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ALGEBRAIC STABILITY: SCHUR LEMMA AND CANONICAL FILTRATIONS.

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Abstract.

The main goal of the article is to give the general definition of algebraic stability that would permit to consider stalility not only for algebraic vector bundles or torsion-free coherent sheaves but for the whole category of coherent sheaves in an unified way.

We present an axiomatic description of the algebraic stability on an abelian category and prove some general results. Then the stability for coherent sheaves on a projective variety is constructed which generalizes Gieseker stability. Stabilities for graded modules and for quiver representations are also discussed. The constructions could be used for other abelian categories as well.

The idea to generalize stability has appealed to the author because it is quite inconvenient when stability considerations were restricted to the torsion-free sheaf subcategory that is not abelian (see for example [OSS], ch.2). Here in the section 2 we present the definition of stability for coherent sheaves in general.[1]

The section 1 is devoted to the definition and basic properties of a general algebraic stability. Then we discuss possible ways to construct stabilities.

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1. General algebraic stability.

Let $\mathcal{A}$ be an abelian category.

Remark. We will discuss later the cases when $\mathcal{A}$ is the category of algebraic coherent sheaves on a projective variety over a field $\Bbbk$, the category finitely generated graded

[1]When a preliminary version of this text had been written the author found the article [M] where stability for "coherent sheaves of pure dimension d" (thus for torsion sheaves as well) is considered. Although definitions of the stability proposed in [M] and in this paper are different there is some commonality between them and the sets of stable sheaves appear to be the same in both approaches. Hence the results of [M] about the moduli spaces for stable coherent sheaves are valid for stable sheaves in our sense as well.

[2]The research was partly supported by INTAS grant.

R-modules over a polynomial $\Bbbk$-algebra R, and the category of representations of a quiver.

The main ingredient needed to define stability in $\mathcal{A}$ is a stability order on the objects of $\mathcal{A}$.

Definition 1.1. An order on nonzero objects on $\mathcal{A}$ is called a stability order if:
Given an exact sequence of nonzero objects

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have
(SS): (seesaw property)

$$\begin{aligned} A \prec B \ &\Leftrightarrow \ A \prec C \ \Leftrightarrow \ B \prec C, \\ A \succ B \ &\Leftrightarrow \ A \succ C \ \Leftrightarrow \ B \succ C, \\ A \asymp B \ &\Leftrightarrow \ A \asymp C \ \Leftrightarrow \ B \asymp C, \end{aligned}$$

Remark. We imply that for $A, B \in Obj\,\mathcal{A}$ either $A \prec B$, or $A \succ B$, or $A \asymp B$ is valid and that it is possible to have $A \asymp B$ even when $A \neq B$.

One can also deduce from the definition the following property.

Lemma 1.2. Given an exact sequence of nonzero objects

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and an object D we have
(CM): (center of mass property)

$$\begin{aligned} A \prec D \text{ and } C \prec D \ &\Rightarrow \ B \prec D, \\ A \succ D \text{ and } C \succ D \ &\Rightarrow \ B \succ D, \\ A \asymp D \text{ and } C \asymp D \ &\Rightarrow \ B \asymp D. \end{aligned}$$

We leave it to the reader to prove the lemma.

Definition 1.3. Let us call B stable when B is nonzero and for a nontrivial subobject $A \subset B$ we have $A \prec B$.

Definition 1.4. Let us B call semi-stable when B is nonzero and for a nontrivial subobject $A \subset B$ we have $A \preccurlyeq B$.

Because of the seesaw property of the order one can use factorobjects in the above definitions as well:
B is stable if and only if $B \prec C$ for a nontrivial factorobject C,
B is semi-stable means $B \preccurlyeq C$ for a nontrivial factorobject C.

In a sense stable objects are similar to irreducible ones and we have a general Schur lemma type result.

