Spatial decay and spectral properties of rotating waves in parabolic systems

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1 Introduction and main result

1.1 Introduction

The field of nonlinear waves has been extended over the last decades. Nonlinear waves are solutions of time dependent partial differential equations that are posed on an unbounded domain, [35, Chapter 18]. In many cases these equations possess symmetry properties which, depending on their type, allow traveling waves, rotating waves or phase-rotating waves. A common feature to all these solutions is that they are completely characterized by a time independent profile which travels, rotates or oscillates at constant velocity. Such solutions arise in different applications from physical, chemical and biological sciences. Equations that exhibit these types of solutions are for instance the complex Ginzburg-Landau equation (see: [64], [76]), the λ-ω system (see: [61], [80]), the Barkley model (see: [10], [11]), the Schrödinger equation (see: [33], [112]) and the Gross-Pitaevskii equation (see: [44]). One important focus of research is to study nonlinear stability of such solutions and relate it to spectral properties of the linearization at the nonlinear wave. For the numerical approximation it is crucial to study truncations to bounded domains. Proving exponential decay of waves is an important issue in this field, since it implies exponentially small truncation errors. This is one major step before investigating further errors caused by spatial and temporal discretizations.

In the present thesis we deal with systems of reaction-diffusion equations

$$
\begin{aligned}
u_t(x,t) =& A\triangle u(x,t) + f(u(x,t)),\, t>0,\, x\in\mathbb{R}^d,\, d\geqslant 2,\\
u(x,0) =& u_0(x) \qquad\qquad\qquad\qquad ,\, t=0,\, x\in\mathbb{R}^d,
\end{aligned}
\tag{1.1}
$$

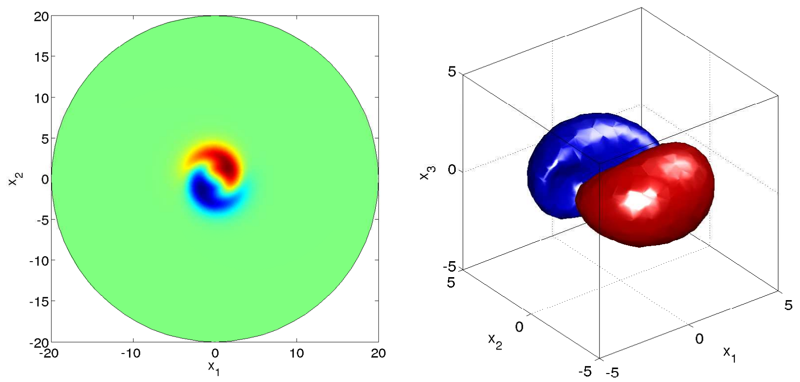
where $A\in\mathbb{R}^{N,N}$ is a diffusion matrix, $f:\mathbb{R}^N\to\mathbb{R}^N$ is a sufficiently smooth nonlinearity, $u_0:\mathbb{R}^d\to\mathbb{R}^N$ are the initial data and $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$ denotes a vector-valued solution which is sought for.

We are mainly interested in rotating wave solutions of (1.1) which are of the form

$$
u_\star(x,t) = v_\star(e^{-tS}x),\, t\geqslant 0,\, x\in\mathbb{R}^d,\, d\geqslant 2 \tag{1.2}
$$

with space-dependent profile $v_\star:\mathbb{R}^d\to\mathbb{R}^N$ and skew-symmetric matrix $S\in\mathbb{R}^{d,d}$.

As an example we discuss in this work the cubic-quintic complex Ginzburg-Landau equation (QCGL), cf. (2.1), where such solutions occur and are called spinning solitons. For more information on spinning solitons see [28]. Figure 1.1(a) shows the real part of a spinning soliton $v_\star$ in two space dimensions. The range of colorbar reaches from -1.6 (blue) to 1.6 (red). Figure 1.1(b) shows the isosurfaces of the real part of a spinning soliton in three space dimensions. The isosurfaces have the values -0.5 (blue) and 0.5 (red).



(a) (b)

Figure 1.1: Spinning soliton of QCGL (2.1) for $d=2$ (a) and $d=3$ (b)

Rotating waves from (1.2) are completely characterized by their time invariant profile $v_\star$ and a skew-symmetric matrix $S\in\mathbb{R}^{d,d}$. The skew-symmetry of S implies that e^{-tS} is a rotation matrix. Therefore, such a solution $u_\star$ rotates at constant velocity while it maintains its shape. Note that rotating waves always come in families: If $u_\star$ from (1.2) solves (1.1), then so does the function $v_\star(e^{-tS}(R^{-1}(x-\tau)))$ for every $(R,\tau)\in\mathrm{SE}(d)$, where $\mathrm{SE}(d)$ denotes the special Euclidean group. Furthermore, the profile $v_\star$ is called localized, if it tends to some constant vector $v_\infty\in\mathbb{R}^N$ as $|x|\to\infty$, and nonlocalized otherwise.

Transforming (1.1) via $u(x,t)=v(e^{-tS}x,t)$ into a co-rotating frame we obtain the evolution equation

$$\begin{aligned} v_t(x,t) =& A\triangle v(x,t)+\langle Sx,\nabla v(x,t)\rangle+f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) =& u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}\tag{1.3}$$

with drift term

$$\langle Sx,\nabla v(x)\rangle := \sum_{i=1}^{d}(Sx)_i D_i v(x).\tag{1.4}$$

Now, the pattern $v_\star$ itself is a stationary solution of (1.3), meaning that $v_\star$ solves the steady state problem

$$A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))=0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\tag{1.5}$$

that involves the Ornstein-Uhlenbeck operator

$$\left[\mathcal{L}_0 v\right](x) := A\triangle v(x)+\langle Sx,\nabla v(x)\rangle\,,\ x\in\mathbb{R}^d.\tag{1.6}$$

An important issue is to investigate the *nonlinear stability* (also called *stability with asymptotic phase*) *of rotating waves*, i.e. to show that for any initial data u_0 sufficiently close to $v_\star$ there exists $(R_\infty,\tau_\infty),(R(t),\tau(t))\in\mathrm{SE}(d)$ such that the solution $u(t)$, $t\geqslant 0$, of (1.1) satisfies $u(t)-v_\star(e^{-tS}(R(t)^{-1}(x-\tau(t))))\to 0$ in a suitable topology and $(R(t),\tau(t))\to(R_\infty,\tau_\infty)$ as $t\to\infty$. A well known task is to derive nonlinear stability from linear stability of the linearized operator

$$\left[\mathcal{L}v\right](x) := \left[\mathcal{L}_0 v\right](x)+Df\left(v_\star(x)\right)v(x),\ x\in\mathbb{R}^d.\tag{1.7}$$

By *Linear stability* (this will be called *strong spectral stability* in Chapter 9) we mean that the essential spectrum and the isolated eigenvalues of $\mathcal{L}$ lie strictly to the left of the imaginary axis except those that are caused by the SE(d)-group action (see Chapter 9 for these eigenvalues). A nonlinear stability result for two dimensional localized rotating patterns was proved by Beyn and Lorenz in [15]. Their proof requires three essential assumptions: The profile $v_\star$ of the rotating wave and their partial derivatives up to order 2 are localized in the above sense. Furthermore, the matrix $Df(v_\infty)$ is stable, meaning that all its eigenvalues have a negative real part. And finally, strong spectral stability in the sense above is assumed. Their result shows that the decay of the rotating wave itself and the spectrum of the linearization are both crucial for investigating nonlinear stability of localized rotating waves. A corresponding result on nonlinear stability of nonlocalized rotating waves, such as spiral waves and scroll waves, is still an open problem. However, the spectrum of the linearization at a spiral wave is well-known and has been extensively studied by Sandstede, Scheel and Fiedler in [92], [38] and [93].

For numerical computations it is essential to truncate equation (1.1) and (1.3) to bounded domains, see Section 1.6. This is motivated by the fact that numerical approximations, e.g. with finite elements, require that the original equation is posed on a bounded domain. The truncation error, that arises by the truncation process, depends on the boundary conditions. Assuming that a rotating wave is (exponentially) localized, we can expect the truncation error to be (exponentially) small as well. For this reason, the *exponential decay of rotating waves* plays a fundamental role in the field of *truncations* and *approximations of rotating waves on bounded domains.*

The basic step before investigating truncations is to study the rotating waves of (1.1) on the whole $\mathbb{R}^d$. This is the topic of the present thesis. For the behavior on bounded domains there are a lot of numerical simulations but the analysis of the limit as $R \to \infty$ is an open problem, see Section 1.6.

The main theme of this work is to derive suitable conditions guaranteeing that every localized rotating wave of (1.1) is already exponentially localized. To be more precise, the main theorem states that every rotating wave that falls below a certain threshold at infinity and that satisfies in addition $v_\star \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ for some $1 < p < \infty$, decays exponentially in space, in the sense that $v_\star$ belongs to some exponentially weighted Sobolev space $W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$. Afterward, we extend this result to complex-valued systems. This is motivated by the exponentially localized spinning solitons arising in the complex Ginzburg-Landau equation, see Figure 1.1.

We follow Mielke and Zelik, [114], and define the exponentially weighted Sobolev spaces $W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$ for some weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate. The main suggestion for our result comes from [15]. In [15, Remark 5], the authors conjecture that the stability of the matrix $Df(v_\infty)$, i.e. $\operatorname{Re}\sigma\,(Df(v_\infty)) < 0$, implies the exponential decay of the rotating wave as $|x| \to \infty$. Assuming in addition that $v_\star$ is localized, they believe that one can also deduce that its partial derivatives up to order 2 are localized. For traveling waves in dimension $d = 1$ such results are well known. There one usually considers x as the time variable, transforms the steady state problem to a first order ODE and applies the theory of exponential dichotomies. But the procedure does not carry over directly to $d \geqslant 2$.

Therefore, we develop in this thesis a new approach that allows to prove exponential decay in higher space dimensions.

Our approach works as follows: In the first step we compute a (complex-valued) heat kernel H_0 for the differential operator $\mathcal{L}_0$. Using this kernel, we introduce the associated semigroup $(T_0(t))_{t\geqslant 0}$ on an appropriate state space X, e.g. $X = L^p(\mathbb{R}^d, \mathbb{C}^N)$, $X = C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{C}^N)$ or $X = C^{\alpha}(\mathbb{R}^d, \mathbb{C}^N)$. We verify that the semigroup is strongly continuous on X (or possibly on a certain subspace of X), which justifies to introduce the infinitesimal generator and to apply semigroup theory, see e.g. Engel and Nagel, [34]. The generator itself can be considered as the abstract version of the formal differential operator $\mathcal{L}_0$. To investigate their relation we must solve the identification problem, which on the one hand yields an explicit representation for the maximal domain and on the other hand shows that the abstract and the formal differential operator coincide on this domain. The identification problem was solved for the scalar real-valued case by Metafune, Pallara and Vespri in [73] for $X = L^p(\mathbb{R}^d, \mathbb{R})$ and by Da Prato and Lunardi in [29] for $X = C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R})$ and $X = C^{\alpha}(\mathbb{R}^d, \mathbb{R})$.

For investigating the asymptotic behavior of solutions of (1.5) we decompose $Df(v_\star(x))$ as follows

$$Df(v_\star(x)) = Df(v_\infty) + Q_\varepsilon(x) + Q_c(x),\ x \in \mathbb{R}^d, \tag{1.8}$$

for some small perturbation $Q_\varepsilon \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$ and for some perturbation $Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$ with compact support. Then we show that it suffices to analyze solutions of the linear operator

$$[\mathcal{L}_0 v](x) + (Df(v_\infty) + Q_\varepsilon(x) + Q_c(x))\, v(x) = 0,\ x \in \mathbb{R}^d.$$

For this reason, we apply semigroup theory to study constant coefficient perturbations as well as small and compactly supported variable coefficient perturbations of $\mathcal{L}_0$.

We are faced with different problems in this work: The main problem is that the rotational term $\langle Sx, \nabla v_\star(x)\rangle$ has unbounded coefficients. Therefore, this term cannot be treated as a lower order term on unbounded domains. Moreover, since we consider complex-valued systems, we have to transfer many results, that are only known for the scalar real-valued case, to complex systems. Furthermore, due to the unbounded coefficients of $\langle Sx, \nabla v_\star(x)\rangle$ it turns out to be hard to solve the identification problem for $\mathcal{L}_0$. And finally, there is the question about a suitable state space X.

Furthermore, we investigate the eigenvalue problem for the linearization (1.7) at a localized rotating wave $v_\star$. We determine the eigenvalues located on the imaginary axis and caused by the SE(d)-group action as follows

$$\sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S),\ \lambda_1 \neq \lambda_2\} \subseteq \sigma_{\mathrm{point}}(\mathcal{L}). \tag{1.9}$$

And we derive the shape of the corresponding eigenfunctions $v : \mathbb{R}^d \to \mathbb{C}^N$, which are of the form

$$v(x) = \left\langle C^{\mathrm{rot}} x + C^{\mathrm{tra}}, \nabla v_\star(x) \right\rangle,\ x \in \mathbb{R}^d,$$

for some explicitly given skew-symmetric $C^{\text{rot}} \in \mathbb{C}^{d,d}$ and $C^{\text{tra}} \in \mathbb{C}^{d}$, where $\langle \cdot, \cdot \rangle$ is defined as in (1.4). In particular, the result shows that for every space dimension $d \geqslant 2$ the eigenvalue $\lambda = 0$ belongs to $\sigma_{\text{point}}(\mathcal{L})$ with associated eigenfunction $v(x) = \langle Sx, \nabla v_{\star}(x) \rangle$. Another application of our main theorem shows that eigenfunctions of the linearized operator decay exponentially in space, provided the corresponding eigenvalues are sufficiently close to the imaginary axis. In addition to eigenvalues, we identify a certain part of the essential spectrum,

$$\left\{ -\lambda(\omega) - i \sum_{l=1}^{k} n_l \sigma_l \mid n_l \in \mathbb{Z},\, \lambda(\omega) \in \sigma(\omega^2 A - Df(v_\infty)),\, \omega \in \mathbb{R} \right\} \subseteq \sigma_{\text{ess}}(\mathcal{L}), \tag{1.10}$$

where $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S. For this purpose we derive a dispersion relation for localized rotating patterns. All these studies are motivated by [15] and [71] and are necessary to investigate nonlinear stability of rotating waves in higher space dimensions.

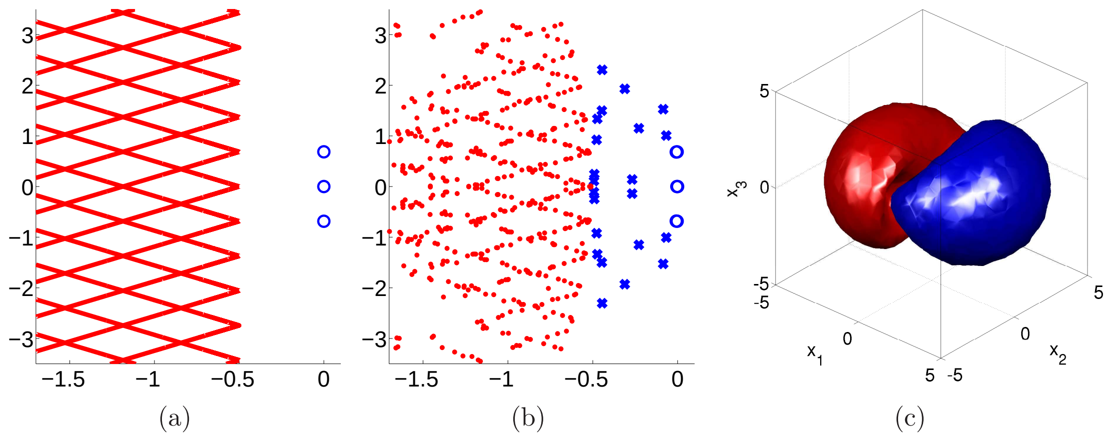
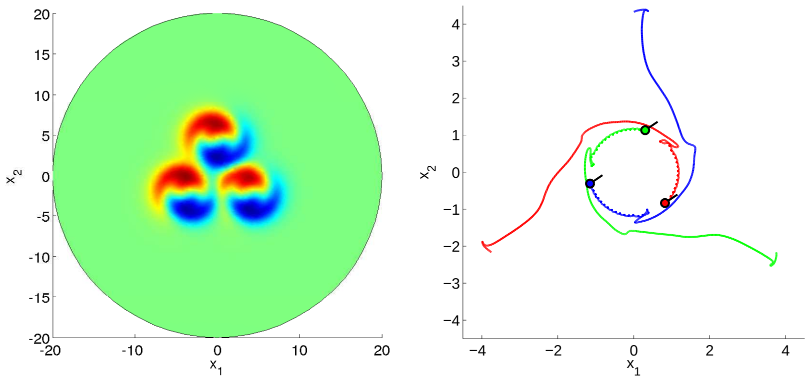


Figure 1.2: Essential and point spectrum (a), numerical spectrum (b) and two isofurfaces of eigenfunction corresponding to the eigenvalue 0 (c) for the three-dimensional spinning soliton of QCGL (2.1) from Figure 1.1(b)

Figure 1.2(a)–1.2(b) illustrates the spectral behavior of the QCGL when linearized at the spinning soliton from Figure 1.1(b). The red lines in Figure 1.2(a) correspond to the part of the essential spectrum from (1.10). They form a zig-zag structure that is parallel to the imaginary axis. The distance of two neighboring tips of the cones equals the rotational velocity $\sigma_1 = 0.68576$. The blue circles correspond to the part of the point spectrum from (1.9), that is caused by the SE(3)-group action. Each of these isolated eigenvalues has multiplicity 2. Figure 1.2(b) shows a numerical approximation of the full spectrum. Red dots approximate the essential spectrum, blue circles the known eigenvalues on the imaginary axis, and blue crosses the remaining point spectrum. Figure 1.2(c) illustrates an approximation of the eigenfunction $\langle Sx, \nabla v_{\star}(x) \rangle$ that corresponds to the zero eigenvalue. The isosurfaces have values -1.5 (blue) and 1.5 (red). Note that the eigenfunction coincides with our drift term and decays exponentially in space.

Finally, we numerically investigate the interaction of several spinning solitons in the cubic-quintic complex Ginzburg-Landau equation in two space dimensions. We are mainly interested in the fate of the single shapes and velocities when solitons collide or repel each other. In order to analyze the interaction of multi-solitons we extend the decompose and freeze method from Beyn, Thümmler and Selle, [17], to higher space dimensions. It writes the solution of (1.1) as a superposition of finite number of solutions (given by the number of patterns) which solve a system of coupled nonlinear partial differential algebraic equations.



(a) (b)

Figure 1.3: Interaction of three spinning solitons in the QCGL (2.1) with $d = 2$ and their positions of centers

Figure 1.3(a) shows the real part of the sum of three spinning solitons of the QCGL for $d = 2$, cf. Figure 1.1(a). Each of these solitons is located on a different vertex of an equilateral triangle and rotates at constant velocity. After some time they collide into a single spinning soliton that rotates at their common velocity. Figure 1.3(b) shows the time evolution for the positions of the 3 spinning solitons, that are obtained from the decompose and freeze method for multi-solitons. Each of the colors represent the motion of a single soliton with a pointer at the end which indicates the current phase position. For a detailed description we refer to Section 10.6.

1.2 Assumptions and main result

Below we give a more technical outline of the basic assumptions and the main result of this thesis:

Consider the steady state problem of the form

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2, \tag{1.11}$$

with **diffusion matrix** $A \in \mathbb{K}^{N,N}$ and a function $f : \mathbb{K}^N \to \mathbb{K}^N$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The **drift term** is defined by a matrix $0 \neq S \in \mathbb{R}^{d,d}$ as

$$\langle Sx, \nabla v(x)\rangle := \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_j D_i v(x), \tag{1.12}$$

where $D_i = \frac{\partial}{\partial x_i}$. The operator $A\triangle v(x) + \langle Sx, \nabla v(x)\rangle$ is usually called the **complex Ornstein-Uhlenbeck operator**, [107].

Our interest is in skew-symmetric matrices $S = -S^T$, in which case (1.12) is a **rotational term** containing angular derivatives

$$\langle Sx, \nabla v(x)\rangle = \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_j D_i - x_i D_j\right) v(x). \tag{1.13}$$

We look for different types of solutions, which satisfy at least $v \in L^p(\mathbb{R}^d, \mathbb{K}^N)$ for some $1 \leqslant p \leqslant \infty$ and $N \in \mathbb{N}$.

Under appropriate conditions our main result states that a solution $v_\star$ of (1.11) and its first order derivatives decay exponentially in space as the radius $|x|$ goes to infinity.

Investigating steady state problems of this type is motivated by the stability theory of rotating patterns in several spatial dimensions, [15]. There one considers reaction diffusion equations

$$\begin{aligned} u_t(x,t) =& A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) =& u_0(x) \qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d, \end{aligned} \tag{1.14}$$

where $A \in \mathbb{K}^{N,N}$ is a diffusion matrix, $f : \mathbb{K}^N \to \mathbb{K}^N$ a nonlinearity and u a solution that maps $\mathbb{R}^d \times [0,\infty[$ into $\mathbb{K}^N$.

We define a rotating wave solution $u_\star$ of (1.14) in the following sense:

Definition 1.1. A function $u_\star : \mathbb{R}^d \times [0,\infty[\to \mathbb{K}^N$ is called a **rotating wave** (or **rotating pattern**) if it has the form

$$u_\star(x,t) = v_\star(e^{-tS}(x - x_\star)),\ x \in \mathbb{R}^d,\ t \in [0,\infty[, \tag{1.15}$$

with **profile** (or **pattern**) $v_\star : \mathbb{R}^d \to \mathbb{K}^N$, a skew-symmetric matrix $0 \neq S \in \mathbb{R}^{d,d}$ and $x_\star \in \mathbb{R}^d$. A rotating wave $u_\star$ satisfying

$$\lim_{|x|\to\infty} e^{\eta|x|}\left|v_\star(x) - v_\infty\right| = 0 \text{ for some } v_\infty \in \mathbb{K}^N \text{ and } \eta \geqslant 0$$

for $\eta = 0$ is called **localized** and **nonlocalized**, otherwise. Moreover, a localized rotating wave $u_\star$ is called **exponentially localized (with decay rate η)** if $\eta > 0$.

The vector $x_\star \in \mathbb{R}^d$ can be considered as the center of rotation for $d = 2$ and as the support vector of the axis of rotation for $d = 3$. In case $d \in \{2,3\}$, S can be considered as the angular velocity tensor associated to the angular velocity vector $\omega \in \mathbb{R}^{\frac{d(d-1)}{2}}$ containing S_{ij}, $i = 1,\dots,d-1$, $j = i+1,\dots,d$.

A transformation into a **co-rotating frame** shows that if $u(x,t)$ solves (1.14) then $v(x,t) = u(e^{tS}x + x_\star, t)$ solves

$$\begin{aligned} v_t(x,t) =& A\triangle v(x,t) + \langle Sx, \nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) =& u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d, \end{aligned} \tag{1.16}$$

where the drift term is given by (1.13). Conversely, if $v(x,t)$ solves (1.16) then $u(x,t) = v(e^{-tS}(x - x_\star), t)$ solves (1.14).

Note that $v_\star$ is a stationary solution of (1.16), meaning that $v_\star$ solves the nonlinear problem (1.11). In Section 2.1 we illustrate such rotating patterns by a series of examples.

In order to investigate exponential decay of the profile $v_\star$, we list a series of assumptions that will be important in the sequel. Throughout, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$:

Assumption 1.2. *For $A \in \mathbb{K}^{N,N}$ consider the following conditions:*

(A1) *A is diagonalizable (over $\mathbb{C}$), (system condition)*

(A2) $\operatorname{Re}\sigma(A) > 0$ *(ellipticity condition)*
where $\sigma(A)$ denotes the spectrum of A,

(A3) $\exists\, \beta_A > 0 :\ \operatorname{Re}\langle w, Aw\rangle \geqslant \beta_A\ \forall\, w \in \mathbb{K}^N,\ |w| = 1,$ *(accretivity condition)*
where $\langle u, v\rangle := \overline{u}^T v$ denotes the standard inner product on $\mathbb{K}^N$,

(A4) *case ($N = 1$, $\mathbb{K} = \mathbb{R}$): $A = a > 0$,*
cases ($N \geqslant 2$, $\mathbb{K} = \mathbb{R}$) and ($N \geqslant 1$, $\mathbb{K} = \mathbb{C}$):
$\mu_1(A) > \dfrac{|p-2|}{p}$ *for some fixed $1 < p < \infty$ (L^p-antieigenvalue condition)*
where $\mu_1(A)$ is the first antieigenvalue of A.

The assumptions (A1)–(A4) satisfy the obvious relations:

$$\text{(A4)} \Rightarrow \text{(A3)} \Rightarrow \text{(A2)}.$$

Condition (A1) ensures that all results for scalar equations can be extended to system cases. It is completely independent of (A2)–(A4). Assumption (A2) guarantees that the diffusion part $A\triangle$ is an elliptic operator and requires that all eigenvalues λ of A are contained in the open right half-plane $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$, where $\sigma(A)$ denotes the spectrum of A. A matrix $C \in \mathbb{K}^{N,N}$ that satisfies $\operatorname{Re}\sigma(C) < 0$ is called a **stable matrix**. Thus, (A2) states that the matrix $-A$ is stable. In particular, (A2) implies that the matrix A is invertible. Condition (A3), states that A is an **strongly accretive matrix**, which is more restrictive than (A2). Assumption (A4) postulates that the **first antieigenvalue of** A, defined by, [48],

$$\mu_1(A) := \inf_{\substack{w\in\mathbb{K}^N \\ w\neq 0 \\ Aw\neq 0}} \frac{\operatorname{Re}\langle w, Aw\rangle}{|w||Aw|} = \inf_{\substack{w\in\mathbb{K}^N \\ |w|=1 \\ Aw\neq 0}} \frac{\operatorname{Re}\langle w, Aw\rangle}{|Aw|},$$

is bounded from below by a non-negative p-dependent constant. This is equivalent to the following p-dependent upper bound for the (**real**) **angle of** A, [47],

$$\Phi_{\mathbb{R}}(A) := \cos^{-1}(\mu_1(A)) < \cos^{-1}\left(\frac{|p-2|}{p}\right) \in]0, \frac{\pi}{2}], \quad 1 < p < \infty.$$

Condition (A4) imposes additional requirements on the spectrum of A and is more restrictive than (A3). For some special cases, the constant $\mu_1(A)$ can be given explicitly in terms of the eigenvalues of A. In the scalar complex case $A = \alpha \in \mathbb{C}$, assumption (A4) leads to a cone condition which requires α to lie in a p-dependent sector in the right half-plane. In the scalar case condition (A4) coincides with the L^p-dissipativity condition from [26].

Assumption 1.3. *The matrix $S \in \mathbb{R}^{d,d}$ satisfies*

(A5) *S is skew-symmetric, i.e. $S = -S^T$, $S \in \mathfrak{so}(d, \mathbb{R})$ (rotational condition).*

Assumption (A5) guarantees that the drift term (1.12) contains only angular derivatives, see (1.13). Our main result will be formulated for the real-valued case.

Assumption 1.4. *The function $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies*

(A6) $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ *(smoothness condition).*

Later on we apply our results also to complex-valued nonlinearities of the form

$$f : \mathbb{C}^N \to \mathbb{C}^N, \quad f(u) = g\left(|u|^2\right) u,$$

where $g : \mathbb{R} \to \mathbb{C}^{N,N}$ is a sufficiently smooth function. Such nonlinearities arise for example in Ginzburg-Landau equations, Schrödinger equations, $\lambda - \omega$ systems and many other equations from physical sciences, see Section 2.1. Note, that in this case, the function f is not holomorphic in $\mathbb{C}$, but its real-valued version in $\mathbb{R}^2$ satisfies (A6) if g is in C^2. For differentiable functions $f : \mathbb{R}^N \to \mathbb{R}^N$, Df denotes the Jacobian matrix in the real sense, see the following conditions (A8) and (A9).

Assumption 1.5. *For $v_\infty \in \mathbb{R}^N$ consider the following conditions:*

(A7) $f(v_\infty) = 0$ *(constant asymptotic state),*

(A8) $A, Df(v_\infty) \in \mathbb{R}^{N,N}$ *are simultaneously diagonalizable (over $\mathbb{C}$) (system condition),*

(A9) $\sigma\left(Df(v_\infty)\right) \subset \mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$ *(spectral condition).*

Condition (A7) states that v_∞ is a zero of the nonlinearity f. Note, that by (A8) assumption (A1) is automatically satisfied. Condition (A9) states that the matrix $Df(v_\infty)$ is stable.

Definition 1.6. A function $v_\star : \mathbb{R}^d \to \mathbb{K}^N$ is called a **classical solution of** (1.11) if

$$v_\star \in C^2(\mathbb{R}^d, \mathbb{K}^N) \cap C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) \tag{1.17}$$

and $v_\star$ solves (1.11) pointwise.

Equation (1.17) requires $v_\star$ to be C^2-smooth and bounded, see Section 3.2 for general function spaces. For a matrix $C \in \mathbb{K}^{N,N}$ we denote by $\sigma(C)$ the **spectrum of** C, by $\rho(C) := \max_{\lambda\in\sigma(C)} |\lambda|$ the **spectral radius of** C and by $s(C) := \max_{\lambda\in\sigma(C)} \operatorname{Re}\lambda$ the **spectral abscissa** (or **spectral bound**) **of** C. Using this notation, we define the constants

$$\begin{aligned} a_{\min} &:= \left(\rho\left(A^{-1}\right)\right)^{-1}, \quad & a_0 &:= -s(-A), \\ a_{\max} &:= \rho(A), & b_0 &:= -s(Df(v_\infty)). \end{aligned} \tag{1.18}$$

Our main tool for investigating exponential decay in space are exponentially weighted function spaces, which we introduce in Section 3 in detail. An essential ingredient for these function spaces is the choice of the weight function, which follows [114, Def. 3.1]:

Definition 1.7. (**1**) A function $\theta \in C(\mathbb{R}^d,\mathbb{R})$ is called a **weight function of exponential growth rate** $\eta \geqslant 0$ provided that

(W1) $\theta(x) > 0 \;\forall\, x \in \mathbb{R}^d$,

(W2) $\exists\, C_\theta > 0:\; \theta(x+y) \leqslant C_\theta \theta(x) e^{\eta|y|} \;\forall\, x,y \in \mathbb{R}^d$.

(**2**) A weight function $\theta \in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ is called **radial** provided that

(W3) $\exists\, \phi : [0,\infty[\to \mathbb{R}:\; \theta(x) = \phi(|x|) \;\forall\, x \in \mathbb{R}^d$.

(**3**) A radial weight function $\theta \in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ is called **non-decreasing** (or **monotonically increasing**) provided that

(W4) $\theta(x) \leqslant \theta(y) \;\forall\, x,y \in \mathbb{R}^d$ with $|x| \leqslant |y|$.

Note, that radial weight functions satisfy $\theta(x) = \theta(y)$ for every $x, y \in \mathbb{R}^d$ with $|x| = |y|$. Standard examples are

$$\theta_1(x) = \exp\left(-\mu|x|\right) \quad \text{and} \quad \theta_2(x) = \cosh\left(\mu|x|\right),$$

as well as their smooth analogs

$$\theta_3(x) = \exp\left(-\mu\sqrt{|x|^2+1}\right) \quad \text{and} \quad \theta_4(x) = \cosh\left(\mu\sqrt{|x|^2+1}\right),$$

for $\mu \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Obviously, all these functions are radial weight functions of exponential growth rate $\eta = |\mu|$ with $C_\theta = 1$. Moreover, θ_1, θ_3 are non-decreasing and θ_2, θ_4 are non-decreasing if $\mu \leqslant 0$. Note, that for $\mu = 0$ the examples include the weight function $\theta(x) = 1$. Furthermore, Definition 1.7 includes (radial) tableau functions, e.g.

$$\theta_5(x) = \begin{cases} 1 & , |x| \leqslant R, \\ \exp(-\mu(|x|-R)) & , |x| \geqslant R, \end{cases}$$

for some $R > 0$, where the constant C_θ depends on the size of the support, but not on the growth rate η.

Associated with weight functions of exponential growth rate are **exponentially weighted Lebesgue** and **Sobolev spaces**

$$\begin{aligned} L^p_\theta(\mathbb{R}^d,\mathbb{K}^N) :=& \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{K}^N) \mid \|\theta u\|_{L^p} < \infty\}, \\ W^{k,p}_\theta(\mathbb{R}^d,\mathbb{K}^N) :=& \{u \in L^p_\theta(\mathbb{R}^d,\mathbb{K}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d,\mathbb{K}^N) \;\forall\, |\beta| \leqslant k\}, \end{aligned}$$

for every $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}_0$. Our main result is the following:

Theorem 1.8 (Exponential decay of $v_\star$)**.** *Let the assumptions* (A4)–(A9) *be satisfied for some* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{R}$. *Then for every* $0 < \vartheta < 1$ *and for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$ *with*

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$$

and $a_{\max}$, a_0, b_0 *from* (1.18), *there exists a constant* $K_1 = K_1(A, f, v_\infty, d, p, \theta, \vartheta) > 0$ *with the following property:*
Every classical solution $v_\star$ *of*

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d, \tag{1.19}$$

such that $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ *and*

$$\sup_{|x|\geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0 \tag{1.20}$$

satisfies

$$v_\star - v_\infty \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N).$$

Roughly speaking, Theorem 1.8 states that every classical solution $v_\star$ which satisfies $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and which is sufficiently close to the steady state v_∞ at infinity, see (1.20), must already decay exponentially in space. The exponential decay is expressed by the fact, that $v_\star - v_\infty$ belongs to an exponentially weighted Sobolev space. Moreover, the theorem gives an explicit bound for the exponential growth rate, that depends only on p, the spectral radius of A and the spectral abscissas of $-A$ and $Df(v_\infty)$.

In the following we outline several implications and extensions of this result.

Complex valued equations. Later on we apply Theorem 1.8 to complex systems with nonlinearities of the form

$$f : \mathbb{C}^N \to \mathbb{C}^N, \quad f(u) = g\left(|u|^2\right) u,$$

where $g : \mathbb{R} \to \mathbb{C}^{N,N}$ is a sufficiently smooth function. For this purpose, we transform the N-dimensional complex-valued system into a $2N$-dimensional real-valued system and show how assumptions on the real-valued version of f translate into the complex case.

Relation to the one-dimensional case. As we will see in Section 1.3, the proof of Theorem 1.8 is fundamentally different from the proofs of exponential decay towards limits at $\pm\infty$ for traveling waves in one space dimension. Consider the one-dimensional reaction-diffusion equation

$$u_t(x,t) = u_{xx}(x,t) + f(u(x,t)),\ x \in \mathbb{R},\ t \geqslant 0,$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$ is a given nonlinearity and $u : \mathbb{R} \times [0,\infty[\to \mathbb{R}^N$ denotes the solution we seek for. Assume a traveling wave solution $u_\star(x,t) = v_\star(x-ct)$ for some $0 \neq c \in \mathbb{R}$, then the profile $v_\star : \mathbb{R} \to \mathbb{R}^N$ is a stationary solution of the co-moving frame, i.e. $v_\star$ solves

$$0 = v_{xx}(x) + cv_x(x) + f(v(x)),\ x \in \mathbb{R}. \tag{1.21}$$

Note that in the one-dimensional case the steady-state equation is an ordinary differential equation. An essential tool for investigating exponential decay and nonlinear stability of traveling waves are exponential dichotomies, [35, Chapter 18

(Sandstede)]. This requires to cast the second-order ODE into a first-order system: Defining $V_1(x) := v(x)$ and $V_2(x) := v_x(x)$, then $V(x) = (V_1(x), V_2(x))^T$ satisfies

$$V_x(x) = \begin{pmatrix} v_x(x) \\ v_{xx}(x) \end{pmatrix} = \begin{pmatrix} v_x(x) \\ -cv_x(x) - f(v(x)) \end{pmatrix} = \begin{pmatrix} V_2(x) \\ -cV_2(x) - f(V_1(x)) \end{pmatrix} =: F(V(x)),$$

for every $x \in \mathbb{R}$. Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $v_\infty^\pm \in \mathbb{R}^N$ be such that $f(v_\infty^\pm) = 0$ and $\operatorname{Re}\sigma(Df(v_\infty^\pm)) < 0$, then $F \in C^1(\mathbb{R}^{2N}, R^{2N})$ and $V_\infty^\pm := (v_\infty^\pm, 0)^T \in \mathbb{R}^{2N}$ are hyperbolic fixed points, i.e. $F(V_\infty^\pm) = 0$ and $\sigma(DF(V_\infty^\pm)) \cap i\mathbb{R} = \emptyset$. Now, the theory of exponential dichotomies yields some constants $K^\pm = K^\pm(f, v_\infty^\pm) > 0$ such that every solution $v_\star \in C^2(\mathbb{R}, \mathbb{R}^N) \cap C_{\mathrm{b}}(\mathbb{R}, \mathbb{R}^N)$ of (1.21) with

$$\left|v_\star(x) - v_\infty^\pm\right| + |v_{\star,x}(x)| \leqslant K^\pm \text{ for every } x \geqslant x_+ \ (x \leqslant x_-)$$

satisfies $v_\star(x) \to v_\infty^\pm$ and $v_{\star,x}(x) \to 0$ exponentially fast as $x \to \pm\infty$, cf. [100, Theorem III.7 (2)] for a time-discrete version.

To explicate the analogy, let us consider the Ornstein-Uhlenbeck operator $\mathcal{L}_0 v$ instead of $v_{xx} + cv_x$. The smoothness assumption for f now corresponds to assumption (A6). If we consider v_∞ instead of $v_\infty^\pm$, we see that $f(v_\infty^\pm) = 0$ and $\operatorname{Re}\sigma(Df(v_\infty^\pm)) < 0$ is expressed by assumption (A7) and (A9), respectively. The threshold condition now corresponds to (1.20). Finally, we emphasize that in the general case with Av_{xx} instead of v_{xx}, compare (1.21), the assumptions (A2), (A8) and (A9) imply the hyperbolicity condition $\sigma(DF(V_\infty^\pm)) \cap i\mathbb{R} = \emptyset$.

A common feature of the one- and multi-dimensional case is that one considers small and compactly supported perturbations in both situations.

1.3 Decomposition of linear differential operators

In the following we explain the decomposition of the linear differential operators that leads to the proof of Theorem 1.8.

Far-Field Linearization. Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Let $v_\infty \in \mathbb{R}^N$ be the constant asymptotic state satisfying (A7). Assume that $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ as in (A6), then Taylor's theorem yields

$$f(v_\star(x)) = \underbrace{f(v_\infty)}_{=0} + \underbrace{\int_0^1 Df(v_\infty + t(v_\star(x) - v_\infty))dt}_{=:a(x)}(v_\star(x) - v_\infty),\ x \in \mathbb{R}^d,$$

where $a \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$ since $v_\star \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^N)$ and $v_\star$ is a classical solution. Since $v_\infty \in \mathbb{R}^N$ is constant, the difference $w(x) := v_\star(x) - v_\infty$ satisfies the linearized equation

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + a(x)w(x) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

In order to study the behavior of solutions as $|x| \to \infty$ we decompose the variable coefficient. In the following we decompose a rather than $Df(v_\star(x))$ in a fashion similar to (1.8). For a direct application of (1.8) see Chapter 9.

Decomposition of a. Let $a(x) = Df(v_\infty) + Q(x)$ with Q defined by

$$Q(x) = \int_0^1 Df\left(v_\infty + tw(x)\right) - Df\left(v_\infty\right) dt,\, x \in \mathbb{R}^d.$$

This yields $Q \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$ and

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + \left(Df(v_\infty) + Q(x)\right) w(x) = 0,\, x \in \mathbb{R}^d,\, d \geqslant 2.$$

Decomposition of Q. Let $Q(x) = Q_\varepsilon(x) + Q_c(x)$, where $Q_\varepsilon \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$ is small w.r.t. $\|\cdot\|_{C_{\mathrm{b}}}$ and $Q_c \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$ is compactly supported on $\mathbb{R}^d$, see Figure 1.4. Then we arrive at

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + \left(Df(v_\infty) + Q_\varepsilon(x) + Q_c(x)\right) w(x) = 0,\, x \in \mathbb{R}^d. \tag{1.22}$$

If we omit the term $Q_\varepsilon + Q_c$ in (1.22), the equation (1.22) is called the **far-field linearization**.

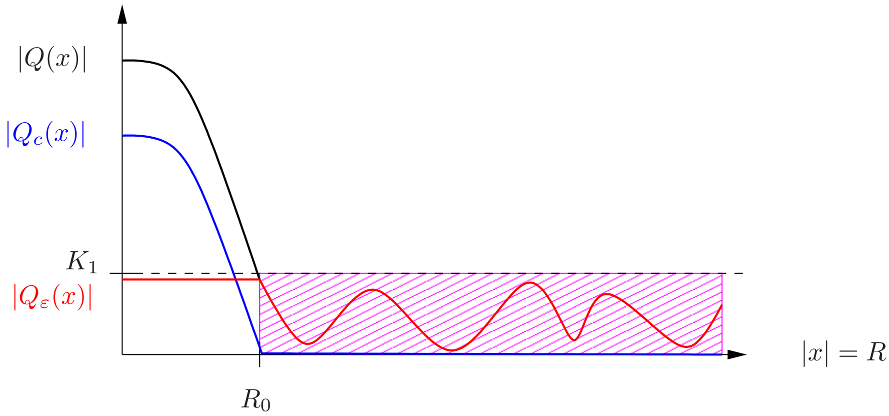


Figure 1.4: Decomposition of Q with data R_0 and K_1 from Theorem 1.8

Perturbations of Ornstein-Uhlenbeck operator. In order to show exponential decay for the solution $v_\star$ of the nonlinear steady state problem (1.11), it is sufficient to analyze the solutions of the linear system (1.22). Abbreviating $B := -Df(v_\infty)$, we will study the following linear differential operators:

$$\begin{aligned}
\left[\mathcal{L}_Q v\right](x) &= A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x) + Q_\varepsilon(x)v(x) + Q_c(x)v(x),\\
\left[\mathcal{L}_{Q_\varepsilon} v\right](x) &= A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x) + Q_\varepsilon(x)v(x),\\
\left[\mathcal{L}_\infty v\right](x) &= A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x),\\
\left[\mathcal{L}_0 v\right](x) &= A\triangle v(x) + \langle Sx, \nabla v(x)\rangle .
\end{aligned}$$

The **Ornstein-Uhlenbeck operator** $\mathcal{L}_0$, is the sum of the **diffusion term** $\left[\mathcal{L}_0^{\mathrm{diff}} v\right](x) = A\triangle v(x)$ and the **drift term** $\left[\mathcal{L}_0^{\mathrm{drift}} v\right](x) = \langle Sx, \nabla v(x)\rangle$.The drift term has unbounded (in fact linearly increasing) coefficients. Later on, it will be convenient to allow complex coefficients for the operators $\mathcal{L}_0$, $\mathcal{L}_\infty$, $\mathcal{L}_{Q_\varepsilon}$ and $\mathcal{L}_Q$. Therefore, we rewrite the assumptions (A8) and (A9) as follows:

Assumption 1.9. *Let $B \in \mathbb{K}^{N,N}$ be such that*

(A8$_B$) $A, B \in \mathbb{K}^{N,N}$ *are simultaneously diagonalizable (over $\mathbb{C}$), i.e.*
$$\exists\, Y \in \mathbb{C}^{N,N} \text{ invertible}: \; Y^{-1}AY = \Lambda_A \text{ and } Y^{-1}BY = \Lambda_B$$
where $\Lambda_A = \operatorname{diag}\left(\lambda_1^A, \ldots, \lambda_N^A\right), \Lambda_B = \operatorname{diag}\left(\lambda_1^B, \ldots, \lambda_N^B\right) \in \mathbb{C}^{N,N}$
(system condition),

(A9$_B$) $\operatorname{Re}\sigma(B) > 0$ *(spectral condition).*

In this context b_0 is defined by $b_0 := -s(-B)$, cf. (1.18). Note that in case of $B = 0$ assumption (A8$_B$) coincides with (A1).

1.4 Detailed outline of the thesis

In Chapter 2 we recall the derivation of the real scalar Ornstein-Uhlenbeck operator from an underlying stochastic ordinary differential equation (SODE). After that we motivate the complex Ornstein-Uhlenbeck operator in scalar and system cases. In the second part of Chapter 2 we give a series of examples from physical and biological sciences, where the Ornstein-Uhlenbeck operator appears in the theory of rotating patterns. Further, we give a short summary of known results concerning the real-valued Ornstein-Uhlenbeck operator.

In Chapter 3 we discuss the special Euclidean group, [37], and the exponentially weighted Lebesgue and Sobolev spaces, [114], as well as some general notation that will be used throughout this work.

Complex-valued Ornstein-Uhlenbeck kernel. In Chapter 4 we extend the approach from [14], [4] and [22] and use assumptions (A1), (A2), (A5) and (A8$_B$) to determine a heat kernel of the complex-valued operator $\mathcal{L}_\infty$ for the case, where A and B are complex simultaneously diagonalizable matrices. This leads to the following heat kernel matrix

$$H(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}} \exp\left(-Bt - (4tA)^{-1}\left|e^{tS}x - \xi\right|^2\right)$$

of $\mathcal{L}_\infty$, which we will denote later by H_∞. The choice $B = 0$ provides us with a heat kernel, denoted by H_0, for the complex Ornstein-Uhlenbeck operator $\mathcal{L}_0$. Further, we show that H satisfies a Chapman-Kolmogorov formula, needed for the subsequent semigroup theory. In the remaining section we prove some integral properties for the modified kernel $K(\psi,t) = H(x, e^{tS}x - \psi, t)$, which will be needed in the sequel for the exponential decay and the application of semigroup theory.

