety into irreducible, locally closed subsets (see [Kir], Introduction).

Definition 3.3 *For a HN type d^* , we denote by $R_{d^*}^{HN} \subset R_d$ the subset of representations whose HN filtration is of type d^* . $R_{d^*}^{HN}$ is called the HN stratum for the HN type d^* . More generally, we denote by $R_d^{d^*}$ the subset of representations possessing some filtration of type d^* , i.e. $\underline{\dim} X_k / X_{k-1} = d^k$ for $k = 1 \dots s$ in the notation of Definition 2.4.*

Proposition 3.4 *The HN strata for HN types d^* of weight d define a stratification of R_d into irreducible, locally closed subvarieties. The codimension of $R_{d^*}^{HN}$ in R_d is given by $-\sum_{1 \leq k < l \leq s} \langle d^k, d^l \rangle$.*

Proof: Let $F^* : 0 = F^0 \subset F^1 \subset \dots \subset F^s$ be a flag of type d^* in the I -graded vector space $\mathbf{k}^d = \oplus_{i \in I} \mathbf{k}^{d_i}$, i.e. $F^k / F^{k-1} \simeq \mathbf{k}^{d^k}$ for $k = 1 \dots s$, and

denote by F_i^k the i -component of F^k . Denote by Z the closed subvariety of R_d of representations X which are compatible with F^* , i.e. $X_\alpha(F_i^k) \subset F_j^k$ for $k = 1 \dots s$ and for all arrows $\alpha : i \rightarrow j$ in Q . It is easy to see that Z is a trivial vector bundle of rank $\sum_{k < l} \sum_{i \rightarrow j} d_i^l d_j^k$ over $R_{d^1} \times \dots \times R_{d^s}$, via the projection p mapping $X \in Z$ to the sequence of subquotients with respect to F^* . Thus, the inverse image of $R_{d^1}^{ss} \times \dots \times R_{d^s}^{ss}$ under p is an open subvariety Z_0 of Z . The action of G_d on R_d induces actions of the parabolic subgroup P_{d^*} of G_d , consisting of elements fixing the flag F^* , on Z_0 and Z . The image of the associated fibre bundle $G_d \times^{P_{d^*}} Z$ under the action morphism m equals $R_d^{d^*}$, which is thus a closed subvariety of R_d . The image of $G_d \times^{P_{d^*}} Z_0$ under m equals $R_{d^*}^{HN}$, and $G_d \times^{P_{d^*}} Z_0$ is the full preimage. By the uniqueness of the HN filtration, the morphism m is bijective over $R_{d^*}^{HN}$, which therefore is an irreducible, open subvariety of $R_d^{d^*}$. The codimension is now easily computed as $-\sum_{k < l} \langle d^k, d^l \rangle$, using the identity $\langle d, d \rangle = \dim G_d - \dim R_d$ and the above description of Z . \square

From this description, we can derive a recursive criterion for the existence of semistable representations, complementing a criterion for the existence of stable representations in [Kin].

Corollary 3.5 *A dimension type $d \in \mathbf{NI}$ is semistable if and only if there exists no HN type d^* of weight d such that $\langle d^k, d^l \rangle = 0$ for all $1 \leq k < l \leq s$.*

Proof: A dimension type d is semistable if and only if there exists no dense HN stratum, i.e. a stratum $R_{d^*}^{HN}$ of codimension 0, which by the above codimension formula means $\langle d^k, d^l \rangle = 0$ for all $1 \leq k < l \leq s$. \square

We have seen in the proof of Proposition 3.4 that the closure of the HN stratum $R_{d^*}^{HN}$ equals $R_d^{d^*}$. In general, this is not anymore a union of HN strata (see the example below). But at least it is contained in such a union, and the strata involved in this union can be controlled using polygonal combinatorics. We proceed as in the vector bundle situation, following [Sh].

Definition 3.6 *Given an arbitrary tuple $d^* = (d^1, \dots, d^s)$ in \mathbf{NI} , we denote by $P(d^*)$ the polygon in \mathbf{N}^2 with vertices $(\sum_{l=1}^k \dim d^l, \sum_{l=1}^k \Theta(d^l))$ for $k = 0 \dots s$. For two such tuples d^*, e^* , define $d^* \leq e^*$ (resp. $d^* < e^*$) if $P(d^*)$ lies on or below (resp. strictly below) $P(e^*)$ (see [Sh]). We call d^* convex if the polygon $P(d^*)$ is convex.*

Note that, by definition, HN types are always convex.

Proposition 3.7 *The closure $R_d^{d^*}$ of the HN stratum for the HN type d^* is contained in the union of the HN strata for HN types $e^* \geq d^*$.*

Proof: Let X be a representation in R_d^{d*} , and let e^* be the HN type of X . Since the polygon $P(e^*)$ is convex by definition, it suffices to prove that the slope $\mu(\underline{\dim} U)$ of an arbitrary subrepresentation $U \subset X$ lies on or below $P(e^*)$. We proceed by induction over the length t of e^* ; in case $t = 1$ there is nothing to prove, since then X is semistable. By induction, the slope of $(U + X_1)/X_1$ lies on or below $P(e^{*+1})$, where e^{*+1} is the HN type of X/X_1 . It follows that the slope of $U + X_1$ lies on or below $P(e^*)$. Using the obvious exact sequences relating U , X_1 and $U + X_1$, we get by the maximality property of X_1 :

$$\mu(U + X_1) \leq \mu(X_1) \leq \mu(X_1/(U \cap X_1)), \text{ thus } \mu(U) \leq \mu(U + X_1),$$