Theorem 1. *Let A, B be semi-stable objects from $\mathcal{A}$ such that $A \succcurlyeq B$ and suppose there is a nonzero morphism $\varphi : A \to B$. Then:*

(a) *$A \asymp B$,*
(b) *if B is stable then φ is an epimorphism,*
(c) *if A is stable then φ is a monomorphism,*
(d) *if both A, B are stable then φ is an isomorphism.*

Corollary (Schur lemma). *Suppose that* $\mathrm{Hom}(A, B)$ *are finite dimensional vector spaces over a field* $\Bbbk$ *and that* $\Bbbk$ *is algebraically closed. Let* A, B *be stable objects such that* $A \succcurlyeq B$*. Then*

$$\textit{if}\quad \mathrm{Hom}(A, B) \neq 0 \quad \textit{then} \quad A \simeq B \quad \textit{and} \quad \mathrm{Hom}(A, B) = \mathrm{Hom}(A, A) = \Bbbk.$$

Remark. For our examples of coherent sheaves and graded R-modules Hom-s are finite dimensional vector spaces so the Schur lemma is valid.

To derive Corollary from the theorem we need only to mention the classical fact that a finite dimensional associative algebra, where a nonzero element is invertible, over an algebraically closed field is necessary the field itself.

Proof of Theorem 1. Let us consider the usual ker-im and im-coker exact sequences for φ

$$0 \longrightarrow K \longrightarrow A \longrightarrow I \longrightarrow 0, \qquad 0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0.$$

As $\varphi \neq 0$ so $I \neq 0$. By the definition of semi-stability

$$I \preccurlyeq B, \quad \text{and} \quad A \preccurlyeq I \quad \text{so} \quad A \preccurlyeq B.$$

But $A \succcurlyeq B$, so $A \asymp I \asymp B$, thus (a) is proved.

For (b) we need to mention that $I \neq B$ implies $I \prec B$ (because B is stable) in contradiction with $I \asymp B$ that we have got above. We proceed similarly with (c) and (d). □

We can also generalize the Harder-Narasimhan theorem for algebraic vector bundles in the following way.

Let us use in the following the convenient shorthand notations like $A \subset; \preccurlyeq B$ instead of writing $A \subset B$ and $A \preccurlyeq B$ (with obvious variations).

As usual we call B noetherian if an ascending chain in B stabilizes and say $\mathcal{A}$ is noetherian when any object of $\mathcal{A}$ is noetherian.

Definition 1.5. Let us call B quasi-noetherian (or q-noetherian) if a chain

$$A_1 \subset; \preccurlyeq A_2 \subset; \preccurlyeq \dots$$

in B has to stabilize.

Of course the condition of being q-noetherian is weaker than being noetherian.

Definition 1.6. Let us call B weakly artinian (or w-artinian) if

(wa1): a chain

$$A_1 \supset; \prec A_2 \supset; \prec \dots$$

in B has to be finite;

(wa2): a chain

$$A_1 \supset; \asymp A_2 \supset; \asymp \dots$$

in B has to stabilize.

We call $\mathcal{A}$ w-artinian if any object A in $\mathcal{A}$ is w-artinian.

Theorem 2. *Suppose $\mathcal{A}$ is w-artinian and noetherian and B is an object of $\mathcal{A}$. Then B has a filtration*

$$B = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} = 0$$

such that:

(i) *factors $G^i = F^i/F^{i+1}$ are semistable,*
(ii) $\quad G^0 \prec G^1 \prec \cdots \prec G^m$,

and the filtration is uniquelly defined by the properties (i),(ii).

We need to prove some propositions to get the theorem.

Proposition 1.7. *Let B be q-noetherian and w-artinian then it exist a subobject $B^{\#}$ in B such that:*

(a) *if $A \subset B$ is a subobject in B then $A \preccurlyeq B^{\#}$,*
(b) *if $A \subset B$ and $A \asymp B^{\#}$ then $A \subset B$,*

and it is defined uniquelly by these properties.

Clearly $B^{\#}$ would be semi-stable and B is semi-stable iff $B = B^{\#}$.

Let B be under conditions of Proposition 1.7 further on.

Lemma 1.8. *Let $A \subset B$. Then either A is semi-stable or there is a semi-stable $A' \subset B$ such that $A' \succ A$.*

Proof of the lemma. Let $A_1 = A$. If A_1 is not semi-stable then there is A_2 such that

$$A_1 \supset; \prec A_2$$

The same is valid for A_2 and so on. We have to come to a semi-stable subobject after a finite number of steps because the infinite chain

$$A_1 \supset; \prec A_2 \supset; \prec \dots$$

does not exist in the w-artinian B.