Ornstein-Uhlenbeck semigroup. Assuming (A1), (A2) and (A5) we will study in Chapter 5 the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ defined by the heat kernel of $\mathcal{L}_0$ as

$$\left[T_0(t)v_0\right](x) := \int_{\mathbb{R}^d} H_0(x,\xi,t)v_0(\xi)d\xi,\; t > 0,\; x \in \mathbb{R}^d.$$

Here we show that the semigroup $(T_0(t))_{t\geqslant 0}$ (also known as the transition semigroup) is strongly continuous in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for every $1 \leqslant p < \infty$. Hence, we can define the infinitesimal generator A_p of $(T_0(t))_{t\geqslant 0}$. Using abstract semigroup theory, [34], we derive solvability and uniqueness results for the resolvent equation and resolvent estimates. Moreover, we show that the Schwartz space $\mathcal{S}$ is dense in the domain of A_p with respect to the graph norm of A_p for every $1 \leqslant p < \infty$. This shows that A_p and $\mathcal{L}_0$ coincide on $\mathcal{S}$. To prove that A_p is indeed the maximal realization (extension) of $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$, we must restrict p to $1<p<\infty$ and require in addition (A3) and the L^p-antieigenvalue condition (A4) for $\mathcal{L}_0$. Then, we derive some resolvent estimates for $\mathcal{L}_0$ in

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) := \left\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p(\mathbb{R}^d,\mathbb{C}^N) \mid \mathcal{L}_0 v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}$$

for $1<p<\infty$, [73]. This enables us to conclude that the maximal domain $\mathcal{D}(A_p)$ of A_p is equal to $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and that A_p and $\mathcal{L}_0$ coincide on $D^p_{\mathrm{loc}}(\mathcal{L}_0)$ for every $1<p<\infty$.

Using exponentially weighted Sobolev spaces with radial weight functions of exponential growth, we then obtain exponential decay of the solutions for the resolvent equation and its derivatives up to order 1, even if (A3) and (A4) are not satisfied. In order to show that the maximal domain of the Ornstein-Uhlenbeck operator $\mathcal{L}_0=\mathcal{L}_0^{\mathrm{diff}}+\mathcal{L}_0^{\mathrm{drift}}$ coincides with the intersection of the domains of its diffusion and drift term, i.e.

$$\mathcal{D}\left(\mathcal{L}_0^{\mathrm{diff}}+\mathcal{L}_0^{\mathrm{drift}}\right)=\mathcal{D}\left(\mathcal{L}_0^{\mathrm{diff}}\right)\cap\mathcal{D}\left(\mathcal{L}_0^{\mathrm{drift}}\right),$$

we analyze the homogeneous and inhomogeneous Cauchy problem for $\mathcal{L}_0$, following the approach in [73] for the scalar real-valued case, and show for $1<p<\infty$ that the domain $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ coincides with

$$\mathcal{D}^p_{\max}(\mathcal{L}_0) := \left\{v\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N) \mid \langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.$$

Constant coefficient perturbations. In Chapter 6 we perturb the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ by adding the term $Bv(x)$ with constant coefficients, that leads us to the operator $\mathcal{L}_\infty$. To find a realization of $\mathcal{L}_\infty$, we assume (A1), (A2), (A5) and perturb the generator A_p by adding the operator $E_p v := -Bv$. Then the bounded perturbation $B_p := A_p + E_p$, equipped with the same domain as A_p, generates a C^0-semigroup $(T_\infty(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$. If we require in addition the assumptions (A3) and (A4), then the infinitesimal generator B_p is indeed the maximal realization of $\mathcal{L}_\infty$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$ and the domain equals $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. Note, that in general we do not have an explicit formula for the semigroup $(T_\infty(t))_{t\geqslant 0}$ any more. But if A and B satisfy in addition to (A1), (A2), (A5) the assumption (A8$_B$), we are able to derive an explicit representation for the new semigroup $(T_\infty(t))_{t\geqslant 0}$, given by

$$\left[T_\infty(t)v\right](x) := \int_{\mathbb{R}^d} H_\infty(x,\xi,t)v_0(\xi)d\xi,\ t>0,\ x\in\mathbb{R}^d.$$

Here, the function H_∞ coincides with the heat kernel for $\mathcal{L}_\infty$ computed in Section 4. Again, under the assumptions (A1), (A2), (A5) and (A8$_B$) we are able to derive solvability and uniqueness results for the resolvent equation and resolvent estimates.

In particular, assuming (A9$_B$), we can derive an explicit representation of Green's function for B_p, as the time-integral over the heat kernel,

$$G(x,\xi) = -\int_0^\infty H_\infty(x,\xi,s)ds,$$

cf. [4] for such a representation. If in addition (A3) and (A4) are satisfied, this turns out to be also a Green's function for $\mathcal{L}_\infty$. Again, we can prove exponential decay of solutions of the resolvent equation for B_p and its derivatives up to order 1.

Small and compactly supported variable coefficient perturbations. Perturbing the operator $\mathcal{L}_\infty$ by adding the term $Q(x)v(x)$ with variable coefficients $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, leads us in Chapter 7 to the operator $\mathcal{L}_Q$. In order to find a realization of $\mathcal{L}_Q$, we assume (A1), (A2), (A5), (A8$_B$) and perturb this time the generator B_p and obtain the new generator $C_p := B_p + F_p$, $F_p v := Qv$, for the full C^0-semigroup $(T_Q(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 \leqslant p < \infty$. Again, if we require in addition assumptions (A3) and (A4), then the infinitesimal generator C_p is the maximal realization of $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 < p < \infty$ and its domain equals $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. Under the assumptions (A1), (A2), (A5), (A8$_B$) and arbitrary $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, we derive solvability and uniqueness results for the resolvent equation and resolvent estimates in weighted spaces. We then apply this theory to perturbations $Q = Q_\varepsilon$, where Q_ε is assumed to be small with respect to $\|\cdot\|_{L^\infty}$, and to perturbations of the form $Q = Q_\varepsilon + Q_c$, where Q_c is compactly supported.
Finally, assuming in addition (A3), (A4) as well as

$$\operatorname*{ess\,sup}_{|x|\geqslant R} |Q(x)|_2 \to 0 \text{ as } R \to \infty,$$

and following [15] and [71], we compute the essential spectrum of the operator $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1 < p < \infty$. This shows that neither $\mathcal{L}_Q$ nor C_p is sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $(T_Q(t))_{t\geqslant 0}$ does not generate an analytic semigroup in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 < p < \infty$.

Spatial decay of rotating waves. In Chapter 8 we analyze the steady state problem (1.11) and prove the main result from Theorem 1.8, stating that $v_\star - v_\infty$ and its derivatives up to order 1 decay exponentially in space at a certain rate, whenever $v_\star$ is a classical solution of (1.11). Afterward, we extend Theorem 1.8 to complex systems.

Spectral properties of linearization at rotating waves. Generalizing [15] from $d = 2$ to $d \geqslant 2$, we investigate in Chapter 9 the linearization of the nonlinear problem (1.11) of the Ornstein-Uhlenbeck operator on $\mathbb{R}^d$. We analyze the point spectrum and determine the eigenvalues on the imaginary axis, that are caused by the SE(d)-action, as well as an explicit expression for their associated eigenfunctions. Further, we prove that, whenever an isolated eigenvalue is sufficiently close to the imaginary axis, the corresponding eigenfunction and its first order derivatives decay exponentially in space. As a byproduct we conclude that also the rotational term has an exponential decay in space. Moreover, we compute the essential spectrum for exponentially localized rotating patterns.

Freezing method and numerical results. In Chapter 10 we introduce some general theory about equivariant evolution equations following [25], [39] and [43]. Afterward, we introduce the well known freezing method, see [18] and [16]. The main idea of this method is to approximate relative equilibria such as rotating waves in reaction-diffusion systems. We then apply this method numerically to compute the profiles and the velocities of rotating waves for a series of examples. Moreover, we also investigate numerically the spectra of linearizations at rotating waves. Finally, we introduce the decompose and freeze approach, see [17] and [16]. The main idea of this approach is to approximate profiles and velocities of multi-structures, such as multi-solitons. At the end of the chapter, we apply this method to investigate numerically interaction processes of several spinning solitons in the two-dimensional cubic-quintic complex Ginzburg-Landau equation, see [78].

1.5 A guide through the present work

The three main results of this work are classified into spatial decay of rotating waves, spectral properties of rotating waves and numerical results. These issues can be found in the following sections:

Spatial decay:	Chapter 3, Chapter 4, Section 5.1-5.6, Chapter 6, Section 7.1-7.3, Chapter 8,
Spectral properties:	Section 7.4, Chapter 9,
Numerical results:	Section 2.1, Chapter 10.

Of course, these topics are closely related, but the material in the corresponding sections can be read more or less independently. The Sections 2.2 and 2.3 serve as background material for the Ornstein-Uhlenbeck operator. The Sections 5.7 and 5.8 provide a useful preparation for the theoretical part of Sections 10.1, 10.2 and 10.5.

1.6 Extensions and further results

Next we summarize some extensions of our theory, which so far have only partially be completed. Details of these results have been left out in order to keep the size of the present thesis within reasonable bounds.
Extension to the space of bounded continuous functions. A reasonable state space, suggested by the work [29], is the following, cf. Section 3.2,

$$C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{K}^N)=\{u\in C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{K}^N)\mid \lim_{t\to 0}\left\|u(e^{tS}\cdot)-u(\cdot)\right\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)}=0\},$$

where $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{K}^N)$ denotes the space of bounded uniformly continuous functions. Let the assumptions (A1), (A2) and (A5) be satisfied and consider the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ on the function spaces

$$C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)\subseteq C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{C}^N)\subseteq C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{C}^N),$$

that are introduced in Section 3.2. Then $(T_0(t))_{t\geqslant 0}$ generates a semigroup on every of these function spaces. The semigroup $(T_0(t))_{t\geqslant 0}$ is discontinuous on $C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{C}^N)$, weakly continuous on $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{C}^N)$ and strongly continuous on $C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$, in fact $(T_0(t))_{t\geqslant 0}$ is neither strongly continuous on $C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{C}^N)$ nor on $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{C}^N)$. Therefore, we consider the semigroup $(T_0(t))_{t\geqslant 0}$ only on the closed subspace $C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$. Introducing its infinitesimal generator $A_\infty : C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)\supseteq\mathcal{D}(A_\infty)\to C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$, we obtain the unique solvability of the resolvent equation for A_∞ in $C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$ by application of semigroup theory. Moreover, the semigroup $(T_0(t))_{t\geqslant 0}$ is not analytic in $C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$. All these facts were observed in [29] for the first time, but only for the scalar real-valued case. In order to investigate the relation between the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ and the infinitesimal generator A_∞, we must solve the identification problem in $C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$: Defining

$$\mathcal{D}^\infty(\mathcal{L}_0):=\left\{v\in C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)\cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\;\forall\,p\geqslant 1\mid \mathcal{L}_0v\in C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)\right\},$$

we believe that $\mathcal{D}(A_\infty)=\mathcal{D}^\infty(\mathcal{L}_0)$ holds with $A_\infty v=\mathcal{L}_0 v$ for every $v\in\mathcal{D}(A_\infty)$, i.e. A_∞ is the maximal realization of $\mathcal{L}_0$ in $C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$. This result is proved in [29, Proposition 3.5] for the scalar real-valued case, where the authors use local elliptic regularity to verify $\mathcal{D}(A_\infty)\subseteq\mathcal{D}^\infty(\mathcal{L}_0)$ and a maximum principle to show $\mathcal{D}(A_\infty)\supseteq\mathcal{D}^\infty(\mathcal{L}_0)$. The perturbation theory for A_∞ is straightforward and works in a way similar to the L^p-case. We think that Theorem 1.8 extends to $C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$ if $v_\star - v_\infty\in C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)$ and without assumptions (A3) and (A4). The result will then be

$$v_\star - v_\infty\in C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{C}^N)\cap C^1_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{C}^N)\cap C^1_{\mathrm{b},\theta}(\mathbb{R}^d,\mathbb{C}^N),$$

but the details have not been fully worked out yet.

Fourier-Bessel method on unbounded domains. We next present a further possibility to determine a heat kernel and a Green's function of the complex Ornstein-Uhlenbeck operator for skew-symmetric matrices S. Consider the steady state problem

$$[\mathcal{L}_\infty v](x):=\alpha\triangle v(x)+\langle Sx,\nabla v(x)\rangle-\beta v(x)=g(x),\; x\in\mathbb{R}^2, \tag{1.23}$$

where $A=\alpha\in\mathbb{C}$, $B=\beta\in\mathbb{C}$, $g:\mathbb{R}^2\to\mathbb{C}$ is continuous, $N=1$ and $d=2$. The matrix $0\neq S\in\mathbb{R}^{2,2}$ is assumed to be skew-symmetric and thus we have $\pm i\sigma_1\in\sigma(S)$ for some $\sigma_1\in\mathbb{R}$. Equation (1.23) reads in polar coordinates

$$\alpha\left[\partial_{rr}+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_{\phi\phi}\right]\hat{v}(r,\phi)-\sigma_1\partial_\phi\hat{v}(r,\phi)-\beta\hat{v}(r,\phi)=\hat{g}(r,\phi), \tag{1.24}$$

for $r>0$ and $\phi\in[-\pi,\pi[$, where $\pm i\sigma_1\in\sigma(S)$. Representing $\hat{v}$ and $\hat{g}$ by a complex Fourier series w.r.t. ϕ

$$\hat{v}(r,\phi)=\sum_{n=-\infty}^{\infty}v_n(r)e^{in\phi},\quad \hat{g}(r,\phi)=\sum_{n=-\infty}^{\infty}g_n(r)e^{in\phi} \tag{1.25}$$

and inserting (1.25) into (1.24), a comparison of the Fourier coefficients yields

$$\alpha\left(v_n''(r)+\frac{1}{r}v_n'(r)-\frac{n^2}{r^2}v_n(r)\right)-(\beta+in\sigma_1)\,v_n(r)=g_n(r),\; r>0,\; n\in\mathbb{Z}. \tag{1.26}$$

This can easily be transformed into a modified Bessel equation. Let us define the left hand side in (1.26) as $[\mathcal{L}_{\infty,n}v_n](r)$ and let $w_n\in\mathbb{C}$ be the square root of $\alpha^{-1}(\beta+in\sigma_1)$ with $\operatorname{Re} w_n>0$ (using assumptions (A2), (A5) and (A9$_B$)), then one can show that the Green's function for $\mathcal{L}_{\infty,n}$ is given by

$$G_n(r,s)=-s\int_0^\infty\frac{1}{2\alpha t}\exp\left(-(\beta+in\sigma_1)t-\frac{r^2+s^2}{4\alpha t}\right)I_n\left(\frac{rs}{2\alpha t}\right)dt$$

for $0<r,s\in\mathbb{R}$ with $r\neq s$, where $I_n(z)$ denotes the modified Bessel function of the first kind, [111]. The Green's function for $\mathcal{L}_\infty$ in polar coordinates turns out to be

$$\hat{G}((r,\phi),(s,\varphi))=-s\int_0^\infty\frac{1}{4\pi\alpha t}\exp\left(-\beta t-\frac{r^2+s^2}{4\alpha t}+\frac{rs}{2\alpha t}\cos(-\sigma_1t+\phi-\varphi)\right)dt$$

which in Cartesian coordinates corresponds to

$$G(x,\xi)=-\int_0^\infty\frac{1}{4\pi\alpha t}\exp\left(-\beta t-\frac{1}{4\alpha t}\left|e^{tS}x-\xi\right|^2\right)dt.$$

Therefore, the solution of (1.23) can be represented by

$$v(x)=\int_{\mathbb{R}^2}G(x,\xi)g(\xi)d\xi,\; x\in\mathbb{R}^2.$$

In particular, we observe that the Green's function of $\mathcal{L}_\infty$ coincides with the time-integral of the heat kernel of $\mathcal{L}_\infty$, [4],

$$G(x,\xi)=-\int_0^\infty H(x,\xi,t)dt. \tag{1.27}$$

Moreover, using an orthogonal transformation for the skew-symmetric matrix S into several planar polar coordinates, we expect that this approach extends to $d>2$ and $N>1$.

Fourier-Bessel method on circular domains. An essential advantage of the Fourier-Bessel approach is that it can be applied to bounded domains, which is an important issue when investigating truncations. Consider the boundary value problem

$$\begin{aligned}
&(1.28\text{a}) && [\mathcal{L}_{\infty,R}v_R](x):=\alpha\triangle v_R(x)+\langle Sx,\nabla v_R(x)\rangle-\beta v_R(x)=g_R(x),\; x\in\overset{\bullet}{B}_R(0),\\
&(1.28\text{b}) && \lim_{|x|\to0}\nabla v_R(x)\cdot x=0,\\
&(1.28\text{c}) && av_R(x)+b\frac{\partial}{\partial\mathbf{n}}v_R(x)=b_R(x),\; x\in\partial B_R(0),
\end{aligned}$$

where $B_R(0) := \{x \in \mathbb{R}^2 \mid |x| < R\}$ and $\overset{\bullet}{B}_R(0) = B_R(0)\backslash\{0\}$ for some $R > 0$, $A = \alpha \in \mathbb{C}$, $B = \beta \in \mathbb{C}$, $S \in \mathbb{R}^{2,2}$ skew-symmetric, $g_R : \overline{B_R(0)} \to \mathbb{C}$ continuous, $b_R : \partial B_R(0) \to \mathbb{C}$ continuous and $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 > 0$. The condition (1.28b) guarantees that the solution is bounded as $|x| \to 0$. Moreover, the Robin boundary condition from (1.28c) contains Dirichlet ($a = 1$, $b = 0$) and Neumann ($a = 0$, $b = 1$) boundary conditions. Using (A5), equation (1.28) reads in polar coordinates

$$\alpha \left[\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\phi\phi}\right] \hat{v}_R(r,\phi) - \sigma_1 \partial_\phi \hat{v}_R(r,\phi) - \beta \hat{v}_R(r,\phi) = \hat{g}_R(r,\phi), \tag{1.29a}$$

$$\lim_{r\to 0} \partial_r \hat{v}_R(r,\phi) = 0, \tag{1.29b}$$

$$a\hat{v}_R(R,\phi) + b\left[\frac{\partial}{\partial \mathbf{n}}\hat{v}_R(r,\phi)\right]_{r=R} = \hat{b}_R(\phi), \tag{1.29c}$$

for $0 < r < R$ and $\phi \in [-\pi, \pi[$, where $\pm i\sigma_1 \in \sigma(S)$. Representing $\hat{v}_R$, $\hat{g}_R$ and $\hat{b}_R$ by complex Fourier series w.r.t. ϕ

$$\begin{aligned} \hat{v}_R(r,\phi) &= \sum_{n=-\infty}^{\infty} v_{R,n}(r)e^{in\phi}, \quad \hat{g}_R(r,\phi) = \sum_{n=-\infty}^{\infty} g_{R,n}(r)e^{in\phi}, \\ \hat{b}_R(\phi) &= \sum_{n=-\infty}^{\infty} b_{R,n}e^{in\phi} \end{aligned} \tag{1.30}$$

and inserting (1.30) into (1.29), a comparison of Fourier coefficients yields

$$\alpha\left(v''_{R,n}(r) + \frac{1}{r}v'_{R,n}(r) - \frac{n^2}{r^2}v_n(r)\right) - (\beta + in\sigma_1)\, v_{R,n}(r) = g_{R,n}(r), \tag{1.31a}$$

$$\lim_{r\to 0} r v'_{R,n}(r) = 0, \tag{1.31b}$$

$$a v_{R,n}(R) + b v'_{R,n}(R) = b_{R,n}, \tag{1.31c}$$

for every $n \in \mathbb{Z}$, where (1.31a) holds for $0 < r < R$. Similar as above, (1.31a) can be transformed into a modified Bessel equation. For the general theory of Bessel functions and related material see [111] and [81]. Once more, we define the right hand side of (1.31a) by $[\mathcal{L}_{\infty,R,n}v_{R,n}](r)$ and requiring (A2), (A5) and (A9$_B$) we choose $w_n \in \mathbb{C}$ with $\operatorname{Re} w_n > 0$ as above, then one can show that the Green's function $G_{R,n}$ and the Poisson kernel $P_{R,n}$ for $\mathcal{L}_{\infty,R,n}$ are given by

$$\begin{aligned} G_{R,n}(r,s) &= G_n(r,s) + \frac{s}{\alpha}F_{R,n}I_n(w_n r)I_n(w_n s),\ 0 < r \neq s < R, \\ F_{R,n} &:= \frac{aRK_n(w_nR) - bw_nRK_{n+1}(w_nR) + bnK_n(w_nR)}{aRI_n(w_nR) + bw_nRI_{n+1}(w_nR) + bnI_n(w_nR)}, \\ P_{R,n}(r) &= \frac{I_n(w_n r)}{aI_n(w_nR) + \frac{bw_n}{2}\left(I_{n+1}(w_nR) + I_{n+1}(w_nR)\right)},\ 0 < r < R, \end{aligned}$$

where $I_n(z)$ and $K_n(z)$ denote the modified Bessel function of the first and second kind, respectively, and $G_n(r,s)$ denotes Green's function of $\mathcal{L}_{\infty,n}$ from above on the whole domain $[0,\infty[$. The solution of (1.31) is now given by

$$v_{R,n}(r) = \int_0^R G_{R,n}(r,s)g_{R,n}(s)ds + P_{R,n}(r)b_{R,n},\ 0 < r < R.$$

Thus, the Green's function and the Poisson kernel for $\mathcal{L}_{\infty,R}$ in polar coordinates are

$$\hat{G}_R((r,\phi),(s,\varphi)) = \hat{G}((r,\phi),(s,\varphi)) + \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{s}{\alpha}F_{R,n}I_n(w_n r)I_n(w_n s)e^{-in\varphi}e^{-in\phi}, \tag{1.32}$$

$$\hat{P}_R((r,\phi),\varphi) = \sum_{n=-\infty}^{\infty} P_{R,n}(r)e^{-in\varphi}e^{-in\phi}. \tag{1.33}$$

Therefore, the solution of (1.29) can be represented by

$$\hat{v}_R(r,\phi) = \int_0^R \int_{-\pi}^{\pi} \hat{G}_R((r,\phi),(s,\varphi))\hat{g}_R(s,\varphi)d\varphi ds + \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{P}_R((r,\phi),\varphi)\hat{b}_R(\varphi)d\varphi.$$

In contrast to the Fourier-Bessel method on the whole $\mathbb{R}^2$, it is not possible to find closed expressions for the Fourier series from (1.32) and (1.33). It remains as an open problem to derive suitable estimates of $\hat{G}_R$ and $\hat{P}_R$ which hold uniformly in R. A relation, similar to (1.27), between the heat kernel and the Green's function as well as the Poisson kernel for $\mathcal{L}_{\infty,R}$, seems not to be known in this case.

Phase-rotating waves and space-state-dependent nonlinearities. We currently extend the theory to include **phase-rotating** (or **oscillating**) **waves**

$$u_\star(x,t) = e^{-i\omega t}v_\star(x),\ x\in\mathbb{R}^d,\ t\in[0,\infty[,$$

with phase velocity $0 \neq \omega \in \mathbb{R}$.

Such phase-rotating waves arise for instance in cubic-quintic complex Ginzburg Landau equations, but also in Schrödinger equations (see: [44], [45], [33], [112]) and Gross-Pitaevskii equations (see: [44]). Note that in the latter equations x-dependent potentials occur. This suggest still another extension to state dependent nonlinearities $f(x,u)$.

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2 Applications and origin of the Ornstein-Uhlenbeck operator

2.1 Rotating waves in reaction diffusion systems

In Section 1.2 we have already motivated the nonlinear steady state problem (1.11) for the complex Ornstein-Uhlenbeck operator by the existence of rotating wave solutions. Such rotating waves arise in many applications from physical, chemical and biological sciences. In the following, we list a set of examples, where such rotating wave solutions exist. All the computations where done with Comsol Multiphysics$^{\text{TM}}$, [1].

Example 2.1 (Ginzburg-Landau equation)**.** Consider the **cubic-quintic complex Ginzburg-Landau equation (QCGL)**, [64],

$$u_t = \alpha \triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right) \tag{2.1}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$, $\alpha,\beta,\gamma,\mu \in \mathbb{C}$ and $\operatorname{Re}\alpha > 0$. The real-valued version of this equation reads as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix} \triangle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with

$$f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (u_1\mu_1 - u_2\mu_2) + (u_1\beta_1 - u_2\beta_2)\left(u_1^2+u_2^2\right) + (u_1\gamma_1 - u_2\gamma_2)\left(u_1^2+u_2^2\right)^2 \\ (u_1\mu_2 + u_2\mu_1) + (u_1\beta_2 + u_2\beta_1)\left(u_1^2+u_2^2\right) + (u_1\gamma_2 + u_2\gamma_1)\left(u_1^2+u_2^2\right)^2 \end{pmatrix},$$

$u = u_1 + iu_2$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$ and $u_i, \alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ for $i = 1,2$. This equation describes different aspects of signal propagation in heart tissue, superconductivity, superfluidity, nonlinear optical systems, see [79], photonics, plasmas, physics of lasers, Bose-Einstein condensation, liquid crystals, fluid dynamics, chemical waves, quantum field theory, granular media and is used in the study of hydrodynamic instabilities, see [76]. It shows a variety of coherent structures like stable and unstable pulses, fronts, sources and sinks in 1D, see [109], [102], [6] and [106], vortex solitons, see [27], spinning solitons, see [28], rotating spiral waves, propagating clusters, see [84], and exploding dissipative solitons, see [101] in 2D as well as scroll waves and spinning solitons in 3D, see [77].

Let us discuss the assumptions (A1)–(A9): Assumption (A1) is satisfied for every $\alpha \in \mathbb{C}$, assumptions (A2) and (A3) if $\operatorname{Re}\alpha = a_1 > 0$ and (A4) if

$$\left|\arg\alpha\right| < \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right), \text{ for some } 1 < p < \infty.$$

The condition (A5) is satisfied with

$$S = \begin{pmatrix} 0 & S_{12} \\ -S_{12} & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & S_{12} & S_{13} \\ -S_{12} & 0 & S_{23} \\ -S_{13} & -S_{23} & 0 \end{pmatrix} \tag{2.2}$$

for $d = 2$ and $d = 3$, respectively. In the examples below we specify the entries S_{12}, S_{13}, $S_{23} \in \mathbb{R}$ and the point $x_\star \in \mathbb{R}^d$, that will be the center of rotation if $d = 2$ and the support vector of the axis of rotation if $d = 3$. All these informations come actually from a simulation. First we simulate the original system for some time then we switch to the freezing method, which then yields the profile $v_\star$ as well as the values for the rotational and translational velocities. This will be done in Example 10.9, 10.10 and 10.11. For general theory about the freezing method we refer to [16], [18], [20], [19] and [103]. The specific values of these variables will be given in the examples below. Note that in case $d = 2$ we have a clockwise rotation, if $S_{12} > 0$, and a counter clockwise rotation, if $S_{12} < 0$. Assumption (A6) is obviously satisfied. Using, for instance, $v_\infty = (0, 0)^T$ then assumption (A7) is satisfied. Then, we have

$$Df(v_\infty) = \begin{pmatrix} \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix}$$

and assumption (A8) is also satisfied. Assumption (A9) is only satisfied if $\operatorname{Re}\mu < 0$. The bound for the rate of the exponential decay from Theorem 1.8 reads

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{\operatorname{Re}\alpha\,(-\operatorname{Re}\mu)}{|\alpha|^2 p^2} \tag{2.3}$$

for some $0 < \vartheta < 1$. Let us now consider some specific examples:

(**1**): For the parameter values

$$\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i, \quad \mu = -\frac{1}{2} \tag{2.4}$$

this equation exhibits so called **spinning soliton** solutions for space dimensions $d = 2$ and $d = 3$, see Figure 2.1.

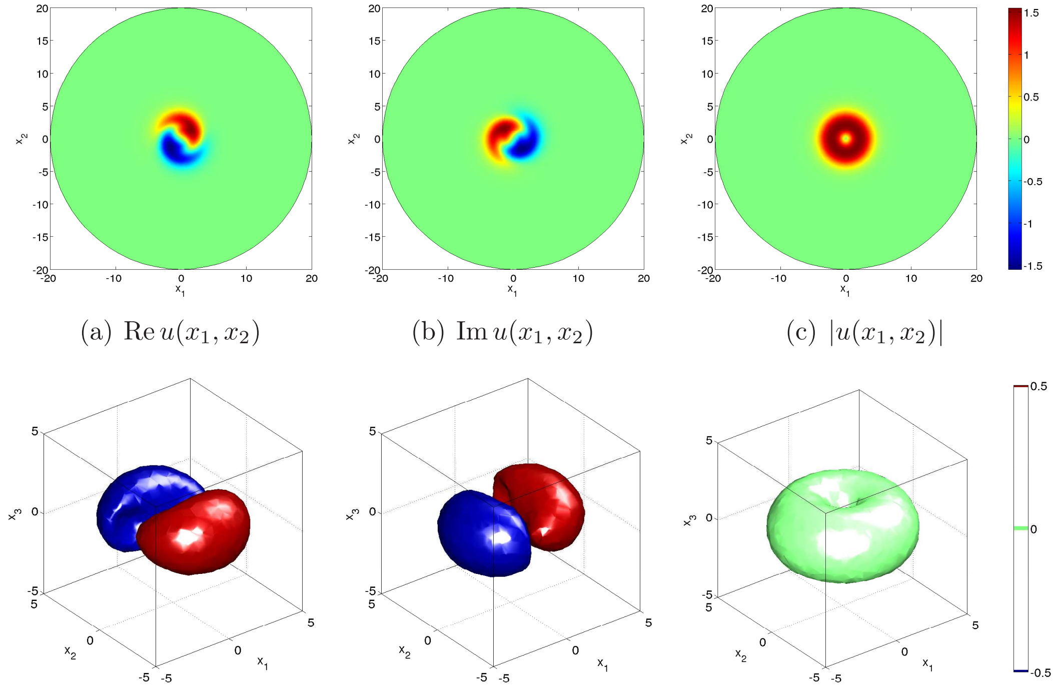
Figure 2.1(a)–2.1(c) shows the spinning soliton in $\mathbb{R}^2$ as the solution of (2.1) on a circular disk of radius $R = 20$ centered in the origin at time $t = 150$. For the computation we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.25$, the BDF method of order 2 with absolute tolerance $\mathrm{atol} = 10^{-5}$, relative tolerance $\mathrm{rtol} = 10^{-4}$ and maximal stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and initial data

$$u_0^{2D}(x_1, x_2) = \frac{1}{5}\,(x_1 + ix_2) \exp\left(-\frac{x_1^2 + x_2^2}{49}\right).$$

Figure 2.1(d)–2.1(f) shows the spinning soliton in $\mathbb{R}^3$ as the solution of (2.1) on a cube with edge length $L = 20$ centered in the origin at time $t = 100$. For the computation we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.8$, the BDF method of order 2 with absolute tolerance atol $= 10^{-4}$, relative tolerance rtol $= 10^{-2}$ and maximal stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and (discontinuous) initial data

$$u_0^{3D}(x_1,x_2,x_3) = u_0^{2D}(x_1,x_2)$$

for $|x_3| < 9$ and 0 otherwise.



(a) $\operatorname{Re} u(x_1,x_2)$ (b) $\operatorname{Im} u(x_1,x_2)$ (c) $|u(x_1,x_2)|$

(d) $\operatorname{Re} u(x_1,x_2,x_3)$ (e) $\operatorname{Im} u(x_1,x_2,x_3)$ (f) $|u(x_1,x_2,x_3)|$

Figure 2.1: Spinning soliton of QCGL for $d = 2$ (above) and $d = 3$ (bottom)

The parameter values (2.4) satisfy our assumptions (A1)–(A9) for every p with

$$1.1716 \approx \frac{4}{2+\sqrt{2}} < p < \frac{4}{2-\sqrt{2}} \approx 6.8284,$$

e.g. $p = 2,3,4,5,6$. At time $t = 400$ we have the rotational velocity S_{12} with center of rotation $x_\star$ given by

$$S_{12} = 1.027, \quad x_\star = \begin{pmatrix} -0.016465 \\ -0.002849 \end{pmatrix}$$

in case $d = 2$ and the rotational velocities S_{12}, S_{13}, S_{23} with support vector $x_\star$ given by

$$\begin{pmatrix} S_{12} \\ S_{13} \\ S_{23} \end{pmatrix} = \begin{pmatrix} 0.6855 \\ -0.01558 \\ 0.01086 \end{pmatrix}, \quad x_\star = \begin{pmatrix} 0.179489 \\ 0.191649 \\ -0.007199 \end{pmatrix}$$

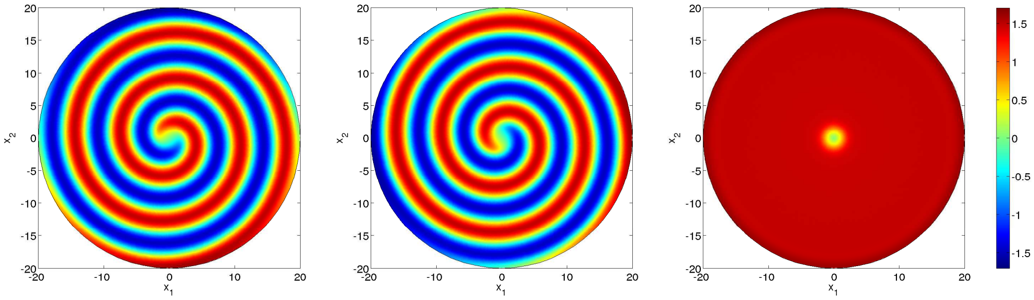
at time $t = 500$ in case $d = 3$, compare Example 10.9. The solitons are localized in the sense of Theorem 1.8 with the bound

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{1}{3p^2} < \frac{1}{3p^2} \quad \text{for} \quad p \in]4-2\sqrt{2}, 4+2\sqrt{2}[.$$

(**2**): For the parameter values

$$\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{13}{5} + i, \quad \gamma = -1 - \frac{1}{10}i, \quad \mu = -\frac{1}{2} \tag{2.5}$$

this equation exhibits so called **rotating spiral wave** solutions, see Figure 2.2.



(a) $\operatorname{Re} u(x_1, x_2)$ (b) $\operatorname{Im} u(x_1, x_2)$ (c) $|u(x_1, x_2)|$

Figure 2.2: Rotating spiral wave of QCGL for $d = 2$

Figure 2.2(a)–2.2(c) shows the spiral wave in $\mathbb{R}^2$ as the solution of (2.1) on a circular disk of radius $R = 20$ centered in the origin at time $t = 150$. For the computation we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.25$, the BDF method of order 2 with absolute tolerance atol $= 10^{-5}$, relative tolerance rtol $= 10^{-4}$ and maximal stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and initial data u_0^{2D} from above.

The only difference in the choice of parameters in (2.5) when compared to (2.4), is the real part of β, which is now slightly larger. The parameter values satisfy our assumptions (A1)–(A9) also for $p \in]4 - 2\sqrt{2}, 4 + 2\sqrt{2}[$. At time $t = 400$ we have the rotational velocity S_{12} with center of rotation $x_\star$ given by

$$S_{12} = 1.323, \quad x_\star = \begin{pmatrix} -0.007763 \\ -0.019773 \end{pmatrix}.$$

The spiral wave seems not to be localized in the sense of Theorem 1.8 since condition (1.20) is not satisfied. We observe that enlarging β from $\frac{5}{2}$ to $\frac{13}{5}$ generates a pattern with a higher rotational velocity.

(**3**): For the parameter values

$$\alpha = 1, \quad \beta = -(1 + i), \quad \gamma = 0, \quad \mu = 1 \tag{2.6}$$

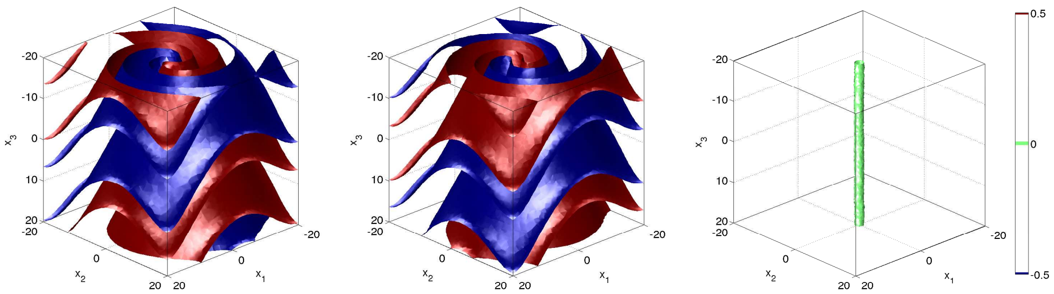
this equation exhibits so called **twisted** and **untwisted scroll wave** as well as **scroll ring** solutions, see Figure 2.3.

Figure 2.3(a)–2.3(c) shows the untwisted scroll ring in $\mathbb{R}^3$ as the solution of (2.1) and (2.7), respectively, on a cube with edge length $L = 40$ centered in the origin at time $t = 150$. For the computation we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 1.6$, the BDF method of order 2 with absolute tolerance atol $= 10^{-3}$, relative tolerance rtol $= 10^{-2}$ and maximal stepsize

$\triangle t = 0.5$, homogeneous Neumann boundary conditions on the lateral surfaces, periodic boundary conditions on the faces for $x_3 = \mp 20$ and (discontinuous) initial data

$$u_0^{3D}(x_1, x_2, x_3) = \frac{1}{5}(x_1 + ix_2)\exp\left(-\frac{x_1^2 + x_2^2}{49} + iK\frac{2\pi z}{L_{x_3}}\right)$$

for $|x_3| \leqslant 16$ with winding number $K = 1$ and edge length in x_3-direction $L_{x_3} = L = 40$ and 0 otherwise. Due to the periodic boundary conditions on the x_3-slices the untwisted scroll wave can be considered as an untwisted scroll ring on a torus.



(a) $\operatorname{Re} u(x_1, x_2, x_3)$ (b) $\operatorname{Im} u(x_1, x_2, x_3)$ (c) $|u(x_1, x_2, x_3)|$

Figure 2.3: Untwisted scroll ring of QCGL and of the λ-ω system for $d = 3$

The parameter values (2.6) satisfy only the assumptions (A1)–(A8) for every $1 < p < \infty$ but not condition (A9), since the real part of μ is not negative. In this case the pattern is not localized in the sense of Theorem 1.8. At time $t = 850$ we have the rotational velocities S_{12}, S_{13}, S_{23} with support vector $x_\star$ given by

$$\begin{pmatrix} S_{12} \\ S_{13} \\ S_{23} \end{pmatrix} = \begin{pmatrix} -0.8934 \\ 0.002114 \\ -0.001088 \end{pmatrix}, \quad x_\star = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Example 2.2 (λ-ω system)**.** Consider the λ-ω **system**, [61], [80],

$$u_t = \alpha \triangle u + u\left(\lambda\left(|u|^2\right) + i\omega\left(|u|^2\right)\right) \tag{2.7}$$

with $u : \mathbb{R}^d \times [0, \infty[\to \mathbb{C}$, $d \in \{2, 3\}$, $\alpha \in \mathbb{C}$, $\lambda : [0, \infty[\to \mathbb{R}$ and $\omega : [0, \infty[\to \mathbb{R}$. The real-valued version of this equation reads as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix} \triangle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with

$$f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1\lambda\left(u_1^2 + u_2^2\right) - u_2\omega\left(u_1^2 + u_2^2\right) \\ u_1\omega\left(u_1^2 + u_2^2\right) + u_2\lambda\left(u_1^2 + u_2^2\right) \end{pmatrix},$$

$u = u_1 + iu_2$, $\alpha = \alpha_1 + i\alpha_2$ and $u_i, \alpha_i \in \mathbb{R}$ for $i = 1, 2$. This equation describes chemical reaction processes, see [61] and [60], physiological processes in the study of cardiac arrhythmias, time evolution of biological systems, see [80], and is often used to analyze the mechanism of pattern formation as well as to study the onset of turbulent behavior. An example of an emerging technological application based on pattern forming systems is given by memory devices using magnetic domain patterns. This model exhibits rotating spirals as well as scroll wave and scroll ring solutions, see [32] and [36].

Let us again discuss the assumptions (A1)–(A9): Assumption (A1) is satisfied for every $\alpha \in \mathbb{C}$, assumptions (A2) and (A3) if $\operatorname{Re}\alpha = a_1 > 0$ and (A4) for some $1 < p < \infty$ if

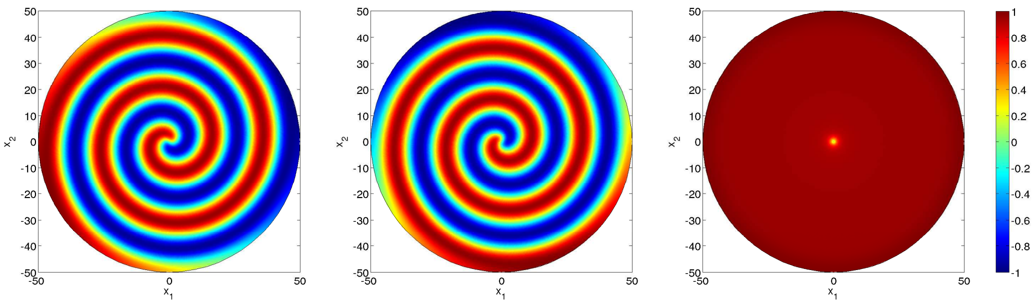
$$|\arg\alpha| \leqslant \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right).$$

The condition (A5) is satisfied with S from (2.2). Assumption (A6) is satisfied if $\lambda, \omega \in C^2([0,\infty[,\mathbb{R})$. Since the assumptions (A7)–(A9) depends on the choice of λ and ω, we explain these conditions in the following example.

(**1**): For the parameter settings

$$\alpha = 1, \quad \lambda\left(|u|^2\right) = 1 - |u|^2, \quad \omega\left(|u|^2\right) = -|u|^2 \tag{2.8}$$

this equation exhibits so called **rigidly rotating spiral wave** solutions, see Figure 2.4, as well as **twisted** and **untwisted scroll wave** and **scroll ring** solutions, see Figure 2.3 for an untwisted scroll ring.



(a) $\operatorname{Re} u(x_1, x_2)$ (b) $\operatorname{Im} u(x_1, x_2)$ (c) $|u(x_1, x_2)|$

Figure 2.4: Rigidly rotating spiral wave of λ-ω system for $d = 2$

Figure 2.4(a)–2.4(c) shows the spiral wave in $\mathbb{R}^2$ as the solution of (2.7) on a circular disk of radius $R = 50$ centered in the origin at time $t = 150$. For the computation we used continuous piecewise linear finite elements with stepsize $\triangle x = 0.5$, the BDF method of order 2 with absolute tolerance atol $= 10^{-4}$, relative tolerance rtol $= 10^{-3}$ and stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and initial data

$$u_0(x, y) = \frac{1}{20}\left(x_1, x_2\right)^T.$$

For a discussion about the scroll ring from Figure 2.3(a)–2.3(c) we refer to Example 2.1(3).

The parameter values (2.8) satisfy only the assumptions (A1)–(A8) for every $1 < p < \infty$ and with $v_\infty = (0,0)^T$ but not condition (A9), since $Df(0,0)$ has the eigenvalue 1 with algebraic multiplicity 2. In this case the pattern is not localized in the sense of Theorem 1.8. The rotational velocity S_{12} and the center of rotation $x_\star$ of the spiral are

$$S_{12} = -0.9091, \quad x_\star = \begin{pmatrix} -0.001770 \\ 0.000650 \end{pmatrix}$$

at time $t = 550$. Since the λ-ω system (2.7) equipped with the parameter-values (2.8) is indeed a special case of the cubic-quintic complex Ginzburg-Landau equation (2.1), namely for $\beta = -(1+i)$, $\gamma = 0$ and $\mu = 1$, compare (2.6), we refer for a discussion about the assumptions also to Example 2.1(3).

Example 2.3 (Barkley model)**.** Consider the **Barkley model**, [10], [11], [12]

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \triangle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\varepsilon} u_1 (1-u_1)\left(u_1 - \frac{u_2+b}{a}\right) \\ g(u_1) - u_2 \end{pmatrix} \tag{2.9}$$

with $u = (u_1, u_2)^T$, $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{R}^2$, $d \in \{2,3\}$, $0 \leqslant D << 1$, $0 < \varepsilon << 1$, $0 < a, b \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$. This equation describes excitable media, oscillatory media, see[10], catalytic surface reactions, see [9], the interaction of a fast activator u and a slow inhibitor v (in this case $g(u)$ describes a delayed production of the inhibitor) and is often used as a qualitative model in pattern forming systems (e.g. Belousov-Zhabotinsky reaction). This model exhibits rotating spiral wave and scroll wave solutions, see [11], [18] and [103].

Let us discuss the assumptions (A1)–(A9): Assumption (A1) is satisfied for every $D \in \mathbb{R}$, assumption (A2) and (A3) if $D > 0$ and (A4) for every

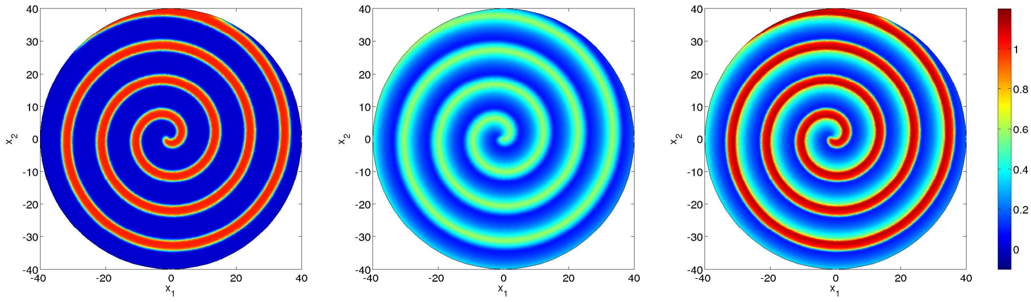
$$\frac{2(D+1)}{2\sqrt{D}+D+1} < p < \frac{2(D+1)}{-2\sqrt{D}+D+1}, \text{ with } 0 < D \leqslant 1.$$

Condition (A4) doesn't hold for $D = 0$. The condition (A5) is satisfied with $S \in \mathbb{R}^{2,2}$ from (2.2). Specific values for S_{12} will be given in the example below. Assumption (A6) is satisfied if $g \in C^2(\mathbb{R},\mathbb{R})$. The zeros of the nonlinearity are $(0, g(0))$, $(1, g(1))$ and another one. Using, for instance, $v_\infty = (0, g(0))^T$ then assumption (A7) is satisfied and (A8) holds for $D = 1$. Since the eigenvalues of $Df(v_\infty)$ are $\frac{g(0)+b}{a^2}$ and -1, condition (A9) is equivalent to $\frac{g(0)+b}{a^2} < 0$, i.e. $g(0) < -b$. Analogously, using $v_\infty = (1, g(1))^T$ then assumption (A7) is satisfied and (A8) holds for $D = 1$. Since the eigenvalues of $Df(v_\infty)$ are $\frac{g(1)+b-a}{a^2}$ and -1, condition (A9) is equivalent to $\frac{g(1)+b-a}{a^2} < 0$, i.e. $g(1) < a - b$. Let us now consider some specific examples:

(**1**): For the parameter values

$$D = 0, \quad \varepsilon = \frac{1}{50}, \quad a = \frac{7}{10}, \quad b = \frac{1}{100}, \quad g(u_1) = u_1 \tag{2.10}$$

this equation exhibits so called **rigidly rotating spiral wave** solutions, see Figure 2.5.