which proves the desired property. \square

Example: Let Q be the quiver $i \rightarrow j \rightarrow k$, $d = i + j + k$, $\theta = 2i^* + 3j^*$. Thus $R_d \simeq \mathbf{k}^2$, where a point (x, y) corresponds to the representation $\mathbf{k} \xrightarrow{x} \mathbf{k} \xrightarrow{y} \mathbf{k}$. The HN stratum for the HN type $(i, j + k)$ equals $\mathbf{k}^* \times 0 \subset \mathbf{k}^2$, and the HN stratum for the HN type (j, i, k) equals $0 \times \mathbf{k} \subset \mathbf{k}^2$. Thus, the closure of $R_{(i, j+k)}$ is contained in $R_{(i, j+k)} \cup R_{(j, i, k)}$, but is not a union of HN strata.

4 Quantum groups and the HN system

In this and the following sections, \mathbf{k} denotes a finite field, whose cardinality is denoted by v^2 for $v \in \mathbf{C}$.

To get a relation between the methods of the previous sections and (quantized) Kac-Moody algebras, we first define the Hall algebra (see [Rin], [Gr]) in the version of Kapranov (see [Kap]).

Definition 4.1 For $d \in \mathbf{NI}$, define $\mathcal{H}_d = \mathbf{C}^{G_d}[R_d]$, the \mathbf{C} -vector space of (arbitrary) G_d -invariant functions from R_d to \mathbf{C} . Define a \mathbf{NI} -graded \mathbf{C} -vector space $\mathcal{H} = \mathcal{H}(Q) = \bigoplus_{d \in \mathbf{NI}} \mathcal{H}_d$ with the multiplication

$$(f * g)(X) = v^{(e, d)} \sum_{U \subset X} f(U)g(X/U)$$

on homogeneous elements f, g of degree $|f| = d$ (i.e. $f \in \mathcal{H}_d$) and e , respectively. Define a scalar product on \mathcal{H} by

$$(f, g) = (\#G_d)^{-1} \sum_X f(X)g(X)$$

for $f, g \in \mathcal{H}_d$, where $\#S$ denotes the cardinality of a finite set S , and $(\mathcal{H}_d, \mathcal{H}_e) = 0$ for $d \neq e$.

By [Rin], [Gr], \mathcal{H} is an associative, \mathbf{NI} -graded \mathbf{C} -algebra.

An immediate induction yields:

Lemma 4.2 For $f_k \in \mathcal{H}_{d^k}$, $k = 1 \dots s$, we have

$$(f_1 * \dots * f_s)(X) = v^{\langle d^* \rangle} \sum_{0=X_0 \subset X_1 \subset \dots \subset X_s=X} f_1(X_1/X_0) \dots f_s(X_s/X_{s-1}),$$

where

$$\langle d^* \rangle = \sum_{k < l} \langle d^l, d^k \rangle.$$

Next, we define the so-called composition subalgebra of \mathcal{H} .

Definition 4.3 For $d \in \mathbf{NI}$, define $\chi_d = \chi_{R_d}$, the characteristic function of the variety R_d . Define \mathcal{C} as the \mathbf{C} -subalgebra of \mathcal{H} generated by the functions χ_i for $i \in I$.

Lemma 4.4 We have $\chi_d \in \mathcal{C}$ for all $d \in \mathbf{NI}$.

Proof: We order the set of vertices of Q as $I = \{1, \dots, n\}$ such that $i > j$ if there exists an arrow $i \rightarrow j$, which is possible since Q has no oriented cycles. It is easy to see that any representation $X \in R_d$ has a unique filtration $0 = X_0 \subset \dots \subset X_n = X$ such that $\dim X_k / X_{k-1} = d_k k \in \mathbf{NI}$, so that the product $\chi_{d_{11}} * \dots * \chi_{d_{nn}}$ equals χ_d up to some power of v . But since R_{ni} for $n \in \mathbf{N}$, $i \in I$ consists of a single point, χ_{ni} equals a (non-zero) scalar multiple of χ_i^{*n} . This shows that χ_d is generated by the χ_i . \square

The connection to quantum groups is given by the following theorem, due to J. A. Green [Gr], which generalizes a theorem of C. M. Ringel [Rin] in the finite type case.

Theorem 4.5 The composition algebra \mathcal{C} is isomorphic to $\mathcal{U}_v(\mathfrak{n}^+)$, the quantized enveloping algebra of the positive part of the symmetric Kac-Moody algebra corresponding to the quiver Q , specialized at v .

Here, the Kac-Moody algebra corresponding to Q is defined by the generalized Cartan matrix $(\langle i, j \rangle + \langle j, i \rangle)_{i,j \in I}$ as in [Kac].

The link between the HN filtration and the Hall algebra is provided by the following definition.

Definition 4.6 For a HN type $d^* = (d^1, \dots, d^s)$ of weight d , define

$$\chi_{d^*}^{HN} = \chi_{R_{d^*}^{HN}} \in \mathcal{H}_d,$$

the characteristic function of the HN stratum for d^* . In particular, define $\chi_d^{ss} = \chi_{(d)}^{HN}$.

We find the following analogue of the description of the HN stratum in Proposition 3.4.