Lemma 1.9. *Let C be a subobject in B. If there is $A \subset B$ satisfying $A \succ C$ then it exists $C' \subset B$ such that $C' \supset; \succ C$.*

Proof of the lemma. By Lemma 1.8 we can suppose that A is semi-stable. Now we have two standard exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \cap C & \longrightarrow & A & \longrightarrow & U & \longrightarrow & 0, \\ 0 & \longrightarrow & C & \longrightarrow & A + C & \longrightarrow & U & \longrightarrow & 0. \end{array}$$

Because A is semistable, $A \cap C \preccurlyeq A$. Thus $A \preccurlyeq U$ by the seesaw property applied to the first sequence. But $C \prec A$ so $C \prec U$. Hence the second sequence implies that $C \prec (A + C)$ because of the seesaw property.

We see that $C' = A + C$ satisfies the lemma.

Proof of Proposition 1.7. The uniqueness of $B^{\#}$ is clear.

To prove the existence suppose to the contrary that for any subobject $B^{\#}$ in B either (a) or (b) is wrong.

Let B_0 be a subobject in B. If (a) is wrong for B_0 then by Lemma 1.9 it exists $B_1 \supset; \succ B_0$ and B_1 is strictly larger then B_0.

If (a) is valid for B_0 but (b) is wrong then it exists A, $A \asymp B_0$, A is not a subobject in B_0 and we can suppose that A is semi-stable by Lemma 1.8. Let $B_1 = B_0 + A$. Again it is easy to show that $B_1 \succcurlyeq B_0$ and B_1 is also strictly large than B_0.

So we have got $B_0 \subset; \preccurlyeq B_1$ anyway with B_1 is strictly larger then B_0. Repeating these arguments we find B_2, $B_3, \dots$, such that

$$B_0 \subset; \preccurlyeq B_1 \subset; \preccurlyeq B_2 \dots$$

with strict inclusion on every step. This is impossible because B is q-noetherian. □

Suppose that $\mathcal{A}$ satisfies the conditions of Theorem 2.

Proposition 1.10. *Let B have a filtration with the properties (i),(ii) from Theorem 2. Then $B^\# = F^m$.*

Proof of the proposition. We can proceed by induction on m. For $m = 0$ the statement is trivial. So let us consider the general case.

Let A be a subobject in B. By induction $F^{m-1}/F^m = (A/F^m)^\#$, thus

$$A/(F^m \cap A) \preccurlyeq F^{m-1}/F^m \ = \ G^{m-1}.$$

But $G^{m-1} \prec G^m$ so $A/(F^m \cap A) \prec F^m$.

Notice that $(F^m \cap A) \preccurlyeq F^m$ because F^m is semi-stable. Then by the property (CM) we have

$$A \preccurlyeq F^m,$$

so F^m satisfies the condition (a) from Proposition 1.7.

To prove that F^m satisfies (b) consider $A \asymp F^m$. Now we have $(F^m \cap A) \preccurlyeq F^m \asymp A$. By (SS)-property this implies

$$A/(F^m \cap A) \succcurlyeq A,$$

provided that $A/(F^m \cap A) \neq 0$. But $A \asymp F^m \ = \ G^m \succ G^{m-1}$, hence

$$A/(F^m \cap A) \succ G^{m-1},$$

which is impossible by induction. Whence $A/(F^m \cap A) = 0$ and $F^m \cap A = A$. Thus we conclude that $A \subset F^m$ so F^m satisfies (b), and the uniqueness statement from Proposition 1.7 gives us exactly what is needed. □

Proof of Theorem 2. To construct the filtration let us define

$$F^0 = 0, \ \ F^{-1} = B^\# \ \text{ and } \ F^{-(i+1)} = \text{preimage } (B/F^{-i})^\#.$$

Clearly a factor $G^{-(i+1)} = (B/F^{-i})^\#$ is semi-stable and $G^{-(i+2)} \prec G^{-i+1}$ by (SS)-property applied to the sequence

$$0 \longrightarrow G^{-i+2} \longrightarrow F^{-i+2}/F^{-i} \longrightarrow G^{-i+1} \longrightarrow 0.$$

Since B is noetherian so $F^{-(m+1)} = B$ for some m and we have only to shift the indices to get the filtration as it is needed for the theorem.