(a) $u_1(x_1, x_2)$ (b) $u_2(x_1, x_2)$ (c) $|u(x_1, x_2)|$

Figure 2.5: Rigidly rotating spiral wave of Barkley model for $d = 2$

Figure 2.5(a)–2.5(c) shows the rotating spiral wave in $\mathbb{R}^2$ as the solution of (2.9) on a circular disk of radius $R = 40$ centered in the origin at time $t = 150$. For the computation we used continuous piecewise linear finite elements with stepsize $\triangle x = 0.5$, the BDF method of order 2 with absolute tolerance atol $= 10^{-5}$, relative tolerance rtol $= 10^{-2}$ and stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and initial data

$$u_0^{(1)}(x,y) = \begin{cases} 1 & , x > 0 \\ 0 & , x \leqslant 0 \end{cases}, \quad u_0^{(2)}(x,y) = \begin{cases} \frac{a}{2} & , y > 0 \\ 0 & , y \leqslant 0 \end{cases}.$$

The parameter values (2.10) satisfy the assumptions (A1), (A6) since g is twice continuously differentiable, (A7) for $v_\infty = (0,0)^T$, $(1,1)^T$ and $\left(\frac{b}{a-1}, \frac{b}{a-1}\right)^T$. At time $t = 650$ we found the rotational velocity S_{12} for the matrix S from (A5) and the center of rotation $x_\star$ given by

$$S_{12} = 2.067, \quad x_\star = \begin{pmatrix} -1.1717 \\ 0.6628 \end{pmatrix}.$$

All other assumptions are not satisfied. $D = 0$ violates assumption (A2), (A3), (A4) and (A8). For $v_\infty = (0,0)^T$ condition (A9) needs $\frac{b}{a^2} < 0$, which is not satisfied, and for $v_\infty = (1,1)^T$ assumption (A9) needs $\frac{1+b-a}{a^2} < 0$, which is not true in this case. Assumption (A9) is also not satisfied for $v_\infty = \left(\frac{b}{a-1}, \frac{b}{a-1}\right)^T$ with the parameters above.

2.2 The origin of the Ornstein-Uhlenbeck from stochastic ODEs

In this section we recall the origin of the Ornstein-Uhlenbeck operator from stochastic differential equations. For this purpose we consider a stochastic ordinary differential equation (SODE) and derive a second-order partial differential equation (PDE), that is called the Kolmogorov equation. For a detailed treatment about the transformation of a single SODE to a second-order PDE we refer to [55, Chapter 24], [56, Section 5.7], [57, Section 4.8] and [70, Section 2.8]. The corresponding differential operator of the Kolmogorov equation is called the Kolmogorov operator, which comes originally from [58]. Different types of Kolmogorov operators were treated in [3], [23] and [58]. Applications of Kolmogorov operators in physics and finance can be found in [63] and [82]. The Ornstein-Uhlenbeck operator, which is an elliptic operator with unbounded linearly growing coefficients, is a special type of a Kolmogorov operator. For a motivation of the Ornstein-Uhlenbeck operator from SODE's we refer to [66, Chapter 9]. Note that Section 2.2 and Section 2.3 are not relevant to understand the following theory concerning the exponential decay and thus they can also be skipped.

2.2.1 From ODE to first-order PDE

Let $d \in \mathbb{N}$ and let $\mu \in C^{\infty}(\mathbb{R}^d, \mathbb{R}^d)$ be a function, which is at most linearly growing, i.e.

$$\exists\, C > 0 : \ |\mu(x)| \leqslant C\,(1 + |x|) \quad \forall\, x \in \mathbb{R}^d.$$

Then there exists a family

$$\Phi(\cdot; x) : [0, \infty[\to \mathbb{R}^d,\ x \in \mathbb{R}^d,$$

of unique smooth functions, satisfying the ordinary differential equation

$$\text{(ODE)} \qquad \begin{aligned} \frac{\partial}{\partial t}\Phi(t; x) &= \mu\left(\Phi(t; x)\right),\ t \in [0, \infty[,\ x \in \mathbb{R}^d, \\ \Phi(0; x) &= x. \end{aligned}$$

The mapping $\Phi(\cdot; x)$ is known as the **solution flow of** (ODE) satisfying the flow properties. These functions are ∞-times continuously differentiable with respect to x for every fixed $t \in [0, \infty[$, i.e.

$$\Phi(t; \cdot) : \mathbb{R}^d \to \mathbb{R}^d, \quad x \longmapsto \Phi(t; x) \text{ is smooth } \forall\, t \in [0, \infty[.$$

For a similar result we refer to [73, Lemma 3.1] and for its proof to [69, Section 2.1]. The family $T(t) : C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}) \to C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R})$, $t \in [0, \infty[$, of linear operators defined by

$$\left[T(t)u_0\right](x) := u_0\left(\Phi(t; x)\right),\ x \in \mathbb{R}^d,\ t \in [0, \infty[,\ u_0 \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}),$$

is called the **transition semigroup of the** (ODE). $(T(t))_{t \geqslant 0}$ satisfies the semigroup properties

$$T(0) = I \quad \text{and} \quad T(t_1)T(t_2) = T(t_1 + t_2)\ \forall\, t_1, t_2 \in [0, \infty[,$$

which follow immediately by the flow properties of Φ, and it satisfies

$$T(t)C_{\mathrm{b}}^k(\mathbb{R}^d, \mathbb{R}) \subseteq C_{\mathrm{b}}^k(\mathbb{R}^d, \mathbb{R})\ \forall\, t \in [0, \infty[\ \forall\, k \in \mathbb{N}_0 \cup \{\infty\}.$$

Let us fix $u_0 \in C_{\mathrm{b}}^1(\mathbb{R}^d, \mathbb{R})$ and consider $u : \mathbb{R}^d \times [0, \infty[\to \mathbb{R}$ given by

$$u(x, t) := \left[T(t)u_0\right](x) = u_0\left(\Phi(t; x)\right),\ t \in [0, \infty[,\ x \in \mathbb{R}^d,$$

then u is the classical solution of the first-order linear PDE

$$\begin{aligned}(\mathrm{PDE}_{1st})\quad \frac{\partial}{\partial t}u(x,t) &= \sum_{i=1}^{d}\mu_i(x)\frac{\partial}{\partial x_i}u(x,t) =: \langle \mu(x), \nabla u(x,t)\rangle\,,\ x\in\mathbb{R}^d,\ t\in[0,\infty[,\\ u(x,0) &= u_0(x).\end{aligned}$$

As we will see in Section 2.2.2, the **forward advection equation** (PDE_{1st}) is a special case of a Kolmogorov equation. In particular, the solution preserves the smoothness of the initial data, i.e. for every $k\in\mathbb{N}\cup\{\infty\}$ with $k\leqslant r$

$$u(\cdot,0)=u_0(\cdot)\in C_{\mathrm{b}}^k(\mathbb{R}^d,\mathbb{R}) \quad\Rightarrow\quad u(\cdot,t)\in C_{\mathrm{b}}^k(\mathbb{R}^d,\mathbb{R})\ \forall\, t\in[0,\infty[$$

Example 2.4 (Drift term of the Ornstein-Uhlenbeck operator)**.** Let $d\in\mathbb{N}$ and $\mu:\mathbb{R}^d\to\mathbb{R}^d$ with $\mu(x)=Sx$ for some $0\neq S\in\mathbb{R}^{d,d}$, then $\Phi(\cdot;x):[0,\infty[\to\mathbb{R}^d$ with $\Phi(t;x)=e^{tS}x$, $x\in\mathbb{R}^d$, $t\in[0,\infty[$, is the unique smooth solution of

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(t;x) &= S\Phi(t;x),\ t\in[0,\infty[,\ x\in\mathbb{R}^d,\\ \Phi(0;x) &= x.\end{aligned}$$

The corresponding transition semigroup is given by $T(t):C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})\to C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$, $t\in[0,\infty[$, with

$$[T(t)u_0](x) := u_0\left(e^{tS}x\right),\ x\in\mathbb{R}^d,\ t\in[0,\infty[,\ u_0\in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}).$$

If we fix $u_0\in C_{\mathrm{b}}^1(\mathbb{R}^d,\mathbb{R})$, then $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}$ given by

$$u(x,t) := [T(t)u_0](x) = u_0\left(e^{tS}x\right),\ t\in[0,\infty[,\ x\in\mathbb{R}^d,$$

is a classical solution of the first-order linear PDE

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \sum_{i=1}^{d}(Sx)_i\frac{\partial}{\partial x_i}u(x,t) =: \langle Sx, \nabla u(x,t)\rangle\,,\ x\in\mathbb{R}^d,\ t\in[0,\infty[,\\ u(x,0) &= u_0(x).\end{aligned}$$

Later we consider skew-symmetric matrices S, in which case e^{tS} describes rigid rotations. We will denote the semigroup $(T(t))_{t\geqslant 0}$, that generates even a group, by $(R(t))_{t\geqslant 0}$ and call $(R(t))_{t\geqslant 0}$ the **rotation group**.

2.2.2 From SODE to second-order PDE

Let us consider $d,m\in\mathbb{N}$ and two globally Lipschitz continuous functions $\mu\in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and $\sigma\in C^\infty(\mathbb{R}^d,\mathbb{R}^{d,m})$, which are at most linearly growing, i.e.

$$\begin{aligned}\exists\, C>0:\ |\mu(x)|\leqslant C\,(1+|x|)\quad \forall\, x\in\mathbb{R}^d,\\ \exists\, C>0:\ |\sigma(x)|\leqslant C\,(1+|x|)\quad \forall\, x\in\mathbb{R}^d.\end{aligned}$$

Furthermore, let $(\Omega,\mathcal{F},\mathbb{P})$ denote a probability space with a standard Brownian motion

$$W:[0,\infty[\times\Omega\to\mathbb{R}^m,\quad (t,\omega)\mapsto W(t,\omega).$$

Then there exists a family

$$\Phi(\cdot,\cdot;x):[0,\infty[\times\Omega\to\mathbb{R}^d,\quad (t,\omega)\mapsto\Phi(t,\omega;x),\quad x\in\mathbb{R}^d,$$

of solution processes of the stochastic ordinary differential equation

$$\text{(SODE)}\qquad \begin{aligned} &d\Phi(t;x)=\mu\left(\Phi(t;x)\right)dt+\sigma\left(\Phi(t;x)\right)dW(t),\; t\in[0,\infty[,\; x\in\mathbb{R}^d,\\ &\Phi(0;x)=x.\end{aligned}$$

This is an abbreviating notation for the integral equation

$$\Phi(t;x)=x+\int_0^t\mu\left(\Phi(s;x)\right)ds+\int_0^t\sigma\left(\Phi(s;x)\right)dW(s).$$

It is well known from [70, Section 2.3], that the solution processes are unique up to indistinguishability. As usual, we suppress the dependency on $\omega\in\Omega$ and write $\Phi(t;x)$ instead of $\Phi(t,\omega;x)$. Note, that the (SODE) describes for instance the random motion of a particle in a fluid, [107]. The family $T(t):C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})\to C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$, $t\in[0,\infty[$, of linear operators defined by

$$\left[T(t)u_0\right](x):=\mathbb{E}\left[u_0\left(\Phi(t;x)\right)\right],\; x\in\mathbb{R}^d,\; t\in[0,\infty[,\; u_0\in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}),$$

is called the **transition semigroup of the** (SODE). $(T(t))_{t\geqslant 0}$ satisfies the semigroup properties

$$T(0)=I\quad\text{and}\quad T(t_1)T(t_2)=T(t_1+t_2)\;\forall\,t_1,t_2\in[0,\infty[$$

and

$$T(t)C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})\subseteq C_{\mathrm{b}}^{\infty}(\mathbb{R}^d,\mathbb{R})\;\forall\,t\in]0,\infty[.$$

Such smoothing properties were established by Hörmander in 1967 under the Hörmander condition, [54]. This condition is for example satisfied, if

$$\operatorname{span}\left\{\sigma_1(x),\ldots,\sigma_m(x)\right\}=\mathbb{R}^d\;\forall\,x\in\mathbb{R}^d.$$

Let us fix $u_0\in C_{\mathrm{b}}^2(\mathbb{R}^d,\mathbb{R})$ and consider $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}$ given by

$$u(x,t):=\left[T(t)u_0\right](x)=\mathbb{E}\left[u_0\left(\Phi(t;x)\right)\right],\; t\in[0,\infty[,\; x\in\mathbb{R}^d.$$

If $u(\cdot,t)$ is smooth for all $t\in]0,\infty[$, then u is the classical solution of the second-order linear PDE

(PDE$_{2nd}$)

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t)&=\sum_{i=1}^d\mu_i(x)\frac{\partial}{\partial x_i}u(x,t)+\frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d\left(\sigma(x)\sigma^T(x)\right)_{ij}\frac{\partial^2}{\partial x_i\partial x_j}u(x,t),\\ &=:\left\langle\mu(x),\nabla u(x,t)\right\rangle+\frac{1}{2}\operatorname{tr}\left(\sigma(x)\sigma^T(x)D^2u(x,t)\right)\\ u(x,0)&=u_0(x),\end{aligned}$$

for $x \in \mathbb{R}^d$ and $t \in]0,\infty[$. The smoothness of u is guaranteed for instance by [57, Theorem 4.8.6] assuming further properties on μ, σ and u_0, or alternatively by [66, Theorem 14.2.7] in case of a linear drift. (PDE_{2nd}) is called the **forward Kolmogorov equation**. The second-order differential operator

$$[\mathcal{L}_{\mathrm{Kol}} u](x,t) := \frac{1}{2}\mathrm{tr}\left(\sigma(x)\sigma^T(x)D^2u(x,t)\right) + \langle \mu(x), \nabla u(x,t)\rangle,\ x\in\mathbb{R}^d,\ t\in]0,\infty[$$

is called **Kolmogorov operator** with **diffusion term** $\frac{1}{2}\mathrm{tr}\left(\sigma(x)\sigma^T(x)D^2u(x,t)\right)$ and **drift term** $\langle \mu(x), \nabla u(x,t)\rangle$, $x \in \mathbb{R}^d$, $t \in]0,\infty[$. Note that the Kolmogorov operator $\mathcal{L}_{\mathrm{Kol}}$ can be considered as the infinitesimal generator of the transition semigroup of (SODE).

Example 2.5 (Ornstein-Uhlenbeck operator)**.** Let $m = d \in \mathbb{N}$, $\mu : \mathbb{R}^d \to \mathbb{R}^d$ with $\mu(x) = Sx$ for some $0 \neq S \in \mathbb{R}^{d,d}$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d,d}$ with $\sigma(x) = \sqrt{Q}$ for some symmetric and positive definite matrix $Q \in \mathbb{R}^{d,d}$, where $\sqrt{Q}$ denotes the unique symmetric and positive definite square root of Q. Then σ satisfies $\sigma(x)\sigma^T(x) = Q$ for every $x \in \mathbb{R}^d$. Furthermore, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a standard Brownian motion $W : [0,\infty[\times\Omega \to \mathbb{R}^d$. Then the family $\Phi(\cdot,\cdot;x) : [0,\infty[\times\Omega \to \mathbb{R}^d$ given by

$$\Phi(t;x) = e^{tS}x + \int_0^t e^{(t-\tau)S}dW(\tau),\ t\in[0,\infty[,\ x\in\mathbb{R}^d,$$

are the 'up to indistinguishability' unique solution processes of

$$\begin{aligned} d\Phi(t;x) &= S\Phi(t;x)dt + \sqrt{Q}dW(t),\ t\in[0,\infty[,\ x\in\mathbb{R}^d,\\ \Phi(0;x) &= x. \end{aligned}$$

The solution process $\Phi(\cdot,x)$ is called the **Ornstein-Uhlenbeck process on** $\mathbb{R}^d$ and the corresponding SODE is also known as the **Langevin equation**. A prototype of this equation, $u_t = u + xu_x + u_{xx}$, was considered by Ornstein and Uhlenbeck in 1930, [107]. The corresponding transition semigroup, or sometimes called the **Ornstein-Uhlenbeck semigroup**, is given by $T(t) : C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}) \to C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$, $t \in [0,\infty[$, with

$$\begin{aligned} [T(t)u_0](x) :=& \mathbb{E}\left[u_0(\Phi(t;x))\right]\\ =& \begin{cases} (4\pi)^{-\frac{d}{2}}(\det Q_t)^{-\frac{1}{2}}\int_{\mathbb{R}^d} e^{-\frac{1}{4}\langle Q_t^{-1}\psi,\psi\rangle}v_0(e^{tS}x-\psi)d\psi &, t>0,\\ u_0(x) &, t=0, \end{cases}\\ =& \begin{cases} \int_{\mathbb{R}^d} H(x,\xi,t)v_0(\xi)d\xi &, t>0,\\ u_0(x) &, t=0, \end{cases} \end{aligned}$$

for $x \in \mathbb{R}^d$, $t \in [0,\infty[$ and $u_0 \in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$ where

$$H(x,\xi,t) = (4\pi)^{-\frac{d}{2}}(\det Q_t)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\left\langle Q_t^{-1}(e^{tS}x-\xi),(e^{tS}x-\xi)\right\rangle\right),$$

for $x, \xi \in \mathbb{R}^d$, $t \in]0, \infty[$ and

$$Q_t = \int_0^t e^{\tau S} Q \left(e^{\tau S}\right)^T d\tau,$$

for $t \in]0, \infty[$. The explicit representation of $(T(t))_{t \geqslant 0}$ is due to Kolmogorov, [58]. The function $H : \mathbb{R}^d \times \mathbb{R}^d \times]0, \infty[\to \mathbb{R}$ denotes the heat kernel of the Ornstein-Uhlenbeck operator and is called the **Kolmogorov kernel**, or sometimes the **Ornstein-Uhlenbeck kernel**. Since $Q \in \mathbb{R}^{d,d}$ is symmetric and positive definite, the following relation holds between the heat kernel and the d-dimensional Gaussian measure $\mathcal{N}_d$, see [66, Chapter 9.1] and [13, Satz 30.4],

$$\mathcal{N}_d \left(e^{tS} x, 2Q_t\right) (d\xi) = H(x, \xi, t) d\xi, \; x \in \mathbb{R}^d, \; t > 0,$$

i.e. $H(x, \cdot, t)$ is the density function of the normal distribution $\mathcal{N}_d \left(e^{tS} x, 2Q_t\right)$ with respect to the Lebesgue measure. $2Q_t$ denotes the covariance matrix and $e^{tS} x$ the mean value vector. Let us fix $u_0 \in C_{\mathrm{b}}^2(\mathbb{R}^d, \mathbb{R})$ and let us define $u : \mathbb{R}^d \times [0, \infty[\to \mathbb{R}$ by

$$u(x,t) := [T(t) u_0] (x) = \mathbb{E} \left[u_0 \left(\Phi(t; x)\right)\right], \; t \in [0, \infty[, \; x \in \mathbb{R}^d,$$

then, if $u(\cdot, t)$ is smooth for all $t \in]0, \infty[$, u is the classical solution of the Kolmogorov equation

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) &= \sum_{i=1}^d (Sx)_i \frac{\partial}{\partial x_i} u(x,t) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u(x,t) \\ &= \langle Sx, \nabla u(x,t) \rangle + \frac{1}{2} \mathrm{tr} \left(Q D^2 u(x,t)\right), \; x \in \mathbb{R}^d, \; t \in]0, \infty[, \\ u(x,0) &= u_0(x). \end{aligned}$$

The smoothness of u follows for instance directly from [66, Theorem 9.1.1] even if $u_0 \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R})$. The second-order differential operator

$$\left[\mathcal{L}_{\mathrm{OU}} u\right](x,t) := \frac{1}{2} \mathrm{tr} \left(Q D^2 u(x,t)\right) + \langle Sx, \nabla u(x,t) \rangle$$

is called the **Ornstein-Uhlenbeck operator** with **diffusion term** $\frac{1}{2} \mathrm{tr} \left(Q D^2 u(x,t)\right)$ and **drift term** $\langle Sx, \nabla u(x,t) \rangle$. This operator can be considered as the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $(T(t))_{t \geqslant 0}$. In addition, if Q is only assumed to be symmetric and positive semidefinite, $\mathcal{L}_{\mathrm{OU}}$ is called the **degenerate Ornstein-Uhlenbeck operator**. Several interpretations in physics and finance of this operator or its evolutionary counterpart - the **Kolmogorov-Fokker-Planck operator** $\mathcal{L}_{\mathrm{OU}} - \partial_t$ - are explained in the survey by Pascucci [82]. Finally, we observe that for $Q = 2I_d$ we have $\frac{1}{2} \mathrm{tr} \left(Q D^2 u(x,t)\right) = \triangle u(x,t)$, where $\triangle$ denotes the Laplacian on $\mathbb{R}^d$.

2.3 The real-valued Ornstein-Uhlenbeck operator in function spaces

Before we start to investigate nonlinear Ornstein-Uhlenbeck problems in complex systems, let us present some well-known results about the **scalar Ornstein-Uhlenbeck operator**

$$[\mathcal{L}_{\mathrm{OU}}u]\,(x) := \frac{1}{2}\mathrm{tr}\left(QD^2u(x)\right) + \langle Sx, \nabla u(x)\rangle$$

considered in real-valued function spaces, where $Q \in \mathbb{R}^{d,d}$ with $Q = Q^T$, $Q > 0$ and $0 \neq S \in \mathbb{R}^{d,d}$. Note, that the properties of the matrix S play a fundamental role in the study of this operator.

The space $L^p(\mathbb{R}^d, \mathbb{R})$. The Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d, \mathbb{R})$ related to the Lebesgue measure is indeed a semigroup for every $1 \leqslant p \leqslant \infty$. A general problem is to show that $(T(t))_{t\geqslant 0}$ is strongly continuous. On $L^p(\mathbb{R}^d, \mathbb{R})$ one can verify that $(T(t))_{t\geqslant 0}$ is a C^0-semigroup for every $1 \leqslant p < \infty$. A further problem that occurs, caused by the unbounded coefficients in the drift term, is to give an explicit representation for the domain of the infinitesimal generator A_p, which can be considered as the maximal realization of $\mathcal{L}_{\mathrm{OU}}$ in $L^p(\mathbb{R}^d, \mathbb{R})$ for $1 < p < \infty$. In this context it was proved that the maximal domain is given by

$$\mathcal{D}^p_{\max}(\mathcal{L}_{\mathrm{OU}}) = \{v \in W^{2,p}(\mathbb{R}^d, \mathbb{R}) \mid \langle Sx, \nabla v(x)\rangle \in L^p(\mathbb{R}^d, \mathbb{R})\}$$

for every $1 < p < \infty$, which can be shown directly, [73], or with the aid of the Dore-Venni theorem, [83]. In case of $p = 1$ no such representation is available, but it was proved that $\mathcal{D}^1(\mathcal{L}_{\mathrm{OU}})$ is the closure of $C_c^\infty(\mathbb{R}^d, \mathbb{R})$ with respect to the graph norm $\|\cdot\|_{\mathcal{L}_{\mathrm{OU}}} := \|\cdot\|_{L^1} + \|\mathcal{L}_{\mathrm{OU}}\cdot\|_{L^1}$, i.e. $\mathcal{D}^1(\mathcal{L}_{\mathrm{OU}}) = \overline{C_c^\infty}^{\|\cdot\|_{\mathcal{L}_{\mathrm{OU}}}}$. Moreover, it was established that the semigroup $(T(t))_{t\geqslant 0}$ is not analytic on $L^p(\mathbb{R}^d, \mathbb{R})$ for every $1 \leqslant p < \infty$, if $S \neq 0$, which can be verified by analyzing the L^p-spectrum of $\mathcal{L}_{\mathrm{OU}}$, [71]. It was shown that the spectrum of the infinitesimal generator A_p of the Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ considered on $L^p(\mathbb{R}^d, \mathbb{R})$ is given by

$$\sigma(A_p) = \left\{z \in \mathbb{C} \mid \mathrm{Re}\, z \leqslant -\frac{\mathrm{tr}(S)}{p}\right\}$$

for every $1 < p < \infty$, if $\sigma(S) \subset \mathbb{C}_+$, $\sigma(S) \subset \mathbb{C}_-$ or S symmetric and Q and S commute, [71]. Thus, since $(T(t))_{t\geqslant 0}$ is not analytic for every $1 < p < \infty$, the parabolic equation $v_t = \mathcal{L}_{\mathrm{OU}}v$ does not satisfy the standard parabolic regularity properties on $L^p(\mathbb{R}^d, \mathbb{R})$.

The space $L^p(\mathbb{R}^d, \mathbb{R}, \mu)$. Under the additional assumption that $\sigma(S) \subset \mathbb{C}_-$, which is very interesting from the point of view of diffusion processes, the Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ considered on $L^p(\mathbb{R}^d, \mathbb{R}, \mu)$ with uniquely determined invariant probability measure

$$\mu(x) = (4\pi)^{-\frac{d}{2}}\left(\det Q_\infty\right)^{-\frac{1}{2}} e^{-\frac{1}{4}\langle Q_\infty^{-1}x, x\rangle}$$

is a semigroup of positive contractions on $L^p(\mathbb{R}^d,\mathbb{R},\mu)$ for every $1\leqslant p\leqslant\infty$ and a C^0-semigroup for every $1\leqslant p<\infty$. The maximal domain is given by

$$\begin{aligned}\mathcal{D}^p_{\max,\mu}(\mathcal{L}_{\mathrm{OU}}) =&W^{2,p}(\mathbb{R}^d,\mathbb{R},\mu)\\ =&\{v\in L^p(\mathbb{R}^d,\mathbb{R},\mu)\mid D_iv, D_jD_iv\in L^p(\mathbb{R}^d,\mathbb{R},\mu),\, i,j=1,\dots,d\}\end{aligned}$$

for every $1<p<\infty$, [75], [68]. In case of $p=1$ no such representation is available. A major difference to the usual L^p-cases is that $(T(t))_{t\geqslant 0}$ is compact and analytic on $L^p(\mathbb{R}^d,\mathbb{R},\mu)$ for every $1<p<\infty$, [41]. In [72], it was shown for $1<p<\infty$ that the spectrum of the infinitesimal generator A_p of the Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ considered on $L^p(\mathbb{R}^d,\mathbb{R},\mu)$ is a discrete set, independent of p and given by

$$\sigma(A_p)=\left\{\lambda=\sum_{i=1}^{r}n_i\lambda_i \mid n_i\in\mathbb{N}_0,\, i=1,\dots,r\right\},$$

where $\lambda_1,\dots,\lambda_r$ denote the distinct eigenvalues of S. This is in strong contrast to the L^p-case. The eigenvalues are semisimple if and only if S is diagonalizable over $\mathbb{C}$. Moreover, the eigenfunctions of A_p are polynomials of degree at most $\frac{\mathrm{Re}\,\lambda}{s(S)}$. In case $p=1$ the situation changes drastically and the spectrum is given by $\sigma(A_1)=\mathbb{C}_-\cup i\mathbb{R}$.

The space $C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$. The Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ is a semigroup on $C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$. To guarantee the strong continuity of $(T(t))_{t\geqslant 0}$ one usually considers the semigroup on the closed subspace $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$ if the operator has constant or smooth bounded coefficients. But in case of the Ornstein-Uhlenbeck operator this space leads only to a weakly continuous semigroup, since the rotational term $\langle Sx,\nabla v(x)\rangle$ has smooth but unbounded coefficients, and hence, the space $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$ is too large in order to guarantee strong continuity of $(T(t))_{t\geqslant 0}$. One can show that $T(t)v_0$ tends to v_0 in $C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$ as t tends to 0^+, if and only if $v_0\in C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$ and $v_0(e^{tS}\cdot)$ tends to v_0 uniformly in $\mathbb{R}^d$ as t tends to 0^+. Hence, $(T(t))_{t\geqslant 0}$ is a C^0-semigroup on the much smaller subspace

$$C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{R}):=\left\{f\in C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})\mid f(e^{tS}\cdot)\to f(\cdot)\text{ as }t\to 0^+\text{ uniformly in }\mathbb{R}^d\right\},$$

[29], [30, see I.6]. The domain is completely characterized by

$$\mathcal{D}(\mathcal{L}_{\mathrm{OU}})=\{v\in C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{R})\cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R})\;\forall\, p\geqslant 1\mid \mathcal{L}_{\mathrm{OU}}v\in C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{R})\},$$

[29]. Therein, it was also observed that $(T(t))_{t\geqslant 0}$ in not analytic on $C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{R})$ and hence not analytic on $C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$ and $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$. Further investigations of the Ornstein-Uhlenbeck operator in spaces of Hölder-continuous functions can also be found in [29].

In Table 2.1, we summarize these facts. For a detailed treatment of the Ornstein-Uhlenbeck operator we refer the reader e.g. to [66, Chapter 9].

Table 2.1: Properties of the Ornstein-Uhlenbeck semigroup

$T(t)$	**semigroup**	C^0**-semigroup**	**analytic semigroup**
$L^p(\mathbb{R}^d,\mathbb{R})$	$1 \leqslant p \leqslant \infty$	$1 \leqslant p < \infty$	no
$L^p(\mathbb{R}^d,\mathbb{R},\mu)$	$1 \leqslant p \leqslant \infty$	$1 \leqslant p < \infty$	$1 < p < \infty$, if $\sigma(S) \subset \mathbb{C}_-$
$C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R})$	yes	no	no
$C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$	yes	no	no
$C_{\mathrm{rub}}(\mathbb{R}^d,\mathbb{R})$	yes	yes	no

3 Notations and definitions

In this chapter we introduce the basic definitions and notations that we use throughout the present thesis.

In Section 3.1 we summarize general facts about the special Euclidean group. Details about the special Euclidean group can also be found in [37]. For general theory about matrix analysis and matrix computations we refer to [53] and [42], respectively.

In Section 3.2 we introduce the exponentially weighted Sobolev spaces, which we will use for all estimates in the sequel. For the weight functions of exponential growth rate, see Definition 1.7, and we follow [114, Section 3] for the exponentially weighted Sobolev spaces.

3.1 Special Euclidean group $\mathrm{SE}(d)$

We denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{Z}$ the set of integers, by $\mathbb{Q}$ the set of rational numbers, by $\mathbb{R}$ the set of real numbers and by $\mathbb{C}$ the set of complex numbers. For an element $z \in \mathbb{C}$ we denote by $\mathrm{Re}\, z$ the real part of z, by $\mathrm{Im}\, z$ the imaginary part of z and by $\arg z$ the argument of z.

Let $d \in \mathbb{N}$ with $d \geqslant 2$ and let

$$\mathrm{SE}(d) = \mathrm{SO}(d) \ltimes \mathbb{R}^d$$

denote the **special Euclidean group** consisting of all pairs

$$g = (R, \tau) \in \mathrm{SE}(d),\ R \in \mathrm{SO}(d),\ \tau \in \mathbb{R}^d$$

with the **group operation**

$$g_2 \circ g_1 = (R_2, \tau_2) \circ (R_1, \tau_1) = (R_2 R_1, \tau_2 + R_2 \tau_1),$$

the **unit element** $(I_d, 0)$ and **inverse element** $(R, \tau)^{-1} = (R^{-1}, -R^{-1}\tau)$. Here

$$\mathrm{SO}(d) = \{R \in \mathbb{R}^{d,d} \mid R^T = R^{-1} \text{ and } \det(R) = 1\}$$

denotes the **special orthogonal group**. $\mathrm{SE}(d)$ is a Lie group of dimension $\frac{d(d+1)}{2}$, which is the sum of $\dim(\mathrm{SO}(d)) = \frac{d(d-1)}{2}$ and $\dim(\mathbb{R}^d) = d$. The associated **Lie algebra of** $\mathrm{SE}(d)$, given by

$$\mathfrak{se}(d) = \mathfrak{so}(d) \times \mathbb{R}^d,$$

is the product of $\mathbb{R}^d$ and the space

$$\mathfrak{so}(d) = \{S \in \mathbb{R}^{d,d} \mid S^T = -S\}$$

of skew-symmetric matrices, which generate rotations by the exponential mapping. The exponential mapping

$$\exp : (\mathfrak{so}(d), +) \to (\mathrm{SO}(d), \cdot),\ S \mapsto \exp(S) := \sum_{j=1}^{\infty} \frac{1}{j!} S^j$$

is onto, i.e.

$$\forall\, R \in \mathrm{SO}(d)\ \exists\, S \in \mathfrak{so}(d):\ \exp(S) = R.$$

Thus, the inverse of the matrix R satisfies

$$R^{-1} = (\exp(S))^{-1} = \exp(-S) = \exp\left(S^T\right) = (\exp(S))^T,$$

and the determinant

$$\det\left(\exp(S)\right) = \det(R) = 1.$$

Since multiplication with orthogonal matrices preserve the vector lengths we have

$$|\exp(S)x| = |Rx| = |x| \quad \forall\, x \in \mathbb{R}^d,$$

where $|\cdot| = \|\cdot\|_2$ denotes the Euclidean norm. Moreover, note that the matrix exponential maps the unit element $0 \in \mathfrak{so}(d)$ to the unit element $I_d \in \mathrm{SO}(d)$, i.e. $\exp(S + S^T) = \exp(0) = I_d$. Let $e_l \in \mathbb{R}^d$ denote the l-th unit vector in $\mathbb{R}^d$ for $l = 1, \ldots, d$. Defining the matrices $I_{ij} := e_i e_j^T \in \mathbb{R}^{d,d}$ for every $i = 1, \ldots, d-1$ and $j = i+1, \ldots, d$ with entry 1 in the i-th row and j-th column and 0 otherwise, the matrices $I_{ij} - I_{ji}$ form a basis of $\mathfrak{so}(d)$ from which we deduce the unique representation

$$\forall\, S \in \mathfrak{so}(d)\ \exists\ (S_{ij})_{\substack{i=1,\ldots,d-1 \\ j=i+1,\ldots,d}} \in \mathbb{R}:\ S = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij}\left(I_{ij} - I_{ji}\right).$$

This yields

$$\forall\, R \in \mathrm{SO}(d)\ \exists\ (S_{ij})_{\substack{i=1,\ldots,d-1 \\ j=i+1,\ldots,d}} \in \mathbb{R}:\ R = \exp\left(\sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij}\left(I_{ij} - I_{ji}\right)\right)$$

and we conclude

$$R^{-1} = \exp\left(-\sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij}\left(I_{ij} - I_{ji}\right)\right).$$

Later on, we also need the derivative of the matrix exponential in the following form

$$\frac{d}{dt} e^{X(t)} = \int_0^1 e^{(1-\alpha)X(t)} \left[\frac{d}{dt} X(t)\right] e^{\alpha X(t)} d\alpha. \tag{3.1}$$

Note that for every real skew-symmetric matrix $S \in \mathfrak{so}(d)$ the eigenvalues lie on the imaginary axis, i.e. $\sigma(S) \subset i\mathbb{R}$, and they appear in complex conjugate pairs. Thus, for odd space dimensions d we have $0 \in \sigma(S)$. Let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S, i.e. $0 \neq \sigma_j \in \mathbb{R}$ for every $j = 1, \ldots, k$ and $1 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$. Since every real skew-symmetric matrix is a normal matrix, an application of the spectral theorem yields that S is unitarily diagonalizable (over $\mathbb{C}$), i.e.

$$\exists\, U \in \mathbb{C}^{d,d} \text{ unitary (i.e. } \bar{U}^T U = U\bar{U}^T = I_d\text{)}: \Lambda_S = \bar{U}^T S U, \tag{3.2}$$

with $\Lambda_S = \operatorname{diag}\left(\lambda_1^S, \ldots, \lambda_d^S\right) \in \mathbb{C}^{d,d}$ and $\lambda_1^S, \ldots, \lambda_d^S \in \sigma(S)$. Because of $\sigma(S) \subset i\mathbb{R}$, it is in general not possible to diagonalize S by a real-valued matrix U. However, we can transform every $S \in \mathfrak{so}(d)$ into a block diagonal form by an orthogonal transformation, i.e.

$$\exists\, P \in \mathbb{R}^{d,d} \text{ orthogonal matrix}: \; S = P\Lambda^S_{\text{block}}P^T,$$

where $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S with $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, $1 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$,

$$\Lambda^S_{\text{block}} = \begin{pmatrix} \Lambda_1^S & & & & & 0 \\ & \ddots & & & & \\ & & \Lambda_k^S & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix} \in \mathbb{R}^{d,d}, \quad \Lambda_j^S = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \in \mathbb{R}^{2,2},$$

for every $j = 1, \ldots, k$. The singular value decomposition (SVD) of Λ_j^S is given by

$$\Lambda_j^S = L_j \Sigma_j R_j^T, \; \Sigma_j = |\sigma_j|\, I_2, \; L_j := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; R_j := \operatorname{sgn}(\sigma_j) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus, the SVD for Λ^S_{block} is

$$\Lambda^S_{\text{block}} = L\Sigma R^T, \; \Sigma := \begin{pmatrix} \Sigma_1 & & & & & 0 \\ & \ddots & & & & \\ & & \Sigma_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix} \in \mathbb{R}^{d,d}$$

with orthogonal matrices

$$L := \begin{pmatrix} L_1 & & & & & 0 \\ & \ddots & & & & \\ & & L_k & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}, \; R := \begin{pmatrix} R_1 & & & & & 0 \\ & \ddots & & & & \\ & & R_k & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}.$$

This yields the **singular value decomposition** for any real skew-symmetric matrix $S \in \mathfrak{so}(d)$

$$S = U\Sigma V^T,\ U := PL,\ V := PR. \tag{3.3}$$

3.2 Exponentially weighted function spaces

Sobolev Spaces. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $N \in \mathbb{N}$, $p \in \mathbb{R}$ with $1 \leqslant p \leqslant \infty$ and let $\theta \in C(\mathbb{R}^d, \mathbb{R})$ be a weight function of exponential growth rate $\eta \geqslant 0$ in the sense of Definition 1.7. We define the **exponentially weighted L^p–spaces** and their associated norms by

$$\begin{aligned}
&L^p_\theta(\mathbb{R}^d, \mathbb{K}^N) := \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{K}^N) \mid \|u\|_{L^p_\theta} < \infty\},\\
&\|u\|_{L^p_\theta} := \left(\int_{\mathbb{R}^d} \theta^p(x)\,|u(x)|^p\,dx\right)^{\frac{1}{p}},\ 1 \leqslant p < \infty,\\
&\|u\|_{L^\infty_\theta} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \theta(x)\,|u(x)|\,,\ p = \infty.
\end{aligned}$$

By definition $(L^p_\theta(\mathbb{R}^d, \mathbb{K}^N), \|\cdot\|_{L^p_\theta})$ is a Banach space.

Let $k \in \mathbb{N}_0$ and $1 \leqslant p \leqslant \infty$, then we define the **exponentially weighted Sobolev spaces of order k with exponent** p and their associated norms by

$$\begin{aligned}
&W^{k,p}_\theta(\mathbb{R}^d, \mathbb{K}^N) := \{u \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^N)\ \forall\ |\beta| \leqslant k\},\\
&\|u\|_{W^{k,p}_\theta} := \left(\sum_{|\beta|\leqslant k} \left\|D^\beta u\right\|^p_{L^p_\theta}\right)^{\frac{1}{p}},\ 1 \leqslant p < \infty,\\
&\|u\|_{W^{k,\infty}_\theta} := \max_{|\beta|\leqslant k} \left\|D^\beta u\right\|_{L^\infty_\theta},\ p = \infty.
\end{aligned}$$

Let $l \in \mathbb{N}_0$, $1 \leqslant p < \infty$, $T > 0$, $\Omega_T = \mathbb{R}^d \times]0, T[$, then we define the **space-time Sobolev space of order** $(2l, l)$ **with exponent** p and their associated norms by

$$\begin{aligned}
&W^{(2l,l),p}(\Omega_T, \mathbb{K}^N) := \{u \in L^p(\Omega_T, \mathbb{K}^N) \mid \|u\|_{W^{(2l,l),p}} < \infty\},\\
&\|u\|_{W^{(2l,l),p}(\Omega_T,\mathbb{K}^N)} := \left(\sum_{0\leqslant 2r+|\beta|\leqslant 2l} \left\|D^r_t D^\beta_x u\right\|^p_{L^p(\Omega_T,\mathbb{K}^N)}\right)^{\frac{1}{p}},
\end{aligned}$$

[62, p. 5, (1.4)]. The summation $\sum_{0\leqslant 2r+|\beta|\leqslant 2l}$ is taken over all nonnegative integers $r \in \mathbb{N}_0$ and all multiindices $\beta \in \mathbb{N}^d$ satisfying the condition $0 \leqslant 2r + |\beta| \leqslant 2l$. In the special case $l = 1$ we have

$$\begin{aligned}
\|u\|_{W^{(2,1),p}(\Omega_T,\mathbb{K}^N)} = \Bigg(&\|u\|^p_{L^p(\Omega_T,\mathbb{K}^N)} + \|D_t u\|^p_{L^p(\Omega_T,\mathbb{K}^N)} + \sum_{|\beta|=1} \left\|D^\beta_x u\right\|^p_{L^p(\Omega_T,\mathbb{K}^N)}\\
&+ \sum_{|\beta|=2} \left\|D^\beta_x u\right\|^p_{L^p(\Omega_T,\mathbb{K}^N)}\Bigg)^{\frac{1}{p}},\ 1 \leqslant p < \infty.
\end{aligned}$$

Let $\Omega = \mathbb{R}^d$ or $\Omega = \Omega_T$ and $1 \leqslant p \leqslant \infty$, then we define the **local L^p–spaces** by

$$L^p_{\mathrm{loc}}(\Omega, \mathbb{K}^N) := \{u : \Omega \to \mathbb{K}^N \text{ measurable} \mid \|u\|_{L^p(A,\mathbb{K}^N)} < \infty \; \forall\, A \subset \Omega \text{ compact}\}.$$

The **local Sobolev space** $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{K}^N)$ can be defined in the same way.

Spaces of continuous functions. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $N \in \mathbb{N}$. We define the **space of bounded continuous functions** and its associated norm by

$$\begin{aligned} C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) &:= \{u \in C(\mathbb{R}^d, \mathbb{K}^N) \mid \|u\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)} < \infty\}, \\ \|u\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)} &:= \|u\|_\infty := \sup_{x\in\mathbb{R}^d} |u(x)| . \end{aligned}$$

By definition $(C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N), \|\cdot\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)})$ is a Banach space.

Let $k \in \mathbb{N}_0$, then we define the **space of k-times continuously-differentiable functions**, that are bounded up to order k, and its associated norm by

$$\begin{aligned} C^k_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) &:= \{u \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) \mid D^\beta u \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) \; \forall\, |\beta| \leqslant k\}, \\ \|u\|_{C^k_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)} &:= \|u\|_{k,\infty} := \max_{|\beta|\leqslant k} \left\|D^\beta u\right\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)} . \end{aligned}$$

Further, we define the **space of bounded uniformly continuous functions** and the space of k-times continuously-differentiable functions, that are bounded and uniformly continuous up to order k, by

$$\begin{aligned} C_{\mathrm{ub}}(\mathbb{R}^d, \mathbb{K}^N) &:= \{u \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) \mid u \text{ is uniformly continuous on } \mathbb{R}^d\}, \\ C^k_{\mathrm{ub}}(\mathbb{R}^d, \mathbb{K}^N) &:= \{u \in C_{\mathrm{ub}}(\mathbb{R}^d, \mathbb{K}^N) \mid D^\beta u \in C_{\mathrm{ub}}(\mathbb{R}^d, \mathbb{K}^N) \; \forall\, |\beta| \leqslant k\}. \end{aligned}$$

Then, $(C_{\mathrm{ub}}(\mathbb{R}^d, \mathbb{K}^N), \|\cdot\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)})$, $(C^k_{\mathrm{ub}}(\mathbb{R}^d, \mathbb{K}^N), \|\cdot\|_{C^k_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)})$ are Banach spaces.

Let $S \in \mathbb{R}^{d,d}$ be skew-symmetric, then we define the spaces

$$\begin{aligned} C_{\mathrm{rub}}(\mathbb{R}^d, \mathbb{K}^N) &:= \{u \in C_{\mathrm{ub}}(\mathbb{R}^d, \mathbb{K}^N) \mid \lim_{t\to 0} \left\|u(e^{tS}\cdot) - u(\cdot)\right\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)} = 0\}, \\ C^k_{\mathrm{rub}}(\mathbb{R}^d, \mathbb{K}^N) &:= \{u \in C_{\mathrm{rub}}(\mathbb{R}^d, \mathbb{K}^N) \mid D^\beta u \in C_{\mathrm{rub}}(\mathbb{R}^d, \mathbb{K}^N) \; \forall\, |\beta| \leqslant k\}. \end{aligned}$$

Then, $(C_{\mathrm{rub}}(\mathbb{R}^d, \mathbb{K}^N), \|\cdot\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)})$, $(C^k_{\mathrm{rub}}(\mathbb{R}^d, \mathbb{K}^N), \|\cdot\|_{C^k_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)})$ are Banach spaces.

Let $\theta \in C(\mathbb{R}^d, \mathbb{R})$ be a weight function of exponential growth rate $\eta \geqslant 0$. We define the **exponentially weighted space of bounded continuous functions** and its associated norm by

$$\begin{aligned} C_{\mathrm{b},\theta}(\mathbb{R}^d, \mathbb{K}^N) &:= \{u \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) \mid \|u\|_{C_{\mathrm{b},\theta}(\mathbb{R}^d,\mathbb{K}^N)} < \infty\}, \\ \|u\|_{C_{\mathrm{b},\theta}(\mathbb{R}^d,\mathbb{K}^N)} &:= \|u\|_{\infty,\theta} := \|\theta u\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)} . \end{aligned}$$

and the **exponentially weighted space of k-times continuously-differentiable functions**, that are exponentially bounded up to order k, by

$$\begin{aligned} C^k_{\mathrm{b},\theta}(\mathbb{R}^d, \mathbb{K}^N) &:= \{u \in C^k_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) \mid \theta D^\beta u \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N) \; \forall\, |\beta| \leqslant k\}, \\ \|u\|_{C^k_{\mathrm{b},\theta}(\mathbb{R}^d,\mathbb{K}^N)} &:= \|u\|_{k,\infty,\theta} := \max_{|\beta|\leqslant k} \left\|\theta D^\beta u\right\|_{C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{K}^N)} . \end{aligned}$$

We define the **space of smooth functions** $C^\infty(\mathbb{R}^d, \mathbb{K}^N)$, i.e. it contains functions that are continuously differentiable of arbitrary order. Finally, we define the **space of bump functions** $C_c^\infty(\mathbb{R}^d, \mathbb{K}^N)$, i.e. of all smooth functions having a compact support. The **support** of a function $u : \mathbb{R}^d \to \mathbb{K}^N$ is defined by $\operatorname{supp}(u) := \overline{\{x \in \mathbb{R}^d \mid u(x) \neq 0\}}$.