Lemma 4.7 *For a HN type $d^* = (d^1, \dots, d^s)$, we have in \mathcal{H} the following identity:*

$$\chi_{d^*}^{HN} = v^{-\langle d^* \rangle} \chi_{d^1}^{ss} * \dots * \chi_{d^s}^{ss}.$$

Proof: Any filtration X_* of type d^* with semistable subquotients is a HN filtration by definition, which is unique by Proposition 2.5. The statement follows from the description Lemma 4.2 of the convolution product in $\mathcal{H}(Q)$. \square

We easily derive the Harder-Narasimhan recursion, in analogy to [AB] and [HN].

Proposition 4.8 *For all $d \in \mathbf{NI}$, we have*

$$\chi_d^{ss} = \chi_d - \sum_{d^*} v^{-\langle d^* \rangle} \chi_{d^1}^{ss} * \dots * \chi_{d^s}^{ss},$$

where the sum runs over all HN types $d^* \neq (d)$ of weight d . We have $\chi_d^{ss} = \chi_d$ if and only if Θ is constant on $\text{supp } d = \{i \in I : d_i \neq 0\}$, the support of d in I .

Proof: In case Θ is constant on $\text{supp } d$, any representation $X \in R_d$ is obviously semistable, thus $\chi_d^{ss} = \chi_d$. Otherwise, we have a disjoint union $R_d = \bigcup_{d^*} R_{d^*}^{HN}$ (the union being over all HN types of weight d), which translates into the identity $\chi_d = \sum_{d^*} \chi_{d^*}^{HN}$. Using Lemma 4.7, the recursion is proved. \square

As an immediate consequence, we get:

Theorem 4.9 *For all HN types d^* , the element $\chi_{d^*}^{HN}$ already belongs to the composition algebra \mathcal{C} . The scalar product of two such elements is given by*

$$(\chi_{d^*}^{HN}, \chi_{e^*}^{HN}) = \delta_{d^*, e^*} (\#G_d)^{-1} \#R_{d^*}^{HN}.$$

If $\mu(d^s) > \mu(e^1)$, then $\chi_{d^*}^{HN} * \chi_{e^*}^{HN} = \chi_{(d^1 \dots d^s e^1 \dots e^t)}^{HN}$.

Proof: The first statement follows from Proposition 4.8 by induction on the dimension type, using Lemma 4.4. The second statement just reformulates the disjointness of different HN strata, using the definition of the scalar product on \mathcal{H} . For the third part, we just have to note that, by assumption, the concatenation $(d^1, \dots, d^s, e^1, \dots, e^t)$ is again a HN type. \square

Definition 4.10 *The set of elements $\chi_{d^*}^{HN}$ for various HN types d^* is called the Harder-Narasimhan system in $\mathcal{C} \simeq \mathcal{U}_v(\mathfrak{n}^+)$.*

We thus have found a natural orthogonal system of elements in $\mathcal{U}_v(\mathfrak{n}^+)$, which is necessarily linearly independent by orthogonality and the fact that $\chi_{d^*}^{HN}$ is non-zero by definition of a HN type. This system is recursively computable by combining Lemma 4.7, Proposition 4.8 and Corollary 3.5. Furthermore, it is partially multiplicative. The HN system can be viewed as a replacement for PBW type bases (which have 'good' properties only for finite type cases) in the Kac-Moody situation.

We end this section with a compact reformulation of the HN recursion 4.8.

Definition 4.11 *Consider the skew Laurent polynomial ring $\widehat{\mathcal{H}} = \mathcal{H}[[T_i, : i \in I]]$, where $\chi_i * T_j = v^{\langle i, j \rangle} T_j * \chi_i$ for all $i, j \in I$. Define generating functions*

$$X(T) = \sum_{d \in \mathbf{NI}} \chi_d * T^d, \quad X_\mu^{ss}(T) = \sum_{\substack{d \in \mathbf{NI} \\ \mu(d) = \mu}} \chi_d^{ss} * T^d,$$

where $T^d = \prod_{i \in I} T_i^{d_i}$ for $d \in \mathbf{NI}$.

Using these definitions, the recursion 4.8 allows the following notation:

Proposition 4.12 *In $\widehat{\mathcal{H}}$, the generation function $X(T)$ equals the 'descending product' $\prod_{\mu \in \mathbf{Q}}^{\leftarrow} X_\mu^{ss}(T) := \sum_{\mu_1 > \dots > \mu_s} X_{\mu_1}^{ss}(T) * \dots * X_{\mu_s}^{ss}(T)$.*

Proof: First note that for each $f \in \mathcal{C} \cap \mathcal{H}_d$, we have $f * T^e = v^{\langle d, e \rangle} T^e * f$ by definition. We calculate using Lemma 4.7:

$$\begin{aligned} \prod_{\mu \in \mathbf{Q}}^{\leftarrow} X_\mu^{ss}(T) &= \sum_{\mu_1 > \dots > \mu_s} X_{\mu_1}^{ss}(T) * \dots * X_{\mu_s}^{ss}(T) \\ &= \sum_{d^*} \chi_{d^1}^{ss} * T^{d^1} * \dots * \chi_{d^s}^{ss} * T^{d^s} \\ &= \sum_{d^*} v^{-\sum_{k < l} \langle d^l, d^k \rangle} \chi_{d^1}^{ss} * \dots * \chi_{d^s}^{ss} * T^{d^1} * \dots * T^{d^s} \\ &= \sum_{d^*} \chi_{d^*}^{HN} * T^{|d^*|} = \sum_{d \in \mathbf{NI}} \chi_d * T^d = X(T), \end{aligned}$$

where the sums run over all HN types d^* . □

5 Resolving the recursion

Motivated by similar results in [Z],[LR] (but with different methods), we will now derive a resolution of the recursion 4.8. This gives an explicit formula for the elements χ_d^{ss} :

Theorem 5.1 For all $d \in \mathbf{NI}$, we have:

$$\chi_d^{ss} = \sum_{d^*} (-1)^{s-1} v^{-\langle d^* \rangle} \chi_{d^1} * \dots * \chi_{d^s},$$

where the sum runs over all tuples of non-zero dimension types $d^* = (d^1 \dots d^s)$ of weight d such that $d^* = (d)$ or $d^* > (d)$, i.e. $\mu(\sum_{l=1}^k d^l) > \mu(d)$ for $k = 1 \dots s-1$.