To prove the uniqueness let us notice first that the last term of a filtration is uniquely defined by Proposition 1.10. ¿From this it is easy to get the result by induction. □

One can also constract a Jordan-Hölder filtration in a semi-stable object.

Theorem 3. *Suppose $\mathcal{A}$ is w-artinian and noetherian and B is a semi-stable object of $\mathcal{A}$. Then B has a filtration*

$$B = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} = 0$$

such that:

(i) *factors $G^i = F^i/F^{i+1}$ are stable,*

(ii) $G^0 \asymp G^1 \asymp \cdots \asymp G^m$,

and the set $\{G_i\}$ of factors is uniquelly defined by the properties (i),(ii).

Proof of the theorem. Clearly the subobjects X in B such that $X \asymp B$ satisfy the ascending and descending chain conditions. So the result becomes the standard fact of basic algebra. $\square$

2. Polynomial stability.

It is well known that the category of algebraic coherent sheaves on a projective variety is noetherian. The same is the category of finitely generated graded R-modules where the algebra R is commutative and finitely generated over a field $\Bbbk$. We would like to construct a natural stability order for these categories.

In both cases an object of a category has "a characteristic function". For a sheaf A on a variety X it is:

$$P_{[A]}(n) = \dim_{\Bbbk} \mathrm{H}^0(X, A(n)).$$

For a graded module $A = \oplus_{q \in \mathbb{Z}} A_q$ let it be the Hilbert-Samuel function:

$$P_{[A]}(n) = \dim_{\Bbbk} \oplus_{q > -\infty}^{q \le n} A_q.$$

This justifies the following definition.

Definition 2.1. We say that a category $\mathcal{A}$ has a characteristic function if for any object A a function $P_{[A]} : \mathbb{Z} \to \mathbb{Z}$ is defined with the properties:

a) given an exact sequence $0 \to A \to B \to C \to 0$ we have

$$P_{[B]}(n) = P_{[A]}(n) + P_{[C]}(n) \quad \text{for} \quad n >> 0;$$

b) $P_{[A]} = 0$ iff $A = 0$;

c) for $n >> 0$ the function $P_{[A]}$ becomes a polynomial which has a positive highest coefficient when $A \neq 0$.

Remark. The functions discussed above for coherent sheaves and R-modules have these properties.

It follows from the definition that if $A \subset B$ then

$$P_{[A]}(n) \le P_{[B]}(n) \quad \text{for} \quad n >> 0.$$

Without loss of generality we can suppose from now on that $P_{[A]}$ denotes the polynomial obtained via condition c) of the definition.

Definition 2.2. Let A, B be nonzero objects of $\mathcal{A}$ and

$$P_{[A]}(n) = \sum_{i=0}^{m} a_i n^i, \qquad P_{[B]}(n) = \sum_{i=0}^{m} b_i n^i$$

be the corresponding polynomials (m being unspecified large number). Denote

$$\lambda_{i,j} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$$

and let

$$\Lambda_{(A,B)} = (\lambda_{m,m-1}, \lambda_{m,m-2}, \dots, \lambda_{m,0}, \lambda_{m-1,m-2}, \dots, \lambda_{2,1})$$

be the line of 2x2-minors of the matrix $\begin{bmatrix} a_m, & a_{m-1}, & \dots, & a_0 \\ b_m, & b_{m-1}, & \dots, & b_0 \end{bmatrix}$.
The polynomial order is define by conditions:

$A \asymp B \quad \Leftrightarrow \quad \Lambda_{(A,B)} = 0$
$A \prec B \quad \Leftrightarrow \quad$ the first nonzero term in $\Lambda_{(A,B)}$ is positive.

We have to check transitivity and the (SS) property.

Lemma 2.3. *If* $\deg P_{[A]} > \deg P_{[B]}$ *then* $A \prec B$.

Clearly the first nonzero minor in $\Lambda_{(A,B)}$ will be equal to the product of the highest cofficients of $P_{[A]}$ and $P_{[B]}$ which are positive.