Schwartz space. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $N \in \mathbb{N}$. A function $\phi : \mathbb{R}^d \to \mathbb{K}^N$ is said to be rapidly decreasing if it is infinitely many times differentiable, i.e. $\phi \in C^\infty(\mathbb{R}^d, \mathbb{K}^N)$ and

$$\lim_{|x|\to\infty} x^\alpha D^\beta \phi(x) = 0 \in \mathbb{K}^N \quad \forall\, \alpha, \beta \in \mathbb{N}_0^d. \tag{3.4}$$

The space

$$\mathcal{S}(\mathbb{R}^d, \mathbb{K}^N) := \left\{\phi \in C^\infty(\mathbb{R}^d, \mathbb{K}^N) \mid \phi \text{ is rapidly decreasing}\right\}$$

is called the **Schwartz space**, [34, VI.5.1 Definition]. When endowed with the family of seminorms

$$|\phi|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} \left| x^\alpha D^\beta \phi(x) \right|$$

the space $\mathcal{S}(\mathbb{R}^d, \mathbb{K}^N)$ becomes a Fréchet space containing $C_c^\infty(\mathbb{R}^d, \mathbb{K}^N)$ as a dense subspace.

4 Heat kernel for operators of Ornstein-Uhlenbeck type in complex systems

In this chapter we derive a complex-valued heat kernel matrix for the operator

$$[\mathcal{L}_\infty v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x),\ x \in \mathbb{R}^d,\ d \geqslant 2, \tag{4.1}$$

with $v : \mathbb{R}^d \to \mathbb{C}^N$, $A, B \in \mathbb{C}^{N,N}$, skew-symmetric $S \in \mathbb{R}^{d,d}$ and $N \in \mathbb{N}$.

In Section 4.1 we extend the approach from [14], [4] and [22, Chapter 13] to determine a heat kernel of $\mathcal{L}_\infty$, where A and B are assumed to be simultaneously diagonalizable matrices. Assuming (A1), (A2), (A5) and (A8$_B$) for $\mathbb{K} = \mathbb{C}$, Theorem 4.4 states that the heat kernel matrix of $\mathcal{L}_\infty$ is given by

$$H(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}} \exp\left(-Bt - (4tA)^{-1}\left|e^{tS}x - \xi\right|^2\right),\ x, \xi \in \mathbb{R}^d,\ t > 0.$$

To clarify the connection with the differential operator $\mathcal{L}_\infty$, we denote the kernel in the following chapters by H_∞. For the choice $B = 0$, we denote the kernel by H_0 and the differential operator by $\mathcal{L}_0$. In this context, H_0 is called the **complex-valued Ornstein-Uhlenbeck kernel** and $\mathcal{L}_0$ the **Ornstein-Uhlenbeck operator**. In general, having an explicit expression for the heat kernel of a differential operator, one can introduce the corresponding semigroup of the underlying differential operator in a simple way. This will be the starting point in the next chapter.

In Section 4.2 we collect some necessary information about the heat kernel H that are relevant to apply the semigroup theory to the associated semigroup. For this purpose, we first show in Lemma 4.5 that H satisfies the Chapman-Kolmogorov formula

$$\int_{\mathbb{R}^d} H(x,\tilde{\xi},t_1)H(\tilde{\xi},\xi,t_2)d\tilde{\xi} = H(x,\xi,t_1+t_2),\ x,\xi \in \mathbb{R}^d,\ t_1,t_2 > 0,$$

that is used in the next chapter to verify the semigroup properties. In Lemma 4.6 we derive exponentially weighted integral estimates involving the spectral norm of the modified kernel $K(\psi,t) = H(x, e^{tS}x - \psi, t)$, that are used later on to prove boundedness as well as strong continuity of the associated semigroup. The three integrals calculated in Lemma 4.7 are used to verify that the Schwartz space is a core for the infinitesimal generator of the semigroup.

In Section 4.3, we show in Lemma 4.8 some integral estimates, that are necessary to prove exponential decay for solutions of the resolvent equation.

4.1 Complex-valued Ornstein-Uhlenbeck kernel

We introduce the definition for a heat kernel of $\mathcal{L}_\infty$, [22, Section 1.2]:

Definition 4.1. A **heat kernel** (or a **fundamental solution**) **of** $\mathcal{L}_\infty$ given by (4.1) is a function

$$H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^* \to \mathbb{C}^{N,N},\ (x,\xi,t) \mapsto H(x,\xi,t)$$

with $\mathbb{R}_+^* :=]0,\infty[$ such that

(H1) $\quad H \in C^{2,2,1}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^*, \mathbb{C}^{N,N})$,

(H2) $\quad \frac{\partial}{\partial t} H(x,\xi,t) = \mathcal{L}_\infty H(x,\xi,t) \qquad \forall\, x,\xi \in \mathbb{R}^d,\ t>0,$

(H3) $\quad \lim_{t\downarrow 0} H(x,\xi,t) = \delta_x(\xi) I_N \qquad \forall\, x,\xi \in \mathbb{R}^d,$

where the convergence in (H3) is meant in the sense of distributions and $\delta_x(\xi) = \delta(x-\xi)$ denotes the Dirac delta function. A heat kernel H with $N>1$ is called a **heat kernel matrix** (or **matrix fundamental solution**) **of** $\mathcal{L}_\infty$.

The next theorem provides an explicit representation for the heat kernel of $\mathcal{L}_\infty$ in the scalar complex-valued case. The proof contains a formal derivation of this heat kernel, which could also be of interest for the computation of heat kernels for more general complex-valued heat operators. For the scalar real-valued case a formal derivation of this kernel can be found in [14], [4] and [22, Section 13.2].

Theorem 4.2 (Scalar case)**.** *Let the assumptions* (A2) *and* (A5) *be satisfied for* $\mathbb{K}=\mathbb{C}$ *and* $N=1$*, then the function* $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^* \to \mathbb{C}$ *defined by*

$$H(x,\xi,t) = (4\pi\alpha t)^{-\frac{d}{2}} \exp\left(-\delta t - (4\alpha t)^{-1}\left|e^{tS}x - \xi\right|^2\right) \tag{4.2}$$

is a heat kernel of $\mathcal{L}_\infty$ *given by*

$$\left[\mathcal{L}_\infty v\right](x) := \alpha \triangle v(x) + \langle Sx, \nabla v(x)\rangle - \delta v(x). \tag{4.3}$$

Remark. In the scalar case $N=1$ we write α and δ instead of A and B, respectively.

Proof. Before we verify that the heat kernel from (4.2) satisfies the properties (H1)–(H3) we discuss a formal derivation of this kernel. To compute the heat kernel (4.2) of (4.3) we generalize the approach from [14], [4] to the complex case and use the complexified ansatz

$$H(x,\xi,t) = \varphi(t)\cdot \exp\left(-\frac{1}{2}\left\langle M(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right) \tag{4.4}$$

where

$$\varphi : \mathbb{R}_+^* \to \mathbb{C},\ t \mapsto \varphi(t),$$
$$M : \mathbb{R}_+^* \to \mathbb{C}^{2d,2d},\ t \mapsto M(t)$$

have to be determined and $\langle u, v\rangle := \overline{u}^T v$ denotes the Euclidean inner product on $\mathbb{C}^{2d}$. Note at this point that it is sufficient to determine the symmetric part of the complex-valued matrix M which we denote by N, i.e.

$$\begin{aligned}
&N : \mathbb{R}_+^* \to \mathbb{C}^{2d\times 2d},\ t \mapsto N(t) := \frac{1}{2}\left(M(t) + M^T(t)\right) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix},\\
&A, B, C, D : \mathbb{R}_+^* \to \mathbb{C}^{d\times d},\ t \mapsto A(t), B(t), C(t), D(t).
\end{aligned}$$

Note that N is a symmetric but in general not a Hermitian matrix. In particular A and D are symmetric and $B^T = C$. Since $x, \xi \in \mathbb{R}^d$ we have

$$\begin{aligned}
H(x,\xi,t) &= \varphi(t)\cdot\exp\left(-\frac{1}{2}\left\langle M(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right)\\
&= \varphi(t)\cdot\exp\left(-\frac{1}{2}\left\langle \frac{1}{2}\left(M(t) + M^T(t)\right)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right)\\
&= \varphi(t)\cdot\exp\left(-\frac{1}{2}\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right).
\end{aligned}$$

Since the heat kernel must satisfy (H2) we introduce the extended matrices

$$\tilde{P} = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix},\ \tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2d,2d}$$

and obtain from the general Leibniz rule, the chain rule and the symmetry of N

$$H_t(x,\xi,t) = H(x,\xi,t)\left[\frac{\varphi_t(t)}{\varphi(t)} - \frac{1}{2}\left\langle N_t(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right],$$

$$\begin{aligned}
\frac{\partial}{\partial x_i}H(x,\xi,t) &= H(x,\xi,t)\frac{\partial}{\partial x_i}\left[-\frac{1}{2}\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right]\\
&= H(x,\xi,t)\left[-\frac{1}{2}\left(\overline{\left\langle \overline{N}^T(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, e_i\right\rangle} + \left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, e_i\right\rangle\right)\right]\\
&= -H(x,\xi,t)\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, e_i\right\rangle,
\end{aligned}$$

$$\frac{\partial^2}{\partial x_i^2}H(x,\xi,t) = H(x,\xi,t)\left[\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, e_i\right\rangle^2 - \langle N(t)e_i, e_i\rangle\right],$$

$$\begin{aligned}
\alpha\triangle H(x,\xi,t) &= \alpha H(x,\xi,t)\left[\sum_{i=1}^{d}\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, e_i\right\rangle^2 - \sum_{i=1}^{d}\langle N(t)e_i, e_i\rangle\right]\\
&= \alpha H(x,\xi,t)\left[\sum_{i=1}^{d}\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, e_i\right\rangle\overline{\left\langle e_i, N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle} - \operatorname{tr}\left(\overline{A}^T(t)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \alpha H(x,\xi,t)\left[\begin{pmatrix} x \\ \xi \end{pmatrix}^T \overline{N}^T(t)\left(\sum_{i=1}^{d} e_i e_i^T\right)\overline{N}(t)\begin{pmatrix} x \\ \xi \end{pmatrix} - \operatorname{tr}\left(\overline{A}(t)\right)\right] \\
&= \alpha H(x,\xi,t)\left[\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}\overline{N}(t)\begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle - \operatorname{tr}\left(\overline{A}(t)\right)\right] \\
&= H(x,\xi,t)\left[\left\langle \overline{\alpha} N(t)\tilde{P}N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle - \alpha\operatorname{tr}\left(\overline{A}(t)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\langle Sx, \nabla H(x,\xi,t)\rangle &= \left(\frac{\partial}{\partial x_1}H(x,\xi,t),\ldots,\frac{\partial}{\partial x_d}H(x,\xi,t)\right) Sx \\
&= -H(x,\xi,t)\left(\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, e_1\right\rangle,\ldots,\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, e_d\right\rangle\right) Sx \\
&= -H(x,\xi,t)\left\langle N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle \\
&= -H(x,\xi,t)\left\langle \tilde{S}^T N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle \\
&= -H(x,\xi,t)\frac{1}{2}\left[\left\langle \tilde{S}^T N(t)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right. \\
&\quad \left. + \left\langle \begin{pmatrix} x \\ \xi \end{pmatrix}, \overline{N}^T(t)\tilde{S}\begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right] \\
&= -H(x,\xi,t)\left\langle \frac{1}{2}\left(\tilde{S}^T N(t) + N(t)\tilde{S}\right)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle,
\end{aligned}
$$

for every $i = 1,\ldots,d$. Therefore, we end up with

$$
\begin{aligned}
0 =& H(x,\xi,t)\left[\frac{\varphi_t(t)}{\varphi(t)} + \alpha\operatorname{tr}\left(\overline{A}(t)\right) + \delta\right. \\
&\left. + \left\langle\left(-\frac{1}{2}N_t(t) - \overline{\alpha}N(t)\tilde{P}N(t) + \frac{1}{2}\tilde{S}^T N(t) + \frac{1}{2}N(t)\tilde{S}\right)\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle\right].
\end{aligned}
$$

Thus, the kernel satisfies (H2) if the following differential equations hold

$$\varphi_t(t) = -\left(\alpha\operatorname{tr}\left(\overline{A}(t)\right) + \delta\right)\varphi(t) \qquad , t > 0, \tag{4.5}$$

$$N_t(t) = -2\overline{\alpha}N(t)\tilde{P}N(t) + \tilde{S}^T N(t) + N(t)\tilde{S} \qquad , t > 0. \tag{4.6}$$

Since (4.5) depends on the solution of (4.6), we will first solve the matrix-Riccati equation (4.6), see [5, sec. 3.1]. It is obvious that the solutions of (4.5) and (4.6) are not unique but one can select appropriate initial values, see [14] and [4].

Let us first eliminate linear terms in (4.6) by the following transformation

$$
\begin{aligned}
\hat{N}(t) &= \exp\left(-t\tilde{S}^T\right)N(t)\exp\left(-t\tilde{S}\right) \\
&= \begin{pmatrix} \exp\left(-tS^T\right) & 0 \\ 0 & I_d \end{pmatrix}\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}\begin{pmatrix} \exp\left(-tS\right) & 0 \\ 0 & I_d \end{pmatrix} \\
&= \begin{pmatrix} \exp(-tS^T)A(t)\exp(-tS) & \exp(-tS^T)B(t) \\ C(t)\exp(-tS) & D(t) \end{pmatrix} \\
&=: \begin{pmatrix} \hat{A}(t) & \hat{B}(t) \\ \hat{C}(t) & \hat{D}(t) \end{pmatrix}.
\end{aligned} \tag{4.7}
$$

Differentiating $\hat{N}$ with respect to t and using (A5), $N = N^T$ and (4.6) we obtain

$$\begin{aligned}\hat{N}_t(t) =& - \tilde{S}^T \exp\left(-t\tilde{S}^T\right) N(t) \exp\left(-t\tilde{S}\right) \\ &+ \exp\left(-t\tilde{S}^T\right) N_t(t) \exp\left(-t\tilde{S}\right) \\ &- \exp\left(-t\tilde{S}^T\right) N(t)\tilde{S} \exp\left(-t\tilde{S}\right) \\ =& - 2\overline{\alpha}\hat{N}(t) \exp\left(t\tilde{S}\right) \tilde{P} \exp\left(t\tilde{S}^T\right) \hat{N}(t)\end{aligned}$$

and hence

$$\hat{N}_t(t) = -2\overline{\alpha}\hat{N}(t) \exp\left(t\tilde{S}\right) \tilde{P} \exp\left(t\tilde{S}^T\right) \hat{N}(t) \qquad , t > 0. \tag{4.8}$$

Writing this equation blockwise

$$\begin{aligned}\hat{N}_t(t) &= -2\overline{\alpha}\hat{N}(t) \exp\left(t\tilde{S}\right) \tilde{P} \exp\left(t\tilde{S}^T\right) \hat{N}(t) \\ &= -2\overline{\alpha} \begin{pmatrix} \hat{A}(t) \exp\left(t\left(S + S^T\right)\right) \hat{A}(t) & \hat{A}(t) \exp\left(t\left(S + S^T\right)\right) \hat{B}(t) \\ \hat{C}(t) \exp\left(t\left(S + S^T\right)\right) \hat{A}(t) & \hat{C}(t) \exp\left(t\left(S + S^T\right)\right) \hat{B}(t) \end{pmatrix} \\ &= \begin{pmatrix} -2\overline{\alpha}\hat{A}^2(t) & -2\overline{\alpha}\hat{A}(t)\hat{B}(t) \\ -2\overline{\alpha}\hat{C}(t)\hat{A}(t) & -2\overline{\alpha}\hat{C}(t)\hat{B}(t) \end{pmatrix} =: \begin{pmatrix} \hat{A}_t(t) & \hat{B}_t(t) \\ \hat{C}_t(t) & \hat{D}_t(t) \end{pmatrix}\end{aligned}$$

we arrive at the matrix ODE systems

$$\hat{A}_t(t) = -2\overline{\alpha}\hat{A}^2(t) \qquad , t > 0, \tag{4.9}$$

$$\hat{B}_t(t) = -2\overline{\alpha}\hat{A}(t)\hat{B}(t) \qquad , t > 0, \tag{4.10}$$

$$\hat{C}_t(t) = -2\overline{\alpha}\hat{C}(t)\hat{A}(t) \qquad , t > 0, \tag{4.11}$$

$$\hat{D}_t(t) = -2\overline{\alpha}\hat{C}(t)\hat{B}(t) \qquad , t > 0. \tag{4.12}$$

Note that $\hat{A} = \hat{A}^T$, $\hat{D} = \hat{D}^T$ and $\hat{B}^T = \hat{C}$ due to the corresponding properties of A, B, C and D. Therefore, solving (4.10) gives us automatically a solution of (4.11). Now we will successively solve the equations (4.9)–(4.12):

(4.9): Using the transformation $\tilde{A}(t) = \left(\hat{A}(t)\right)^{-1}$ we obtain

$$\begin{aligned}\tilde{A}_t(t) &= \frac{d}{dt}\left(\hat{A}(t)\right)^{-1} = -\left(\hat{A}(t)\right)^{-1} \hat{A}_t(t) \left(\hat{A}(t)\right)^{-1} \\ &= 2\overline{\alpha}\left(\hat{A}(t)\right)^{-1} \left(\hat{A}(t)\right)^2 \left(\hat{A}(t)\right)^{-1} = 2\overline{\alpha}I_d.\end{aligned}$$

Componentwise integration of both sides from 0 to t w.r.t. t yields

$$\tilde{A}(t) - A_0 = \tilde{A}(t) - \tilde{A}(0) = \int_0^t \tilde{A}_s(s)ds = \int_0^t 2\overline{\alpha}I_d ds = 2\overline{\alpha}tI_d.$$

Using the transformation once more yields the solution of (4.9)

$$\hat{A}(t) = \left(2\overline{\alpha}tI_d + A_0\right)^{-1}, \, t > 0.$$

Note that the initial data $A_0 \in \mathbb{C}^{d,d}$ must fulfill the relation $A_0 = A_0^T$ due to the symmetry of $\hat{A}(t)$ for $t > 0$.

(4.10): Obviously, the general solution of (4.10) is of the form $\hat{B} = \hat{A}B_0$ for some constant matrix $B_0 \in \mathbb{C}^{d,d}$ and hence

$$\hat{B}(t) = (2\overline{\alpha}tI_d + A_0)^{-1} B_0, \; t > 0.$$

(4.11): Thanks to the condition that $\hat{B}^T = \hat{C}$ we easily obtain the general solution of (4.11) by transposing $\hat{B}$ and using the symmetry of A_0

$$\hat{C}(t) = B_0^T (2\overline{\alpha}tI_d + A_0)^{-1}, \; t > 0.$$

(4.12): Finally, the general solution of equation (4.12) has the form $B_0^T \hat{A} B_0 + D_0$ for some constant matrix $D_0 \in \mathbb{C}^{d,d}$ with $D_0 = D_0^T$ due to the symmetry of $\hat{D}$. This can be easily seen by rewriting the system as follows

$$\hat{D}_t(t) = -2\overline{\alpha}\hat{C}(t)\hat{B}(t) = -2\overline{\alpha}B_0^T \hat{A}^2(t) B_0.$$

Hence, we obtain

$$\hat{D}(t) = B_0^T (2\overline{\alpha}tI_d + A_0)^{-1} B_0 + D_0, \; t > 0.$$

As in [14] and [4], we now choose $A_0 = 0$, $B_0 = -I_d$ and $D_0 = 0$ which will guarantee (H3). Inserting the solutions into (4.7) yields

$$\hat{N}(t) = \frac{1}{2\overline{\alpha}t} \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}$$

Transforming $\hat{N}$ to N, cf. (4.7), we obtain by (A5)

$$\begin{aligned} N(t) &= \exp\left(t\tilde{S}^T\right) \hat{N}(t) \exp\left(t\tilde{S}\right) \\ &= \begin{pmatrix} \exp(tS^T)\hat{A}(t)\exp(tS) & \exp(tS^T)\hat{B}(t) \\ \hat{C}(t)\exp(tS) & \hat{D}(t) \end{pmatrix} \\ &= \frac{1}{2\overline{\alpha}t} \begin{pmatrix} I_d & -\exp(tS^T) \\ -\exp(tS) & I_d \end{pmatrix}. \end{aligned} \tag{4.13}$$

Thus, $\operatorname{tr}\left(\overline{A}(t)\right) = \frac{d}{2\alpha t}$ and (4.5) can be written as

$$\varphi_t(t) = -\left(\alpha \operatorname{tr}\left(\overline{A}(t)\right) + \delta\right)\varphi(t) = -\left(\frac{d}{2t} + \delta\right)\varphi(t)$$

Hence, the general solution of (4.5) is given by

$$\varphi(t) = C\exp\left(-\int\left(\frac{d}{2t} + \delta\right)dt\right) = C\exp\left(-\frac{d}{2}\ln(t) - \delta t\right) = Ct^{-\frac{d}{2}}e^{-\delta t} \tag{4.14}$$

where $C \in \mathbb{C}$. Below we choose $C \in \mathbb{C}$ such that the normalization condition

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^d} H(x, \xi, t) d\xi = 1 \quad \forall\, x \in \mathbb{R}^d \tag{4.15}$$

holds. First note that from

$$\left\langle \frac{1}{2\overline{\alpha} t} \begin{pmatrix} I_d & -\exp(tS^T) \\ -\exp(tS) & I_d \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix} \right\rangle = \frac{1}{2\alpha t} \left| e^{tS} x - \xi \right|^2$$

we obtain

$$H(x, \xi, t) = C t^{-\frac{d}{2}} e^{-\delta t - \frac{1}{4\alpha t} \left| e^{tS} x - \xi \right|^2}.$$

Now, integrating over $\mathbb{R}^d$ w.r.t. ξ, we obtain from the transformation theorem and assumption (A2)

$$\begin{aligned} &\int_{\mathbb{R}^d} H(x, \xi, t) d\xi = C t^{-\frac{d}{2}} e^{-\delta t} \int_{\mathbb{R}^d} e^{-\frac{1}{4\alpha t} \left| e^{tS} x - \xi \right|^2} d\xi \\ =& C t^{-\frac{d}{2}} e^{-\delta t} \int_{\mathbb{R}^d} e^{-\frac{1}{4\alpha t} |x - \psi|^2} d\psi = C t^{-\frac{d}{2}} e^{-\delta t} \prod_{j=1}^{d} \int_{-\infty}^{\infty} e^{-\frac{1}{4\alpha t} x_j^2} dx_j \\ =& C t^{-\frac{d}{2}} e^{-\delta t} (4\pi\alpha t)^{\frac{d}{2}} = C (4\pi\alpha)^{\frac{d}{2}} e^{-\delta t} \overset{t \to 0}{\to} C (4\pi\alpha)^{\frac{d}{2}} \overset{!}{=} 1. \end{aligned}$$

Hence, we choose $C = (4\pi\alpha)^{-\frac{d}{2}}$ such that (4.15) is satisfied. Here $\alpha^{-\frac{d}{2}}$ denotes the principal root (main branch) of α^{-d}. Finally, we obtain the heat kernel (4.2) from (4.13) and (4.14). The properties (H1) and (H2) follow directly from the construction of the heat kernel. It remains to verify property (H3). For this we use the integral

$$\int_0^\infty r^{n-1} e^{-zr^2} dr = \frac{z^{-\frac{n}{2}}}{2\Gamma\left(\frac{n}{2}\right)}, \tag{4.16}$$

which holds for $n \in \mathbb{R}$ with $n > 0$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, [2]. Using the transformation theorem (with transformations for d-dimensional polar coordinates and $\Phi(\xi) = 2^{-1} t^{-\frac{1}{2}} \left(e^{tS} x - \xi\right)$) and formula (4.16) (with $n = d$ and $z = \alpha^{-1}$) we obtain, similarly to the proof of [22, Prop. 3.4.1], for every $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$

$$\begin{aligned} &\lim_{t \downarrow 0} H(x, \xi, t)(\phi) = \lim_{t \downarrow 0} \int_{\mathbb{R}^d} H(x, \xi, t) \phi(\xi) d\xi \\ =& \lim_{t \downarrow 0} \int_{\mathbb{R}^d} (4\pi\alpha t)^{-\frac{d}{2}} \exp\left(-\delta t - (4\alpha t)^{-1} \left| e^{tS} x - \xi \right|^2\right) \phi(\xi) d\xi \\ =& \lim_{t \downarrow 0} (4\pi\alpha t)^{-\frac{d}{2}} (4t)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(-\delta t - \alpha^{-1} |\psi|^2\right) \phi(e^{tS} x - 2t^{\frac{1}{2}} \psi) d\psi \\ =& (\pi\alpha)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(-\alpha^{-1} |\psi|^2\right) d\psi \phi(x) \\ =& (\pi\alpha)^{-\frac{d}{2}} 2\pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \int_0^\infty r^{d-1} e^{-\alpha^{-1} r^2} dr \phi(x) \\ =& (\pi\alpha)^{-\frac{d}{2}} 2\pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \frac{\alpha^{\frac{d}{2}}}{2\Gamma\left(\frac{d}{2}\right)} \phi(x) = \phi(x) = \delta_x(\xi)(\phi). \end{aligned}$$

Note that $\operatorname{Re} z = \operatorname{Re}\left(\alpha^{-1}\right) = \frac{\operatorname{Re} \overline{\alpha}}{|\alpha|^2} = \frac{\operatorname{Re} \alpha}{|\alpha|^2} > 0$ is true by assumption (A2). $\square$

The next statement yields a heat kernel representation of $\mathcal{L}_\infty$ for complex-valued diagonal matrices A and B. This follows from an application of Theorem 4.2.

Theorem 4.3 (Case of diagonal matrices). *Let $\Lambda_A, \Lambda_B \in \mathbb{C}^{N,N}$ be two diagonal matrices and let the assumptions* (A2) *and* (A5) *be satisfied for $\mathbb{K} = \mathbb{C}$, then the function $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^* \to \mathbb{C}^{N,N}$ defined by*

$$H(x,\xi,t) = (4\pi t\Lambda_A)^{-\frac{d}{2}} \exp\left(-\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x-\xi\right|^2\right) \tag{4.17}$$

is a heat kernel of $\mathcal{L}_\infty$ given by

$$\left[\mathcal{L}_\infty v\right](x) := \Lambda_A \triangle v(x) + \langle Sx, \nabla v(x)\rangle - \Lambda_B v(x). \tag{4.18}$$

Remark. In case of diagonal matrices we write Λ_A and Λ_B instead of A and B, respectively.

Proof. Let $v = (v_1, \ldots, v_N) \in \mathbb{C}^N$, $\Lambda_A = \operatorname{diag}\left(\lambda_1^A, \ldots, \lambda_N^A\right) \in \mathbb{C}^{N,N}$ and $\Lambda_B = \operatorname{diag}\left(\lambda_1^B, \ldots, \lambda_N^B\right) \in \mathbb{C}^{N,N}$. Since the matrices Λ_A and Λ_B are diagonal, the components of the operator $\mathcal{L}_\infty$ from (4.18) are decoupled, i.e.

$$\left[\mathcal{L}_\infty v\right]_k(x) = \lambda_k^A \triangle v_k(x) + \langle Sx, \nabla v_k(x)\rangle - \lambda_k^B v_k(x),\ k = 1, \ldots, N.$$

Using (A2) and (A5) we infer from Theorem 4.2 that

$$H_k(x,\xi,t) := (4\pi t\lambda_k^A)^{-\frac{d}{2}} \exp\left(-\lambda_k^B t - (4t\lambda_k^A)^{-1}\left|e^{tS}x-\xi\right|^2\right)$$

is a heat kernel for the k-th component of $\mathcal{L}_\infty$. Indeed, an easy computation shows that $H(x,\xi,t) := \operatorname{diag}\left(H_1(x,\xi,t), \ldots, H_N(x,\xi,t)\right)$ is a heat kernel of $\mathcal{L}_\infty$ from (4.18) that coincides with H from (4.17). The properties (H1)–(H3) for the heat kernel H of $\mathcal{L}_\infty$ follow directly from those of H_k for $k = 1, \ldots, N$. ☐

The following theorem is an extension of Theorem 4.3 and provides a heat kernel of $\mathcal{L}_\infty$ for complex-valued simultaneously diagonalizable matrices A and B. Note that assumption (A8$_B$) implies (A1) and that they coincide for $B = 0$.

Theorem 4.4 (Case of simultaneously diagonalizable matrices). *Let the assumptions* (A1), (A2), (A5) *and* (A8$_B$) *be satisfied for $\mathbb{K} = \mathbb{C}$, then the function $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^* \to \mathbb{C}^{N,N}$ defined by*

$$H(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}} \exp\left(-Bt - (4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right) \tag{4.19}$$

is a heat kernel of $\mathcal{L}_\infty$ given by

$$\left[\mathcal{L}_\infty v\right](x) := A \triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x). \tag{4.20}$$

Proof. Let us define the diagonalized operator $\tilde{\mathcal{L}}_\infty := Y^{-1}\mathcal{L}_\infty Y$ with Y from ($\mathrm{A8}_B$). Multiplying (4.20) from left by Y^{-1} and using the transformations $A = Y\Lambda_A Y^{-1}$ and $B = Y\Lambda_B Y^{-1}$, the substitution $u(x) := Y^{-1}v(x)$, the property $Y^{-1}\langle Sx, \nabla v(x)\rangle = \langle Sx, \nabla Y^{-1}v(x)\rangle$ we obtain

$$\begin{aligned}\left[\tilde{\mathcal{L}}_\infty u\right](x) &= \left[Y^{-1}\mathcal{L}_\infty Y u\right](x) = Y^{-1}\left[\mathcal{L}_\infty v\right](x)\\ &= Y^{-1}\left(A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x)\right)\\ &= \Lambda_A Y^{-1}\triangle v(x) + Y^{-1}\langle Sx, \nabla v(x)\rangle - \Lambda_B Y^{-1}v(x)\\ &= \Lambda_A\triangle u(x) + \langle Sx, \nabla u(x)\rangle - \Lambda_B u(x)\end{aligned}$$

In this way we have decoupled the operator $\mathcal{L}_\infty$ from (4.20). Since $\Lambda_A, \Lambda_B \in \mathbb{C}^{N,N}$ are diagonal matrices, $\sigma(\Lambda_A) = \sigma(A) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$ by (A1) and (A2) hold, we deduce from Theorem 4.3 that

$$\tilde{H}(x,\xi,t) = (4\pi t\Lambda_A)^{-\frac{d}{2}}\exp\left(-\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x-\xi\right|^2\right)$$

is a heat kernel of $\tilde{\mathcal{L}}_\infty$. Again, an easy computation shows that $H(x,\xi,t) := Y\tilde{H}(x,\xi,t)Y^{-1}$ is a heat kernel of $\mathcal{L}_\infty$ from (4.20) that coincides with H from (4.19):

$$\begin{aligned}H(x,\xi,t) =& Y\tilde{H}(x,\xi,t)Y^{-1}\\ =& Y(4\pi t\Lambda_A)^{-\frac{d}{2}}Y^{-1}Y\exp\left(-\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x-\xi\right|^2\right)Y^{-1}\\ =& (4\pi t)^{-\frac{d}{2}}Y\Lambda_A^{-\frac{d}{2}}Y^{-1}\exp\left(-Y\left(\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x-\xi\right|^2\right)Y^{-1}\right)\\ =& (4\pi t)^{-\frac{d}{2}}Y\Lambda_A^{-\frac{d}{2}}Y^{-1}\exp\left(-Y\Lambda_B Y^{-1}t - (4t)^{-1}Y\Lambda_A^{-1}Y^{-1}\left|e^{tS}x-\xi\right|^2\right)\\ =& (4\pi tA)^{-\frac{d}{2}}\exp\left(-Bt - (4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right).\end{aligned}$$

The properties (H1)–(H3) for the heat kernel H of $\mathcal{L}_\infty$ follow again directly from those of $\tilde{H}$. $\square$

Simultaneous diagonalization of A and B. Note that the condition ($\mathrm{A8}_B$) in Theorem 4.4 is crucial. For arbitrary matrices $A, B \in \mathbb{C}^{N,N}$ satisfying only (A1) and (A2) the heat kernel of (4.20) is in general not given by (4.19), as we will see later in Theorem 6.1 and Theorem 6.2. To extend Theorem 4.4 for arbitrary matrices $A, B \in \mathbb{C}^{N,N}$, one can try to use the Hadamard lemma or the Baker-Campbell-Hausdorff formula. But in this case one can expect at most a series representation for the heat kernel. This seems to be an open problem.

Ellipticity assumption. As mentioned above, the assumption (A2) in Theorem 4.2, 4.3 and 4.4 states that $\mathcal{L}_\infty$ is an elliptic differential operator. Using the weaker assumption $\operatorname{Re}\sigma(A) \geqslant 0$, which includes coupled parabolic-hyperbolic differential operators as for instance the Barkley model from Example 2.3, no heat kernel representation seems to be known.

Generalized heat kernel ansatz. For the computation of heat kernels with more general operators, Beals used in [14, (2)] - instead of (4.4) - the generalized ansatz

$$H(x,\xi,t) = \varphi(t)\exp\left(-Q_t(x,\xi)\right),\ t > 0,\ x,\xi \in \mathbb{R}^d$$

where Q_t is a quadratic form of $2d$ variables. This formula is motivated by the Trotter product formula and the Feynman-Kac formula, [70, section 2.8]. Such a general ansatz was also used in [22, (13.2.14)] for the construction of heat kernels for degenerate elliptic operators.

Generalized Ornstein-Uhlenbeck operator. Let the assumptions (A1), (A2), (A8$_B$), $Q \in \mathbb{R}^{d,d}$, $Q > 0$, $Q = Q^T$ and $S \in \mathbb{R}^{d,d}$ be satisfied for $\mathbb{K} = \mathbb{C}$ and consider the generalized N-dimensional complex-valued Ornstein-Uhlenbeck operator

$$\begin{aligned}\left[\mathcal{L}_{\mathrm{OU}}v\right](x) =& A\mathrm{tr}\left(QD^2v(x)\right) + \langle Sx, \nabla v(x)\rangle - Bv(x)\\ =& A\sum_{i=1}^{d}\sum_{j=1}^{d} Q_{ij}D_iD_jv(x) + \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_jD_iv(x) - Bv(x),\ x\in\mathbb{R}^d,\end{aligned}$$

Then one can show that

$$H(x,\xi,t) = (4\pi A)^{-\frac{d}{2}}\left(\det Q_t\right)^{-\frac{1}{2}}\exp\left(-Bt - (4A)^{-1}\left\langle Q_t^{-1}(e^{tS}x-\psi), (e^{tS}x-\psi)\right\rangle\right)$$

with

$$Q_t = \int_0^t \exp\left(\tau S\right) Q \exp\left(\tau S^T\right) d\tau$$

is a heat kernel of $\mathcal{L}_{\mathrm{OU}}$. This is true, even if (A5) is not satisfied.

Heat kernel via Fourier-Bessel method. The Fourier-Bessel method, [15], which is summarized in Section 1.6, provides a further possibility to determine a heat kernel for $\mathcal{L}_\infty$ on $\mathbb{R}^2$. There one computes Green's function of $\mathcal{L}_\infty$ and discovers that Green's function equals the time integral over the heat kernel. This method can easily be extended to circular disks (bounded domains) with Dirichlet, Neumann and Robin boundary conditions, see Section 1.6.

Heat kernel for the diffusion operator. Let us emphasize that all results are also valid for $S = 0$ and $B = 0$, which provides us a heat kernel for the diffusion operator $\left[\mathcal{L}_0^{\mathrm{diff}}v\right](x) = A\triangle v(x)$.

4.2 Some properties of the Ornstein-Uhlenbeck kernel

The heat kernel satisfies the following Chapman-Kolmogorov formula, which plays an important role for the generation of semigroups, [66, Proposition C.3.2]. This formula can be understood as the semigroup property on the basis of heat kernels.

Lemma 4.5 (Chapman-Kolmogorov formula)**.** *Let the assumptions* (A1), (A2), (A5) *and* (A8$_B$) *be satisfied for* $\mathbb{K} = \mathbb{C}$. *Then*

$$\int_{\mathbb{R}^d} H(x,\tilde{\xi},t_1)H(\tilde{\xi},\xi,t_2)d\tilde{\xi} = H(x,\xi,t_1+t_2) \quad \forall\, x,\xi\in\mathbb{R}^d,\ \forall\, t_1,t_2>0.$$

Remark. For the proof we need the following integral

$$\begin{aligned}(4.21)\qquad &\int_{-\infty}^{\infty} \exp\left(-c_1 (a-\psi)^2 - c_2 (\psi - b)^2\right) d\psi \\ &= \left(\frac{\pi}{c_1+c_2}\right)^{\frac{1}{2}} \exp\left(-\frac{c_1 c_2}{c_1+c_2}(a-b)^2\right)\end{aligned}$$

for $a, b, c_1, c_2 \in \mathbb{C}$ with $\operatorname{Re} c_1 > 0$, $\operatorname{Re} c_2 > 0$.

Proof. First let us prove the assertion for the diagonalized kernel

$$\tilde{H}(x,\xi,t) = (4\pi t\Lambda_A)^{-\frac{d}{2}} \exp\left(-\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x - \xi\right|^2\right).$$

Because of (A5) we have $\left|e^{tS}x\right| = |x|$ and hence

$$\begin{aligned}(4.22)\qquad &\int_{\mathbb{R}^d} \tilde{H}(x,\tilde{\xi},t_1)\tilde{H}(\tilde{\xi},\xi,t_2)d\tilde{\xi} \\ &= (4\pi t_1\Lambda_A)^{-\frac{d}{2}}(4\pi t_2\Lambda_A)^{-\frac{d}{2}} \exp\left(-\Lambda_B(t_1+t_2)\right) \\ &\cdot \int_{\mathbb{R}^d} \exp\left(-(4t_1\Lambda_A)^{-1}\left|e^{t_1S}x - \tilde{\xi}\right|^2 - (4t_2\Lambda_A)^{-1}\left|\tilde{\xi} - e^{-t_2S}\xi\right|^2\right) d\tilde{\xi}\end{aligned}$$

From (A2) we deduce that $\operatorname{Re}\lambda_j^A > 0$ and hence $\operatorname{Re}\left(\lambda_j^A\right)^{-1} = \operatorname{Re}\frac{\overline{\lambda_j^A}}{\left|\lambda_j^A\right|^2} > 0$ for every $j = 1,\ldots,N$. Using formula (4.21) componentwise with $c_1 = \left(4t_1\lambda_j^A\right)^{-1}$, $c_2 = \left(4t_2\lambda_j^A\right)^{-1}$, $\psi = \tilde{\xi}_i$, $a = \left(e^{t_1S}x\right)_i$, $b = \left(e^{-t_2S}\xi\right)_i$, $i = 1,\ldots,d$ we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} \exp\left(-(4t_1\Lambda_A)^{-1}\left(\left(e^{t_1S}x\right)_i - \tilde{\xi}_i\right)^2 - (4t_2\Lambda_A)^{-1}\left(\tilde{\xi}_i - \left(e^{-t_2S}\xi\right)_i\right)^2\right) d\tilde{\xi}_i \\ &= (4\pi t_1\Lambda_A)^{\frac{1}{2}}(4\pi t_2\Lambda_A)^{\frac{1}{2}}\left(4\pi(t_1+t_2)\Lambda_A\right)^{-\frac{1}{2}} \\ &\cdot \exp\left(-\left(4(t_1+t_2)\Lambda_A\right)^{-1}\left(\left(e^{t_1S}x\right)_i - \left(e^{-t_2S}\xi\right)_i\right)^2\right)\end{aligned}$$

Using this integral and again $\left|e^{tS}x\right| = |x|$ we are able to compute the latter integral in (4.22)

$$\begin{aligned}&\int_{\mathbb{R}^d} \exp\left(-(4t_1\Lambda_A)^{-1}\left|e^{t_1S}x - \tilde{\xi}\right|^2 - (4t_2\Lambda_A)^{-1}\left|\tilde{\xi} - e^{-t_2S}\xi\right|^2\right) d\tilde{\xi} \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(\sum_{i=1}^{d}\left[-(4t_1\Lambda_A)^{-1}\left((e^{t_1S}x)_i - \tilde{\xi}_i\right)^2\right.\right. \\ &\qquad \left.\left. - (4t_2\Lambda_A)^{-1}\left(\tilde{\xi}_i - (e^{-t_2S}\xi)_i\right)^2\right]\right) d\tilde{\xi}_1\cdots d\tilde{\xi}_d \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^{d} \exp\left(-(4t_1\Lambda_A)^{-1}\left((e^{t_1S}x)_i - \tilde{\xi}_i\right)^2\right. \\ &\qquad \left. - (4t_2\Lambda_A)^{-1}\left(\tilde{\xi}_i - (e^{-t_2S}\xi)_i\right)^2\right) d\tilde{\xi}_1\cdots d\tilde{\xi}_d\end{aligned}$$

$$
\begin{aligned}
&=\prod_{i=1}^{d}\int_{-\infty}^{\infty}\exp\left(-\left(4t_1\Lambda_A\right)^{-1}\left((e^{t_1S}x)_i-\tilde{\xi}_i\right)^2-\left(4t_2\Lambda_A\right)^{-1}\left(\tilde{\xi}_i-(e^{-t_2S}\xi)_i\right)^2\right)d\tilde{\xi}_i\\
&=\left(4\pi t_1\Lambda_A\right)^{\frac{d}{2}}\left(4\pi t_2\Lambda_A\right)^{\frac{d}{2}}\left(4\pi\left(t_1+t_2\right)\Lambda_A\right)^{-\frac{d}{2}}\\
&\qquad\cdot\exp\left(-\left(4\left(t_1+t_2\right)\Lambda_A\right)^{-1}\sum_{i=1}^{d}\left(\left(e^{t_1S}x\right)_i-\left(e^{-t_2S}\xi\right)_i\right)^2\right)\\
&=\left(4\pi t_1\Lambda_A\right)^{\frac{d}{2}}\left(4\pi t_2\Lambda_A\right)^{\frac{d}{2}}\left(4\pi\left(t_1+t_2\right)\Lambda_A\right)^{-\frac{d}{2}}\\
&\qquad\cdot\exp\left(-\left(4\left(t_1+t_2\right)\Lambda_A\right)^{-1}\left|e^{(t_1+t_2)S}x-\xi\right|^2\right)
\end{aligned}
$$

Using this in (4.22) we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\tilde{H}(x,\tilde{\xi},t_1)\tilde{H}(\tilde{\xi},\xi,t_2)d\tilde{\xi}\\
=&\left(4\pi\left(t_1+t_2\right)\Lambda_A\right)^{-\frac{d}{2}}\exp\left(-\Lambda_B(t_1+t_2)\right)\exp\left(-\left(4\left(t_1+t_2\right)\Lambda_A\right)^{-1}\left|e^{(t_1+t_2)S}x-\xi\right|^2\right)\\
=&\tilde{H}(x,\xi,t_1+t_2)\quad\forall\,x,\xi\in\mathbb{R}^d,\,\forall\,t_1,t_2>0.
\end{aligned}
$$

Let us now consider the general case: Since $H(x,\xi,t)=Y\tilde{H}(x,\xi,t)Y^{-1}$ with Y from $(\mathrm{A8}_B)$ we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}H(x,\tilde{\xi},t_1)H(\tilde{\xi},\xi,t_2)d\tilde{\xi}\\
=&Y\int_{\mathbb{R}^d}\tilde{H}(x,\tilde{\xi},t_1)\tilde{H}(\tilde{\xi},\xi,t_2)d\tilde{\xi}Y^{-1}\\
=&Y\tilde{H}(x,\xi,t_1+t_2)Y^{-1}=H(x,\xi,t_1+t_2)\quad\forall\,x,\xi\in\mathbb{R}^d,\,\forall\,t_1,t_2>0.
\end{aligned}
$$

□

The first two partial derivatives of H with respect to x are given by

$$
\begin{aligned}
D_iH(x,\xi,t)=&-\left(2tA\right)^{-1}\left\langle e^{tS}x-\xi,e^{tS}e_i\right\rangle H(x,\xi,t),\\
D_jD_iH(x,\xi,t)=&\left(-\left(2tA\right)^{-1}\delta_{ij}+\left(2tA\right)^{-2}\left\langle e^{tS}x-\xi,e^{tS}e_i\right\rangle\left\langle e^{tS}x-\xi,e^{tS}e_j\right\rangle\right)\\
&\cdot H(x,\xi,t)
\end{aligned}
$$

for $i,j=1,\ldots,d$, where we used $(\mathrm{A8}_B)$ once more. Let us define the kernels

$$
\begin{aligned}
&(4.23)\qquad \tilde{K}(\psi,t):=\left(4\pi t\Lambda_A\right)^{-\frac{d}{2}}\exp\left(-\Lambda_Bt-\left(4t\Lambda_A\right)^{-1}|\psi|^2\right),\\
&(4.24)\qquad K(\psi,t):=H(x,e^{tS}x-\psi,t)=Y\tilde{K}(\psi,t)Y^{-1}\\
&\qquad\qquad\qquad=\left(4\pi tA\right)^{-\frac{d}{2}}\exp\left(-Bt-\left(4tA\right)^{-1}|\psi|^2\right),\\
&(4.25)\qquad \tilde{K}^i(\psi,t):=-\left(2t\Lambda_A\right)^{-1}\left\langle\psi,e^{tS}e_i\right\rangle\tilde{K}(\psi,t),\\
&(4.26)\qquad K^i(\psi,t):=\left[D_iH(x,\xi,t)\right]_{\xi=e^{tS}x-\psi}=Y\tilde{K}^i(\psi,t)Y^{-1}\\
&\qquad\qquad\qquad=-\left(2tA\right)^{-1}\left\langle\psi,e^{tS}e_i\right\rangle K(\psi,t),\\
&(4.27)\qquad \tilde{K}^{ji}(\psi,t):=\left(\left(2t\Lambda_A\right)^{-2}\left\langle\psi,e^{tS}e_i\right\rangle\left\langle\psi,e^{tS}e_j\right\rangle-\left(2t\Lambda_A\right)^{-1}\delta_{ij}\right)\tilde{K}(\psi,t),
\end{aligned}
$$

$$
\begin{aligned}
K^{ji}(\psi,t) := & \left[D_j D_i H(x,\xi,t)\right]_{\xi=e^{tS}x-\psi} = Y\tilde{K}^{ji}(\psi,t)Y^{-1} \\
= & \left((2tA)^{-2}\left\langle \psi, e^{tS}e_i\right\rangle \left\langle \psi, e^{tS}e_j\right\rangle - (2tA)^{-1}\delta_{ij}\right) K(\psi,t).
\end{aligned} \tag{4.28}
$$

To prove boundedness of the associated semigroup in exponentially weighted function spaces, that we will both perform in the next chapter, we need some upper bounds of the exponentially weighted integrals over the kernels K, K^i and K^{ji}.