Proof: The proof proceeds along the following lines:

First, we reduce the statement by an explicit calculation to a purely combinatorial formula. This will be interpreted in terms of the Euler characteristic of a simplicial complex (encoding convex coarsenings of a polygon). By explicit combinatorics, we show that this simplicial complex is in fact always contractible, proving the Theorem.

We start with some definitions.

Definition 5.2 Let $d^* = (d^1, \dots, d^s)$ be a tuple of dimension types.

1. For a subset $I = \{s_1 < \dots < s_k\} \subset \{1, \dots, s-1\}$, define the I -coarsening of d^* as

$$c_I^*(d^*) = (d^1 + \dots + d^{s_1}, d^{s_1+1} + \dots + d^{s_2}, \dots, d^{s_k+1} + \dots + d^s).$$

2. The subset I is called d^* -admissible if the following holds:

- (a) $c_I^*(d^*)$ is convex,
- (b) For all $i = 0 \dots k$, we have $(d^{s_i+1}, \dots, d^{s_{i+1}}) \geq (d^{s_i+1} + \dots + d^{s_{i+1}}) = (c_I^i(d^*))$.

3. Define $\mathcal{A}(d^*)$ as the set of all d^* -admissible subsets of $\{1, \dots, s-1\}$.

In polygonal language, a coarsening is thus admissible if it is convex and lies on or below the polygon $P(d^*)$.

Lemma 5.3 For all sequences d^* , the set $\mathcal{A}(d^*)$ is a simplicial complex.

Proof: If I is d^* -admissible, and $J \subset I$ is a subset, then $c_J^*(d^*) = c_J^*(c_I^*(d^*))$. Thus, $c_J^*(d^*)$ is a coarsening of the convex tuple $c_I^*(d^*)$. Therefore, it obviously has to be convex again, and it has to lie on or below $c_I^*(d^*)$, which lies on or below $P(d^*)$ by assumption. We conclude that J is also d^* -admissible. \square

Using these definitions, we can reduce the formula of Theorem 5.1 to a purely combinatorial problem. We adopt Proposition 4.8. If Θ is constant on $\text{supp } d$, then any dimension type $e \leq d$ has the same slope as d , which means that the sum in Theorem 5.1 just runs over the single tuple (d) , and the formula is trivial.

Otherwise, we consider the HN recursion and replace any term $\chi_{d^i}^{ss}$ on its right hand side by the claimed formula:

$$\begin{aligned}\chi_d &= \sum_{d^*} v^{-\langle d^* \rangle} \chi_{d^1}^{ss} * \dots * \chi_{d^s}^{ss} \\ &= \sum_{d^*} \sum_{d^{1,*}, \dots, d^{s,*}} v^{-\langle d^* \rangle} (-1)^{\sum_{i=1}^s (t_i - 1)} v^{-\sum_{i=1}^s \langle d^{i,*} \rangle} \times \\ &\quad \times \chi_{d^{1,1}} * \dots * \chi_{d^{1,t_1}} * \dots * \chi_{d^{s,1}} * \dots * \chi_{d^{s,t_s}}.\end{aligned}$$

In this equation, the outer sum runs over all HN types d^* of weight d , and the inner sums run over all sequences $(d^{1,*}, \dots, d^{s,*})$ of tuples $d^{i,*}$ of weight d^i and length t_i such that $d^{i,*} = (d^i)$ or $d^{i,*} > (d^i)$.

We now want to exchange the order of summation. So denote by e^* the concatenation $e^* = (d^{1,1}, \dots, d^{s,t_s})$ of all the tuples $d^{i,*}$. Obviously, the resulting tuples e^* run over all tuples of weight d such that $e^* = (d)$ or $e^* > (d)$. By the above definitions, the HN type d^* of the outer sum is an admissible coarsening of e^* . The sign and the v -exponent in the above sum are easily computed as $(-1)^{l(e^*) - l(d^*)} v^{-\langle e^* \rangle}$. So the above sum can be rewritten as:

$$\chi_d = \sum_{e^*} (-1)^{l(e^*) - 1} v^{-\langle e^* \rangle} \sum_{d^*} (-1)^{l(d^*) - 1} \chi_{e^1} * \dots * \chi_{e^{l(e^*)}},$$

where the outer sum runs over all tuples e^* of weight d such that $e^* = (d)$ or $e^* \geq d$, and the inner sum runs over all admissible coarsenings of e^* . Theorem 5.1 thus reduces to the following statement:

Lemma 5.4 *For each tuple d^* of weight d such that $d^* = (d)$ or $d^* > (d)$, we have*

$$\sum_{I \in \mathcal{A}(d^*)} (-1)^{\#I} = \begin{cases} 0 & , \quad d^* \neq (d), \\ 1 & , \quad d^* = (d) \end{cases}$$

Proof: We proceed by induction on $s = l(d^*)$. In case $s = 1$, we have $d^* = (d)$, and there is nothing to prove. Otherwise, consider the slopes of the first two entries d^1, d^2 .