Lemma 2.4. *If* $\deg P_{[A]} = \deg P_{[B]} = d$ *then* $A \prec B$ *if and only if*

$$\left(\frac{a_{d-1}}{a_d}, \frac{a_{d-2}}{a_d}, \dots, \frac{a_0}{a_d}\right) <_{lex} \left(\frac{b_{d-1}}{b_d}, \frac{b_{d-2}}{b_d}, \dots, \frac{b_0}{b_d}\right)$$

(where "$<_{lex}$" is used for "lexicographically less").

This amounts to the straight checking according to the definition.

It follows from Lemmas 2.3, 2.4 that the order is transitive.

Lemma 2.5. *The polynomial order is a stability order.*

Proof of the proposition. Let $0 \to A \to B \to C \to 0$ be an exact sequence. Then

$$P_{[B]}(n) = P_{[A]}(n) + P_{[C]}(n).$$

Hence

$$\begin{vmatrix} a_j & a_i \\ b_j & b_i \end{vmatrix} = \begin{vmatrix} a_j & a_i \\ a_j + c_j & a_i + c_i \end{vmatrix} = \begin{vmatrix} a_j & a_i \\ c_j & c_i \end{vmatrix} = \begin{vmatrix} a_j + c_j & a_i + c_i \\ c_j & c_i \end{vmatrix} = \begin{vmatrix} b_j & b_i \\ c_j & c_i \end{vmatrix}$$

and this implies the seesaw property. $\square$

Proposition 2.6. *If the characteristic function with the properties a)-c) is defined for $\mathcal{A}$, then $\mathcal{A}$ is w-artinian.*

Proof of the proposition. By the contrary let us have an infinite chain

$$A_1 \supset; \preccurlyeq A_2 \supset; \preccurlyeq \dots ,$$

with strict inclusions and let

$$P_r = \sum a_i^{[r]} x^i$$

be the corresponding polynomials. As $A_r \supset A_{r+1}$ strictly so

$$P_r(n) > P_{r+1}(n) \quad \text{for} \quad n >> 0.$$

Hence $\deg P_r \geq \deg P_{r+1}$ and therefore $\deg P_r = \deg P_{r+1} = \dots = d$ for large enough r.

Since the polynomials have positive integer values for $n >> 0$ so their highest coefficients $a_d^{[r]}$ belong to $\dfrac{1}{d!}\mathbb{N}$ and $a_d^{[r]} \geq a_d^{[r+1]}$ by the same reason so $a_d^{[r]} = a_d^{[r+1]} = \dots = q$ for some large r.

Then the property $P_r(n) > P_{r+1}(n)$ for $n >> 0$ is equivalent to

$$(q, a_{d-1}^{[r]}, a_{d-2}^{[r]}, \dots , a_0^{[r]}) >_{lex} (q, a_{d-1}^{[r+1]}, a_{d-2}^{[r+1]}, \dots , a_0^{[r+1]})$$

and this is the same as

$$\left(\frac{a_{d-1}^{[r]}}{q}, \frac{a_{d-2}^{[r]}}{q}, \dots , \frac{a_0^{[r]}}{q}\right) >_{lex} \left(\frac{a_{d-1}^{[r+1]}}{q}, \frac{a_{d-2}^{[r+1]}}{q}, \dots , \frac{a_0^{[r+1]}}{q}\right).$$

Because of Lemma 2.4 this means $A_r \succ A_{r+1}$ which contradicts to the presupposition that $A_r \preccurlyeq A_{r+1}$. $\square$

3. Ratio of additive functions stability.

Another, perhaps more usual way to define a stability order ([F],[K],[LT],[OSS]) is via a ratio of two additive functions in a way that we are going to discuss in this section.

Definition 3.1. Let c and r be two additive functions on $\mathcal{A}$ and let $r(A) > 0$ for any nonzero object A of $\mathcal{A}$. We call the ratio

$$\mu(A) = \frac{c(A)}{r(A)}$$

the $(c\!:\!r)$-slope of A and define the slope order by conditions:

$A \prec B \Leftrightarrow \mu(A) < \mu(B)$,
$A \asymp B \Leftrightarrow \mu(A) = \mu(B)$.