Lemma 4.6. *Let the assumptions* (A1), (A2), (A5) *and* (A8$_B$) *be satisfied for* $\mathbb{K}=\mathbb{C}$, $p,\eta\in\mathbb{R}$ *and let* K, K^i, K^{ji} *be given by* (4.24), (4.26), (4.28) *for every* $i,j=1,\ldots,d$, *then*

$$
\begin{aligned}
&\textbf{(1)} \int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K(\psi,t)\right|_2 d\psi \leqslant C_1(t) && , t>0,\\
&\textbf{(2)} \int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K^i(\psi,t)\right|_2 d\psi \leqslant C_2(t) && , t>0,\\
&\textbf{(3)} \int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K^{ji}(\psi,t)\right|_2 d\psi \leqslant C_3(t) && , t>0,
\end{aligned}
$$

where $|\cdot|_2$ *denotes the spectral norm and the functions are given by*

$$
\begin{aligned}
C_1(t) = & M^{\frac{d}{2}}e^{-b_0 t}\left[{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right) + 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right],\\
C_2(t) = & M^{\frac{d+1}{2}}e^{-b_0 t}\left(ta_{\min}\right)^{-\frac{1}{2}}\left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\kappa t\right)\right.\\
& \left. + 2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\kappa t\right)\right],\\
C_3(t) = & M^{\frac{d+2}{2}}e^{-b_0 t}\left(ta_{\min}\right)^{-1}\left[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+2}{2};\frac{1}{2};\kappa t\right)\right.\\
& + 2\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+3}{2};\frac{3}{2};\kappa t\right) + \frac{\delta_{ij}}{2}M^{-1}{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)\\
& \left. + \delta_{ij}M^{-1}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right],
\end{aligned}
$$

with $M:=\frac{a_{\max}^2}{a_{\min}a_0}\geqslant 1$ *and* $\kappa:=\frac{a_{\max}^2\eta^2p^2}{a_0}\geqslant 0$. *Note that* $C_{1+|\beta|}(t)\sim t^{\frac{d-1}{2}}e^{-(b_0-\kappa)t}$ *as* $t\to\infty$ *and* $C_{1+|\beta|}(t)\sim t^{-\frac{|\beta|}{2}}$ *as* $t\to 0$ *for every* $|\beta|=0,1,2$.

Remark. The function ${}_1F_1(a;b;z)$ denotes the Kummer confluent hypergeometric function $M(a,b,z)$ and satisfies the formula

$$
\begin{aligned}
\int_0^\infty s^n e^{-s^2+rs}ds = & \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right){}_1F_1\left(\frac{n+1}{2};\frac{1}{2};\frac{r^2}{4}\right)\\
& + \frac{B}{2}\Gamma\left(\frac{n}{2}+1\right){}_1F_1\left(\frac{n}{2}+1;\frac{3}{2};\frac{r^2}{4}\right)
\end{aligned} \tag{4.29}
$$

for $r \in \mathbb{R}$ with $r \geqslant 0$ and $n \in \mathbb{C}$ with $\mathrm{Re}\, n > -1$, see [2], that we need to prove Lemma 4.6. Moreover, in Lemma 4.8 we will need the connection formula

$$ {}_1F_1\left(a; b; x\right) = e^x\, {}_1F_1\left(b-a; b; -x\right) \tag{4.30}$$

for $a, b, x \in \mathbb{C}$ with $b \neq 0, -1, -2, \ldots$ (see [81] 13.2.39) and the integral

$$\int_0^\infty t^{\alpha-1} e^{-ct}\, {}_1F_1\left(a; b; -t\right) dt = c^{-\alpha}\Gamma\left(\alpha\right) {}_2F_1\left(a, \alpha; b; -\frac{1}{c}\right) \tag{4.31}$$

for $a, b, c, \alpha \in \mathbb{C}$ with $b \neq 0, -1, -2, \ldots$, $\mathrm{Re}\, \alpha > 0$ and $\mathrm{Re}\, c > 0$ (see [81] 16.5.3) where ${}_2F_1\left(a_1, a_2; b_1; z\right)$ denotes the generalized hypergeometric function. To verify the asymptotic behavior of the function ${}_1F_1\left(a, b, z\right)$ at infinity we need the limiting form

$$ {}_1F_1\left(a, b, z\right) \sim \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z, \text{ as } z \to \infty,\ \left|\arg z\right| < \frac{\pi}{2} \tag{4.32}$$

for $z \in \mathbb{C}$ and $a, b \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$ (see [81] 13.2.4 and 13.2.23). Observe that ${}_1F_1\left(a; b; 0\right) = 1$ and ${}_2F_1\left(a_1, a_2; b_1; 0\right) = 1$ which induce a simplification of the constants in Lemma 4.6 in case of $\eta = 0$.

Proof. First note that by ($\mathrm{A8}_B$), (4.24),(4.26),(4.28) it hold

$$\left|K^\beta(\psi, t)\right|_2 = \left|Y\tilde{K}^\beta(\psi, t)Y^{-1}\right|_2 = \left|\tilde{K}^\beta(\psi, t)\right|_2 = \max_{k=1,\ldots,N}\left|\tilde{K}^\beta_{kk}(\psi, t)\right|, \tag{4.33}$$

for every multi-index $\beta \in \mathbb{N}_0^d$ with $|\beta| \leqslant 2$. Note that $\tilde{K}^\beta(\psi, t) \in \mathbb{C}^{N,N}$ is diagonal. (**1**): Using (4.23) a simple computation shows that

$$\max_{k=1,\ldots,N}\left|\tilde{K}_{kk}(\psi, t)\right| \leqslant \left(4\pi t a_{\min}\right)^{-\frac{d}{2}} e^{-b_0 t - \frac{a_0}{4ta_{\max}^2}|\psi|^2} \tag{4.34}$$

for every $\psi \in \mathbb{R}^d$ and $t > 0$. From (4.33) with $|\beta| = 0$, (4.34), the transformation theorem (with transformations for d-dimensional polar coordinates and $\Phi(r) = \left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}} r$) and formula (4.29) (since (A2) is satisfied) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K(\psi, t)\right|_2 d\psi \\
\leqslant &\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left(4\pi t a_{\min}\right)^{-\frac{d}{2}} e^{-b_0 t - \frac{a_0}{4ta_{\max}^2}|\psi|^2} d\psi \\
= &\left(4\pi t a_{\min}\right)^{-\frac{d}{2}} e^{-b_0 t}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty r^{d-1} e^{-\frac{a_0}{4ta_{\max}^2}r^2 + \eta p r} dr \\
= &\left(\frac{a_{\max}^2}{a_{\min} a_0}\right)^{\frac{d}{2}} e^{-b_0 t}\frac{2}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty s^{d-1} e^{-s^2 + \left(\frac{4a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac{1}{2}} s} ds = C_1(t).
\end{aligned}$$

(**2**): Using (4.25) for every $i = 1, \dots, d$, $\psi \in \mathbb{R}^d$ and $t > 0$ we obtain

$$(4.35) \qquad \max_{k=1,\dots,N} \left|\tilde{K}^i_{kk}(\psi,t)\right| \leqslant (2ta_{\min})^{-1} \left|\left\langle \psi, e^{tS} e_i\right\rangle\right| \max_{k=1,\dots,N} \left|\tilde{K}_{kk}(\psi,t)\right|.$$

From (4.33) with $|\beta| = 1$, (4.35) with (4.34), Cauchy-Schwarz inequality with assumption (A5) ($\left|\left\langle \psi, e^{tS} e_i\right\rangle\right| \leqslant |\psi||e^{tS}e_i| = |\psi|$), the transformation theorem (with transformations from (1)) and formula (4.29) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\eta p|\psi|} \left|K^i(\psi,t)\right|_2 d\psi \\
\leqslant &\int_{\mathbb{R}^d} e^{\eta p|\psi|} (2ta_{\min})^{-1} \left|\left\langle \psi, e^{tS} e_i\right\rangle\right| \max_{k=1,\dots,N} \left|\tilde{K}_{kk}(\psi,t)\right| d\psi \\
\leqslant &\int_{\mathbb{R}^d} e^{\eta p|\psi|} (2ta_{\min})^{-1} |\psi| \, (4\pi t a_{\min})^{-\frac{d}{2}} e^{-b_0 t - \frac{a_0}{4ta^2_{\max}}|\psi|^2} d\psi \\
= &(2ta_{\min})^{-1} (4\pi t a_{\min})^{-\frac{d}{2}} e^{-b_0 t} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty r^d e^{-\frac{a_0}{4ta^2_{\max}} r^2 + \eta p r} dr \\
= &\left(\frac{a^2_{\max}}{a_{\min} a_0}\right)^{\frac{d+1}{2}} e^{-b_0 t} \frac{2}{\Gamma\left(\frac{d}{2}\right)} (ta_{\min})^{-\frac{1}{2}} \int_0^\infty s^d e^{-s^2 + \left(\frac{4a^2_{\max}\eta^2 p^2 t}{a_0}\right)^{\frac{1}{2}} s} ds = C_2(t).
\end{aligned}$$

(**3**): Using (4.27), the triangle inequality and Cauchy-Schwarz inequality with assumption (A5) (see (2)) yield for every $i, j = 1, \dots, d$, $\psi \in \mathbb{R}^d$ and $t > 0$

$$(4.36) \quad \max_{k=1,\dots,N} \left|\tilde{K}^{ji}_{kk}(\psi,t)\right| \leqslant \left((2ta_{\min})^{-2} |\psi|^2 + (2ta_{\min})^{-1} \delta_{ij}\right) \max_{k=1,\dots,N} \left|\tilde{K}_{kk}(\psi,t)\right|.$$

From (4.33) with $|\beta| = 2$, (4.36) with (4.34), the transformation theorem (with transformations from (1)) and formula (4.29) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\eta p|\psi|} \left|K^{ji}(\psi,t)\right|_2 d\psi \\
\leqslant &\int_{\mathbb{R}^d} e^{\eta p|\psi|} \left((2ta_{\min})^{-2} |\psi|^2 + (2ta_{\min})^{-1} \delta_{ij}\right) \max_{k=1,\dots,N} \left|\tilde{K}_{kk}(\psi,t)\right| d\psi \\
\leqslant &\int_{\mathbb{R}^d} e^{\eta p|\psi|} \left((2ta_{\min})^{-2} |\psi|^2 + (2ta_{\min})^{-1} \delta_{ij}\right) (4\pi t a_{\min})^{-\frac{d}{2}} e^{-b_0 t - \frac{a_0}{4ta^2_{\max}}|\psi|} d\psi \\
= &(4\pi t a_{\min})^{-\frac{d}{2}} e^{-b_0 t} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \left((2ta_{\min})^{-2} r^2 + (2ta_{\min})^{-1} \delta_{ij}\right) r^{d-1} e^{-\frac{a_0}{4ta^2_{\max}} r^2 + \eta p r} dr \\
= &\left(\frac{a^2_{\max}}{a_{\min} a_0}\right)^{\frac{d+2}{2}} e^{-b_0 t} \frac{2}{\Gamma\left(\frac{d}{2}\right)} (ta_{\min})^{-1} \int_0^\infty s^{d+1} e^{-s^2 + \left(\frac{4a^2_{\max}\eta^2 p^2 t}{a_0}\right)^{\frac{1}{2}} s} ds \\
&+ \delta_{ij} \left(\frac{a^2_{\max}}{a_{\min} a_0}\right)^{\frac{d}{2}} e^{-b_0 t} \frac{1}{\Gamma\left(\frac{d}{2}\right)} (ta_{\min})^{-1} \int_0^\infty s^{d-1} e^{-s^2 + \left(\frac{4a^2_{\max}\eta^2 p^2 t}{a_0}\right)^{\frac{1}{2}} s} ds = C_3(t).
\end{aligned}$$

$\square$

To show that the Schwartz space is a core for the infinitesimal generator of the Ornstein-Uhlenbeck semigroup we need the following lemma.

Lemma 4.7. *Let the assumptions* (A1), (A2) *and* ($A8_B$) *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and let* K *be given by* (4.24), *then for every* $i, j = 1, \dots, d$ *and* $t > 0$ *we have*

$$(\mathbf{1})\ \int_{\mathbb{R}^d} K(\psi, t)d\psi = e^{-Bt},$$

$$(\mathbf{2})\ \int_{\mathbb{R}^d} K(\psi, t)\psi_i d\psi = 0,$$

$$(\mathbf{3})\ \int_{\mathbb{R}^d} K(\psi, t)\psi_i\psi_j d\psi = \begin{cases} 2te^{-Bt}A & , i = j \\ 0 & , i \neq j \end{cases}.$$

Remark. Throughout this proof we will use d-dimensional polar coordinates: Let $x \in \mathbb{R}^d$, $\Omega :=]0, \infty[\times[0, 2\pi[\times[0, \pi]^d$ and $(r, \phi, \theta_1, \dots, \theta_{d-2}) \in \Omega$, then we define

$$\begin{aligned} x_1 =& \Phi_1(r, \phi, \theta_1, \dots, \theta_{d-2}) := r\cos\phi \prod_{k=1}^{d-2} \sin\theta_k, \\ x_2 =& \Phi_2(r, \phi, \theta_1, \dots, \theta_{d-2}) := r\sin\phi \prod_{k=1}^{d-2} \sin\theta_k, \\ x_i =& \Phi_i(r, \phi, \theta_1, \dots, \theta_{d-2}) := r\cos\theta_{i-2} \prod_{k=i-1}^{d-2} \sin\theta_k,\ 3 \leqslant i \leqslant d. \end{aligned} \tag{4.37}$$

The transformation $\Phi : \Omega \to \mathbb{R}^d$ is a C^∞-diffeomorphism, [8, X.8.8 Lemma], satisfying $\Phi(\overline{\Omega}) = \mathbb{R}^d$ and

$$\det D\Phi(r, \phi, \theta_1, \dots, \theta_{d-2}) = (-1)^d r^{d-1} \prod_{k=1}^{d-2} (\sin\theta_k)^k.$$

Proof. First note that (A2), ($A8_B$) and componentwise integration yields for every $n > -1$

$$\begin{aligned} &\int_0^\infty r^n e^{-(4tA)^{-1}r^2} dr = \int_0^\infty r^n e^{-Y(4t\Lambda_A)^{-1}Y^{-1}r^2} dr \\ =& Y \int_0^\infty r^n e^{-(4t\Lambda_A)^{-1}r^2} dr Y^{-1} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2} Y(4t\Lambda_A)^{\frac{n+1}{2}} Y^{-1} \\ =& \frac{\Gamma\left(\frac{n+1}{2}\right)}{2} (4tA)^{\frac{n+1}{2}}. \end{aligned} \tag{4.38}$$

(**1**): From (4.24), (4.38) (with $n = d - 1$), the transformation theorem (with d-dimensional polar coordinates) and ($A8_B$) we directly obtain for $t > 0$

$$\begin{aligned} \int_{\mathbb{R}^d} K(\psi, t)d\psi =& (4\pi tA)^{-\frac{d}{2}} e^{-Bt} \int_{\mathbb{R}^d} e^{-(4tA)^{-1}|\psi|^2} d\psi \\ =& (4\pi tA)^{-\frac{d}{2}} e^{-Bt} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty r^{d-1} e^{-(4tA)^{-1}r^2} dr \end{aligned}$$

$$= (4\pi tA)^{-\frac{d}{2}} e^{-Bt} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma\left(\frac{d}{2}\right)}{2} (4tA)^{\frac{d}{2}} = e^{-Bt}.$$

(**2**): Now we must use d-dimensional polar coordinates. From the transformation theorem we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-(4tA)^{-1}|\psi|^2} \psi_i d\psi \\
&= \int_\Omega e^{-(4tA)^{-1}r^2} \cdot \left\{ \begin{array}{cl} r\cos\phi \prod_{k=1}^{d-2} \sin\theta_k & , i=1 \\ r\sin\phi \prod_{k=1}^{d-2} \sin\theta_k & , i=2 \\ r\cos\theta_{i-2} \prod_{k=i-1}^{d-2} \sin\theta_k & , 3 \leqslant i \leqslant d-2 \end{array} \right\} \\
&\qquad \cdot \left|\det D\Phi(r,\phi,\theta_1,\dots,\theta_{d-2})\right| dr d\phi d\theta_1 \cdots d\theta_{d-2} \\
&= \int_\Omega e^{-(4tA)^{-1}r^2} \cdot \left\{ \begin{array}{cl} r\cos\phi \prod_{k=1}^{d-2} \sin\theta_k & , i=1 \\ r\sin\phi \prod_{k=1}^{d-2} \sin\theta_k & , i=2 \\ r\cos\theta_{i-2} \prod_{k=i-1}^{d-2} \sin\theta_k & , 3 \leqslant i \leqslant d-2 \end{array} \right\} \\
&\qquad \cdot r^{d-1} \prod_{k=1}^{d-2} |\sin\theta_k|^k dr d\phi d\theta_1 \cdots d\theta_{d-2} \\
&= \left(\int_0^\infty r^d e^{-(4tA)^{-1}r^2} dr \right) \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \\
&\qquad \left\{ \begin{array}{cl} \cos\phi \prod_{k=1}^{d-2} \sin\theta_k \prod_{k=1}^{d-2} |\sin\theta_k|^k & , i=1 \\ \sin\phi \prod_{k=1}^{d-2} \sin\theta_k \prod_{k=1}^{d-2} |\sin\theta_k|^k & , i=2 \\ \cos\theta_{i-2} \prod_{k=i-1}^{d-2} \sin\theta_k \prod_{k=1}^{d-2} |\sin\theta_k|^k & , 3 \leqslant i \leqslant d-2 \end{array} \right\} d\phi d\theta_1 \cdots d\theta_{d-2}
\end{aligned}$$

In case of $i = 1$ and $i = 2$ the ϕ-integrals vanishes and in case of $3 \leqslant i \leqslant d-2$ the θ_{i-2}-integral vanishes, since using for example

$$(\sin a)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos\left((n-2k) \left(a - \frac{\pi}{2} \right) \right),\ n \in \mathbb{N},$$

we obtain

$$\int_0^\pi \cos\theta_{i-2} |\sin\theta_{i-2}|^{i-2} d\theta_{i-2} = \int_0^\pi \cos\theta_{i-2} (\sin\theta_{i-2})^{i-2} d\theta_{i-2} = 0. \tag{4.39}$$

Hence, we have for every $i = 1, \dots, d$ and $t > 0$

$$\int_{\mathbb{R}^d} K(\psi,t)\psi_i d\psi = (4\pi tA)^{-\frac{d}{2}} e^{-Bt} \int_{\mathbb{R}^d} e^{-(4tA)^{-1}|\psi|^2} \psi_i d\psi = 0.$$

(**3**): Finally, let us use d-dimensional polar coordinates once more. Similar to (2) from the transformation theorem we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-(4tA)^{-1}|\psi|^2} \psi_i \psi_j d\psi \\
&= \left(\int_0^\infty r^{d+1} e^{-(4tA)^{-1}r^2} dr \right) \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi
\end{aligned}$$

$$\left\{\begin{array}{ll} \cos\phi \prod_{k=1}^{d-2} \sin\theta_k & , i=1 \\ \sin\phi \prod_{k=1}^{d-2} \sin\theta_k & , i=2 \\ \cos\theta_{i-2} \prod_{k=i-1}^{d-2} \sin\theta_k & , 3 \leqslant i \leqslant d-2 \end{array}\right\} \prod_{k=1}^{d-2} |\sin\theta_k|^k$$
$$\left\{\begin{array}{ll} \cos\phi \prod_{k=1}^{d-2} \sin\theta_k & , j=1 \\ \sin\phi \prod_{k=1}^{d-2} \sin\theta_k & , j=2 \\ \cos\theta_{j-2} \prod_{k=j-1}^{d-2} \sin\theta_k & , 3 \leqslant j \leqslant d-2 \end{array}\right\} d\phi d\theta_1 \cdots d\theta_{d-2}$$
$$= \begin{cases} \frac{\pi^{\frac{d}{2}}}{2}(4tA)^{\frac{d}{2}+1} & , i=j \\ 0 & , i \neq j \end{cases}.$$

Accept the last equality, we first deduce from (4.38) with $n = d+1$

$$\int_0^\infty r^{d+1} e^{-(4tA)^{-1}r^2} dr = \frac{\Gamma\left(\frac{d+2}{2}\right)}{2}(4tA)^{\frac{d}{2}+1}. \tag{4.40}$$

Moreover, for $\operatorname{Re} l > -1$, $a, b \in \mathbb{N}_0$ with $a \leqslant b$ it holds

$$\prod_{l=a}^{b} \int_0^\pi (\sin\theta)^l \, d\theta = \prod_{l=a}^{b} \pi^{\frac{1}{2}} \frac{\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{l+2}{2}\right)} = \pi^{\frac{b-a+1}{2}} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{b+2}{2}\right)}. \tag{4.41}$$

Let is first consider the cases $i = j = 1$ and $i = j = 2$. Here we must use

$$\int_0^{2\pi} (\cos\phi)^2 \, d\phi = \pi, \quad \int_0^{2\pi} (\sin\phi)^2 \, d\phi = \pi$$

and (4.41) with $a = 3$ and $b = d$

$$\begin{aligned} \prod_{k=1}^{d-2} \int_0^\pi (\sin\theta_k)^2 |\sin\theta_k|^k \, d\theta_k =& \prod_{k=1}^{d-2} \int_0^\pi (\sin\theta)^{k+2} \, d\theta = \prod_{l=3}^{d} \int_0^\pi (\sin\theta)^l \, d\theta \\ =& \pi^{\frac{d}{2}-1} \frac{\Gamma\left(\frac{4}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)} = \frac{\pi^{\frac{d}{2}-1}}{\Gamma\left(\frac{d+2}{2}\right)}. \end{aligned}$$

Now, let us consider the case $3 \leqslant i = j \leqslant d$. Here we can deduce from (4.41) (with $a = 1$ and $b = i-3$, $a = b = i-2$, $a = b = i$ as well as $a = i+1$ and $b = d$)

$$\prod_{k=1}^{i-3} \int_0^\pi |\sin\theta_k|^k \, d\theta_k = \prod_{k=1}^{i-3} \int_0^\pi (\sin\theta)^k \, d\theta = \pi^{\frac{i-3}{2}} \frac{\Gamma(1)}{\Gamma\left(\frac{i-1}{2}\right)} = \frac{\pi^{\frac{i-3}{2}}}{\Gamma\left(\frac{i-1}{2}\right)},$$

$$\begin{aligned} &\int_0^\pi (\cos\theta_{i-2})^2 |\sin\theta_{i-2}|^{i-2} \, d\theta_{i-2} = \int_0^\pi \left(1 - (\sin\theta_{i-2})^2\right)(\sin\theta_{i-2})^{i-2} \, d\theta_{i-2} \\ &\quad = \int_0^\pi (\sin\theta)^{i-2} \, d\theta - \int_0^\pi (\sin\theta)^i \, d\theta = \pi^{\frac{1}{2}} \left(\frac{\Gamma\left(\frac{i-1}{2}\right)}{\Gamma\left(\frac{i}{2}\right)} - \frac{\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)} \right), \end{aligned}$$

$$\begin{aligned} &\prod_{k=i-1}^{d-2} \int_0^\pi (\sin\theta_k)^2 |\sin\theta_k|^k \, d\theta_k = \prod_{k=i-1}^{d-2} \int_0^\pi (\sin\theta)^{k+2} \, d\theta \\ &\quad = \prod_{l=i+1}^{d} \int_0^\pi (\sin\theta)^l \, d\theta = \pi^{\frac{d-i}{2}} \frac{\Gamma\left(\frac{i+2}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}, \end{aligned}$$

$$\int_0^{2\pi} 1 d\phi = 2\pi.$$

Multiplying these four terms with (4.40) and using $\Gamma(x+1) = x\Gamma(x)$ we obtain $\frac{\pi^{\frac{d}{2}}}{2}(4tA)^{\frac{d}{2}+1}$. Next, we consider the cases $3 \leqslant i < j \leqslant d$ and $3 \leqslant j < i \leqslant d$. Let w.l.o.g. $i < j$, then the term from (4.39) vanishes. For all the other cases exactly one term vanishes, namely

$$\begin{aligned}
&\int_0^{2\pi} \sin\phi\cos\phi d\phi = 0, && \text{if } (i=1, j=2) \text{ or } (i=2, j=1),\\
&\int_0^{2\pi} \cos\phi d\phi = 0, && \text{if } (i=1, 3 \leqslant j \leqslant d) \text{ or } (3 \leqslant i \leqslant d, j=1),\\
&\int_0^{2\pi} \sin\phi d\phi = 0, && \text{if } (i=2, 3 \leqslant j \leqslant d) \text{ or } (3 \leqslant i \leqslant d, j=2).
\end{aligned}$$

□

4.3 Some useful integrals

Using the notation from Section 1.2 and assuming (A2) we define

$$\begin{aligned}
C_4(t) =& C_\theta M^{\frac{d}{2}} e^{-b_0 t}\left[{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right) + 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},\\
C_5(t) =& C_\theta M^{\frac{d+1}{2}} e^{-b_0 t}\left(t a_{\min}\right)^{-\frac{1}{2}}\left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\kappa t\right)\right.\\
&\left. + 2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},\\
C_6(t) =& C_\theta M^{\frac{d+2}{2}} e^{-b_0 t}\left(t a_{\min}\right)^{-1}\left[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+2}{2};\frac{1}{2};\kappa t\right)\right.\\
& + 2\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+3}{2};\frac{3}{2};\kappa t\right) + \frac{\delta_{ij}}{2}M^{-1}{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)\\
&\left. + \delta_{ij}M^{-1}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},
\end{aligned}$$

with $M := \frac{a_{\max}^2}{a_{\min}a_0} \geqslant 1$, $\kappa := \frac{a_{\max}^2\eta^2 p^2}{a_0} \geqslant 0$, $1 \leqslant p \leqslant \infty$ and $\eta \geqslant 0$. In case of $p = \infty$ the constants are given by $C_{4+|\beta|}(t)$ with $p = 1$ for every $|\beta| = 0, 1, 2$. Moreover, in case of $p = 1$ it holds $C_{4+|\beta|}(t) = C_\theta C_{1+|\beta|}(t)$.

In order to show that the solutions of the steady state problem for the Ornstein-Uhlenbeck operator decay exponentially, see Theorem 5.8, we need the following lemma. The upper bound for η^2 can be considered as the maximal decay rate.

Lemma 4.8. *Let the assumption* (A2) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Moreover, let* $0 < \vartheta < 1$, $\tilde{\omega} \in \mathbb{R}$, $\omega := \tilde{\omega} - b_0$, $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $0 \leqslant \eta^2 \leqslant \vartheta \frac{a_0(\operatorname{Re}\lambda - \omega)}{a_{\max}^2 p^2}$, *then we have*

$$(\mathbf{1}) \quad \int_0^\infty e^{-\operatorname{Re}\lambda t} C_4(t)dt \leqslant \frac{C_7}{\operatorname{Re}\lambda - \omega},$$

$$(\mathbf{2}) \quad \int_0^\infty e^{-\operatorname{Re}\lambda t} C_5(t)dt \leqslant \frac{C_8}{(\operatorname{Re}\lambda - \omega)^{\frac{1}{2}}},$$

with $M := \frac{a_{\max}^2}{a_{\min} a_0} \geqslant 1$ *and*

$$\begin{aligned} C_7 =& C_\theta M^{\frac{d}{2}} \left(\frac{1}{1-\vartheta}\right)^{\frac{1}{p}} \left({}_2F_1\left(-\frac{d-1}{2}, 1; \frac{1}{2}; -\frac{\vartheta}{1-\vartheta}\right)\right. \\ &\left. + \pi^{\frac{1}{2}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{1}{2}} {}_2F_1\left(-\frac{d-2}{2}, \frac{3}{2}; \frac{3}{2}; -\frac{\vartheta}{1-\vartheta}\right)\right)^{\frac{1}{p}}, \\ C_8 =& C_\theta M^{\frac{d+1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{a_{\min}^{\frac{1}{2}}} \left(\frac{1}{1-\vartheta}\right)^{\frac{1}{2p}} \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(-\frac{d}{2}, \frac{1}{2}; \frac{1}{2}; -\frac{\vartheta}{1-\vartheta}\right)\right. \\ &\left. + 2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)} \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{1}{2}} {}_2F_1\left(-\frac{d-1}{2}, 1; \frac{3}{2}; -\frac{\vartheta}{1-\vartheta}\right)\right)^{\frac{1}{p}}. \end{aligned}$$

Proof. (**1**): From $c_0 := \operatorname{Re}\lambda - \omega$, Hölder's inequality (with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leqslant p < \infty$), the transformation theorem (with transformation $\Phi(t) = \frac{a_{\max}^2 \eta^2 p^2 t}{a_0}$), formula (4.30) (with $a = \frac{d}{2}$, $b = \frac{1}{2}$, $x = s$ and $a = \frac{d+1}{2}$, $b = \frac{3}{2}$, $x = s$) and formula (4.31) (with $\alpha = 1$, $c = \frac{a_0 c_0 - a_{\max}^2 \eta^2 p^2}{a_{\max}^2 \eta^2 p^2}$, $a = -\frac{d-1}{2}$, $b = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, $c = \frac{a_0 c_0 - a_{\max}^2 \eta^2 p^2}{a_{\max}^2 \eta^2 p^2}$, $a = -\frac{d-2}{2}$, $b = \frac{3}{2}$ – note that because of (A2), $c_0 > 0$ and $\eta^2 < \frac{a_0 c_0}{a_{\max}^2 p^2}$ we have $\operatorname{Re} c > 0$) we obtain

$$\begin{aligned} &\int_0^\infty e^{-\operatorname{Re}\lambda t} C_4(t)dt \\ =& \int_0^\infty C_\theta \left(\frac{a_{\max}^2}{a_{\min} a_0}\right)^{\frac{d}{2}} e^{-c_0 t} \left[{}_1F_1\left(\frac{d}{2}; \frac{1}{2}; \frac{a_{\max}^2 \eta^2 p^2 t}{a_0}\right)\right. \\ &\left. + 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{a_{\max}^2 \eta^2 p^2 t}{a_0}\right)^{\frac{1}{2}} {}_1F_1\left(\frac{d+1}{2}; \frac{3}{2}; \frac{a_{\max}^2 \eta^2 p^2 t}{a_0}\right)\right]^{\frac{1}{p}} dt \\ \leqslant& C_\theta \left(\frac{a_{\max}^2}{a_{\min} a_0}\right)^{\frac{d}{2}} \left(\int_0^\infty e^{-c_0 t} dt\right)^{\frac{1}{q}} \left(\int_0^\infty e^{-c_0 t} {}_1F_1\left(\frac{d}{2}; \frac{1}{2}; \frac{a_{\max}^2 \eta^2 p^2 t}{a_0}\right) dt\right. \\ &\left. + 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \left(\frac{a_{\max}^2 \eta^2 p^2 t}{a_0}\right)^{\frac{1}{2}} e^{-c_0 t} {}_1F_1\left(\frac{d+1}{2}; \frac{3}{2}; \frac{a_{\max}^2 \eta^2 p^2 t}{a_0}\right) dt\right)^{\frac{1}{p}} \\ =& C_\theta M^{\frac{d}{2}} \left(\frac{1}{c_0}\right)^{\frac{1}{q}} \left(\left(\frac{a_{\max}^2 \eta^2 p^2}{a_0}\right)^{-1} \int_0^\infty e^{-\frac{a_0 c_0}{a_{\max}^2 \eta^2 p^2} s} {}_1F_1\left(\frac{d}{2}; \frac{1}{2}; s\right) ds\right. \\ &\left. + 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{a_{\max}^2 \eta^2 p^2}{a_0}\right)^{-1} \int_0^\infty s^{\frac{1}{2}} e^{-\frac{a_0 c_0}{a_{\max}^2 \eta^2 p^2} s} {}_1F_1\left(\frac{d+1}{2}; \frac{3}{2}; s\right) ds\right)^{\frac{1}{p}} \end{aligned}$$

$$
\begin{aligned}
=&C_\theta M^{\frac{d}{2}}\left(\frac{1}{c_0}\right)\left(\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0}\right)^{-1}\int_0^\infty e^{-\left(\frac{a_0c_0}{a_{\max}^2\eta^2p^2}-1\right)s}\,{}_1F_1\left(-\frac{d-1}{2};\frac{1}{2};-s\right)ds\right.\\
&\left.+2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0}\right)^{-1}\int_0^\infty s^{\frac{1}{2}}e^{-\left(\frac{a_0c_0}{a_{\max}^2\eta^2p^2}-1\right)s}\,{}_1F_1\left(-\frac{d-2}{2};\frac{3}{2};-s\right)ds\right)^{\frac{1}{p}}\\
=&C_\theta M^{\frac{d}{2}}\left(\frac{1}{c_0}\right)\left(\frac{a_0c_0}{a_0c_0-a_{\max}^2\eta^2p^2}\right)^{\frac{1}{p}}\left({}_2F_1\left(-\frac{d-1}{2},1;\frac{1}{2};-\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)\right.\\
&\left.+\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)^{\frac{1}{2}}{}_2F_1\left(-\frac{d-2}{2},\frac{3}{2};\frac{3}{2};-\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Finally, to obtain C_7 we must use that ${}_2F_1$ is strictly monotonically decreasing in $]-\infty,0]$ as well as the inequalities

$$
\frac{a_0c_0}{a_0c_0-a_{\max}^2\eta^2p^2}\leqslant\frac{1}{1-\vartheta}\quad\text{and}\quad\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\leqslant\frac{\vartheta}{1-\vartheta}. \tag{4.42}
$$

(**2**): From $c_0:=\operatorname{Re}\lambda-\omega$, Hölder's inequality (with $\frac{1}{p}+\frac{1}{q}=1$ and $1\leqslant p<\infty$), the transformation theorem (with transformation $\Phi(t)=\frac{a_{\max}^2\eta^2p^2t}{a_0}$), formula (4.30) (with $a=\frac{d+1}{2}$, $b=\frac{1}{2}$, $x=s$ and $a=\frac{d+2}{2}$, $b=\frac{3}{2}$, $x=s$) and formula (4.31) (with $\alpha=\frac{1}{2}$, $c=\frac{a_0c_0-a_{\max}^2\eta^2p^2}{a_{\max}^2\eta^2p^2}$, $a=-\frac{d}{2}$, $b=\frac{1}{2}$ and $\alpha=1$, $c=\frac{a_0c_0-a_{\max}^2\eta^2p^2}{a_{\max}^2\eta^2p^2}$, $a=-\frac{d-1}{2}$, $b=\frac{3}{2}$ – note that because of (A2), $c_0>0$ and $\eta^2<\frac{a_0c_0}{a_{\max}^2p^2}$ we have $\operatorname{Re}c>0$) we obtain

$$
\begin{aligned}
&\int_0^\infty e^{-\operatorname{Re}\lambda t}C_5(t)dt\\
=&\int_0^\infty C_\theta\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d+1}{2}}e^{-c_0t}\left(ta_{\min}\right)^{-\frac{1}{2}}\left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)\right.\\
&\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac{1}{2}}{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)\right]^{\frac{1}{p}}dt\\
\leqslant&C_\theta\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d+1}{2}}a_{\min}^{-\frac{1}{2}}\left(\int_0^\infty t^{-\frac{1}{2}}e^{-c_0t}dt\right)^{\frac{1}{q}}\\
&\cdot\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty t^{-\frac{1}{2}}e^{-c_0t}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)dt\right.\\
&\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0}\right)^{\frac{1}{2}}\int_0^\infty e^{-c_0t}{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)dt\right)^{\frac{1}{p}}\\
=&C_\theta M^{\frac{d+1}{2}}\left(\left(\frac{1}{c_0}\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}a_{\min}^{-\frac{1}{2}}\\
&\cdot\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0}\right)^{-\frac{1}{2}}\int_0^\infty s^{-\frac{1}{2}}e^{-\frac{a_0c_0}{a_{\max}^2\eta^2p^2}s}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};s\right)ds\right.\\
&\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0}\right)^{-\frac{1}{2}}\int_0^\infty e^{-\frac{a_0c_0}{a_{\max}^2\eta^2p^2}s}{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};s\right)ds\right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
=&C_\theta M^{\frac{d+1}{2}} \left(\frac{1}{c_0}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{a_{\min}^{\frac{1}{2}}} \\
&\cdot \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)} \left(\frac{a_{\max}^2 \eta^2 p^2}{a_0 c_0}\right)^{-\frac{1}{2}} \int_0^\infty s^{-\frac{1}{2}} e^{-\left(\frac{a_0 c_0}{a_{\max}^2 \eta^2 p^2}-1\right)s} {}_1F_1\left(-\frac{d}{2}; \frac{1}{2}; -s\right) ds \right. \\
&\left. + 2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)} \left(\frac{a_{\max}^2 \eta^2 p^2}{a_0 c_0}\right)^{-\frac{1}{2}} \int_0^\infty e^{-\left(\frac{a_0 c_0}{a_{\max}^2 \eta^2 p^2}-1\right)s} {}_1F_1\left(-\frac{d-1}{2}; \frac{3}{2}; -s\right) ds \right)^{\frac{1}{p}} \\
=&C_\theta M^{\frac{d+1}{2}} \left(\frac{1}{c_0}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{a_{\min}^{\frac{1}{2}}} \left(\frac{a_0 c_0}{a_0 c_0 - a_{\max}^2 \eta^2 p^2}\right)^{\frac{1}{2p}} \\
&\cdot \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(-\frac{d}{2}, \frac{1}{2}; \frac{1}{2}; -\frac{a_{\max}^2 \eta^2 p^2}{a_0 c_0 - a_{\max}^2 \eta^2 p^2}\right) \right. \\
&\left. + 2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)} \left(\frac{a_{\max}^2 \eta^2 p^2}{a_0 c_0 - a_{\max}^2 \eta^2 p^2}\right)^{\frac{1}{2}} {}_2F_1\left(-\frac{d-1}{2}, 1; \frac{3}{2}; -\frac{a_{\max}^2 \eta^2 p^2}{a_0 c_0 - a_{\max}^2 \eta^2 p^2}\right) \right)^{\frac{1}{p}}.
\end{aligned}
$$

Finally, to obtain C_8 we use again that ${}_2F_1$ is strictly monotonically decreasing in $]-\infty, 0]$ and the inequalities (4.42). $\square$

5 The complex Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^d, \mathbb{C}^N)$

In this chapter we apply semigroup theory to the Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle, \; x \in \mathbb{R}^d, \; d \geqslant 2,$$

in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p \leqslant \infty$, where $v : \mathbb{R}^d \to \mathbb{C}^N$, $A \in \mathbb{C}^{N,N}$, $S \in \mathbb{R}^{d,d}$ skew-symmetric and $N \in \mathbb{N}$.

In Section 5.1 we introduce the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ by the heat kernel of $\mathcal{L}_0$ as

$$[T_0(t)v](x) := \int_{\mathbb{R}^d} H_0(x,\xi,t)v(\xi)d\xi, \; t > 0, \; x \in \mathbb{R}^d.$$

Assuming (A1), (A2) and (A5) for $\mathbb{K} = \mathbb{C}$, we show in Theorem 5.1–5.3 that $(T_0(t))_{t\geqslant 0}$ is a strongly continuous semigroup in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for every $1 \leqslant p < \infty$. Hence, we can define $A_p : \mathcal{D}(A_p) \subseteq L^p(\mathbb{R}^d, \mathbb{C}^N) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$, the infinitesimal generator of $(T_0(t))_{t\geqslant 0}$ for every $1 \leqslant p < \infty$. Assuming (A1), (A2) and (A5) for $\mathbb{K} = \mathbb{C}$, we prove in Corollary 5.7 that the resolvent equation for A_p, which is

$$(\lambda I - A_p)\, v = g,$$

admits a unique solution $v_\star \in \mathcal{D}(A_p)$ for every $g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. This follows from some applications of abstract semigroup theory, [34, II.1].

In Section 5.2 we derive a-priori estimates for the resolvent equation for A_p in exponentially weighted L^p-spaces. Assuming (A1), (A2) and (A5) for $\mathbb{K} = \mathbb{C}$, we prove in Theorem 5.8 that the solution $v_\star$ belongs to $W^{1,p}_\theta(\mathbb{R}^d, \mathbb{C}^N)$ for every $g \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. In particular, we conclude that $\mathcal{D}(A_p) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ for every $1 \leqslant p < \infty$.

In Section 5.3–5.6 we analyze the relation between the abstract Ornstein-Uhlenbeck operator A_p and the formal Ornstein-Uhlenbeck operator $\mathcal{L}_0$ and derive a precise characterization of the maximal domain $\mathcal{D}(A_p)$, which means that we solve the identification problem for the Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 < p < \infty$. This approach comes originally from [71] and [73], where such a result was proved for the scalar real-valued Ornstein-Uhlenbeck operator. The procedure is structured as follows:

In Section 5.3, assuming (A1), (A2) and (A5) for $\mathbb{K} = \mathbb{C}$, we prove in Theorem 5.10 that the Schwartz space $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$ is a core for the infinitesimal generator $(A_p, \mathcal{D}(A_p))$ for every $1 \leqslant p < \infty$. The main idea of the proof comes from [71, Proposition 2.2 and 3.2] and partially from [34, II.2.13].

In Section 5.4 we consider the formal complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_0 : \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \subseteq L^p(\mathbb{R}^d,\mathbb{C}^N) \to L^p(\mathbb{R}^d,\mathbb{C}^N)$ on its domain

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) := \left\{ v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N) \cap L^p(\mathbb{R}^d,\mathbb{C}^N) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d,\mathbb{C}^N) \right\}, \quad 1 < p < \infty.$$

Assuming (A3), (A4) and (A5) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$, we prove in Theorem 5.13 that the resolvent equation for $\mathcal{L}_0$, which is given by

$$(\lambda I - \mathcal{L}_0)\, v = g,$$

admits a unique solution $v_\star \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ for every $g \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. The main idea of the proof comes from [73, Theorem 2.2 and Remark 2.3] for the scalar real-valued case. But we refer also to [15, Theorem 3.1] for the special case $d = 2$ with $A \in \mathbb{R}^{N,N}$. In contrast to [73] and [15], our proof requires an additional L^p-dissipativity condition stating that for fixed $1 < p < \infty$ there exists some positive constant $\gamma_A > 0$ such that

$$(5.1) \qquad |z|^2 \operatorname{Re}\langle w, Aw\rangle + (p-2)\operatorname{Re}\langle w, z\rangle \operatorname{Re}\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2 \;\forall\, z, w \in \mathbb{C}^N.$$

This condition seems to be new in the literature and guarantees that the operator $\mathcal{L}_0$ is a dissipative operator in $L^p(\mathbb{R}^d,\mathbb{C}^N)$.

In Section 5.5 we derive a complete characterization of the L^p-dissipativity condition (5.1) in terms of the antieigenvalues of the diffusion matrix A. Assuming $A \in \mathbb{K}^{N,N}$ for $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, we prove in Theorem 5.18 that (5.1) is satisfied if and only if

$$(5.2) \qquad \mu_1(A) := \inf_{\substack{w \in \mathbb{K}^N \\ w \neq 0 \\ Aw \neq 0}} \frac{\operatorname{Re}\langle w, Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p}, \quad 1 < p < \infty,$$

for $N \geqslant 2$ (if $\mathbb{K} = \mathbb{R}$) and $N \geqslant 1$ (if $\mathbb{K} = \mathbb{C}$), where $\mu_1(A)$ denotes the first antieigenvalue of A, see [47]. The antieigenvalue condition (5.2) for the matrix A is nothing but a p-dependent lower bound for the first antieigenvalue of A. In case of $N = 1$ and $\mathbb{K} = \mathbb{R}$, condition (5.1) is equivalent to $A > 0$. The main idea of the proof is to apply the Lagrange multiplier method twice, first in the z component and afterwards in the w component. Concluding, for normal matrices A and for Hermitian positive-definite matrices A we specify well known explicit expressions for $\mu_1(A)$ in terms of the eigenvalues of A. These representations come originally from [49, Theorem 5.1] for normal matrices A and from [53, 7.4.P4] for Hermitian positive-definite matrices A.

In Section 5.6, assuming (A1)–(A5) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$, we prove that the abstract and the formal Ornstein-Uhlenbeck operator A_p and $\mathcal{L}_0$, respectively, coincide on $\mathcal{D}(A_p)$ and that the maximal domain $\mathcal{D}(A_p)$ equals $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. The main idea for the first part of the proof comes from [71, Proposition 2.2 and 3.2], where such a result was proved for the scalar real-valued case.

In Section 5.7–5.8 we derive a second characterization of the maximal domain $\mathcal{D}(A_p)$, which even contains second order derivatives. We stress that this additional characterization is not necessary to prove the main result from Theorem 1.8, but later in Section 10.1 we will apply the result for equivariant evolution equations. The approach is based on the results from Section 5.3–5.6 and comes originally from [73]. The procedure is structured as follows:

In Section 5.7 we investigate abstract Cauchy problems

$$\begin{aligned} v_t(t) =& A_p v(t) + f(t), \; t \in]0,T], \\ v(0) =& v_0, \qquad\qquad\quad t = 0, \end{aligned}$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$, $1 \leqslant p < \infty$, for the infinitesimal generator A_p. Recall that A_p coincides with $\mathcal{L}_0$ if we require the assumptions (A1)–(A5) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$. Assuming (A1), (A2) and (A5) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$ and considering $v_0 \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ and a time-independent inhomogeneity $f \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$, we prove in Theorem 5.22 and Theorem 5.23 spatial L^p_θ-regularity results for the mild solution of the homogeneous and inhomogeneous Cauchy problem. Their proofs follow directly from Theorem 5.1. Assuming (A1)–(A5) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$, we prove in Theorem 5.24 a time-space L^p-regularity result for the mild solution of the inhomogeneous problem. The main idea of the proof comes from [73, Theorem 3.4] for the scalar real-valued case and is based on an application of [67, Proposition 6.1.3] and [62, IV. Theorem 9.1].