If $\mu(d^1) < \mu(d^2)$, we define $I_0 \subset \{1, \dots, s-1\}$ as the subset of all k such that $\mu(d^1) \geq \mu(d^1 + \dots + d^k)$. It is then easy to see from the definitions that $\mathcal{A}(d^*) = \mathcal{A}(c_{I_0}^*(d^*))$. Since $2 \notin I_0$, the Lemma holds for the I_0 -coarsening by induction.

If $\mu(d^1) \geq \mu(d^2)$, we define $d'^* = (d^2, \dots, d^s)$ and $d''^* = (d^1 + d^2, d^3, \dots, d^s)$. Again, it is then easy to see from the definitions that we have a disjoint union

$$\mathcal{A}(d^*) = \mathcal{A}(d''^*) \cup (\{1\} \cup (\mathcal{A}(d'^*) + 1)),$$

where $\{1\} \cup (\mathcal{A}(d'^*) + 1)$ consists of the sets $\{1, s_1 + 1, \dots, s_k + 1\}$ for $\{s_1, \dots, s_k\} \in \mathcal{A}(d'^*)$. This leads to the calculation

$$\sum_{I \in \mathcal{A}(d^*)} (-1)^{\#I} = \sum_{I \in \mathcal{A}(d'^*)} (-1)^{\#I} + \sum_{I \in \mathcal{A}(d'^*)} (-1)^{\#I+1},$$

which thus equals zero by induction. \square

Combining the above calculation and the Lemma, we see that Theorem 5.1 is proved. \square

Remarks:

1. Implicitly in the proof of the above Lemma, we have proved the contractibility of the simplicial complex $\mathcal{A}(d^*)$.
2. The statement is easily generalized to give an explicit formula for arbitrary elements of the HN system, using Lemma 4.7.
3. The summation can be viewed as running over all lattice paths with arbitrary step length in the lattice of all elements $e \in \mathbf{N}I$ such that $e \leq d$ and $\mu(e) > \mu(d)$.

The last remark gives a key to a compact reformulation of the above formula. We adopt a version of the transfer matrix method (see [St]).

Corollary 5.5 *Let \mathcal{T}_d be the quadratic matrix with rows and columns indexed by $\mathcal{I}(d) = \{e \in \mathbf{N}I : e \leq d, \mu(e) > \mu(d)\} \cup \{0, d\}$, and with entries in \mathcal{C} given by $v^{(e-f, e)} \chi_{f-e}$ if $e \leq f$, and zero otherwise. Then the $(0, d)$ -entry of the inverse matrix \mathcal{T}_d^{-1} equals $-\chi_d^{ss}$.*

Proof: Since the matrix \mathcal{T} is upper unitriangular with respect to the ordering \leq on $\mathcal{I}(d)$, an entry of the inverse matrix can be computed as

$$\begin{aligned} -(\mathcal{T}^{-1})_{0,d} &= \sum_{0=e^0 < e^1 < \dots < e^s = d} (-1)^{s-1} \mathcal{T}_{e^0, e^1} \dots \mathcal{T}_{e^{s-1}, e^s} \\ &= \sum_{0=e^0 < e^1 < \dots < e^s = d} (-1)^{s-1} v^{\langle e^0 - e^1, e^0 \rangle + \dots + \langle e^{s-1} - e^s, e^{s-1} \rangle} \times \\ &\quad \times \chi_{e^1 - e^0} * \dots * \chi_{e^s - e^{s-1}} \\ &= \sum_{d^1, \dots, d^s} (-1)^{s-1} v^{-\sum_{k < l} \langle d^l, d^k \rangle} \chi_{d^1} * \dots * \chi_{d^s} \end{aligned}$$

by substituting $d^k = e^k - e^{k-1}$. Noting that the defining properties of $\mathcal{I}(d)$ translate into $\mu(d^k) > \mu(d)$ for all $k = 1 \dots s-1$, we arrive at the formula of

Theorem 5.1. □

This reformulation of the formula will be used in the next section to derive a fast algorithm for the computation of Betti numbers (see Corollary 6.9).

6 Cohomology of quiver moduli

We still keep the assumption that \mathbf{k} is a finite field with v^2 elements. As the main application of the HN system and its explicit formula Theorem 5.1, we derive a formula for the Poincare polynomial of ordinary cohomology of the complex moduli spaces $\mathcal{M}_d^{ss}(\mathbf{C})$ 'in the coprime case' (see below for the definition).

The strategy is similar to the approach of [HN]: First, we count numbers of rational points of varieties. Then, we use results of [M] to prove compatibilities between these numbers. Finally, we relate these numbers to Betti numbers of complex varieties with the aid of the Weil conjectures.

The link between the HN system and numbers of rational points is provided by a twisted character of the Hall algebra \mathcal{H} .

Lemma 6.1 *The map $ev : \mathcal{H} \rightarrow \mathbf{C}$, defined by $ev(f) = (\#G_d)^{-1} \sum_X f(X)$ on \mathcal{H}_d , fulfills*

$$ev(f * g) = v^{-\langle e, d \rangle} ev(f) ev(g)$$

for $|f| = d$, $|g| = e$.