This way stability for algebraic vector bundles is usually defined ([OSS],[M],[LT]).

Lemma 3.2. *The $(c\!:\!r)$-slope order is a stability order.*

Proof of the lemma. Let us notice that

$$\frac{c(A)}{r(A)} - \frac{c(B)}{r(B)} = \frac{1}{r(A)r(B)} \begin{vmatrix} r(B) & c(B) \\ r(A) & c(A) \end{vmatrix}.$$

So the ordering between A and B is determined by the positivity, negativity or nullity of the determinant

$$\begin{vmatrix} r(B) & c(B) \\ r(A) & c(A) \end{vmatrix}.$$

Now it is easy to see that the same transformations of determinants that were used in the proof of Lemma 2.5 also work here. We leave details to the reader. □

Remark. The function c is not obliged to take values in $\mathbb{Z}$. For example, $\mathbb{Q}$, $\mathbb{C}$ or an ordered $\mathbb{Z}$-module could be the target set as well. The latter one was the case for the stability used in ([R]).

A.D.King, [K] has used the notion of stability to construst moduli spaces of the representations of a quiver. In his case stability is discussed only for representations with a fixed K_0-image α and it depends on a choice of an additive function θ such that $\theta(\alpha) = 0$. This approach makes it possible to construct a moduli space but at the same moment it does not allow to compare stable representations with different α as their stabilities often have to be defined with respect to different functions θ.

In order to relate the King's definition with ours let us first remind the definition from the King's paper.

Definition 3.3. ([K],p.516) Let $\mathcal{A}$ be an abelian category and $\theta : K_0(\mathcal{A}) \to \mathbb{R}$ an additive function on the Grothendieck group. An object $M \in \mathcal{A}$ is called θ-semistable if $\theta(M) = 0$ and every subobject $M' \subset M$ satisfies $\theta(M') \geq 0$. Such an M is called θ-stable if the only subobjects M' with $\theta(M') = 0$ are M and 0.

Proposition 3.4. *Given a stability for an abelian category $\mathcal{A}$ that is defined via the $(c\!:\!r)$-slope order and $M \in \mathcal{A}$ let us consider an additive function θ such that*

$$\theta = -c + \frac{c(M)}{r(M)} r.$$

Then $\theta(M) = 0$ and M is stable by the $(c\!:\!r)$-stability if and only if it is θ-stable in the sense Definition 3.3.

Proof. Let us notice that

$$\theta(M') \geq 0 \quad \Leftrightarrow \quad -c(M') + \frac{c(M)}{r(M)} r(M') \geq 0 \quad \Leftrightarrow \quad \frac{c(M')}{r(M')} \leq \frac{c(M)}{r(M)}. \quad \square$$

So the King's results about moduli spaces θ-stable objects are relevant to our stability. The existence theorems from ([K]) for moduli spaces of θ-stable representations of a finite dimensional algebra imply the existence theorems for moduli spaces of $(c\!:\!r)$-stable representations.

Remark. The filtration of Theorem 2 depends on the stability. This is easy to check with the following example.

Let (1) $\longrightarrow$ (2) $\longrightarrow$ (3) be a quiver of type A_3 and

$$V = \{V_1 \longrightarrow V_2 \longrightarrow V_3\}$$

be the representation of the quiver (for the definitions consult for example [K]).

We take $r(V) = \sum \dim V_i$, $c(V) = \sum a_i \dim V_i$. Let V' be the representation where $\dim V_i' = 1$ and the maps are isomorphisms.

The subobjects of V' are the following two:

$$V^{[1]} = \{V_1^{[1]} = 0,\ V_2^{[1]} = 0,\ V_3^{[1]} = V_3'\};$$
$$V^{[2]} = \{V_1^{[2]} = 0,\ V_2^{[2]} = V_2',\ V_3^{[2]} = V_3'\}.$$

As a result we conclude that if $a_1 = 3$, $a_2 = 2$, $a_3 = 1$ then V' is stable. But if $a_i = i$ then V' is not stable and

$$V' \supset V^{[2]} \supset V^{[1]} \supset 0$$

is the Harder-Narasimhan filtration in V'.

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