In Section 5.8 we derive a further and even stronger characterization of the maximal domain $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. Recall the decomposition $\mathcal{L}_0 v = \mathcal{L}_0^{\mathrm{diff}} v + \mathcal{L}_0^{\mathrm{drift}} v$ of the Ornstein-Uhlenbeck operator into diffusion and drift term

$$\left[\mathcal{L}_0^{\mathrm{diff}} v\right](x) := A\triangle v(x), \qquad \left[\mathcal{L}_0^{\mathrm{drift}} v\right](x) := \langle Sx, \nabla v(x)\rangle$$

with domains

$$\begin{aligned} \mathcal{D}^p_{\max}(\mathcal{L}_0^{\mathrm{diff}}) :=& W^{2,p}(\mathbb{R}^d,\mathbb{C}^N), \\ \mathcal{D}^p_{\max}(\mathcal{L}_0^{\mathrm{drift}}) :=& \left\{v \in L^p(\mathbb{R}^d,\mathbb{C}^N) \mid \langle S\cdot, \nabla v\rangle \in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\} \end{aligned}$$

for $1 < p < \infty$, where $\langle S\cdot, \nabla v\rangle$ is meant in the sense of distributions. Assuming (A1)–(A5) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$, we prove in Theorem 5.25 that the maximal domain $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ of the Ornstein-Uhlenbeck operator coincides with the intersection of the domains of its diffusion and drift part

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \mathcal{D}^p_{\max}(\mathcal{L}_0) := \mathcal{D}^p_{\max}(\mathcal{L}_0^{\mathrm{diff}}) \cap \mathcal{D}^p_{\max}(\mathcal{L}_0^{\mathrm{drift}})$$

i.e. $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ coincides with

$$\mathcal{D}^p_{\max}(\mathcal{L}_0) := \left\{v \in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N) \mid \langle S\cdot, \nabla v\rangle \in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.$$

The main idea of this result comes from [73, Theorem 1] for the scalar real-valued case. Assuming (A1)–(A5) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$ and considering the norms

$$\begin{aligned} \|v\|_{A_p} :=& \|A_p v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} = \|\mathcal{L}_0 v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}, \\ \|v\|_{\mathcal{L}_0} :=& \|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} + \|\langle S\cdot, \nabla v\rangle\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}, \end{aligned}$$

for $v \in \mathcal{D}^p_{\max}(\mathcal{L}_0)$, we prove in Corollary 5.26 that these norms are equivalent, i.e. there exist $C_1, C_2 \geqslant 1$ such that

$$C_1 \|v\|_{\mathcal{L}_0} \leqslant \|v\|_{A_p} \leqslant C_2 \|v\|_{\mathcal{L}_0} \quad \forall\, v \in \mathcal{D}^p_{\max}(\mathcal{L}_0).$$

Requiring the same assumptions, we prove in Corollary 5.27 that the unique solution $v_\star \in \mathcal{D}^p_{\max}(\mathcal{L}_0)$ of the resolvent equation for $\mathcal{L}_0$ with $g \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ satisfies

$$\left\|v_\star\right\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C\left\|g\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}, \quad \left\|\langle S\cdot,\nabla v_\star\rangle\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C\left\|g\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.$$

For the sake of completeness note that, assuming (A1)–(A5) for $1<p<\infty$ and $\mathbb{K}=\mathbb{C}$, every $\lambda\in\mathbb{C}$ of the form

$$\lambda = -\lambda(\omega) - i\sum_{l=1}^{k} n_l\sigma_l, \quad n_l\in\mathbb{Z}, \quad \omega\in\mathbb{R}, \quad \lambda(\omega)\in\sigma(\omega^2 A),$$

belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L}_0)$ of $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Hence, $\mathcal{L}_0$ is not sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $(T_0(t))_{t\geqslant 0}$ is not analytic on $L^p(\mathbb{R}^d,\mathbb{C}^N)$, whenever $S\neq 0$. These results will be proved later in Section 7.4 for more general perturbed Ornstein-Uhlenbeck operators. Their proofs combine and extend the results from [71] and [15].

5.1 Application of semigroup theory

Let us consider the **Ornstein-Uhlenbeck kernel** of $\mathcal{L}_0$ from Theorem 4.4 (with $B=0$)

$$H_0(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right)$$

and the family of mappings $(T_0(t))_{t\geqslant 0}$ given by

$$[T_0(t)v](x) := \begin{cases} \int_{\mathbb{R}^d} H_0(x,\xi,t)v(\xi)d\xi & ,t>0 \\ v(x) & ,t=0\end{cases} \quad ,x\in\mathbb{R}^d \tag{5.3}$$

on the (complex-valued) Banach space $\left(L^p(\mathbb{R}^d,\mathbb{C}^N),\|\cdot\|_{L^p}\right)$, $1\leqslant p\leqslant\infty$. In the scalar real-valued case, formula (5.3) is due to Kolmogorov, [58]. The next three theorems show that the family of mappings $(T_0(t))_{t\geqslant 0}$ defined in (5.3) generates a strongly continuous semigroup on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$. In order to show exponential decay of the solutions of the resolvent equation via a-priori estimates, we have to prove the boundedness of T_0 and its derivatives up to order 2 in exponentially weighted function spaces.

Theorem 5.1 (Boundedness on $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1\leqslant p\leqslant\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then for every radial weight function* $\theta\in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta\geqslant 0$ *and for every* $v\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$

$$\left\|T_0(t)v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C_4(t)\left\|v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad ,t\geqslant 0, \tag{5.4}$$

$$\left\|D_iT_0(t)v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C_5(t)\left\|v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad ,t>0,\ i=1,\ldots,d, \tag{5.5}$$

$$\left\|D_jD_iT_0(t)v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C_6(t)\left\|v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad ,t>0,\ i,j=1,\ldots,d,$$

where the constants $C_{4+|\beta|}(t) = C_{4+|\beta|}(t;b_0=0)$ are from Section 4.3 for every $|\beta|=0,1,2$, i.e.

$$\begin{aligned}
C_4(t;b_0=0) =& C_\theta M^{\frac{d}{2}}\left[{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)+2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},\\
C_5(t;b_0=0) =& C_\theta M^{\frac{d+1}{2}}\left(ta_{\min}\right)^{-\frac{1}{2}}\left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\kappa t\right)\right.\\
&\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},\\
C_6(t;b_0=0) =& C_\theta M^{\frac{d+2}{2}}\left(ta_{\min}\right)^{-1}\left[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+2}{2};\frac{1}{2};\kappa t\right)\right.\\
&+2\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+3}{2};\frac{3}{2};\kappa t\right)+\frac{\delta_{ij}}{2}M^{-1}{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)\\
&\left.+\delta_{ij}M^{-1}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}}.
\end{aligned}$$

In case $p=\infty$ they are given by $C_{4+|\beta|}(t;b_0=0)$ with $p=1$, where $M := \frac{a_{\max}^2}{a_{\min}a_0} \geqslant 1$ and $\kappa := \frac{a_{\max}^2\eta^2p^2}{a_0} \geqslant 0$. Note that $C_{4+|\beta|}(t) \sim t^{\frac{-p|\beta|+d+|\beta|-1}{2p}}e^{\frac{\kappa}{p}t}$ as $t\to\infty$ and $C_{4+|\beta|}(t)\sim t^{-\frac{|\beta|}{2}}$ as $t\to 0$ for every $|\beta|=0,1,2$.

Proof. Let $v\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$. In the following $\beta\in\mathbb{N}_0^d$ denotes a d-dimensional multi-index with $|\beta|\leqslant 2$ and we will use the notation

$$D^\beta v = \begin{cases} v & ,|\beta|=0\\ D_iv & ,|\beta|=1\\ D_jD_iv & ,|\beta|=2\end{cases},\; D^\beta H_0 = \begin{cases} H_0 & ,|\beta|=0\\ D_iH_0 & ,|\beta|=1\\ D_jD_iH_0 & ,|\beta|=2\end{cases},\; K_\beta = \begin{cases} K & ,|\beta|=0\\ K^i & ,|\beta|=1\\ K^{ji} & ,|\beta|=2\end{cases}$$

where $i,j=1,\ldots,d$. Note that $H_0(x,\xi,t)=H(x,\xi,t)$ since we have $B=0$. Moreover, in this proof K, K^i and K^{ji} are given by (4.24), (4.26) and (4.28) with $B=0$. To show (5.4), (5.5) and (5.6) for $1\leqslant p<\infty$ we use (5.3), the transformation theorem (with transformation $\Phi(\xi)=e^{tS}x-\xi$ in ξ, $\Phi(x)=e^{tS}x-\psi$ in x), (4.24), (4.26), (4.28), the triangle inequality, Hölder's inequality (with q such that $\frac{1}{p}+\frac{1}{q}=1$), Fubini's theorem, (W1)–(W3), Lemma 4.6 (1),(2),(3)

$$\begin{aligned}
&\left\|D^\beta T_0(t)v\right\|_{L^p_\theta} = \left(\int_{\mathbb{R}^d}\theta^p(x)\left|D^\beta\left[T_0(t)v\right](x)\right|^p dx\right)^{\frac{1}{p}}\\
=&\left(\int_{\mathbb{R}^d}\theta^p(x)\left|\int_{\mathbb{R}^d}\left[D^\beta H_0(x,\xi,t)\right]v(\xi)d\xi\right|^p dx\right)^{\frac{1}{p}}\\
=&\left(\int_{\mathbb{R}^d}\theta^p(x)\left|\int_{\mathbb{R}^d}K^\beta(\psi,t)v(e^{tS}x-\psi)d\psi\right|^p dx\right)^{\frac{1}{p}}\\
\leqslant&\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\theta(x)\left|K^\beta(\psi,t)\right|_2\left|v(e^{tS}x-\psi)\right|d\psi\right)^p dx\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2 d\psi\right)^{\frac{p}{q}}\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2\left(\theta(x)\left|v(e^{tS}x-\psi)\right|\right)^p d\psi dx\right)^{\frac{1}{p}}\\
&=\left(\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2 d\psi\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2\int_{\mathbb{R}^d}\left(\theta(x)\left|v(e^{tS}x-\psi)\right|\right)^p dx d\psi\right)^{\frac{1}{p}}\\
&=\left(\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2 d\psi\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2\int_{\mathbb{R}^d}\left(\theta(e^{-tS}(y+\psi))\left|v(y)\right|\right)^p dy d\psi\right)^{\frac{1}{p}}\\
&\leqslant\left(\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2 d\psi\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^d}C_\theta^p e^{\eta p|\psi|}\left|K^{\beta}(\psi,t)\right|_2\int_{\mathbb{R}^d}\left(\theta(y)\left|v(y)\right|\right)^p dy d\psi\right)^{\frac{1}{p}}\\
&\leqslant C_\theta\left(\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2 d\psi\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}e^{\eta p|\psi|}\left|K^{\beta}(\psi,t)\right|_2 d\psi\right)^{\frac{1}{p}}\|v\|_{L^p_\theta}\\
&\leqslant C_{4+|\beta|}(t;b_0=0)\|v\|_{L^p_\theta}
\end{aligned}
$$

for $t\geqslant 0$, if $|\beta|=0$ and for $t>0$, if $|\beta|=1$ or $|\beta|=2$. Similarly, to show (5.4), (5.5) and (5.6) for $p=\infty$ we use (5.3), the transformation theorem (with transformation $\Phi(\xi)=e^{tS}x-\xi$ in ξ and $\Phi(x)=e^{tS}x-\psi$ in x), (4.24), (4.26), (4.28), the triangle inequality, (W1)–(W3), Lemma 4.6 (1),(2),(3) and obtain

$$
\begin{aligned}
\left\|D^{\beta}T_0(t)v\right\|_{L^\infty_\theta}&=\operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\theta(x)\left|D^{\beta}\left[T_0(t)v\right](x)\right|\\
&=\operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\theta(x)\left|\int_{\mathbb{R}^d}\left[D^{\beta}H_0(x,\xi,t)\right]v(\xi)d\xi\right|\\
&=\operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\theta(x)\left|\int_{\mathbb{R}^d}K^{\beta}(\psi,t)v(e^{tS}x-\psi)d\psi\right|\\
&\leqslant\operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\theta(x)\left|K^{\beta}(\psi,t)\right|_2\left|v(e^{tS}x-\psi)\right|d\psi\\
&\leqslant\int_{\mathbb{R}^d}\operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\theta(x)\left|K^{\beta}(\psi,t)\right|_2\left|v(e^{tS}x-\psi)\right|d\psi\\
&=\int_{\mathbb{R}^d}\left|K^{\beta}(\psi,t)\right|_2\operatorname*{ess\,sup}_{y\in\mathbb{R}^d}\theta(e^{-tS}(y+\psi))\left|v(y)\right|d\psi\\
&\leqslant C_\theta\left(\int_{\mathbb{R}^d}e^{\eta|\psi|}\left|K^{\beta}(\psi,t)\right|_2 d\psi\right)\|v\|_{L^\infty_\theta}\leqslant C_{4+|\beta|}(t;b_0=0)\|v\|_{L^\infty_\theta}.
\end{aligned}
$$

□

Theorem 5.2 (Semigroup on $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1\leqslant p\leqslant\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then the operators* $(T_0(t))_{t\geqslant 0}$ *given by* (5.3) *generate a semigroup on* $L^p(\mathbb{R}^d,\mathbb{C}^N)$, *i.e.* $T_0(t):L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ *is linear and bounded for every* $t\geqslant 0$ *and satisfies the* ***semigroup properties***

$$T_0(0)=I, \tag{5.7}$$

$$T_0(t)T_0(s)=T_0(t+s),\ \forall\, s,t\geqslant 0. \tag{5.8}$$

Proof. The boundedness of $T_0(t)$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $t\geqslant 0$ can be deduced from (5.4) (with $\theta\equiv 1$, $\eta=0$, $C_\theta=1$). The linearity of $T_0(t)$ and property (5.7) follow from the definition of $T_0(t)$ in (5.3). Property (5.8) can easily be verified by using (5.3), Lemma 4.5 (with $B=0$, i.e. with H_0 instead of H) and Fubini's theorem

$$\begin{aligned}
[T_0(t)\,(T_0(s)v)]\,(x) &= \int_{\mathbb{R}^d} H_0(x,\tilde{\xi},t)\,[T_0(s)v]\,(\tilde{\xi})d\tilde{\xi} \\
&= \int_{\mathbb{R}^d} H_0(x,\tilde{\xi},t)\int_{\mathbb{R}^d} H_0(\tilde{\xi},\xi,s)v(\xi)d\xi d\tilde{\xi} \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_0(x,\tilde{\xi},t)H_0(\tilde{\xi},\xi,s)d\tilde{\xi}v(\xi)d\xi \\
&= \int_{\mathbb{R}^d} H_0(x,\xi,t+s)v(\xi)d\xi = [T_0(t+s)v]\,(x),\; x\in\mathbb{R}^d.
\end{aligned}$$

□

The next theorem states that the semigroup $(T_0(t))_{t\geqslant 0}$ is strongly continuous on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$, which justifies to define its infinitesimal generator.

Theorem 5.3 (Strong continuity on $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then* $(T_0(t))_{t\geqslant 0}$ *is a* C^0*-semigroup (or strongly continuous semigroup) on* $L^p(\mathbb{R}^d,\mathbb{C}^N)$*, i.e.*

$$\lim_{t\downarrow 0}\|T_0(t)v-v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}=0\;\forall\, v\in L^p(\mathbb{R}^d,\mathbb{C}^N). \tag{5.9}$$

Proof. 1. Let us define the (d**-dimensional**) **diffusion semigroup** (**Gaussian semigroup**, **heat semigroup**)

$$\begin{aligned}
[G(t,0)v]\,(y) &:= \int_{\mathbb{R}^d} H_0(e^{-tS}y,\xi,t)v(\xi)d\xi \\
&= \int_{\mathbb{R}^d} (4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}|y-\xi|^2\right)v(\xi)d\xi
\end{aligned} \tag{5.10}$$

then we have $[T_0(t)v]\,(x)=[G(t,0)v]\,(e^{tS}x)$. Let $1\leqslant p<\infty$. Motivated by [29], we consider the decomposition

$$\begin{aligned}
\|T_0(t)v-v\|_{L^p} &\leqslant \left\|[G(t,0)v]\,(e^{tS}\cdot)-v(e^{tS}\cdot)\right\|_{L^p}+\left\|v(e^{tS}\cdot)-v(\cdot)\right\|_{L^p} \\
&=: \|v_1(\cdot,t)\|_{L^p}+\|v_2(\cdot,t)\|_{L^p}
\end{aligned}$$

Here and in the sequel of the proof we abbreviate $\|\cdot\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}$ by $\|\cdot\|_{L^p}$.
2. First we compute the v_1-term. Therefore, we use the transformation theorem with $\Phi(x)=e^{tS}x$ and consider the decomposition

$$\begin{aligned}
&\|v_1(\cdot,t)\|_{L^p} = \left\|[G(t,0)v]\,(e^{tS}\cdot)-v(e^{tS}\cdot)\right\|_{L^p} = \|[G(t,0)v]\,(\cdot)-v(\cdot)\|_{L^p} \\
&\leqslant \left\|\int_{\mathbb{R}^d} H_0(e^{-tS}\cdot,\xi,t)\,(v(\xi)-v(\cdot))\,d\xi\right\|_{L^p}+\left\|\left(\int_{\mathbb{R}^d} H_0(e^{-tS}\cdot,\xi,t)d\xi-I_N\right)v(\cdot)\right\|_{L^p} \\
&=: \|v_3(\cdot,t)\|_{L^p}+\|v_4(\cdot,t)\|_{L^p}
\end{aligned}$$

3. Let us consider the v_4-term. Using the transformation theorem (with transformation $\Phi(\xi)=y-\xi$) and Lemma 4.7 (1) (with $B=0$), we obtain

$$
\begin{aligned}
&\left\|v_4(\cdot,t)\right\|_{L^p}\\
&=\left(\int_{\mathbb{R}^d}\left|\left(\int_{\mathbb{R}^d}H_0(e^{-tS}y,\xi,t)d\xi-I_N\right)v(y)\right|^p dy\right)^{\frac{1}{p}}\\
&=\left(\int_{\mathbb{R}^d}\left|\left(\int_{\mathbb{R}^d}(4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}\left|y-\xi\right|^2\right)d\xi-I_N\right)v(y)\right|^p dy\right)^{\frac{1}{p}}\\
&=\left(\int_{\mathbb{R}^d}\left|\left(\int_{\mathbb{R}^d}K(\psi,t)d\psi-I_N\right)v(y)\right|^p dy\right)^{\frac{1}{p}}\\
&\leqslant\left|\int_{\mathbb{R}^d}K(\psi,t)d\psi-I_N\right|_2\left\|v\right\|_{L^p}\\
&=\left|I_N-I_N\right|_2\left\|v\right\|_{L^p}=0\quad\text{for}\quad t>0.
\end{aligned}
$$

4. The v_3-term is much more delicate: First we need the following integral for $b_0=0$ and some constant $\delta_0>0$, compare proof of Lemma 4.6,

$$
\begin{aligned}
&\int_{|\psi|\geqslant\delta_0}\left|K(\psi,t)\right|_2 d\psi\\
\leqslant&\int_{|\psi|\geqslant\delta_0}(4\pi ta_{\min})^{-\frac{d}{2}}e^{-b_0t-\frac{a_0}{4ta_{\max}^2}|\psi|^2}d\psi\\
=&(4\pi ta_{\min})^{-\frac{d}{2}}e^{-b_0t}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_{\delta_0}^{\infty}r^{d-1}e^{-\frac{a_0}{4ta_{\max}^2}r^2}dr\\
=&\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}}e^{-b_0t}\frac{2}{\Gamma\left(\frac{d}{2}\right)}\int_{\left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}}\delta_0}^{\infty}s^{d-1}e^{-s^2}ds=:C(t,\delta_0)
\end{aligned}
$$

where we used the transformation theorem (with transformations for d-dimensional polar coordinates and $\Phi(r)=\left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}}r$). Note, that $C(t,\delta_0)\to 0$ as $t\to 0$ for every fixed $\delta_0>0$. Using the transformation theorem (with transformations $\Phi(\xi)=y-\xi$ and $\Phi(y)=y-\psi$), the triangle inequality, Hölder's inequality (with q such that $\frac{1}{p}+\frac{1}{q}=1$), Fubini's theorem, the L^p-continuity from [7, Satz 2.14(1)], (4.24) and Lemma 4.6(1) (with $\eta=0$ and $b_0=0$) we obtain

$$
\begin{aligned}
&\left\|v_3(\cdot,t)\right\|_{L^p}\\
&=\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}H_0(e^{-tS}y,\xi,t)\left(v(\xi)-v(y)\right)d\xi\right|^p dy\right)^{\frac{1}{p}}\\
&=\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}\left|y-\xi\right|^2\right)\left(v(\xi)-v(y)\right)d\xi\right|^p dy\right)^{\frac{1}{p}}\\
&=\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}K(\psi,t)\left(v(y-\psi)-v(y)\right)d\psi\right|^p dy\right)^{\frac{1}{p}}\\
&\leqslant\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2\left|v(y-\psi)-v(y)\right|d\psi\right)^p dy\right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K(\psi,t)|_2 \, d\psi \right)^{\frac{p}{q}} \int_{\mathbb{R}^d} |K(\psi,t)|_2 \, |v(y-\psi)-v(y)|^p \, d\psi dy \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} |K(\psi,t)|_2 \, d\psi \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |K(\psi,t)|_2 \int_{\mathbb{R}^d} |v(y-\psi)-v(y)|^p \, dy d\psi \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} |K(\psi,t)|_2 \, d\psi \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |K(\psi,t)|_2 \, \|v(\cdot-\psi)-v(\cdot)\|_{L^p}^p \, d\psi \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} |K(\psi,t)|_2 \, d\psi \right)^{\frac{1}{q}} \left(\int_{|\psi|\leqslant\delta_0} |K(\psi,t)|_2 \, \|v(\cdot-\psi)-v(\cdot)\|_{L^p}^p \, d\psi \right. \\
&\quad \left. + \int_{|\psi|\geqslant\delta_0} |K(\psi,t)|_2 \, \|v(\cdot-\psi)-v(\cdot)\|_{L^p}^p \, d\psi \right)^{\frac{1}{p}} \\
&\leqslant \left(\int_{\mathbb{R}^d} |K(\psi,t)|_2 \, d\psi \right)^{\frac{1}{q}} \left(\varepsilon_0^p \int_{|\psi|\leqslant\delta_0} |K(\psi,t)|_2 \, d\psi \right. \\
&\quad \left. + 2^{p-1} \int_{|\psi|\geqslant\delta_0} |K(\psi,t)|_2 \left(\|v(\cdot-\psi)\|_{L^p}^p + \|v\|_{L^p}^p \right) d\psi \right)^{\frac{1}{p}} \\
&\leqslant \left(\int_{\mathbb{R}^d} |K(\psi,t)|_2 \, d\psi \right)^{\frac{1}{q}} \left(\varepsilon_0^p \int_{\mathbb{R}^d} |K(\psi,t)|_2 \, d\psi + 2^p \int_{|\psi|\geqslant\delta_0} |K(\psi,t)|_2 \, d\psi \, \|v\|_{L^p}^p \right)^{\frac{1}{p}} \\
&\leqslant C_1^{\frac{1}{q}}(t) \left(\varepsilon_0^p C_1(t) + 2^p C(t,\delta_0) \, \|v\|_{L^p}^p \right)^{\frac{1}{p}}
\end{aligned}
$$

Hence, $\lim_{t\to 0} \|v_3(\cdot,t)\|_{L^p} \leqslant \varepsilon_0 C_1(0) = \varepsilon_0 M^{\frac{d}{2}}$. Now, choose $\varepsilon_0 > 0$ arbitrary small.
5. Finally, let us consider the v_2-term. Let $\varepsilon > 0$. Since $C_c^\infty(\mathbb{R}^d, \mathbb{C}^N)$ is dense in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$ for every $1 \leqslant p < \infty$, see [7, Satz 2.14(3)], we can choose $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^N)$ such that $\|v - \varphi_\varepsilon\|_{L^p} \leqslant \frac{\varepsilon}{3}$. Since $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^N)$, φ_ε is uniformly continuous on $\mathrm{supp}(\varphi_\varepsilon)$, i.e.

$$
\begin{aligned}
&\forall\, \varepsilon_0 > 0 \; \exists\, \delta_0 = \delta_0(\varepsilon_0) > 0 \; \forall\, x, x_0 \in \mathrm{supp}(\varphi_\varepsilon) \\
&\qquad\qquad\qquad \text{with } |x - x_0| \leqslant \delta_0 : \; |\varphi_\varepsilon(x) - \varphi_\varepsilon(x_0)| \leqslant \varepsilon_0
\end{aligned}
$$

Choosing $x_0 := e^{tS} x$ we have

$$
\exists\, t_0 = t_0(\delta_0) > 0 \; \forall\, 0 \leqslant t \leqslant t_0 : \; \left| e^{tS} x - x \right| \leqslant \delta_0
$$

Thus, choosing $\varepsilon_0 := \varepsilon \left(3 \, |\mathrm{supp}(\varphi_\varepsilon)|^{\frac{1}{p}} \right)^{-1}$ and combining this facts yields

$$
\left\| \varphi_\varepsilon(e^{tS}\cdot) - \varphi_\varepsilon(\cdot) \right\|_{L^p} = \left(\int_{\mathrm{supp}(\varphi_\varepsilon)} \left| \varphi_\varepsilon(e^{tS}x) - \varphi_\varepsilon(x) \right|^p \right)^{\frac{1}{p}} \leqslant \varepsilon \quad \forall\, 0 \leqslant t \leqslant t_0(\varepsilon).
$$

This implies

$$
\begin{aligned}
&\|v_2(\cdot,t)\|_{L^p} = \left\| v(e^{tS}\cdot) - v(\cdot) \right\|_{L^p} \\
\leqslant& \left\| v(e^{tS}\cdot) - \varphi_\varepsilon(e^{tS}\cdot) \right\|_{L^p} + \left\| \varphi_\varepsilon(e^{tS}\cdot) - \varphi_\varepsilon(\cdot) \right\|_{L^p} + \|\varphi_\varepsilon(\cdot) - v(\cdot)\|_{L^p} \\
\leqslant& \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall\, 0 \leqslant t \leqslant t_0(\varepsilon).
\end{aligned}
$$

Hence, $\lim_{t\to 0} \|v_2(\cdot,t)\|_{L^p} \leqslant \varepsilon$. Now, choose $\varepsilon > 0$ arbitrary small. $\square$

Strong continuity on spaces of continuous functions. Note, that the original Ornstein-Uhlenbeck semigroup is also strongly continuous in certain subspaces of bounded uniformly continuous functions and certain subspaces of Hölder spaces. These function spaces were analyzed in [29] for the first time.

Strong continuity on exponentially weighted function spaces. We suggest that the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ is also strongly continuous on the exponentially weighted spaces $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$ and for every radially weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$. Moreover, one can prove that the Ornstein-Uhlenbeck semigroup is strongly continuous on certain exponentially weighted subspaces of bounded uniformly continuous functions.

Now, the **infinitesimal generator** $A_p:\mathcal{D}(A_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ of the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, short $(A_p,\mathcal{D}(A_p))$, can be defined by, [34, II.1.2 Definition],

$$A_p v := \lim_{t\downarrow 0}\frac{T_0(t)v-v}{t},\ 1\leqslant p<\infty$$

for every $v\in\mathcal{D}(A_p)$, where the **domain** (or **maximal domain**) **of** A_p is given by

$$\begin{aligned}\mathcal{D}(A_p) :=& \left\{v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\mid \lim_{t\downarrow 0}\frac{T_0(t)v-v}{t}\text{ exists in }L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}\\ =& \left\{v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\mid A_pv\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.\end{aligned}$$

Note that $\mathcal{D}(A_p)$ is a linear subspace of $L^p(\mathbb{R}^d,\mathbb{C}^N)$.

An application of [34, II.1.3 Lemma, II.1.4 Theorem] yields the following result:

Lemma 5.4. *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$.
(**1**) *The generator* $A_p:\mathcal{D}(A_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ *is a linear, closed and densely defined operator that determines the semigroup* $(T_0(t))_{t\geqslant 0}$ *uniquely.*
(**2**) *For every* $v\in\mathcal{D}(A_p)$ *and* $t\geqslant 0$ *we have*

$$\begin{aligned}&T_0(t)v\in\mathcal{D}(A_p)\\ &\frac{d}{dt}T_0(t)v=T_0(t)A_pv=A_pT_0(t)v\end{aligned}$$

(**3**) *For every* $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ *and every* $t\geqslant 0$ *we have*

$$\int_0^t T_0(s)vds\in\mathcal{D}(A_p)$$

(**4**) *For every* $t\geqslant 0$ *we have*

$$\begin{aligned}T_0(t)v-v =&A_p\int_0^t T_0(s)vds && \text{, for } v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\\ =&\int_0^t T_0(s)A_pvds && \text{, for } v\in\mathcal{D}(A_p)\end{aligned}$$

Since $(A_p, \mathcal{D}(A_p))$ is a closed operator on the Banach space $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for every $1 \leqslant p < \infty$, we can introduce

$$
\begin{aligned}
\sigma(A_p) &:= \{\lambda \in \mathbb{C} \mid \lambda I - A_p \text{ is not bijective}\} && \textbf{spectrum of } A_p,\\
\rho(A_p) &:= \mathbb{C} \backslash \sigma(A_p) && \textbf{resolvent set of } A_p,\\
R(\lambda, A_p) &:= (\lambda I - A_p)^{-1}\,, \text{ for } \lambda \in \rho(A_p) && \textbf{resolvent of } A_p.
\end{aligned}
$$

In particular, $\left(\mathcal{D}(A_p), \|\cdot\|_{A_p}\right)$ is a Banach space w.r.t. the **graph norm of** A_p

$$\|v\|_{A_p} := \|A_p v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\,, \quad v \in \mathcal{D}(A_p),$$

see [34, B.1 Definition]. The next identities follow from [34, II.1.9 Lemma].

Lemma 5.5. *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Then for every* $\lambda \in \mathbb{C}$ *and* $t > 0$,

$$
\begin{aligned}
e^{-\lambda t} T_0(t) v - v &= (A_p - \lambda I) \int_0^t e^{-\lambda s} T_0(s) v ds && , \textit{for } v \in L^p(\mathbb{R}^d, \mathbb{C}^N),\\
&= \int_0^t e^{-\lambda s} T_0(s) \left(A_p - \lambda I\right) v ds && , \textit{for } v \in \mathcal{D}(A_p).
\end{aligned}
$$

By (5.4) from Theorem 5.1 (with $\theta \equiv 1$, $\eta = 0$ and $C_\theta = 1$) we have

$$\exists\, \omega_0 \in \mathbb{R} \;\wedge\; \exists\, M_0 \geqslant 1 : \; \|T_0(t)\|_{\mathcal{L}(L^p,L^p)} \leqslant M_0 e^{\omega_0 t} \quad \forall\, t \geqslant 0, \tag{5.11}$$

where $M_0 := \left(\frac{a_{\max}^2}{a_{\min} a_0}\right)^{\frac{d}{2}}$ and $\omega_0 := 0$. For the next statement we refer to [34, II.1.10 Theorem].

Theorem 5.6. *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$.
(**1**) *If* $\lambda \in \mathbb{C}$ *is such that* $R(\lambda)v := \int_0^\infty e^{-\lambda s} T_0(s) v ds$ *exists for every* $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$, *then*

$$\lambda \in \rho(A_p) \quad \textit{and} \quad R(\lambda, A_p) = R(\lambda).$$

(**2**) *If* $\lambda \in \mathbb{C}$ *satisfies* $\operatorname{Re} \lambda > \omega_0$, *then*

$$\lambda \in \rho(A_p), \quad R(\lambda, A_p) = R(\lambda)$$

and

$$\|R(\lambda, A_p)\|_{\mathcal{L}(L^p,L^p)} \leqslant \frac{M_0}{\operatorname{Re} \lambda - \omega_0}.$$

Theorem 5.6(2) states that the complete right half–plane $\operatorname{Re} \lambda > \omega_0$ belongs to the resolvent set $\rho(A_p)$. Therefore, the spectrum $\sigma(A_p)$ is contained in the left half–plane $\operatorname{Re} \lambda \leqslant \omega_0$. The **spectral bound** $s(A_p)$ **of** A_p, [34, II.1.12 Definition], defined by

$$-\infty \leqslant s(A_p) := \sup_{\lambda \in \sigma(A_p)} \operatorname{Re} \lambda \leqslant \omega_0 = 0 < +\infty$$

can be considered as the smallest value $\omega\in\mathbb{R}$ such that the spectrum is contained in the half–plane $\operatorname{Re}\lambda\leqslant\omega$. This value is an important characteristic for linear operators.

A direct consequence of Theorem 5.6 is the following:

Corollary 5.7 (Solvability and uniqueness in $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Moreover, let* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda>\omega_0$. *Then for every* $g\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ *the resolvent equation*

$$(\lambda I-A_p)\,v=g$$

admits a unique solution $v_\star\in\mathcal{D}(A_p)$, *which is given by the integral expression*

$$\begin{aligned}v_\star=R(\lambda)g&=\int_0^\infty e^{-\lambda s}T_0(s)gds\\&=\int_0^\infty e^{-\lambda s}\int_{\mathbb{R}^d}H_0(\cdot,\xi,s)g(\xi)d\xi ds.\end{aligned}$$

Moreover, the following resolvent estimate holds

$$\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{M_0}{\operatorname{Re}\lambda-\omega_0}\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\,.$$

5.2 Exponential decay

In this section we prove a-priori estimates for the solution v of the resolvent equation $(\lambda I-A_p)\,v=g$ in exponentially weighted L^p-spaces. We show that the solution $v_\star\in\mathcal{D}(A_p)$ decays exponentially (at least) with the same rate as the inhomogeneity g. Note, that this result needs neither an explicit representation for the domain $\mathcal{D}(A_p)$ nor for the infinitesimal generator A_p. The proof requires only the integral expression for $v_\star$ from Corollary 5.7.

Theorem 5.8 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Moreover, let* $0<\vartheta<1$ *and* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda>\omega_0$. *Then for every radially nondecreasing weight function* $\theta\in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta\geqslant 0$ *with* $0\leqslant\eta^2\leqslant\vartheta\frac{a_0(\operatorname{Re}\lambda-\omega_0)}{a_{\max}^2p^2}$ *and for every* $g\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *we have* $v_\star\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *with*

$$\|v_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{C_7}{\operatorname{Re}\lambda-\omega_0}\|g\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\,,\tag{5.12}$$

$$\|D_iv_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{C_8}{(\operatorname{Re}\lambda-\omega_0)^{\frac{1}{2}}}\|g\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\,,\;i=1,\ldots,d,\tag{5.13}$$

where $v_\star\in\mathcal{D}(A_p)$ *denotes the unique solution of* $(\lambda I-A_p)v=g$ *in* $L^p(\mathbb{R}^d,\mathbb{C}^N)$ *and the* λ*-independent constants* C_7, C_8 *are given by Lemma 4.8 (with* $b_0=0$ *and* $\omega=\omega_0$*).*

Proof. By Corollary 5.7 we have the representation

$$v_\star(x) = \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^d} H_0(x,\xi,t) g(\xi) d\xi dt, \tag{5.14}$$

where $H_0(x,\xi,t) = H(x,\xi,t)$ since we have $B = 0$. In the following we make use of the notation from Theorem 5.1 once more. To show (5.12) and (5.13) for $1 \leqslant p < \infty$ we use (5.14), the transformation theorem (with transformation $\Phi(\xi) = e^{tS}x - \xi$ in ξ and $\Phi(x) = e^{tS}x - \psi$ in x), (4.24) and (4.26) (with $B = 0$), the triangle inequality, Hölder's inequality (with q such that $\frac{1}{p} + \frac{1}{q} = 1$), Fubini's theorem, (W1)–(W4), Lemma 4.8 (with $b_0 = 0$ and $\omega = \omega_0$) and obtain for every $\beta \in \mathbb{N}_0^d$ with $|\beta| \in \{0,1\}$

$$\begin{aligned}
&\left\| D^\beta v_\star \right\|_{L^p_\theta} = \left(\int_{\mathbb{R}^d} \theta^p(x) \left| D^\beta v_\star(x) \right|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} \theta^p(x) \left| \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^d} \left[D^\beta H_0(x,\xi,t) \right] g(\xi) d\xi dt \right|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} \theta^p(x) \left| \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^d} K^\beta(\psi,t) g(e^{tS}x - \psi) d\psi dt \right|^p dx \right)^{\frac{1}{p}} \\
&\leqslant \int_0^\infty e^{-\operatorname{Re}\lambda t} \left(\int_{\mathbb{R}^d} \theta^p(x) \left| \int_{\mathbb{R}^d} K^\beta(\psi,t) g(e^{tS}x - \psi) d\psi \right|^p dx \right)^{\frac{1}{p}} dt \\
&\leqslant \int_0^\infty e^{-\operatorname{Re}\lambda t} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \theta(x) \left| K^\beta(\psi,t) \right|_2 \left| g(e^{tS}x - \psi) \right| d\psi \right)^p dx \right)^{\frac{1}{p}} dt \\
&\leqslant \int_0^\infty e^{-\operatorname{Re}\lambda t} \left(\int_{\mathbb{R}^d} Z^{\frac{p}{q}}(t) \int_{\mathbb{R}^d} \left| K^\beta(\psi,t) \right|_2 \left(\theta(x) \left| g(e^{tS}x - \psi) \right| \right)^p d\psi dx \right)^{\frac{1}{p}} dt \\
&= \int_0^\infty e^{-\operatorname{Re}\lambda t} Z^{\frac{1}{q}}(t) \left(\int_{\mathbb{R}^d} \left| K^\beta(\psi,t) \right|_2 \int_{\mathbb{R}^d} \left(\theta(x) \left| g(e^{tS}x - \psi) \right| \right)^p dx d\psi \right)^{\frac{1}{p}} dt \\
&= \int_0^\infty e^{-\operatorname{Re}\lambda t} Z^{\frac{1}{q}}(t) \left(\int_{\mathbb{R}^d} \left| K^\beta(\psi,t) \right|_2 \int_{\mathbb{R}^d} \left(\theta(e^{-tS}(y + \psi)) \left| g(y) \right| \right)^p dy d\psi \right)^{\frac{1}{p}} dt \\
&\leqslant \int_0^\infty e^{-\operatorname{Re}\lambda t} Z^{\frac{1}{q}}(t) \left(\int_{\mathbb{R}^d} C_\theta^p e^{\eta p |\psi|} \left| K^\beta(\psi,t) \right|_2 \int_{\mathbb{R}^d} \theta^p(y) \left| g(y) \right|^p dy d\psi \right)^{\frac{1}{p}} dt \\
&= \int_0^\infty e^{-\operatorname{Re}\lambda t} C_\theta \left(\int_{\mathbb{R}^d} \left| K^\beta(\psi,t) \right|_2 d\psi \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} e^{\eta p |\psi|} \left| K^\beta(\psi,t) \right|_2 d\psi \right)^{\frac{1}{p}} dt \left\| g \right\|_{L^p_\theta} \\
&\leqslant \int_0^\infty e^{-\operatorname{Re}\lambda t} C_{4+|\beta|}(t; b_0 = 0) dt \left\| g \right\|_{L^p_\theta} \leqslant \frac{C_{7+|\beta|}}{(\operatorname{Re}\lambda - \omega_0)^{1 - \frac{|\beta|}{2}}} \left\| g \right\|_{L^p_\theta},
\end{aligned}$$

where we used the abbreviation

$$Z(t) := \int_{\mathbb{R}^d} \left| K^\beta(\psi,t) \right|_2 d\psi.$$

□

A superset of the domain of A_p. An application of Theorem 5.8 for $\theta \equiv 1$ (with $\eta = 0$ and $C_\theta = 1$) shows that

$$\mathcal{D}(A_p) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N), \text{ for every } 1 \leqslant p < \infty.$$

Second order derivatives. With the procedure as in the proof of Theorem 5.8 it is in general not possible to specify also an estimate for $\|D_jD_iv_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}$ since $C_{4+|\beta|}(t) \sim t^{-\frac{|\beta|}{2}}$ as $t \to 0$, cf. Theorem 5.1, and consequently we have the singularity t^{-1} at $t = 0$ for $|\beta| = 2$. In Hölder spaces, for instance, there one uses interpolation theory to derive estimates for the second order derivatives, [67].

A-priori estimates for $\mathcal{L}_0$. We suggest that if we require additionally $1 < p < \infty$, assumption (A3) and the L^p–antieigenvalue condition (A4), then we can actually replace A_p by $\mathcal{L}_0$ and $\mathcal{D}(A_p)$ by $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ in Theorem 5.8, compare Theorem 5.19 below.

5.3 A core for the infinitesimal generator

In the next three sections we investigate the relation between the **formal** Ornstein-Uhlenbeck operator $\mathcal{L}_0$ and the **abstract** Ornstein-Uhlenbeck operator A_p and derive a precise characterization of the maximal domain $\mathcal{D}(A_p)$, that is necessary to prove our main result. The approach is motivated by [73] and [71], where this was performed for the scalar real-valued Ornstein-Uhlenbeck operator.

In general, it is very difficult to identify maximal domains of infinitesimal generators, such as $\mathcal{D}(A_p)$. In the case of the Ornstein-Uhlenbeck operator (on $\mathbb{R}^d$) we suggest that there arises an additional complication caused by the unbounded (in fact linearly growing) coefficients, that are contained in the drift term.

A useful concept to analyze subspaces of $\mathcal{D}(A_p)$ is the following, see [34, II.1.6 Definition].

Definition 5.9. A subspace $D \subset \mathcal{D}(A_p)$ of the maximal domain $\mathcal{D}(A_p)$ of the linear operator $A_p : \mathcal{D}(A_p) \subseteq L^p(\mathbb{R}^d,\mathbb{C}^N) \to L^p(\mathbb{R}^d,\mathbb{C}^N)$ with $1 \leqslant p < \infty$ is called a **core for** $(A_p,\mathcal{D}(A_p))$ if D is dense in $\mathcal{D}(A_p)$ with respect to the graph norm of A_p

$$\|v\|_{A_p} := \|A_pv\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}, \quad v \in \mathcal{D}(A_p).$$

The next theorem states that the Schwartz space $\mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$ is a core for the infinitesimal generator $(A_p,\mathcal{D}(A_p))$ of the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$. Moreover, it turns out that the formal Ornstein-Uhlenbeck operator $\mathcal{L}_0$ and the abstract Ornstein-Uhlenbeck operator A_p coincide on the Schwartz space $\mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$. This is an extension of the real-valued scalar result in [71, Proposition 2.2 and 3.2] to complex valued systems.