Proof: Without loss of generality, we can assume f (resp. g) to be the characteristic function of an orbit \mathcal{O}_M (resp. \mathcal{O}_N). By the definitions, we have $ev(\chi_{\mathcal{O}_M}) = \#\text{Aut}(M)^{-1}$, and $\chi_{\mathcal{O}_M} * \chi_{\mathcal{O}_N} = v^{\langle e, d \rangle} \sum_{[X]} F_{N,M}^X \chi_{\mathcal{O}_X}$, where $F_{N,M}^X$ denotes the number of subrepresentations of X which are isomorphic to M , with quotient isomorphic to N . By a formula of C. Riedtmann [Rie], we have

$$F_{N,M}^X = v^{-2 \dim \text{Hom}(N, M)} \frac{\#\text{Aut}(X)}{\#\text{Aut}(M) \cdot \#\text{Aut}(N)} \#\text{Ext}^1(N, M)_X,$$

where $\text{Ext}^1(N, M)_X$ denotes the set of extension classes with middle term isomorphic to X . Using this formula, the Lemma follows by an easy calculation. □

Applying the evaluation map ev to both sides of the formula Theorem 5.1, using its definition and the previous lemma, we get immediately:

Corollary 6.2 *For all $d \in \mathbf{NI}$, we have:*

$$\frac{\#R_d^{ss}}{\#G_d} = \sum_{d^*} (-1)^{s-1} v^{-2\langle d^*, d \rangle} \prod_{k=1}^s \frac{\#R_{d^k}}{\#G_{d^k}},$$

where the sum runs over all tuples of non-zero dimension types $d^* = (d^1 \dots d^s)$ of weight d such that $\mu(\sum_{l=1}^k d^l) > \mu(d)$ for $k = 1 \dots s-1$.

We thus have to relate the number of points in R_d^{ss} to the number of points in the geometric quotient \mathcal{M}_d^{ss} . This is directly possible only in the following case.

Definition 6.3 *A dimension type $d \in \mathbf{NI}$ is called coprime if the numbers $\Theta(d), \dim d \in \mathbf{Z}$ are coprime.*

(This property seems to be very restrictive at first sight; see however section 7 for enough interesting examples.)

Lemma 6.4 *For coprime d , we have $R_d^{ss} = R_d^s$, and $\text{End}_{\mathbf{k}Q}(X) \simeq \mathbf{k}$ for all $X \in R_d^{ss}$.*

Proof: If $X \in R_d^{ss}$ is not stable, there exists a proper subrepresentation $U \subset X$ such that $\Theta(U)/\dim U = \Theta(X)/\dim X$, contradicting coprimality. Consider now the scalar extension $\overline{X} = \overline{\mathbf{k}} \otimes_{\mathbf{k}} X$ to the algebraic closure of \mathbf{k} ; we claim that it is still semistable. In fact, its HN filtration is unique, hence stable under Frobenius, and thus it descends to the HN filtration of X , which is trivial by semistability of X . We conclude that \overline{X} is semistable, having trivial HN filtration. By the first part of the Lemma, \overline{X} is already stable, thus its endomorphism ring reduces to scalars by a Schur's Lemma type argument. Thus the same holds for X . □

Lemma 6.5 *For coprime d , the action of $PG_d = G_d/\mathbf{k}^*$ on R_d^{ss} is free in the sense of Mumford (see ([M], 0.8. iv)).*

Proof: Since PG_d acts set-theoretically free on R_d^{ss} by Lemma 6.4, the natural map $\Psi : PG_d \times R_d^{ss} \rightarrow R_d^{ss} \times R_d^{ss}$ is injective; we have to prove that it is a closed immersion. Consider the map $\Phi : R_d^{ss} \times R_d^{ss} \rightarrow \text{Hom}_{\mathbf{k}}(E_d, R_d)$ given by

$$\Phi(X, Y)(\phi_i)_{i \in I} = (\phi_j X_\alpha - Y_\alpha \phi_i)_{\alpha: i \rightarrow j}.$$

Since the kernel of $\Phi(X, Y)$ can be identified with the space of $\mathbf{k}Q$ -homomorphisms from X to Y , the image Z of Ψ is precisely the set of pairs (X, Y) such that $\Phi(X, Y)$ has non-trivial kernel (i.e. where $\Phi(X, Y)$ has a fixed rank r) by Lemma 6.4. On the open subset of Z where a fixed $r \times r$ minor of $\Phi(X, Y)$ is non-vanishing, we can thus recover from the pair (X, Y) a matrix $\neq 0$ in E_d intertwining X and Y , i.e. we can recover algebraically the unique element of PG_d mapping X to Y . Thus, we have constructed locally an inverse morphism $\Psi^{-1} : Z \rightarrow PG_d \times R_d^{ss}$. □

Proposition 6.6 *For coprime d , we have the following formula for the number of \mathbf{k} -rational points:*

$$\#\mathcal{M}_d^{ss} = \frac{\#R_d^{ss}}{\#PG_d}.$$

Proof: From the proof of Lemma 6.4 above we see that semistability is stable under base change. By coprimality, the same holds for stability. Together with Proposition 1.14. of [M], this implies that R_d^{ss} , the variety of semistable representations over \mathbf{k} , coincides with the semistable locus of the \mathbf{k} -variety R_d . Thus, there exists a uniform geometric quotient $\pi : R_d^{ss} \rightarrow \mathcal{M}_d^{ss}$ (in the sense of ([M], 0.7.)) of R_d^{ss} by PG_d . Using Lemma 6.5, we can apply ([M], 0.9.) to conclude that π turns R_d^{ss} into a principal PG_d -bundle over \mathcal{M}_d^{ss} , in the sense that $PG_d \times R_d^{ss} \simeq R_d^{ss} \times_{\mathcal{M}_d^{ss}} R_d^{ss}$. We conclude that each fibre of π is isomorphic to PG_d , and the formula for the number of \mathbf{k} -rational points follows. \square