Theorem 5.10 (Core for the infinitesimal generator)**.** *Let the assumptions* (A1)*,* (A2) *and* (A5) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*. Then:*
(**1**) $\mathcal{S} \subset L^p(\mathbb{R}^d,\mathbb{C}^N)$ *is dense w.r.t. the* L^p*–norm* $\|\cdot\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}$*.*
(**2**) $\mathcal{S}$ *is a subspace of* $\mathcal{D}(A_p)$*, i.e.* $\mathcal{S} \subset \mathcal{D}(A_p)$*, and*

$$A_p\phi = \mathcal{L}_0\phi \text{ for every } \phi \in \mathcal{S}.$$

(**3**) *$\mathcal{S}$ is invariant under the semigroup $(T_0(t))_{t\geqslant 0}$, i.e.*

$$T_0(t)\mathcal{S} \subseteq \mathcal{S} \text{ for every } t \geqslant 0.$$

(**4**) *$\mathcal{S} \subset \mathcal{D}(A_p)$ is a core for $(A_p, \mathcal{D}(A_p))$, i.e.*

$$\mathcal{D}(A_p) = \overline{\mathcal{S}}^{\|\cdot\|_{A_p}}.$$

Proof. (**1**): Due to the inclusion

$$C_{\mathrm{c}}^{\infty}(\mathbb{R}^d, \mathbb{C}^N) \subset \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N) \subset L^p(\mathbb{R}^d, \mathbb{C}^N)$$

and since $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d, \mathbb{C}^N)$ is dense in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$ for every $1 \leqslant p < \infty$, we deduce that $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$ is also dense in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$ for every $1 \leqslant p < \infty$.
(**2**): Let $\phi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$ be arbitrary. In order to prove $\mathcal{S} \subset \mathcal{D}(A_p)$ we must show that

$$\phi \in L^p(\mathbb{R}^d, \mathbb{C}^N),\ \mathcal{L}_0\phi \in L^p(\mathbb{R}^d, \mathbb{C}^N),\ \lim_{t\downarrow 0} \frac{1}{t}\left(T_0(t)\phi - \phi\right) \text{ exists in } L^p(\mathbb{R}^d, \mathbb{C}^N).$$

1. Since $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$ is a subspace of $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$, we deduce $\phi \in L^p(\mathbb{R}^d, \mathbb{C}^N)$. Therefore, it is sufficient to show $\mathcal{L}_0\phi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$. Then we deduce $\mathcal{L}_0\phi \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ by the same argument. Since $\phi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N) \subseteq C^{\infty}(\mathbb{R}^d, \mathbb{C}^N)$ and since $\mathcal{L}_0$ has smooth coefficients we infer that $\mathcal{L}_0\phi \in C^{\infty}(\mathbb{R}^d, \mathbb{C}^N)$. Considering the operator

$$\left[\mathcal{L}_0\phi\right](x) = A\triangle\phi(x) + \langle Sx, \nabla\phi(x)\rangle = A\sum_{i=1}^{d} D_i^2\phi(x) + \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_j D_i\phi(x)$$

and

$$x^{\tilde{\alpha}} D^{\tilde{\beta}}\left[\mathcal{L}_0\phi\right](x) = A\sum_{i=1}^{d} x^{\tilde{\alpha}} D^{\tilde{\beta}} D_i^2\phi(x) + \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x^{\tilde{\alpha}}x_j D^{\tilde{\beta}} D_i\phi(x)$$

for $\tilde{\alpha}, \tilde{\beta} \in \mathbb{N}_0^d$ and using the fact that ϕ is rapidly decreasing, we conclude from (3.4) with $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta} + 2e_i$ and $\alpha = \tilde{\alpha} + e_j$, $\beta = \tilde{\beta} + e_i$, that every term on the right hand side vanishes as $|x|$ goes to infinity. Hence, $\mathcal{L}_0\phi \in \mathcal{S}$. It remains to verify that the limit exists in $L^p(\mathbb{R}^d, \mathbb{C}^N)$.
2. We first give a motivation how the limit looks like: Using the heat kernel properties (H2) and (H3) with $B = 0$, in this case we have $H(x, \xi, t) = H_0(x, \xi, t)$ and $\mathcal{L}_{\infty} = \mathcal{L}_0$, a formal computation shows

$$\begin{aligned}
\left[A_p\phi\right](x) :=& \lim_{t\downarrow 0} \frac{\left[T_0(t)\phi\right](x) - \phi(x)}{t} = \lim_{t\downarrow 0}\left[\frac{T_0(t) - T_0(0)}{t - 0}\right]\phi(x) \\
=& \left[\frac{d}{dt}\left[T_0(t)\phi\right](x)\right]_{t=0} = \left[\int_{\mathbb{R}^d} \frac{d}{dt} H_0(x, \xi, t)\phi(\xi)d\xi\right]_{t=0} \\
=& \left[\mathcal{L}_0 \int_{\mathbb{R}^d} H_0(x, \xi, t)\phi(\xi)d\xi\right]_{t=0} = \mathcal{L}_0 \int_{\mathbb{R}^d} \delta_x(\xi)\phi(\xi)d\xi = \left[\mathcal{L}_0\phi\right](x).
\end{aligned}$$

This suggests that the limit tends (pointwise) to $A_p\phi(x) := \mathcal{L}_0\phi(x) \in L^p(\mathbb{R}^d,\mathbb{C}^N)$, provided that all steps in the calculation are justified. We next prove that the limit even exists in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$, which is indeed much more involved, [71, Proposition 2.2 and 3.2].
3. Our aim is to apply Lebesgue's dominated convergence theorem in L^p from [7, Satz 1.23] with

$$f_t(x) := \frac{[T_0(t)\phi](x) - \phi(x)}{t}, \quad f(x) := [\mathcal{L}_0\phi](x)$$

to deduce that $f_t, f \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $t > 0$ and $f_t \to f$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ as $t \downarrow 0$. We then directly conclude $\phi \in \mathcal{D}(A_p)$, thus $\mathcal{S}(\mathbb{R}^d,\mathbb{C}^N) \subset \mathcal{D}(A_p)$. In particular, we have $A_p\phi := \mathcal{L}_0\phi$ for every $\phi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$. To justify the application of dominated convergence we must show that

(a) $f_t(x) \to f(x)$ pointwise for a.e. $x \in \mathbb{R}^d$ as $t \downarrow 0$,

(b) $|f_t(x)| \leqslant g(x)$ pointwise for a.e. $x \in \mathbb{R}^d$ and for every $0 < t \leqslant t_0$,

(c) $g \in L^p(\mathbb{R}^d,\mathbb{R})$,

where the function g will be determined during the proof. Before we start to verify the properties (a)–(c) we simplify the term f_t, [71, Proposition 2.2 and 3.2]: Since $\phi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$, Taylor's formula up to order 2 yields

$$\begin{aligned}\phi(e^{tS}x-\psi) =& \phi(x) + \sum_{i=1}^{d}\left(e^{tS}x-x-\psi\right)_i D_i\phi(x) \\ &+ \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{2}\left(e^{tS}x-x-\psi\right)_i\left(e^{tS}x-x-\psi\right)_j D_jD_i\phi(x) \\ &+ R_{x,2}\left(e^{tS}x-x-\psi\right)\end{aligned}$$

with remainder

$$R_{x,2}(z-x) = \sum_{|\beta|=3}\frac{|\beta|}{\beta!}(z-x)^{\beta}\int_0^1(1-\tau)^{|\beta|-1}D^{\beta}\phi(x+\tau(z-x))\,d\tau$$

for $z := e^{tS}x - \psi$ satisfying

$$|R_{x,2}(z-x)| \leqslant C_{\beta}C_{\phi}|z-x|^3, \tag{5.15}$$

where $C_\beta := \sum_{|\beta|=3}\frac{1}{\beta!}$ and $C_\phi := \max_{|\beta|=3}\sup_{y\in\mathbb{R}^d}\left|D^\beta\phi(y)\right|$. Thus, using (5.3), the transformation theorem (with transformation $\Phi(\xi) = e^{tS}x - \xi$) and (4.24) with $B = 0$ we obtain

$$\begin{aligned}f_t(x) :=& \frac{[T_0(t)\phi](x)-\phi(x)}{t} = \frac{1}{t}\left[\int_{\mathbb{R}^d}H_0(x,\xi,t)\phi(\xi)d\xi - \phi(x)\right] \\ =& \frac{1}{t}\left[\int_{\mathbb{R}^d}K(\psi,t)\phi\left(e^{tS}x-\psi\right)d\psi - \phi(x)\right]\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{t}\left[\int_{\mathbb{R}^d} K(\psi,t)d\psi - I_N\right]\phi(x) + \frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)\sum_{i=1}^{d}\left(e^{tS}x - x - \psi\right)_i D_i\phi(x)d\psi \\
&+ \frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{2}\left(e^{tS}x - x - \psi\right)_i\left(e^{tS}x - x - \psi\right)_j D_jD_i\phi(x)d\psi \\
&+ \frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)R_{x,2}\left(e^{tS}x - x - \psi\right)d\psi =: \sum_{i=1}^{4} T_i(x,t),\ t>0.
\end{aligned}
$$

T_1: Using Lemma 4.7 (1) with $B=0$ the term T_1 vanishes for every $t>0$

$$
T_1(x,t) = \frac{1}{t}\left[\int_{\mathbb{R}^d} K(\psi,t)d\psi - I_N\right]\phi(x) = 0.
$$

T_2: A decomposition of T_2 leads to

$$
\begin{aligned}
T_2(x,t) =& \frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)\sum_{i=1}^{d}\left(e^{tS}x - x - \psi\right)_i D_i\phi(x)d\psi \\
=&\frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)d\psi\sum_{i=1}^{d}\left(e^{tS}x - x\right)_i D_i\phi(x) - \frac{1}{t}\sum_{i=1}^{d}\int_{\mathbb{R}^d} K(\psi,t)\psi_i d\psi D_i\phi(x) \\
=&\sum_{i=1}^{d}\left(\frac{e^{tS}x - x}{t}\right)_i D_i\phi(x)
\end{aligned}
$$

for every $t>0$, where we used Lemma 4.7 (1) with $B=0$ for the first and Lemma 4.7 (2) with $B=0$ for the second term.
T_3: Similarly, a decomposition of T_3 leads to

$$
\begin{aligned}
T_3(x,t) =& \frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{2}\left(e^{tS}x - x - \psi\right)_i\left(e^{tS}x - x - \psi\right)_j D_jD_i\phi(x)d\psi \\
=&\frac{1}{2t}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_{\mathbb{R}^d} K(\psi,t)\psi_i\psi_j d\psi D_jD_i\phi(x) \\
&+ \frac{1}{2t}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_{\mathbb{R}^d} K(\psi,t)d\psi\left(e^{tS}x - x\right)_i\left(e^{tS}x - x\right)_j D_jD_i\phi(x) \\
&- \frac{1}{2t}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_{\mathbb{R}^d} K(\psi,t)\left[\left(e^{tS}x - x\right)_i\psi_j + \left(e^{tS}x - x\right)_j\psi_i\right]d\psi D_jD_i\phi(x) \\
=&\frac{1}{2t}\sum_{i=1}^{d} 2tAD_i^2\phi(x) + \frac{t}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{e^{tS}x - x}{t}\right)_i\left(\frac{e^{tS}x - x}{t}\right)_j D_jD_i\phi(x) \\
&- \frac{1}{2t}\sum_{i,j=1}^{d}\left[\int_{\mathbb{R}^d} K(\psi,t)\psi_j d\psi(e^{tS}x - x)_i + \int_{\mathbb{R}^d} K(\psi,t)\psi_i d\psi(e^{tS}x - x)_j\right]D_jD_i\phi(x) \\
=&A\triangle\phi(x) + \frac{t}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{e^{tS}x - x}{t}\right)_i\left(\frac{e^{tS}x - x}{t}\right)_j D_jD_i\phi(x)
\end{aligned}
$$

for every $t>0$, where we used Lemma 4.7 (3), (1) and (2) with $B=0$ for the first, second and third term, respectively.
This yields a simplified representation for $f_t(x)$ for every $t>0$ given by

$$\begin{aligned} f_t(x) =& A\triangle\phi(x) + \sum_{i=1}^{d}\left(\frac{e^{tS}x-x}{t}\right)_i D_i\phi(x) \\ &+\frac{t}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{e^{tS}x-x}{t}\right)_i\left(\frac{e^{tS}x-x}{t}\right)_j D_jD_i\phi(x) \\ &+\frac{1}{t}\int_{\mathbb{R}^d}K(\psi,t)R_{x,2}(e^{tS}x-x-\psi)d\psi. \end{aligned} \tag{5.16}$$

(a): Using $\lim_{t\downarrow 0}\frac{1}{t}\left(e^{tS}x-x\right)=Sx$ we deduce that

$$\begin{aligned} \left[A_p\phi\right](x) =& \lim_{t\downarrow 0}\frac{\left[T_0(t)\phi\right](x)-\phi(x)}{t} = \lim_{t\downarrow 0} f_t(x) \\ =& A\triangle\phi(x) + \lim_{t\downarrow 0}\sum_{i=1}^{d}\left(\frac{e^{tS}x-x}{t}\right)_i D_i\phi(x) \\ &+\lim_{t\downarrow 0}\frac{t}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{e^{tS}x-x}{t}\right)_i\left(\frac{e^{tS}x-x}{t}\right)_j D_jD_i\phi(x) \\ &+\lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^d}K(\psi,t)R_{x,2}(e^{tS}x-x-\psi)d\psi \\ =& A\triangle\phi(x)+\langle Sx,\nabla\phi(x)\rangle = \left[\mathcal{L}_0\phi\right](x) = f(x), \end{aligned}$$

i.e. $f_t(x)\to f(x)$ pointwise for a.e. $x\in\mathbb{R}^d$ as $t\downarrow 0$, provided that the last limit tends to zeros. This can be seen as follows: Using (5.15), the inequality

$$\left(\left|e^{tS}x-x\right|+|\psi|\right)^3 \leqslant 4\left(\left|e^{tS}x-x\right|^3+|\psi|^3\right)$$

and

$$\int_{\mathbb{R}^d}|K(\psi,t)|_2\,|\psi|^k = M^{\frac{d}{2}}e^{-b_0t}\left(\frac{4a_{\max}^2}{a_0}\right)^{\frac{k}{2}}\frac{\Gamma\left(\frac{d+k}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}t^{\frac{k}{2}},\ t>0,\ k\in\mathbb{N}_0 \tag{5.17}$$

with $M:=\frac{a_{\max}^2}{a_{\min}a_0}$, $B=0$ for $k=0$ and $k=3$, we obtain

$$\begin{aligned} &\left|\frac{1}{t}\int_{\mathbb{R}^d}K(\psi,t)R_{x,2}(e^{tS}x-x-\psi)d\psi\right| \\ \leqslant&\frac{1}{t}\int_{\mathbb{R}^d}|K(\psi,t)|_2\left|R_{x,2}(e^{tS}x-x-\psi)\right|d\psi \\ \leqslant&\frac{C_\beta C_\phi}{t}\int_{\mathbb{R}^d}|K(\psi,t)|_2\left|e^{tS}x-x-\psi\right|^3 d\psi \\ \leqslant&\frac{C_\beta C_\phi}{t}\int_{\mathbb{R}^d}|K(\psi,t)|_2\left(\left|e^{tS}x-x\right|+|\psi|\right)^3 d\psi \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{4C_\beta C_\phi}{t}\left[\int_{\mathbb{R}^d} |K(\psi,t)|_2\, d\psi \left|e^{tS}x-x\right|^3 + \int_{\mathbb{R}^d} |K(\psi,t)|_2\, |\psi|^3\, d\psi\right] \\
&=4C_\beta C_\phi M^{\frac{d}{2}}\left[t^2\left|\frac{e^{tS}x-x}{t}\right|^3 + \frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{4a_{\max}^2}{a_0}\right)^{\frac{3}{2}} t^{\frac{1}{2}}\right]
\end{aligned}
$$

for every $t > 0$. Therefore, using $\lim_{t\downarrow 0}\frac{1}{t}\left(e^{tS}x-x\right) = Sx$ once more, the right hand side vanishes for a.e. $x \in \mathbb{R}^d$ as $t \downarrow 0$. Note, that the inequality above follows from a discrete version of Hölder's inequality. The equality (5.17) can be proved in the same way as in Lemma 4.6.
(b): Given some $\varepsilon > 0$ we choose $t_0 = t_0(\varepsilon) > 0$ such that $\left|\frac{1}{t}\left(e^{tS}-I_d\right)\right| \leqslant |S| + \varepsilon$ for every $0 < t \leqslant t_0$. Then (5.16) yields

$$
\begin{aligned}
|f_t(x)| \leqslant& A\sum_{i=1}^d \left|D_i^2\phi(x)\right| + \sum_{i=1}^d \left|\frac{e^{tS}-I_d}{t}\right| |x|\,|D_i\phi(x)| \\
&+ \frac{t}{2}\sum_{i=1}^d\sum_{j=1}^d \left|\frac{e^{tS}-I_d}{t}\right|^2 |x|^2\,|D_jD_i\phi(x)| \\
&+ \left|\frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)R_{x,2}(e^{tS}x-x-\psi)d\psi\right| \\
\leqslant& A\sum_{i=1}^d \left|D_i^2\phi(x)\right| + \sum_{i=1}^d (|S|+\varepsilon)\,|x|\,|D_i\phi(x)| \qquad (5.18)\\
&+ \frac{t_0}{2}\sum_{i=1}^d\sum_{j=1}^d (|S|+\varepsilon)^2 |x|^2\,|D_jD_i\phi(x)| \\
&+ \left|\frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)R_{x,2}(e^{tS}x-x-\psi)d\psi\right|
\end{aligned}
$$

for every $0 < t \leqslant t_0$. Now the first three terms do not depend on t any more. In particular, since $\phi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$, the first three terms belong to $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Therefore, it remains to estimate the last term in such a way, that the bound doesn't depend on t any more and belongs to $L^p(\mathbb{R}^d,\mathbb{C}^N)$ as a function of x. For this purpose, we must handle the last term very carefully.

$$
\begin{aligned}
&\left|\frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)R_{x,2}(e^{tS}x-x-\psi)d\psi\right| \\
=&\left|\frac{1}{t}\int_{\mathbb{R}^d} K(\psi,t)\sum_{|\beta|=3}\frac{|\beta|}{\beta!}(z-x)^\beta\int_0^1 (1-\tau)^2 D^\beta\phi\left(x+\tau\left(z-x\right)\right)d\tau d\psi\right| \\
\leqslant&\frac{1}{t}\sum_{|\beta|=3}\frac{|\beta|}{\beta!}\int_{\mathbb{R}^d}|K(\psi,t)|_2\left|(z-x)^\beta\right|\left|\int_0^1(1-\tau)^2\left|D^\beta\phi\left(x+\tau\left(z-x\right)\right)\right|d\tau d\psi\right. \\
\leqslant&\frac{4C_\beta}{t}\int_{\mathbb{R}^d}|K(\psi,t)|_2\left(\left|e^{tS}x-x\right|^3+|\psi|^3\right)\max_{|\beta|=3}\sup_{\tau\in[0,1]}\left|D^\beta\phi\left(x+\tau\left(z-x\right)\right)\right|d\psi \\
=&\frac{4C_\beta}{t}\int_{\mathbb{R}^d}|K(\psi,t)|_2\left(\left|e^{tS}x-x\right|^3+|\psi|^3\right)\max_{|\beta|=3}\sup_{\tau\in[0,1]}\left|D^\beta\phi\left(x+\tau\left(z-x\right)\right)\right|d\psi,
\end{aligned}
$$

where $z := e^{tS}x-\psi$, $C_\beta = \sum_{|\beta|=3}\frac{1}{\beta!}$. We now must distinguish between four cases: Let $R \geqslant 1$ be arbitrary.

Case 1: ($|x| \geqslant R$, $|\psi| \leqslant \frac{|x|}{4}$). In this case we use $\phi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$, i.e. that ϕ has nice behavior in the far-field. Given $\varepsilon > 0$ we now choose $t_0 = t_0(\varepsilon) > 0$ such that both $\left|\frac{1}{t}\left(e^{tS}-I_d\right)\right| \leqslant |S|+\varepsilon$ and $t\left(|S|+\varepsilon\right) \leqslant \frac{1}{2}$ is satisfied for every $0 < t \leqslant t_0$, then we have

$$\begin{aligned}
&\left|x+\tau\left(e^{tS}x-x-\psi\right)\right| \geqslant |x| - \tau\left|e^{tS}x-x\right| - \tau|\psi| \geqslant |x| - \left|e^{tS}x-x\right| - |\psi| \\
\geqslant&\left(1-t\left|\frac{e^{tS}-I_d}{t}\right|\right)|x| - |\psi| \geqslant \left(1-t\left(|S|+\varepsilon\right)\right)|x| - |\psi| \geqslant \frac{|x|}{2} - |\psi| \geqslant \frac{|x|}{4}.
\end{aligned}$$

Moreover, since $\phi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$, we have

$$\forall\,\alpha,\beta\in\mathbb{N}_0^d\;\exists\,C_{\alpha,\beta}>0:\;\left|y^\alpha D^\beta\phi(y)\right| \leqslant C_{\alpha,\beta}\;\forall\,y\in\mathbb{R}^d,$$

and therefore, for arbitrary $R_0 > 0$

$$\left|D^\beta\phi(y)\right| \leqslant C_{\alpha,\beta}\,|y|^{-|\alpha|}\;\;\forall\,y\in\mathbb{R}^d,\;|y|\geqslant R_0. \tag{5.19}$$

Thus, using (5.17) with $B=0$ for $k=0$ and $k=3$, we obtain, $z := e^{tS}x-\psi$

$$\begin{aligned}
&\frac{4C_\beta}{t}\int_{|\psi|\leqslant\frac{|x|}{4}} |K(\psi,t)|_2\left(\left|e^{tS}x-x\right|^3+|\psi|^3\right)\max_{\substack{|\beta|=3\\ \tau\in[0,1]}}\left|D^\beta\phi\left(x+\tau\left(z-x\right)\right)\right|d\psi \\
\leqslant&4C_\beta\int_{|\psi|\leqslant\frac{|x|}{4}} |K(\psi,t)|_2\left(t^2\left|\frac{e^{tS}-I_d}{t}\right|^3|x|^3+\frac{1}{t}|\psi|^3\right) \\
&\cdot\max_{|\beta|=3}\sup_{\tau\in[0,1]} C_{\alpha,\beta}\left|x+\tau\left(e^{tS}x-x-\psi\right)\right|^{-|\alpha|}d\psi \\
\leqslant&4C_\beta\int_{|\psi|\leqslant\frac{|x|}{4}} |K(\psi,t)|_2\left(t^2\left(|S|+\varepsilon\right)^3|x|^3+\frac{1}{t}|\psi|^3\right)\max_{|\beta|=3}C_{\alpha,\beta}4^{|\alpha|}|x|^{-|\alpha|}d\psi \\
\leqslant&4^{|\alpha|+1}C_\beta C_\phi\Bigg[t^2\left(|S|+\varepsilon\right)^3|x|^{-(|\alpha|-3)}\int_{\mathbb{R}^d}|K(\psi,t)|_2\,d\psi \\
&+\frac{1}{t}|x|^{-|\alpha|}\int_{\mathbb{R}^d}|K(\psi,t)|_2|\psi|^3\,d\psi\Bigg] \\
=&4^{|\alpha|+1}C_\beta C_\phi M^{\frac{d}{2}}\left[t^2\left(|S|+\varepsilon\right)^3|x|^{-(|\alpha|-3)}+t^{\frac{1}{2}}|x|^{-|\alpha|}\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{4a_{\max}^2}{a_0}\right)^{\frac{3}{2}}\right] \\
\leqslant&4^{|\alpha|+1}C_\beta C_\phi M^{\frac{d}{2}}\left[t_0^2\left(|S|+\varepsilon\right)^3+t_0^{\frac{1}{2}}\frac{1}{R^3}\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{4a_{\max}^2}{a_0}\right)^{\frac{3}{2}}\right]|x|^{-(|\alpha|-3)} =: h_1(x)
\end{aligned}$$

for every $0 < t \leqslant t_0$ and $|x| \geqslant R$, where $C_\phi := \max_{|\beta|=3} C_{\alpha,\beta}$. Here, we must choose $|\alpha| > \frac{d}{p}+3$ to guarantee the L^p–integrability of $h_1(x)$ in $|x| \geqslant R$, since

$$\int_a^\infty s^{-n}ds = \frac{a^{1-n}}{n-1},\;n\in\mathbb{N}\text{ with }n>1,\;a\in\mathbb{R}\text{ with }a>0, \tag{5.20}$$

and

$$\int_{|x|\geqslant R}|x|^{-(|\alpha|-3)p}dx = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_R^\infty r^{-((|\alpha|-3)p-(d-1))}dr.$$

Case 2: ($|x| \geqslant R$, $|\psi| \geqslant \frac{|x|}{4}$). In this case we must use that $K(\cdot,t) \in \mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$, i.e. the kernel $K(\cdot,t)$ is a Schwartz function and therefore, it has nice behavior in the far-field. First of all, using $e^{-s^2} \in \mathcal{S}(\mathbb{R},\mathbb{R})$, i.e.

$$\forall\, m \in \mathbb{N}_0 \;\forall\, R > 0 \;\exists\, C_{R,m} > 0 : \; \left|e^{-s^2}\right| \leqslant C_{R,m} \left|s\right|^{-m} \;\; \forall \left|s\right| \geqslant R$$

and (5.20), we deduce

$$\begin{aligned}
&\int_{|\psi|\geqslant\frac{|x|}{4}} \left|K(\psi,t)\right|_2 \left|\psi\right|^k d\psi \leqslant \int_{|\psi|\geqslant\frac{|x|}{4}} (4\pi t a_{\min})^{-\frac{d}{2}} e^{-\frac{a_0}{4ta_{\max}^2}|\psi|^2} \left|\psi\right|^k d\psi \\
=& (4\pi t a_{\min})^{-\frac{d}{2}} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_{\frac{|x|}{4}}^{\infty} r^{d-1} e^{-\frac{a_0}{4ta_{\max}^2}r^2} r^k dr \\
=& \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d+k}{2}} (4ta_{\min})^{\frac{k}{2}} \frac{2}{\Gamma\left(\frac{d}{2}\right)} \int_{\left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}}\frac{|x|}{4}}^{\infty} s^{d-1} e^{-s^2} s^k ds \\
\leqslant& \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d+k}{2}} (4ta_{\min})^{\frac{k}{2}} \frac{2}{\Gamma\left(\frac{d}{2}\right)} \int_{\left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}}\frac{|x|}{4}}^{\infty} s^{d-1+k-m} ds \\
=& \frac{2M^{\frac{d+k}{2}} (4ta_{\min})^{\frac{k}{2}}}{(m-d-k)\Gamma\left(\frac{d}{2}\right)} \left[\left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}} \frac{|x|}{4}\right]^{-(m-d-k)} =: Ct^{\frac{m-d}{2}} \left|x\right|^{-(m-d-k)}
\end{aligned}$$

whenever $m \geqslant d+k+1$. Therefore, we obtain for $0 < t \leqslant t_0$, $z := e^{tS}x - \psi$

$$\begin{aligned}
&\frac{4C_\beta}{t} \int_{|\psi|\geqslant\frac{|x|}{4}} \left|K(\psi,t)\right|_2 \left(\left|e^{tS}x - x\right|^3 + \left|\psi\right|^3\right) \max_{\substack{|\beta|=3\\ \tau\in[0,1]}} \left|D^\beta \phi\left(x + \tau\left(z-x\right)\right)\right| d\psi \\
\leqslant& \frac{4C_\beta C_\phi}{t} \int_{|\psi|\geqslant\frac{|x|}{4}} \left|K(\psi,t)\right|_2 \left(t^3 \left|\frac{e^{tS}-I_d}{t}\right|^3 4^3 + 1\right) \left|\psi\right|^3 d\psi \\
\leqslant& \frac{4C_\beta C_\phi}{t} \left(\frac{4^3}{2^3} + 1\right) \int_{|\psi|\geqslant\frac{|x|}{4}} \left|K(\psi,t)\right|_2 \left|\psi\right|^3 d\psi \\
\leqslant& 4C_\beta C_\phi \left(\frac{4^3}{2^3} + 1\right) Ct^{\frac{m-d-2}{2}} \left|x\right|^{-(m-d-3)} \\
\leqslant& 4C_\beta C_\phi \left(\frac{4^3}{2^3} + 1\right) Ct_0^{\frac{m-d-2}{2}} \left|x\right|^{-(m-d-3)} =: h_2(x)
\end{aligned}$$

for every $0 < t \leqslant t_0$ and $|x| \geqslant R$, where $C_\phi := \max_{|\beta|=3} \sup_{y\in\mathbb{R}^d} \left|D^\beta\phi(y)\right|$. Here, we must choose $m > \frac{d}{p} + d + 3$ to guarantee L^p-integrability in $|x| \geqslant R$.

Case 3: ($|x| \leqslant R$, $|\psi| \geqslant \frac{|x|}{4}$). In this case we use that Schwartz functions, as e.g. ϕ and their derivatives, are bounded on compact sets, e.g. on $B_R(0)$. Using (5.17) with $B = 0$ for $k = 3$, we obtain with $z := e^{tS}x - \psi$

$$\frac{4C_\beta}{t} \int_{|\psi|\geqslant\frac{|x|}{4}} \left|K(\psi,t)\right|_2 \left(\left|e^{tS}x - x\right|^3 + \left|\psi\right|^3\right) \max_{\substack{|\beta|=3\\ \tau\in[0,1]}} \left|D^\beta \phi\left(x + \tau\left(z-x\right)\right)\right| d\psi$$

$$
\begin{aligned}
&\leqslant\frac{4C_\beta C_\phi}{t}\int_{|\psi|\geqslant\frac{|x|}{4}}|K(\psi,t)|_2\left(t^3\left|\frac{e^{tS}-I_d}{t}\right|^3|x|^3+|\psi|^3\right)d\psi\\
&\leqslant\frac{4C_\beta C_\phi}{t}\left(\frac{4^3}{2^3}+1\right)\int_{|\psi|\geqslant\frac{|x|}{4}}|K(\psi,t)|_2\,|\psi|^3\,d\psi\\
&\leqslant\frac{4C_\beta C_\phi}{t}\left(\frac{4^3}{2^3}+1\right)\int_{\mathbb{R}^d}|K(\psi,t)|_2\,|\psi|^3\,d\psi\\
&=4C_\beta C_\phi\left(\frac{4^3}{2^3}+1\right)M^{\frac{d}{2}}\left(\frac{4a_{\max}^2}{a_0}\right)^{\frac{3}{2}}\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}t^{\frac{1}{2}}\\
&\leqslant4C_\beta C_\phi\left(\frac{4^3}{2^3}+1\right)M^{\frac{d}{2}}\left(\frac{4a_{\max}^2}{a_0}\right)^{\frac{3}{2}}\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}t_0^{\frac{1}{2}}=:h_3
\end{aligned}
$$

for every $0<t\leqslant t_0$ and $|x|\leqslant R$, where $C_\phi:=\max_{|\beta|=3}\sup_{y\in\mathbb{R}^d}\left|D^\beta\phi(y)\right|$.

Case 4: ($|x|\leqslant R$, $|\psi|\leqslant\frac{|x|}{4}$). This case is similar to case 3. Using (5.17) with $B=0$ for $k=0$ and $k=3$, we obtain for $z:=e^{tS}x-\psi$

$$
\begin{aligned}
&\frac{4C_\beta}{t}\int_{|\psi|\leqslant\frac{|x|}{4}}|K(\psi,t)|_2\left(\left|e^{tS}x-x\right|^3+|\psi|^3\right)\max_{\substack{|\beta|=3\\ \tau\in[0,1]}}\left|D^\beta\phi\left(x+\tau\left(z-x\right)\right)\right|d\psi\\
&\leqslant4C_\beta C_\phi\int_{|\psi|\leqslant\frac{|x|}{4}}|K(\psi,t)|_2\left(t^2\left|\frac{e^{tS}-I_d}{t}\right|^3|x|^3+\frac{1}{t}|\psi|^3\right)d\psi\\
&\leqslant4C_\beta C_\phi\left[t_0^2\left(|S|+\varepsilon\right)^3R^3\int_{\mathbb{R}^d}|K(\psi,t)|_2\,d\psi+\frac{1}{t}\int_{\mathbb{R}^d}|K(\psi,t)|_2\,|\psi|^3\,d\psi\right]\\
&=4C_\beta C_\phi M^{\frac{d}{2}}\left[t_0^2\left(|S|+\varepsilon\right)R^3+\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{4a_{\max}^2}{a_0}\right)^{\frac{3}{2}}t^{\frac{1}{2}}\right]\\
&\leqslant4C_\beta C_\phi M^{\frac{d}{2}}\left[t_0^2\left(|S|+\varepsilon\right)R^3+\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{4a_{\max}^2}{a_0}\right)^{\frac{3}{2}}t_0^{\frac{1}{2}}\right]=:h_4
\end{aligned}
$$

for every $0<t\leqslant t_0$ and $|x|\leqslant R$, where $C_\phi:=\max_{|\beta|=3}\sup_{y\in\mathbb{R}^d}\left|D^\beta\phi(y)\right|$.

Now choosing $|\alpha|=\frac{d}{p}+4$ and $m=\frac{d}{p}+d+4$ and defining

$$
h:\mathbb{R}^d\to\mathbb{R},\quad h(x):=\begin{cases}\max\{h_3,h_4\} & ,\ |x|\leqslant R\\ \max\{h_1(x),h_2(x)\} & ,\ |x|\geqslant R\end{cases}
$$

we deduce from (5.18)

$$
\begin{aligned}
|f_t(x)|\leqslant&A\sum_{i=1}^d\left|D_i^2\phi(x)\right|+\sum_{i=1}^d\left(|S|+\varepsilon\right)|x|\,|D_i\phi(x)|\\
&+\frac{t_0}{2}\sum_{i=1}^d\sum_{j=1}^d\left(|S|+\varepsilon\right)^2|x|^2\,|D_jD_i\phi(x)|+h(x)=:g(x)
\end{aligned}
$$

for every $0 < t \leqslant t_0$.
(c): Using the decomposition

$$\|g\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)} = \int_{|x|\geqslant R} |g(x)|^p \, dx + \int_{|x|\leqslant R} |g(x)|^p \, dx$$

and (5.19) since $\phi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$, we deduce $g \in L^p(\mathbb{R}^d, \mathbb{R})$ and the application of dominated convergence is justified.
(**3**): The proof can partially be found in [34, II.2.13]. Let $\phi \in \mathcal{S} := \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$.
1. Recall the (d-dimensional) diffusion semigroup $(G(t,0))_{t\geqslant 0}$ from (5.10)

$$\begin{aligned} \left[G(t,0)\phi\right](y) &:= \int_{\mathbb{R}^d} H_0(e^{-tS}y, \xi, t)\phi(\xi)d\xi \\ &= \int_{\mathbb{R}^d} (4\pi t A)^{-\frac{d}{2}} \exp\left(-(4tA)^{-1}|y-\xi|^2\right)\phi(\xi)d\xi \end{aligned}$$

and recall the kernel K from (4.24) with $B = 0$

$$K(\psi, t) = (4\pi t A)^{-\frac{d}{2}} \exp\left(-(4tA)^{-1}|\psi|^2\right),$$

which satisfies $K(\cdot, t) \in \mathcal{S}$ for every $t > 0$, see [34, VI.5.3 Example]. Then we have

$$\left[T_0(t)\phi\right](x) = \left[G(t,0)\phi\right](e^{tS}x), \quad \left[G(t,0)\phi\right](x) = \left[K(t) * \phi\right](x)$$

and hence

$$\left[T_0(t)\phi\right](x) = \left[G(t,0)\phi\right](e^{tS}x) = \left[K(t) * \phi\right](e^{tS}x). \tag{5.21}$$

2. First we show that

$$\left[\mathcal{F}\phi(e^{tS}\cdot)\right](\xi) = \left[\mathcal{F}\phi(\cdot)\right](e^{tS}\xi) \quad \forall\, \phi \in \mathcal{S}, \tag{5.22}$$

where $\mathcal{F}\phi$ denotes the Fourier transform of $\phi \in \mathcal{S}$. From the transformation theorem (with transformation $\Phi(x) = e^{tS}x$), (A5) and the definition of the Fourier transform [34, VI.5.2 Definition] we obtain

$$\begin{aligned} \left[\mathcal{F}\phi(e^{tS}\cdot)\right](\xi) &:= \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle}\phi(e^{tS}x)dx = \int_{\mathbb{R}^d} e^{-i\left\langle e^{-tS}y,\xi\right\rangle}\phi(y)dy \\ &= \int_{\mathbb{R}^d} e^{-i\left\langle y, e^{tS}\xi\right\rangle}\phi(y)dy = \left[\mathcal{F}\phi(\cdot)\right](e^{tS}\xi). \end{aligned}$$

3. Next we show that

$$\left[\mathcal{F}\left[T_0(t)\phi\right](\cdot)\right](\xi) = \left[\mathcal{F}K(\cdot,t)\right](e^{tS}\xi) \cdot \left[\mathcal{F}\phi\right](e^{tS}\xi). \tag{5.23}$$

From (5.21) and (5.22) we obtain for every $t > 0$

$$\begin{aligned} \left[\mathcal{F}\left[T_0(t)\phi\right](\cdot)\right](\xi) &= \left[\mathcal{F}\left[K(t)*\phi\right](e^{tS}\cdot)\right](\xi) = \left[\mathcal{F}\left[K(t)*\phi\right](\cdot)\right](e^{tS}\xi) \\ &= \left[(\mathcal{F}K(t))(\cdot)\cdot(\mathcal{F}\phi)(\cdot)\right](e^{tS}\xi) = \left[\mathcal{F}K(t)\right](e^{tS}\xi)\cdot\left[\mathcal{F}\phi\right](e^{tS}\xi). \end{aligned}$$

4. Since $\phi\in\mathcal{S}$ it follows that $[\mathcal{F}\phi](\cdot)\in\mathcal{S}$ and thus $[\mathcal{F}\phi](e^{tS}\cdot)\in\mathcal{S}$ for every $t\geqslant 0$. Analogously, since $K(\cdot,t)\in\mathcal{S}$ for every $t>0$ it follows that $[\mathcal{F}K(t)](\cdot)\in\mathcal{S}$ and hence $[\mathcal{F}K(t)](e^{tS}\cdot)\in\mathcal{S}$ for every $t>0$. Using (5.23) we deduce that $[\mathcal{F}[T_0(t)\phi](\cdot)](\cdot)\in\mathcal{S}$ for every $t>0$ (since $\mathcal{S}$ is closed under pointwise multiplication), i.e. $\mathcal{F}(T_0(t)\mathcal{S})\subset\mathcal{S}$ for every $t>0$ and hence $T_0(t)\mathcal{S}\subset\mathcal{F}^{-1}(\mathcal{S})=\mathcal{S}$ for every $t>0$, see [90, II.7.7 The inversion theorem]. The case $t=0$ follows directly from the definition of T_0 in (5.3), that gives $T(0)\mathcal{S}=\mathcal{S}$.
(**4**): Using Theorem 5.10 (1)-(3) the assertion can be deduced by [34, II.1.7 Proposition]. □

Remark. Indeed, one can show that also $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d,\mathbb{C}^N)$ is a core for $(A_p,\mathcal{D}(A_p))$, but not with the same argument as before. Since $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d,\mathbb{C}^N)$ is not invariant under the semigroup $(T_0(t))_{t\geqslant 0}$, we cannot apply [34, II.1.7 Proposition]. In this case one must perform a direct proof as in [71, Proposition 3.2].

5.4 Resolvent estimates

In this section we prove resolvent estimates for $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$. For this purpose, we consider the formal Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x)=A\triangle v(x)+\langle Sx,\nabla v(x)\rangle\,,\ x\in\mathbb{R}^d$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$ and define the domain

$$\begin{aligned}\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)&:=\left\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p(\mathbb{R}^d,\mathbb{C}^N)\mid A\triangle v+\langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}\\&=\left\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p(\mathbb{R}^d,\mathbb{C}^N)\mid \mathcal{L}_0 v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.\end{aligned}$$

The following lemma states that $\mathcal{L}_0:\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ is a closed operator in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$, which justifies to define the resolvent of $\mathcal{L}_0$. A proof for the real-valued case can be found in [74, Lemma 3.1], that uses a local elliptic L^p-regularity result from [40, Theorem 9.11].

Lemma 5.11. *Let the assumption* (A3) *be satisfied for $\mathbb{K}=\mathbb{C}$, then the operator $\mathcal{L}_0:\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ is closed in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$.*

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be such that $v_n\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ converges to $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $\mathcal{L}_0 v_n$ converges to $u\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ both w.r.t. $\|\cdot\|_{L^p}$. To show the closedness of $\mathcal{L}_0$ we must verify that $v\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and $\mathcal{L}_0 v=u$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$.

Let $\Omega\subset\mathbb{R}^d$ be an open bounded set. From $\mathcal{L}_0 v_n\to u$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ we infer that $\mathcal{L}_0 v_n|_\Omega\to u|_\Omega$ in $L^p(\Omega,\mathbb{C}^N)$ and therefore, $(\mathcal{L}_0 v_n|_\Omega)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega,\mathbb{C}^N)$. Analogously, we deduce from $v_n\to v$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ that $v_n|_\Omega\to v|_\Omega$ in $L^p(\Omega,\mathbb{C}^N)$ and thus $(v_n|_\Omega)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega,\mathbb{C}^N)$. Since Sx is bounded in Ω by the boundedness of Ω, [40, Theorem 9.11] yields that for every $\Omega'\subset\subset\Omega$ there exists some constant $C=C(\Omega',\Omega,p,A,S,d)>0$ such that

$$\begin{aligned}&\|v_n|_{\Omega'}-v_m|_{\Omega'}\|_{W^{2,p}(\Omega',\mathbb{C}^N)}\\\leqslant&C\left(\|v_n|_\Omega-v_m|_\Omega\|_{L^p(\Omega,\mathbb{C}^N)}+\|\mathcal{L}_0 v_n|_\Omega-\mathcal{L}_0 v_m|_\Omega\|_{L^p(\Omega,\mathbb{C}^N)}\right)\leqslant\varepsilon.\end{aligned}$$

Therefore, $(v_n|_{\Omega'})_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{2,p}(\Omega',\mathbb{C}^N)$ and consequently, there exists some $v^{\Omega'} \in W^{2,p}(\Omega',\mathbb{C}^N)$ such that $v_n|_{\Omega'} \to v^{\Omega'}$ in $W^{2,p}(\Omega',\mathbb{C}^N)$ and hence in particular in $L^p(\Omega',\mathbb{C}^N)$. Moreover, since $v_n \to v$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ we deduce $v_n|_{\Omega'} \to v|_{\Omega'}$ in $L^p(\Omega',\mathbb{C}^N)$. Therefore, $v^{\Omega'} = v|_{\Omega'}$ in $L^p(\Omega',\mathbb{C}^N)$ and we further infer that $v_n|_{\Omega'} \to v|_{\Omega'}$ in $W^{2,p}(\Omega',\mathbb{C}^N)$ and $v|_{\Omega'} \in W^{2,p}(\Omega',\mathbb{C}^N)$.

Now, by the arbitrariness of Ω and Ω' we deduce that $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)$. Moreover, $v_n|_{\Omega'} \to v|_{\Omega'} \in W^{2,p}(\Omega',\mathbb{C}^N)$ implies $\mathcal{L}_0 v_n|_{\Omega'} \to \mathcal{L}_0 v|_{\Omega'}$ in $L^p(\Omega',\mathbb{C}^N)$ and hence $\mathcal{L}_0 v|_{\Omega'} = u|_{\Omega'}$ in $L^p(\Omega',\mathbb{C}^N)$. By arbitrariness of Ω and Ω' we deduce $\mathcal{L}_0 v = u \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and thus $v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. $\square$

Since $(\mathcal{L}_0,\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0))$ is a closed operator on the Banach space $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$, we can introduce

$$\begin{aligned}
\sigma(\mathcal{L}_0) &:= \{\lambda\in\mathbb{C} \mid \lambda I - \mathcal{L}_0 \text{ is not bijective}\} && \textbf{spectrum of } \mathcal{L}_0,\\
\rho(\mathcal{L}_0) &:= \mathbb{C}\backslash\sigma(\mathcal{L}_0) && \textbf{resolvent set of } \mathcal{L}_0,\\
R(\lambda,\mathcal{L}_0) &:= (\lambda I - \mathcal{L}_0)^{-1}, \text{ for } \lambda\in\rho(\mathcal{L}_0) && \textbf{resolvent of } \mathcal{L}_0.
\end{aligned}$$

The following Lemma 5.12 is crucial in order to derive an optimal L^p-dissipativity condition as well as resolvent estimates for $\mathcal{L}_0$. This is a complex-valued version of [73, Lemma 2.1].

Lemma 5.12. *Let the assumption* (A3) *be satisfied for* $\mathbb{K}=\mathbb{C}$. *Moreover, let* $\Omega\subset\mathbb{R}^d$ *be a bounded domain with a* C^2*-boundary or* $\Omega=\mathbb{R}^d$, $1<p<\infty$, $v\in W^{2,p}(\Omega,\mathbb{C}^N)\cap W^{1,p}_0(\Omega,\mathbb{C}^N)$ *and* $\eta\in C^1_b(\Omega,\mathbb{R})$ *be nonnegative, then*

$$\begin{aligned}
&-\operatorname{Re}\int_\Omega \eta\overline{v}^T|v|^{p-2}A\triangle v\\
\geqslant&(p-1)\operatorname{Re}\int_\Omega \eta|v|^{p-2}\sum_{j=1}^d \overline{D_j v}^T A D_j v \mathbb{1}_{\{v\neq 0\}} + \operatorname{Re}\int_\Omega \overline{v}^T|v|^{p-2}\sum_{j=1}^d D_j\eta A D_j v\\
&+(p-2)\operatorname{Re}\int_\Omega \eta|v|^{p-4}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_j v}^T v\right)\overline{v}^T - |v|^2\overline{D_j v}^T\right]AD_j v\mathbb{1}_{\{v\neq 0\}}.
\end{aligned}$$

Remark. For the parameter regime $2\leqslant p<\infty$ Lemma 5.12 follows directly from the integration by parts formula and therefore, the estimate is satisfied with equality. In this case, the real parts in front of the integrals can also be dropped and the assumption (A3) is not used. If $1<p<2$, then Lemma 5.12 is satisfied only with inequality, which is a direct consequence of Fatou's lemma. In particular, in this case we need the assumption (A3) that yields positivity of the quadratic term, that is necessary for the application of Fatou's lemma.

Proof. We only provide the proof for $\Omega\subset\mathbb{R}^d$ bounded. In case $\Omega=\mathbb{R}^d$ integration by parts yields no boundary terms due to decay at infinity and thus it can be treated in an analogous way but without boundary integrals. Let $\Omega\subset\mathbb{R}^d$ be bounded with C^2-boundary $\partial\Omega$.