Using the Weil conjectures [De], we can conclude:

Theorem 6.7 *Assume that d is coprime, and let $\mathcal{M}_d^{ss}(\mathbf{C})$ be the moduli space of semistable representations of Q over the field $\mathbf{k} = \mathbf{C}$. Then the Poincare polynomial of the cohomology with complex coefficients of $\mathcal{M}_d^{ss}(\mathbf{C})$ is given by*

$$\sum_{i \in \mathbf{Z}} \dim_{\mathbf{C}} H^i(\mathcal{M}_d^{ss}(\mathbf{C})) v^i = (v^2 - 1) \sum_{d^*} (-1)^{s-1} v^{-2\langle d^* \rangle} \prod_{k=1}^s \frac{\#R_{d^k}}{\#G_{d^k}},$$

where the sum runs over all tuples of non-zero dimension types $d^* = (d^1 \dots d^s)$ of weight d such that $\mu(\sum_{l=1}^k d^l) > \mu(d)$ for $k = 1 \dots s-1$.

Proof: By Corollary 6.2, the right hand side equals $\#R_d^{ss}/\#G_d$, which by Proposition 6.6 equals $\frac{1}{v^2-1} \# \mathcal{M}_d^{ss}$. A standard argument (see ([Kir], 15.), ([Gö], 1.2.)), together with the generic compatibility of formation of invariants and base change (see ([CBV], Lemma B.4.)) shows that the number of rational points (viewed as a function of v) is precisely the Poincare polynomial of the cohomology of $\mathcal{M}_d^{ss}(\mathbf{C})$. \square

Using the obvious formulas for $\#R_d$ and $\#G_d$, the above formula can be simplified and made more explicit. For $N \in \mathbf{N}$, denote $[N] = \frac{v^{2N}-1}{v^2-1}$ and $[N]! = [1][2] \dots [N]$.

Corollary 6.8 *For coprime $d = \sum_{i \in I} d_i i$, we have*

$$\begin{aligned} \sum_{i \in \mathbf{Z}} \dim_{\mathbf{C}} H^i(\mathcal{M}_d^{ss}(\mathbf{C})) v^i &= (v^2 - 1)^{1 - \sum_i d_i} v^{-\sum_i d_i(d_i-1)} \times \\ &\times \sum_{d^*} (-1)^{s-1} v^{2 \sum_{k \leq l} \sum_{i \rightarrow j} d_i^l d_j^k} \prod_{k=1}^s \prod_i ([d_i^k]!)^{-1}, \end{aligned}$$

where the sum runs over all tuples of non-zero dimension types $d^* = (d^1 \dots d^s)$ of weight d such that $\mu(\sum_{l=1}^k d^l) > \mu(d)$ for $k = 1 \dots s-1$.

We can also apply the evaluation map ev to the transfer matrix analogue 5.5 of the resolved recursion. We easily get:

Corollary 6.9 *Let T_d be the quadratic matrix with rows and columns indexed by $\mathcal{I}(d)$, and with entries in $\mathbf{C}(v)$ given by $v^{2\langle e-f, e \rangle} \frac{\#R_{f-e}}{\#G_{f-e}}$ if $e \leq f$, and zero otherwise. Then for coprime d , the $(0, d)$ -entry of the inverse matrix T_d^{-1} equals $(1 - v^2)^{-1} \sum_{i \in \mathbf{Z}} \dim_{\mathbf{C}} H^i(\mathcal{M}_d^{ss}, \mathbf{C}) v^i$.*

This last corollary gives a simple and fast algorithm for computing the Poincare polynomials, since we just have to solve a linear equation defined by the upper unitriangular matrix T_d . This is clearly a problem of polynomial order. More precisely, it is of quadratic order in the size of the set $\mathcal{I}(d)$, which is approximately quadratic in the product $\prod_i (d_i + 1)$. In contrast to this, both the HN recursion 4.8 and the explicit formula 5.1 are of exponential order compared to the size of the entries of d , since the summations run over certain classes of lattice paths.

7 Applications and examples

As a first (potential) application of the HN system, we consider the case where Q is of Dynkin type, i.e. the underlying unoriented graph is a disjoint union of Dynkin diagrams of type A, D, E . Generalizing example A of section 2, we have the following:

Conjecture 7.1 *If Q is of Dynkin type, there exists a weight Θ such that the stable representations are precisely the indecomposables.*

By Gabriel's theorem (see [ARS]), the dimension types of the indecomposables are precisely the positive roots for the corresponding root system, and the conjecture can be reduced to a purely combinatorial problem.

Provided the conjecture holds, the HN strata for the corresponding slope function μ are precisely the (finitely many) G_d -orbits in R_d . The HN system is thus a basis for $\mathcal{U}_v(\mathfrak{n}^+)$. By orthogonality, it has to coincide with a PBW basis (in the sense of [L]) up to scalars. The formula 5.1 thus gives an explicit description of a PBW basis, and in particular of root elements in $\mathcal{U}_v(\mathfrak{n}^+)$.

Note however that the HN system can never be a basis for infinite types due to the non-trivial root multiplicities of the corresponding Kac-Moody algebra.

Another impact of the developed methods is on the structure of the Hall algebra $\mathcal{H}(Q)$ itself, which is much larger than $\mathcal{C} \simeq \mathcal{U}_v(\mathfrak{n}^+)$ if Q is not of Dynkin type. In fact, \mathcal{H} is a specialization of the quantized enveloping algebra of a Borcherds algebra by [SV]; the structure of this Borcherds algebra remains unknown. Using the concepts of sections 2, 4, we can define μ -local Hall algebras by $\mathcal{H}_\mu = \bigoplus_{d: \mu(d)=\mu} \mathbf{C}^{G_d}[R_d^{ss}]$. This is a subalgebra of \mathcal{H} by Lemma 2.2 and Definition 4.1. Using [SV], it should again be possible to relate \mathcal{H}_μ to a Borcherds algebra, whose structure should be intimately related to the geometry of the

moduli spaces \mathcal{M}_d^{ss} .

After these conjectural applications, we turn to the examples of section 2 and make the formula 6.8 explicit in some cases.

In the case of example B, assume that m and n are coprime. Then we can apply formula 6.8 to this particular case. After some elementary reformulations, we get the Poincare polynomial of cohomology of the quotient $(\mathbf{P}^{m-1})_{\text{stable}}^n / \text{PGL}_m$ as (note the multinomial coefficient):

$$(v^2 - 1)^{1-m-n} v^{-m(m-1)} \sum_{m_*, n_*} (-1)^{s-1} \binom{n}{n_1 \dots n_s} v^{2 \sum_{k \leq l} m_k n_l} \prod_k ([m_k]!)^{-1},$$

where the sum runs over all tuples $m_* = (m_1 \dots m_s)$, $n_* = (n_1 \dots n_s)$ such that $\sum_k m_k = m$, $\sum_k n_k = n$, $(m_k, n_k) \neq (0, 0)$ for all k , and $(m_1 + \dots + m_k)/m < (n_1 + \dots + n_k)/n$ for all $k = 1 \dots s-1$. This formula generalizes the formulas of ([Kir], 16)

Similarly, we can deal with example D. Considering the dimension vector $ai + bj$ for the quiver $Q = i \xrightarrow{(n)} j$ such that a and b are coprime, the Poincare polynomial $P_{a,b}^n(v)$ of the quotient $W_{a,b}^n = \text{Hom}(\mathbf{C}^a, \mathbf{C}^b)_{\text{stable}}^n / \text{GL}_a \times \text{GL}_b$ is given by:

$$(v^2 - 1)^{1-a-b} v^{-a(a-1)-b(b-1)} \sum_{a_*, b_*} (-1)^{s-1} v^{2n \sum_{k \leq l} a_l b_k} \prod_k ([a_k]! [b_k]!)^{-1},$$

where the sum runs over all tuples $a_* = (a_1 \dots a_s)$, $b_* = (b_1 \dots b_s)$ such that $\sum_k a_k = a$, $\sum_k b_k = b$, $(a_k, b_k) \neq (0, 0)$ for all k , and $(a_1 + \dots + a_k)/a > (b_1 + \dots + b_k)/b$ for all $k = 1 \dots s-1$.

It is possible to make this formula more tractable; the necessary calculations are elementary, but quite tedious, so they will be omitted here. The idea is to separate the zero and non-zero entries among the a_k , and to use some standard identities for the v -binomial coefficients $\begin{bmatrix} M \\ N \end{bmatrix} = \frac{[M+N]!}{[M]! [N]!}$ (see e.g. [L]). The final result is

$$P_{a,b}^n(v) = (v^2 - 1)^{1-a} v^{-a(a-1)} \sum_{a_*, b_*} (-1)^{s-1} v^{2 \sum_{k < l} (na_l - b_l) b_k} \prod_{k=1}^s ([a_k]!)^{-1} \begin{bmatrix} na_k \\ b_k \end{bmatrix},$$

where the sum runs over all tuples $a_* = (a_1 \dots a_s)$, $b_* = (b_1 \dots b_s)$ such that $\sum_k a_k = a$, $\sum_k b_k = b$, $a_k \neq 0$ for all k , and $(a_1 + \dots + a_k)/a > (b_1 + \dots + b_k)/b$ for all $k = 1 \dots s-1$.

Since the summation runs only over non-zero a_k , this formula has the advantage of being easily computable for small a :

For $a = 1$, we just get $\begin{bmatrix} n \\ b \end{bmatrix}$, the Poincare polynomial of the cohomology of the

Grassmanian $\text{Gr}_b^n \simeq W_{1,b}^n$. For $a = 2$, we get

$$P_{2,b}^n(v) = (v^2 - 1)^{-1} v^{-2} \left(\frac{1}{v^2 + 1} \begin{bmatrix} 2n \\ b \end{bmatrix} - \sum_{k=0}^{(b-1)/2} v^{2(n-b+k)k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ b-k \end{bmatrix} \right),$$

generalizing results of [Dr]. This leads to a formula for the Euler characteristic (which can not be read off directly from the general formulas):

$$\chi(W_{2,b}^n) = \frac{bn-1}{4} \binom{2n}{b} - n \sum_{k=0}^{(b-1)/2} k \binom{n}{k} \binom{n}{b-k},$$

generalizing results of [ES]. Similar results can be obtained for example B.

Finally, let us remark that the algorithm 6.9 opens the possibility for computer experiments in many non-trivial cases. These experiments suggest several formulas for generating functions and asymptotical behaviours of Poincare polynomials and Euler characteristics. One may hope that such experiments lead to further insights into the geometry of quiver moduli.

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