Case 1: $(2 \leqslant p < \infty)$. Multiplying $-A\triangle v$ from left by $\eta\overline{v}^T\,|v|^{p-2}$, integrating over Ω and using integration by parts formula we obtain

$$\begin{aligned}
&-\int_\Omega \eta\overline{v}^T|v|^{p-2}A\triangle v = -\sum_{j=1}^d\int_\Omega \eta\overline{v}^T|v|^{p-2}AD_j^2 v\\
=&\sum_{j=1}^d\int_\Omega (D_j\eta)\overline{v}^T|v|^{p-2}AD_jv+\sum_{j=1}^d\int_\Omega \eta D_j(\overline{v}^T|v|^{p-2})AD_jv\\
=&\sum_{j=1}^d\int_\Omega (D_j\eta)\overline{v}^T|v|^{p-2}AD_jv+(p-1)\sum_{j=1}^d\int_\Omega \eta|v|^{p-2}\overline{D_jv}^TAD_jv\mathbb{1}_{\{v\neq 0\}}\\
&+(p-2)\sum_{j=1}^d\int_\Omega \eta|v|^{p-4}\left[\mathrm{Re}\,(\overline{D_jv}^Tv)\overline{v}^T-|v|^2\overline{D_jv}^T\right]AD_jv\mathbb{1}_{\{v\neq 0\}}\\
=&(p-1)\int_\Omega \eta|v|^{p-2}\sum_{j=1}^d\overline{D_jv}^TAD_jv\mathbb{1}_{\{v\neq 0\}}+\int_\Omega \overline{v}^T\,|v|^{p-2}\sum_{j=1}^d D_j\eta AD_jv\\
&+(p-2)\int_\Omega \eta|v|^{p-4}\sum_{j=1}^d\left[\mathrm{Re}\,\left(\overline{D_jv}^Tv\right)\overline{v}^T-|v|^2\overline{D_jv}^T\right]AD_jv\mathbb{1}_{\{v\neq 0\}}.
\end{aligned}$$

Now applying real parts we deduce the desired estimates with equality. In the computations above we used the following auxiliaries: The relation $z+\overline{z}=2\mathrm{Re}\,z$ yields

$$\begin{aligned}
&D_j\left(|v|^p\right)=D_j\left(\left(|v|^2\right)^{\frac{p}{2}}\right)=\frac{p}{2}\left(|v|^2\right)^{\frac{p}{2}-1}D_j\left(|v|^2\right)=\frac{p}{2}|v|^{p-2}D_j(\overline{v}^Tv)\\
=&\frac{p}{2}|v|^{p-2}\left[\overline{D_jv}^Tv+\overline{v}^TD_jv\right]=\frac{p}{2}|v|^{p-2}\left[\overline{D_jv}^Tv+\overline{\overline{D_jv}^Tv}^T\right]\\
=&p|v|^{p-2}\mathrm{Re}\,\left(\overline{D_jv}^Tv\right)
\end{aligned}\tag{5.24}$$

for every $v\in\mathbb{C}^N$, $p\geqslant 2$ and $j=1,\ldots,d$. This formula remains valid for every $p\geqslant 0$ and $v\neq 0$. Using the formula (5.24) we obtain for every $v\neq 0$ and $p\geqslant 2$

$$\begin{aligned}
&D_j\left(\overline{v}^T|v|^{p-2}\right)=\overline{D_jv}^T|v|^{p-2}+\overline{v}^TD_j\left(|v|^{p-2}\right)\\
=&\overline{D_jv}^T|v|^{p-2}+(p-2)\overline{v}^T|v|^{p-4}\mathrm{Re}\,\left(\overline{D_jv}^Tv\right)\\
=&(p-1)|v|^{p-2}\overline{D_jv}^T+(p-2)|v|^{p-4}\left[\mathrm{Re}\,\left(\overline{D_jv}^Tv\right)\overline{v}^T-|v|^2\overline{D_jv}^T\right].
\end{aligned}$$

Case 2: $(1<p<2)$. This case is much more involved and one has to be very careful, since the expression $|v|^p$ is not differentiable at $v=0$ for $1<p<2$. We prove the assertion in three steps.
1. First we consider $v\in C^2(\overline{\Omega},\mathbb{C}^N)\cap C_{\mathrm{c}}(\Omega,\mathbb{C}^N)$. Multiplying $-A\triangle v$ from left by $\eta\overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}$ for some $\varepsilon>0$, integrating over Ω and using integration by parts formula we obtain

$$
\begin{aligned}
&-\int_\Omega \eta\overline{v}^T \left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1} A\triangle v = -\sum_{j=1}^d \int_\Omega \eta\overline{v}^T \left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1} AD_j^2 v\\
=&\sum_{j=1}^d \left[\int_\Omega D_j\left(\eta\overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\right)AD_jv - \int_{\partial\Omega}\eta\overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}AD_jv\nu^j dS\right]\\
=&\sum_{j=1}^d \left[\int_\Omega (D_j\eta)\overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}AD_jv + \int_\Omega \eta D_j\left(\overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\right)AD_jv\right]\\
=&\int_\Omega \eta\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\left((p-1)|v|^2+\varepsilon\right)\sum_{j=1}^d \overline{D_jv}^T AD_jv\\
&+\int_\Omega \overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\sum_{j=1}^d D_j\eta AD_jv\\
&+(p-2)\int_\Omega \eta\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\sum_{j=1}^d\left[\operatorname{Re}(\overline{D_jv}^T v)\overline{v}^T - |v|^2\overline{D_jv}^T\right]AD_jv.
\end{aligned}
$$

The boundary integral vanishes because from $v \in C_{\mathrm{c}}(\Omega,\mathbb{C}^N)$ follows $\overline{v}(x)=0$ for every $x\in\partial\Omega$. Moreover, we used the relations

$$
\begin{aligned}
&D_j\left(\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\right) = \left(\frac{p}{2}-1\right)\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}D_j\left(|v|^2+\varepsilon\right)\\
=&(p-2)\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\operatorname{Re}\left(\overline{D_jv}^T v\right),
\end{aligned}
$$

cf. (5.24) for $p=2$, and

$$
\begin{aligned}
&D_j\left(\overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\right) = \overline{D_jv}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1} + \overline{v}^T D_j\left(\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\right)\\
=&\overline{D_jv}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1} + (p-2)\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\overline{v}^T\operatorname{Re}\left(\overline{D_jv}^T v\right)\\
=&\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\left[\overline{D_jv}^T\left(|v|^2+\varepsilon\right)+(p-2)\overline{v}^T\operatorname{Re}\left(\overline{D_jv}^T v\right)\right]\\
=&\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\left((p-1)|v|^2+\varepsilon\right)\overline{D_jv}^T\\
&+\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}(p-2)\left[\operatorname{Re}\left(\overline{D_jv}^T v\right)\overline{v}^T - |v|^2\overline{D_jv}^T\right].
\end{aligned}
$$

Note that both formulas are valid for $1<p<2$, $v\in\mathbb{C}$ and $j=1,\ldots,d$ if $\varepsilon>0$ and for $1<p<2$, $v\neq 0$ and $j=1,\ldots,d$ if $\varepsilon=0$.
2. We now apply Lebesgue's dominated convergence theorem, see [7, A1.21]: Putting the last two terms of the equation from step 1 to the left hand side, taking the limit $\varepsilon\to 0$ and applying dominated convergence twice we obtain

$$
\begin{aligned}
&(p-1)\int_\Omega \eta|v|^{p-2}\sum_{j=1}^d \overline{D_jv}^T AD_jv\mathbb{1}_{\{v\neq 0\}}\\
=&\lim_{\varepsilon\to 0}\int_\Omega \eta\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\left((p-1)|v|^2+\varepsilon\right)\sum_{j=1}^d\overline{D_jv}^T AD_jv
\end{aligned}
$$

$$
\begin{aligned}
=&-\lim_{\varepsilon\to 0}\Bigg[\int_\Omega \eta\overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}A\triangle v+\int_\Omega \overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\sum_{j=1}^d D_j\eta AD_jv\\
&+(p-2)\int_\Omega \eta\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_jv}^Tv\right)\overline{v}^T-|v|^2\overline{D_jv}^T\right]AD_jv\Bigg]\\
=&-\int_\Omega \eta\overline{v}^T|v|^{p-2}A\triangle v-\int_\Omega\overline{v}^T|v|^{p-2}\sum_{j=1}^d D_j\eta AD_jv\\
&-(p-2)\int_\Omega\eta|v|^{p-4}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_jv}^Tv\right)\overline{v}^T-|v|^2\overline{D_jv}^T\right]AD_jv\mathbb{1}_{\{v\neq 0\}}.
\end{aligned}
$$

To justify the applications of Lebesgue's theorem, we discuss the assumptions in both cases: First, we define

$$
\begin{aligned}
f_\varepsilon:=&\eta\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\left((p-1)|v|^2+\varepsilon\right)\sum_{j=1}^d\overline{D_jv}^TAD_jv\\
f:=&(p-1)\eta|v|^{p-2}\sum_{j=1}^d\overline{D_jv}^TAD_jv.
\end{aligned}
$$

Using $v\in C^2(\overline{\Omega},\mathbb{C}^N)\cap C_{\mathrm{c}}(\Omega,\mathbb{C}^N)$, $\eta\in C_{\mathrm{b}}(\Omega,\mathbb{R})$ and $(|v|^2+\varepsilon)^{\frac{p}{2}-k}\leqslant|v|^{p-2k}$ for $k=1,2$ and $1<p<2$ we obtain that f_ε is dominated by g as follows

$$
\begin{aligned}
|f_\varepsilon|=&\left|\eta\left((p-2)|v|^2\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}+\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\right)\sum_{j=1}^d\overline{D_jv}^TAD_jv\right|\\
\leqslant&|\eta|\left(|p-2|+1\right)|v|^{p-2}|A|\sum_{j=1}^d|D_jv|^2\\
=&|p-3||A||\eta||v|^{p-2}|\nabla v|^2\mathbb{1}_{\{v\neq 0\}}\\
\leqslant&|p-3||A|\left\|\eta\right\|_\infty\left\|v\right\|_\infty^{p-2}\left\|\nabla v\right\|_\infty^2\mathbb{1}_{\{v\neq 0\}}=:g.
\end{aligned}
$$

Since v is compactly supported, i.e. $\mathbb{1}_{\{v\neq 0\}}$ is compact, g belongs to $L^1(\Omega,\mathbb{R})$. In particular, $f_\varepsilon\to f$ pointwise a.e. as $\varepsilon\to 0$. Thus, by dominated convergence, $f_\varepsilon,f\in L^1(\Omega,\mathbb{C}^N)$ and $f_\varepsilon\to f$ in $L^1(\Omega,\mathbb{C}^N)$ as $\varepsilon\to 0$. Next, consider

$$
\begin{aligned}
f_\varepsilon:=&\overline{v}^T\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1}\left(\eta A\triangle v+\sum_{j=1}^d D_j\eta AD_jv\right)\\
&+(p-2)\eta\left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_jv}^Tv\right)\overline{v}^T-|v|^2\overline{D_jv}^T\right]AD_jv.\\
f:=&\overline{v}^T|v|^{p-2}\left(\eta A\triangle v+\sum_{j=1}^d D_j\eta AD_jv\right)\\
&+(p-2)\eta|v|^{p-4}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_jv}^Tv\right)\overline{v}^T-|v|^2\overline{D_jv}^T\right]AD_jv
\end{aligned}
$$

Using $v \in C^2(\overline{\Omega}, \mathbb{C}^N) \cap C_c(\Omega, \mathbb{C}^N)$, $\eta \in C_b^1(\Omega, \mathbb{R})$ and $(|v|^2+\varepsilon)^{\frac{p}{2}-k} \leqslant |v|^{p-2k}$ for $k=1,2$ and $1<p<2$ we obtain that f_ε is dominated by g as follows

$$\begin{aligned}
|f_\varepsilon| \leqslant & |v| \left(|v|^2+\varepsilon\right)^{\frac{p}{2}-1} \left(|\eta||A||\triangle v| + |A| \sum_{j=1}^d |D_j\eta||D_j v|\right) \\
& + |p-2||\eta| \left(|v|^2+\varepsilon\right)^{\frac{p}{2}-2} \sum_{j=1}^d \left[\left|\mathrm{Re}\left(\overline{D_j v}^T v\right)\right| |v| + |v|^2 |D_j v|\right] |A||D_j v| \\
\leqslant & |v|^{p-1} \left(|\eta||A||\triangle v| + |A| \sum_{j=1}^d |D_j\eta||D_j v|\right) \mathbb{1}_{\{v\neq 0\}} \\
& + 2|p-2||\eta||v|^{p-2} \sum_{j=1}^d |D_j v|^2 |A| \mathbb{1}_{\{v\neq 0\}} \\
\leqslant & \Big[|A| \left\|\eta\right\|_\infty \left\|v\right\|_\infty^{p-1} \left\|\triangle v\right\|_\infty + d|A| \left\|\eta\right\|_{1,\infty} \left\|v\right\|_{1,\infty} \\
& 2d|p-2||A| \left\|\eta\right\|_\infty \left\|v\right\|_\infty^{p-2} \left\|v\right\|_{1,\infty}^2 \Big] \mathbb{1}_{\{v\neq 0\}} =: g.
\end{aligned}$$

Since v is compactly supported, we deduce once more that g belongs to $L^1(\Omega, \mathbb{R})$. In particular, $f_\varepsilon \to f$ pointwise a.e. as $\varepsilon \to 0$. Thus, by dominated convergence, $f_\varepsilon, f \in L^1(\Omega, \mathbb{C}^N)$ and $f_\varepsilon \to f$ in $L^1(\Omega, \mathbb{C}^N)$ as $\varepsilon \to 0$.
3. Now let $v \in W^{2,p}(\Omega, \mathbb{C}^N) \cap W_0^{1,p}(\Omega, \mathbb{C}^N)$. In this case we use a density argument and Fatou's lemma, that yields the inequality. Note that we have to take real parts on both sides in order to apply Fatou's lemma. Since $C^2(\overline{\Omega}, \mathbb{C}^N) \cap C_c(\Omega, \mathbb{C}^N)$ is a dense subspace of $W^{2,p}(\Omega, \mathbb{C}^N) \cap W_0^{1,p}(\Omega, \mathbb{C}^N)$ w.r.t. $\|\cdot\|_{W^{2,p}}$, there exists a sequence $v_n \in C^2(\overline{\Omega}, \mathbb{C}^N) \cap C_c(\Omega, \mathbb{C}^N)$ such that $v_n \to v$ w.r.t. $\|\cdot\|_{W^{2,p}}$ as $n \to \infty$, $n \in \mathbb{N}$. Furthermore, there exists a subset $\mathbb{N}' \subset \mathbb{N}$ such that $v_n \to v$ and $\nabla v_n \to \nabla v$ pointwise a.e. as $n \to \infty$, $n \in \mathbb{N}'$. In the following we consider this subsequence $(v_n)_{n\in\mathbb{N}'} \subset C^2(\overline{\Omega}, \mathbb{C}^N) \cap C_c(\Omega, \mathbb{C}^N)$: Inserting v_n into the equation from step 2, taking real parts and the limit inferior $n \to \infty$ $(n \in \mathbb{N}')$ on both sides and applying Fatou's lemma on the left hand side we obtain

$$\begin{aligned}
& (p-1)\mathrm{Re} \int_\Omega \eta |v|^{p-2} \sum_{j=1}^d \overline{D_j v}^T A D_j v \mathbb{1}_{\{v\neq 0\}} \\
= & \int_\Omega \lim_{n\to\infty} (p-1)\eta |v_n|^{p-2} \mathrm{Re} \sum_{j=1}^d \overline{D_j v_n}^T A D_j v_n \mathbb{1}_{\{v_n\neq 0\}} \\
= & \int_\Omega \liminf_{n\to\infty} (p-1)\eta |v_n|^{p-2} \mathrm{Re} \sum_{j=1}^d \overline{D_j v_n}^T A D_j v_n \mathbb{1}_{\{v_n\neq 0\}} \\
\leqslant & \liminf_{n\to\infty} \int_\Omega (p-1)\eta |v_n|^{p-2} \mathrm{Re} \sum_{j=1}^d \overline{D_j v_n}^T A D_j v_n \mathbb{1}_{\{v_n\neq 0\}}
\end{aligned}$$

$$
\begin{aligned}
=&\liminf_{n\to\infty}\Bigg[-\operatorname{Re}\int_\Omega \eta\overline{v_n}^T|v_n|^{p-2}A\triangle v_n-\operatorname{Re}\int_\Omega \overline{v_n}^T|v_n|^{p-2}\sum_{j=1}^d D_j\eta AD_jv_n\\
&-(p-2)\operatorname{Re}\int_\Omega \eta|v_n|^{p-4}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_jv_n}^T v_n\right)\overline{v_n}^T-|v_n|^2\overline{D_jv_n}^T\right]AD_jv_n\mathbb{1}_{\{v_n\neq 0\}}\Bigg]\\
=&\lim_{n\to\infty}\Bigg[-\operatorname{Re}\int_\Omega \eta\overline{v_n}^T|v_n|^{p-2}A\triangle v_n-\operatorname{Re}\int_\Omega \overline{v_n}^T|v_n|^{p-2}\sum_{j=1}^d D_j\eta AD_jv_n\\
&-(p-2)\operatorname{Re}\int_\Omega \eta|v_n|^{p-4}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_jv_n}^T v_n\right)\overline{v_n}^T-|v_n|^2\overline{D_jv_n}^T\right]AD_jv_n\mathbb{1}_{\{v_n\neq 0\}}\Bigg]\\
=&-\operatorname{Re}\int_\Omega \eta\overline{v}^T|v|^{p-2}A\triangle v-\operatorname{Re}\int_\Omega \overline{v}^T|v|^{p-2}\sum_{j=1}^d D_j\eta AD_jv\\
&-(p-2)\operatorname{Re}\int_\Omega \eta|v|^{p-4}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_jv}^T v\right)\overline{v}^T-|v|^2\overline{D_jv}^T\right]AD_jv\mathbb{1}_{\{v\neq 0\}}.
\end{aligned}
$$

In the first equality we used the fact that $v_n\to v$ and $\nabla v_n\to\nabla v$ pointwise a.e. as $n\to\infty$, $n\in\mathbb{N}'$. The last equality can be accepted as follows: Let $f_n\to f$ in L^q and $g_n\to g$ in L^p with $\frac{1}{p}+\frac{1}{q}=1$, i.e. $q=\frac{p}{p-1}$, then $\int f_ng_n\to\int fg$ by Hölder's inequality, since

$$
\begin{aligned}
\int(f_ng_n-fg)&=\int(f_n-f)g+\int f(g_n-g)\\
&\leqslant \|f_n-f\|_{L^q}\|q_n\|_{L^p}+\|f\|_{L^q}\|g_n-g\|_{L^p}\to 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\overline{v_n}^T|v_n|^{p-2}\overset{L^q}{\to}\overline{v}^T|v|^{p-2}, && A\triangle v_n\overset{L^p}{\to}A\triangle v,\\
&\overline{v_n}^T|v_n|^{p-2}\overset{L^q}{\to}\overline{v}^T|v|^{p-2}, && AD_jv_n\overset{L^p}{\to}AD_jv,\\
&|v_n|^{p-4}\operatorname{Re}\left(\overline{D_jv_n}^T v_n\right)\overline{v_n}^T\overset{L^q}{\to}|v|^{p-4}\operatorname{Re}\left(\overline{D_jv}^T v\right)\overline{v}^T, && AD_jv_n\overset{L^p}{\to}AD_jv,\\
&|v_n|^{p-2}\overline{D_jv_n}^T\overset{L^q}{\to}|v|^{p-2}\overline{D_jv}^T, && AD_jv_n\overset{L^p}{\to}AD_jv,
\end{aligned}
$$

together with $\eta\in C_{\mathrm{b}}^1(\mathbb{R}^d,\mathbb{R})$ yields the last equality in the above equation. It remains to justify the application of Fatou's lemma, [7, A1.20]: Consider

$$
f_n:=(p-1)\eta|v_n|^{p-2}\operatorname{Re}\sum_{j=1}^d\overline{D_jv_n}^T AD_jv_n\mathbb{1}_{\{v_n\neq 0\}},\ n\in\mathbb{N}'.
$$

By Hölder's inequality we have already seen that $\liminf_{n\to\infty}f_n<\infty$ is satisfied. Moreover, $f_n\geqslant 0$ pointwise a.e., since A satisfies assumption (A3) and η is nonnegative. Finally, $f_n\in L^1(\Omega,\mathbb{R})$, since $v_n\in C^2(\overline{\Omega},\mathbb{C}^N)\cap C_{\mathrm{c}}(\Omega,\mathbb{C}^N)$ and $\eta\in C_{\mathrm{b}}^1(\mathbb{R}^d,\mathbb{R})$. Thus, by Fatou's, $\liminf_{n\to\infty}f_n\in L^1(\Omega,\mathbb{R})$ and

$$
\int_\Omega\liminf_{n\to\infty}f_n\leqslant\liminf_{n\to\infty}\int_\Omega f_n,
$$

that proves the lemma. Note, that it is in general not possible to apply Lebesgue's theorem in case of $v \in W^{2,p}(\Omega, \mathbb{C}^N) \cap W_0^{1,p}(\Omega, \mathbb{C}^N)$, since one cannot determine a n-independent bound for $|f_n| \leqslant g$ a.e. for every $n \in \mathbb{N}'$. In fact, we only know positivity of f_n due to (A3), that justifies the application of Fatou's lemma and generates an inequality for $1 < p < 2$.

$\square$

We are now able to prove sharp resolvent estimates for the formal Ornstein-Uhlenbeck operator $\mathcal{L}_0$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 < p < \infty$, which then yield the unique solvability of the resolvent equation for $\mathcal{L}_0$ in $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. The main idea of the proof comes from [73, Theorem 2.2, Remark 2.3] for the scalar real-valued case and from [15, Theorem 3.1] for $d = 2$. In our situation, the proof requires the additional L^p-dissipativity condition (5.1), that seems to be new in the literature. This condition seems to be the optimal choice in order to derive resolvent estimates for $\mathcal{L}_0$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 < p < \infty$. It contains an additional, more restrictive requirement of the spectrum of the diffusion matrix A, even through it looks slightly complicated. We show later on in Theorem 5.18 that the L^p-dissipativity condition (5.1) is equivalent to the L^p-antieigenvalue condition (A4).

Theorem 5.13 (Resolvent Estimates for $\mathcal{L}_0$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ with $1 < p < \infty$)**.** *Let the assumptions* (A3)*,* (A4) *and* (A5) *be satisfied for* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*. Moreover, let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$ *and let* $v_\star \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ *denote a solution of*

$$(\lambda I - \mathcal{L}_0)\, v = g$$

in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ *for some* $g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$*. Then* $v_\star$ *is the unique solution in* $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ *and satisfies the resolvent estimate*

$$\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{1}{\operatorname{Re}\lambda - \omega_0} \|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} .$$

In addition, for $1 < p \leqslant 2$ *the following gradient estimate is satisfied*

$$|v_\star|_{W^{1,p}(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{d^{\frac{1}{p}}}{\gamma_A^{\frac{1}{2}} \left(\operatorname{Re}\lambda - \omega_0\right)^{\frac{1}{2}}} \|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} .$$

Remark. (**1**) Note that the proof deals with cut-off functions. These are necessary because $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N)$ implies only that $\triangle v \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N)$. What this really means is that v is not p-integrable over the whole $\mathbb{R}^d$ and therefore, we must restrict the solution to a bounded domain.
(**2**) The gradient estimate is proved only for $1 < p \leqslant 2$ but not for $p > 2$. Its proof is based on Hölder's inequality that requires exactly $1 < p \leqslant 2$.
(**3**) An L^p-dissipativity condition for the operator $\nabla^T (Q\nabla v) + \langle b, \nabla v\rangle + av$ in $L^p(\Omega, \mathbb{C})$ with $1 < p < \infty$ can be found in [26], namely for constant coefficients $Q \in \mathbb{C}^{d,d}$, $b \in \mathbb{C}^d$, $a \in \mathbb{C}$ with $\Omega \subseteq \mathbb{R}^d$ open in [26, Theorem 2], and for variable coefficients Q_{ij}, $b_j \in C^1(\overline{\Omega}, \mathbb{C})$, $a \in C^0(\overline{\Omega}, \mathbb{C})$ with $\Omega \subset \mathbb{R}^d$ bounded in [26, Lemma 2].

Proof. Assume $v_\star \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ satisfies

$$(\lambda I - \mathcal{L}_0)\, v_\star = g \tag{5.25}$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for some $g \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ with $1<p<\infty$. Let us define

$$\eta_n(x)=\eta\left(\frac{x}{n}\right),\quad \eta\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d,\mathbb{R}),\quad \eta(x)=\begin{cases}1 & ,\ |x|\leqslant 1\\ \in[0,1],\text{ smooth} & ,\ 1<|x|<2\\ 0 & ,\ |x|\geqslant 2\end{cases}.$$

1. Multiplying (5.25) from left by $\eta_n^2\overline{v_\star}^T|v_\star|^{p-2}$ with $1<p<\infty$, integrating over $\mathbb{R}^d$ and taking real parts yields

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^d}\eta_n^2|v_\star|^{p-2}\,\overline{v_\star}^T g =&(\mathrm{Re}\,\lambda)\int_{\mathbb{R}^d}\eta_n^2|v_\star|^p-\mathrm{Re}\int_{\mathbb{R}^d}\eta_n^2\overline{v_\star}^T|v_\star|^{p-2}A\triangle v_\star\\ &-\mathrm{Re}\int_{\mathbb{R}^d}\eta_n^2\overline{v_\star}^T|v_\star|^{p-2}\sum_{j=1}^d(Sx)_jD_jv_\star.\end{aligned}$$

2. Using (A5), i.e. $-S=S^T$, then integration by parts formula and (5.24) imply

$$\begin{aligned}0=&\frac{1}{p}\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{j=1}^d S_{jj}\right)|v_\star|^p=\frac{1}{p}\int_{\mathbb{R}^d}\eta_n^2\mathrm{div}\,(Sx)\,|v_\star|^p\\ =&\frac{1}{p}\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{j=1}^d D_j\left((Sx)_j\right)\right)|v_\star|^p=\frac{1}{p}\sum_{j=1}^d\int_{\mathbb{R}^d}\eta_n^2D_j\left((Sx)_j\right)|v_\star|^p\\ =&-\frac{1}{p}\sum_{j=1}^d\int_{\mathbb{R}^d}D_j\left(\eta_n^2\right)(Sx)_j|v_\star|^p-\frac{1}{p}\sum_{j=1}^d\int_{\mathbb{R}^d}\eta_n^2(Sx)_jD_j\left(|v_\star|^p\right)\\ =&-\frac{2}{p}\sum_{j=1}^d\int_{\mathbb{R}^d}\eta_n(D_j\eta_n)(Sx)_j|v_\star|^p-\sum_{j=1}^d\int_{\mathbb{R}^d}\eta_n^2(Sx)_j\mathrm{Re}\left(\overline{D_jv_\star}^Tv_\star\right)|v_\star|^{p-2}\\ =&-\frac{2}{p}\int_{\mathbb{R}^d}\eta_n|v_\star|^p\sum_{j=1}^d(D_j\eta_n)(Sx)_j-\mathrm{Re}\int_{\mathbb{R}^d}\eta_n^2\overline{v_\star}^T|v_\star|^{p-2}\sum_{j=1}^d(Sx)_jD_jv_\star.\end{aligned}$$

An application of Lemma 5.12 (with $\Omega=\mathbb{R}^d$ and $\eta=\eta_n^2$) yields

$$\begin{aligned}&\mathrm{Re}\int_{\mathbb{R}^d}\eta_n^2|v_\star|^{p-2}\,\overline{v_\star}^Tg\\ \geqslant&(\mathrm{Re}\,\lambda)\int_{\mathbb{R}^d}\eta_n^2|v_\star|^p+\mathrm{Re}\int_{\mathbb{R}^d}2\eta_n\overline{v_\star}^T|v_\star|^{p-2}\sum_{j=1}^dD_j\eta_nAD_jv_\star\\ &+(p-1)\mathrm{Re}\int_{\mathbb{R}^d}\eta_n^2|v_\star|^{p-2}\sum_{j=1}^d\overline{D_jv_\star}^TAD_jv_\star+\frac{2}{p}\int_{\mathbb{R}^d}\eta_n|v_\star|^p\sum_{j=1}^d(D_j\eta_n)(Sx)_j\\ &+(p-2)\mathrm{Re}\int_{\mathbb{R}^d}\eta_n^2|v_\star|^{p-4}\sum_{j=1}^d\left[\mathrm{Re}\left(\overline{D_jv_\star}^Tv_\star\right)\overline{v_\star}^T-|v_\star|^2\overline{D_jv_\star}^T\right]AD_jv_\star\end{aligned}$$

3. Putting the 2nd and 4th term from the right hand to the left hand side yields

$$
\begin{aligned}
&(\operatorname{Re}\lambda)\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^p+(p-1)\operatorname{Re}\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-2}\sum_{j=1}^d\overline{D_jv_\star}^T AD_jv_\star\\
&+(p-2)\operatorname{Re}\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-4}\sum_{j=1}^d\left[\operatorname{Re}\left(\overline{D_jv_\star}^T v_\star\right)\overline{v_\star}^T-|v_\star|^2\overline{D_jv_\star}^T\right]AD_jv_\star\\
\leqslant&\operatorname{Re}\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-2}\,\overline{v_\star}^T g-\operatorname{Re}\int_{\mathbb{R}^d}2\eta_n\overline{v_\star}^T\,|v_\star|^{p-2}\sum_{j=1}^d D_j\eta_nAD_jv_\star\\
&-\frac{2}{p}\int_{\mathbb{R}^d}\eta_n\,|v_\star|^p\sum_{j=1}^d(D_j\eta_n)(Sx)_j
\end{aligned}
$$

For the 1st term on the right hand side we use $\operatorname{Re}z\leqslant|z|$ and Hölder's inequality (with q such that $\frac{1}{p}+\frac{1}{q}=1$)

$$
\begin{aligned}
&\operatorname{Re}\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-2}\,\overline{v_\star}^T g=\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-2}\operatorname{Re}\left(\overline{v_\star}^T g\right)\\
\leqslant&\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-1}\,|g|\leqslant\left(\int_{\mathbb{R}^d}\left(\eta_n^{\frac{2(p-1)}{p}}\,|v_\star|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\left(\eta_n^{\frac{2}{p}}\,|g|\right)^p\right)^{\frac{1}{p}}\\
=&\left(\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\eta_n^2\,|g|^p\right)^{\frac{1}{p}}
\end{aligned}
$$

For the 2nd term we use $\operatorname{Re}z\leqslant|z|$, Hölder's inequality (with $p=q=2$) and Cauchy's inequality (with $\varepsilon>0$)

$$
\begin{aligned}
&-\operatorname{Re}\int_{\mathbb{R}^d}2\eta_n\overline{v_\star}^T\,|v_\star|^{p-2}\sum_{j=1}^d D_j\eta_nAD_jv_\star\\
\leqslant&2|A|\int_{\mathbb{R}^d}\eta_n\,|v_\star|^{p-1}\sum_{j=1}^d|D_j\eta_n|\,|D_jv_\star|\leqslant\frac{2|A|\,\|\eta\|_{1,\infty}}{n}\sum_{j=1}^d\int_{\mathbb{R}^d}\eta_n\,|D_jv_\star|\,|v_\star|^{p-1}\\
\leqslant&\frac{2|A|\,\|\eta\|_{1,\infty}}{n}\sum_{j=1}^d\left(\int_{\mathbb{R}^d}\eta_n^2\,|D_jv_\star|^2\,|v_\star|^{p-2}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|v_\star|^p\right)^{\frac{1}{2}}\\
\leqslant&\frac{2|A|\,\|\eta\|_{1,\infty}\,\varepsilon}{n}\sum_{j=1}^d\int_{\mathbb{R}^d}\eta_n^2\,|D_jv_\star|^2\,|v_\star|^{p-2}+\frac{2d|A|\,\|\eta\|_{1,\infty}}{4n\varepsilon}\int_{\mathbb{R}^d}|v_\star|^p\,.
\end{aligned}
$$

Here we used that for every $x\in\mathbb{R}^d$ and $j=1,\ldots,d$

$$
|D_j\eta_n(x)|=\left|D_j\left(\eta\left(\frac{x}{n}\right)\right)\right|=\frac{1}{n}\left|(D_j\eta)\left(\frac{x}{n}\right)\right|\leqslant\frac{1}{n}\max_{j=1,\ldots,d}\max_{y\in\mathbb{R}^d}|D_j\eta(y)|=\frac{\|\eta\|_{1,\infty}}{n}
$$

For the 3rd term we use that $\eta_n(x)=0$ for $|x|\geqslant 2n$ and $\eta_n(x)=1$ for $|x|\leqslant n$. Hence $D_j\eta_n(x)=0$ for $|x|\leqslant n$ and we obtain

$$
-\frac{2}{p}\int_{\mathbb{R}^d}\eta_n\,|v_\star|^p\sum_{j=1}^d(D_j\eta_n)(Sx)_j\leqslant\frac{2}{p}\sum_{j=1}^d\int_{\mathbb{R}^d}\eta_n\,|v_\star|^p\,|(Sx)_j|\,|D_j\eta_n|
$$

$$=\frac{2}{p}\sum_{j=1}^{d}\int_{n\leqslant|x|\leqslant 2n}\eta_n\left|v_\star\right|^p\left|(Sx)_j\right|\left|D_j\eta_n\right|\leqslant\frac{4d\,|S|\,\|\eta\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\left|v_\star\right|^p.$$

The last inequality is justified by $\eta_n(x)\leqslant 1$ and

$$\begin{aligned}\left|(Sx)_j\right|\left|D_j\eta_n(x)\right|&=\frac{1}{n}\left|(Sx)_j\right|\left|(D_j\eta)\left(\frac{x}{n}\right)\right|\leqslant\frac{1}{n}|S||x|\left|(D_j\eta)\left(\frac{x}{n}\right)\right|\\ &\leqslant\frac{|S|}{n}\left(\sup_{n\leqslant|x|\leqslant 2n}|x|\right)\max_{j=1,\ldots,d}\max_{y\in\mathbb{R}^d}\left|D_j\eta(y)\right|=2\,|S|\,\|\eta\|_{1,\infty}.\end{aligned}$$

Altogether, combining the 2nd and 3rd term on the left hand side and using the notation $\langle u,v\rangle:=\overline{u}^Tv$ for the Euclidean inner product on $\mathbb{C}^N$, we obtain

$$\begin{aligned}&(\operatorname{Re}\lambda)\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^p+\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^{p-4}\sum_{j=1}^{d}\Big[\left|v_\star\right|^2\operatorname{Re}\left\langle D_jv_\star,AD_jv_\star\right\rangle\\ &\quad+(p-2)\operatorname{Re}\left\langle D_jv_\star,v_\star\right\rangle\operatorname{Re}\left\langle v_\star,AD_jv_\star\right\rangle\Big]\\ &\leqslant\left(\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\eta_n^2\left|g\right|^p\right)^{\frac{1}{p}}+\frac{2|A|\,\|\eta\|_{1,\infty}\,\varepsilon}{n}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\eta_n^2\left|D_jv_\star\right|^2\left|v_\star\right|^{p-2}\\ &\quad+\frac{2d|A|\,\|\eta\|_{1,\infty}}{4n\varepsilon}\int_{\mathbb{R}^d}\left|v_\star\right|^p+\frac{4d\,|S|\,\|\eta\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\left|v_\star\right|^p.\end{aligned}$$

4. The L^p-antieigenvalue condition (A4) yields some constant $\gamma_A>0$ such that

$$|z|^2\operatorname{Re}\langle w,Aw\rangle+(p-2)\operatorname{Re}\langle w,z\rangle\operatorname{Re}\langle z,Aw\rangle\geqslant\gamma_A|z|^2|w|^2\;\forall\,z,w\in\mathbb{C}^N$$

(see Theorem 5.18 below), which guarantees positivity of the term appearing in brackets $[\cdots]$. Therefore, putting the 2nd term from the right hand to the left hand side in the latter inequality from step 3 we obtain

$$\begin{aligned}&(\operatorname{Re}\lambda)\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\eta_n^2\left(\gamma_A-\frac{2|A|\,\|\eta\|_{1,\infty}\,\varepsilon}{n}\right)\left|D_jv_\star\right|^2\left|v_\star\right|^{p-2}\\ \leqslant&(\operatorname{Re}\lambda)\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^p+\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^{p-4}\sum_{j=1}^{d}\Big[\left|v_\star\right|^2\operatorname{Re}\left\langle D_jv_\star,AD_jv_\star\right\rangle\\ &+(p-2)\operatorname{Re}\left\langle D_jv_\star,v_\star\right\rangle\operatorname{Re}\left\langle v_\star,AD_jv_\star\right\rangle\Big]-\frac{2|A|\,\|\eta\|_{1,\infty}\,\varepsilon}{n}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\eta_n^2\left|D_jv_\star\right|^2\left|v_\star\right|^{p-2}\\ \leqslant&\left(\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\eta_n^2\left|g\right|^p\right)^{\frac{1}{p}}+\frac{2d|A|\,\|\eta\|_{1,\infty}}{4n\varepsilon}\int_{\mathbb{R}^d}\left|v_\star\right|^p\\ &+\frac{4d\,|S|\,\|\eta\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\left|v_\star\right|^p.\end{aligned}$$

5. Choosing $\varepsilon>0$ such that $\gamma_A-\frac{2|A|\|\eta\|_{1,\infty}\varepsilon}{n}>0$ for every $n\in\mathbb{N}$, using $\omega_0=0$ from (5.11), i.e. $\operatorname{Re}\lambda=\operatorname{Re}\lambda-\omega_0$, and taking the limit inferior for $n\to\infty$, an

application of Lebesgue's dominated convergence theorem and Fatou's lemma yield

$$
\begin{aligned}
&(\operatorname{Re}\lambda-\omega_0)\left\|v_\star\right\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \leqslant (\operatorname{Re}\lambda)\int_{\mathbb{R}^d}\left|v_\star\right|^p+\gamma_A\sum_{j=1}^{d}\int_{\mathbb{R}^d}\left|D_jv_\star\right|^2\left|v_\star\right|^{p-2}\\
=&(\operatorname{Re}\lambda)\int_{\mathbb{R}^d}\lim_{n\to\infty}\eta_n^2\left|v_\star\right|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\liminf_{n\to\infty}\eta_n^2\left(\gamma_A-\frac{2|A|\left\|\eta\right\|_{1,\infty}\varepsilon}{n}\right)\left|D_jv_\star\right|^2\left|v_\star\right|^{p-2}\\
\leqslant&\liminf_{n\to\infty}\left[(\operatorname{Re}\lambda)\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\eta_n^2\left(\gamma_A-\frac{2|A|\left\|\eta\right\|_{1,\infty}\varepsilon}{n}\right)\left|D_jv_\star\right|^2\left|v_\star\right|^{p-2}\right]\\
\leqslant&\liminf_{n\to\infty}\Bigg[\left(\int_{\mathbb{R}^d}\eta_n^2\left|v_\star\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\eta_n^2\left|g\right|^p\right)^{\frac{1}{p}}+\frac{2d|A|\left\|\eta\right\|_{1,\infty}}{4n\varepsilon}\int_{\mathbb{R}^d}\left|v_\star\right|^p\\
&+\frac{4d\left|S\right|\left\|\eta\right\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\left|v_\star\right|^p\Bigg]\\
=&\left(\int_{\mathbb{R}^d}\lim_{n\to\infty}\eta_n^2\left|v_\star\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\lim_{n\to\infty}\eta_n^2\left|g\right|^p\right)^{\frac{1}{p}}+\frac{2d|A|\left\|\eta\right\|_{1,\infty}}{4\varepsilon}\int_{\mathbb{R}^d}\lim_{n\to\infty}\frac{1}{n}\left|v_\star\right|^p\\
&+\frac{4d\left|S\right|\left\|\eta\right\|_{1,\infty}}{p}\int_{\mathbb{R}^d}\lim_{n\to\infty}\left|v_\star\right|^p\mathbb{1}_{\{n\leqslant|x|\leqslant 2n\}}\\
=&\left(\int_{\mathbb{R}^d}\left|v_\star\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\left|g\right|^p\right)^{\frac{1}{p}}=\left\|v_\star\right\|^{p-1}_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\left\|g\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.
\end{aligned}
$$

Finally, using $\operatorname{Re}\lambda-\omega_0>0$ the L^p–resolvent estimate follows by dividing both sides by $\operatorname{Re}\lambda-\omega_0$ and $\|v_\star\|^{p-1}_{L^p(\mathbb{R}^d,\mathbb{C}^N)}$. Indeed, we must check that the assumptions of Lebesgue's theorem and Fatou's lemma are satisfied. We suggest that first one must apply Lebesgue's theorem, which then yields that the assumptions of Fatou's lemma are satisfied. For the application of Lebesgue's theorem we have the pointwise convergence $\eta_n^2|v_\star|^p\to|v_\star|^p$, $\eta_n^2|g|^p\to|g|^p$, $\frac{1}{n}|v_\star|^p\to 0$ and $|v_\star|^p\,\mathbb{1}_{\{n\leqslant|x|\leqslant 2n\}}\to 0$ a.e. as $n\to\infty$. Furthermore, they are dominated by $|\eta_n^2|v_\star|^p|\leqslant|v_\star|^p$, $|\eta_n^2|g|^p|\leqslant|g|^p$, $\frac{1}{n}|v_\star|^p\leqslant|v_\star|^p$, $|v_\star|^p\,\mathbb{1}_{\{n\leqslant|x|\leqslant 2n\}}\leqslant|v_\star|^p$ and the bounds belong to $L^1(\mathbb{R}^d,\mathbb{R})$ since $v_\star,g\in L^p(\mathbb{R}^d,\mathbb{C}^N)$. For the application of Fatou's lemma we observe that $\eta_n^2|v_\star|^p$ and $\eta_n^2\left(\gamma_A-\frac{2|A|\|\eta\|_{1,\infty}\varepsilon}{n}\right)\left|D_jv_\star\right|^2\left|v_\star\right|^{p-2}$ belong to $L^1(\mathbb{R}^d,\mathbb{R})$, are positive and the limit inferior of their integrals is bounded by Lebesgue's theorem.

6. To show uniqueness in $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$, let both $u_\star,v_\star\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ be a solution of

$$
(\lambda I-\mathcal{L}_0)\,u_\star=g\quad\text{and}\quad(\lambda I-\mathcal{L}_0)\,v_\star=g
$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Then $w_\star:=v_\star-u_\star\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ is a solution of the homogeneous problem $(\lambda I-\mathcal{L}_0)\,w_\star=0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. From the L^p–resolvent estimate we obtain $\|w_\star\|_{L^p}\leqslant 0$, hence $u_\star$ and $v_\star$ coincide in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Since $u_\star,v_\star\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)\subset L^p(\mathbb{R}^d,\mathbb{C}^N)$ we deduce that $v_\star=u_\star$ in $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$.

7. From step 5 we obtain for every $j=1,\ldots,N$

$$
\int_{\mathbb{R}^d}\left|D_jv_\star\right|^2\left|v_\star\right|^{p-2}\leqslant\frac{1}{\gamma_A}\left\|v_\star\right\|^{p-1}_{L^p}\left\|g\right\|_{L^p}.
$$

Using the L^p–resolvent estimate, we deduce from Hölder's inequality for $1<p\leqslant 2$

$$\begin{aligned}\left\|D_j v_\star\right\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)} &= \int_{\mathbb{R}^d}\left|D_j v_\star\right|^p = \int_{\mathbb{R}^d}\left|D_j v_\star\right|^p \left|v_\star\right|^{-\frac{p(2-p)}{2}}\left|D_j v_\star\right|^{\frac{p(2-p)}{2}}\\ \leqslant\left(\int_{\mathbb{R}^d}\left|D_j v_\star\right|^2\left|v_\star\right|^{p-2}\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^d}\left|v_\star\right|^p\right)^{\frac{2-p}{2}} &\leqslant \frac{1}{\gamma_A^{\frac{p}{2}}}\frac{1}{\left(\operatorname{Re}\lambda-\omega_0\right)^{\frac{p}{2}}}\left\|g\right\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.\end{aligned}$$

Taking the sum over j from 1 to d and the pth root we end up with

$$\left|v_\star\right|_{W^{1,p}(\mathbb{R}^d,\mathbb{C}^N)} = \left(\sum_{j=1}^{d}\left\|D_j v_\star\right\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right)^{\frac{1}{p}} \leqslant \frac{d^{\frac{1}{p}}}{\gamma_A^{\frac{1}{2}}\left(\operatorname{Re}\lambda-\omega_0\right)^{\frac{1}{2}}}\left\|g\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.$$

□

Recall the following definition of a dissipative operator, [34, II.3.13 Definition].

Definition 5.14. The operator $\mathcal{L}_0 : \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)\subset L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ with $1<p<\infty$, is called **L^p-dissipative** (or **dissipative** in $L^p(\mathbb{R}^d,\mathbb{C}^N)$) if

$$\left\|(\lambda-\mathcal{L}_0)\,v\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \geqslant \lambda\left\|v\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)},\quad \forall\,\lambda>0\;\forall\,v\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0).$$

A direct consequence of Theorem 5.13 is that the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ is a dissipative operator in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$.

Corollary 5.15. *Let the assumptions* (A3), (A4) *and* (A5) *be satisfied for* $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then,* $\mathcal{L}_0 : \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)\subset L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ *is* L^p*-dissipative.*

5.5 The L^p-dissipativity condition

In this section we give a complete characterization of the optimal L^p-dissipativity condition (5.1) for the complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$. For this purpose, recall the following definitions.

Definition 5.16. Let $A\in\mathbb{K}^{N,N}$ with $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and $N\in\mathbb{N}$, then A is called

- **accretive**, if $\inf\limits_{\substack{w\in\mathbb{K}^N\\|w|=1}}\operatorname{Re}\langle w,Aw\rangle\geqslant 0$
- **strongly accretive**, if $\inf\limits_{\substack{w\in\mathbb{K}^N\\|w|=1}}\operatorname{Re}\langle w,Aw\rangle> 0$
- **dissipative**, if $\sup\limits_{\substack{w\in\mathbb{K}^N\\|w|=1}}\operatorname{Re}\langle w,Aw\rangle\leqslant 0$
- **strongly dissipative**, if $\sup\limits_{\substack{w\in\mathbb{K}^N\\|w|=1}}\operatorname{Re}\langle w,Aw\rangle< 0$

For selfadjoint matrices A, replace accretive and dissipative by **positive** and **negative**, respectively.

Definition 5.17. Let $A \in \mathbb{K}^{N,N}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $N \in \mathbb{N}$. Then we define by

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{K}^N \\ w \neq 0 \\ Aw \neq 0}} \frac{\operatorname{Re}\langle w, Aw\rangle}{|w||Aw|} = \inf_{\substack{w \in \mathbb{K}^N \\ |w|=1 \\ Aw \neq 0}} \frac{\operatorname{Re}\langle w, Aw\rangle}{|Aw|} \tag{5.26}$$

the **first antieigenvalue of** A. A vector $0 \neq w \in \mathbb{K}^N$ with $Aw \neq 0$ for which the infimum is attained, is called an **antieigenvector of** A. Moreover, we define the **(real) angle of** A by

$$\Phi_{\mathbb{R}}(A) := \cos^{-1}\left(\mu_1(A)\right).$$

The Definitions 5.16 and 5.17 come originally from [47]. Related to the Definition 5.16, we suggest that the assumption (A3) is satisfied if and only if A is strongly accretive. The following lower and upper bounds for the first antieigenvalue of A are well known from [31]

$$\frac{C_{\mathrm{accr}}(A)}{|A|} = \frac{1}{|A|} \inf_{\substack{w \in \mathbb{K}^N \\ |w|=1}} \operatorname{Re}\langle w, Aw\rangle \leqslant \mu_1(A) \leqslant \frac{1}{|A|} \sup_{\substack{w \in \mathbb{K}^N \\ |w|=1}} \operatorname{Re}\langle w, Aw\rangle = \frac{C_{\mathrm{diss}}(A)}{|A|},$$

where we call $C_{\mathrm{accr}}(A)$ and $C_{\mathrm{diss}}(A)$ the **accretivity** and **dissipativity constant of** A, respectively. They describe the inner and outer real numerical radius of A, respectively. In Definition 5.17, $\mu_1(A)$ measures the maximum (real) turning capability of A. The expression for $\mu_1(A)$ is sometimes denoted by $\cos A$ and is called the **cosine of** A. In [59] the expression for $\mu_1(A)$ is denoted by $\operatorname{dev} A$ and is called the **deviation of** A.

The next theorem shows that the L^p-dissipativity condition is equivalent to a lower bound for the first antieigenvalue of the diffusion matrix A. Later, the theorem is applied to $b := p - 2$ for $1 < p < \infty$.

Theorem 5.18 (L^p-dissipativity condition vs. L^p-antieigenvalue condition)**.** *Let $A \in \mathbb{K}^{N,N}$ for $K = \mathbb{R}$ if $N \geqslant 2$ and $\mathbb{K} = \mathbb{C}$ if $N \geqslant 1$, and let $b \in \mathbb{R}$ with $b > -1$.*
(**a**) *Given some $\gamma_A > 0$, then the following statements are equivalent:*

(**1**) $|z|^2 \operatorname{Re}\langle w, Aw\rangle + b\operatorname{Re}\langle w, z\rangle \operatorname{Re}\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2 \quad \forall\, w, z \in \mathbb{K}^N$,

(**2**) $\operatorname{Re}\langle w, Aw\rangle + b\operatorname{Re}\langle w, z\rangle \operatorname{Re}\langle z, Aw\rangle \geqslant \gamma_A \quad \forall\, w, z \in \mathbb{K}^N,\ |z| = |w| = 1$,

(**3**) $\left(1 + \frac{b}{2}\right) \operatorname{Re}\langle w, Aw\rangle - \frac{|b|}{2} |Aw| \geqslant \gamma_A \quad \forall\, w \in \mathbb{K}^N,\ |w| = 1$.

(**b**) *Moreover, the following statements are equivalent*

(**4**) $\exists\, \gamma_A > 0 : \left(1 + \frac{b}{2}\right) \operatorname{Re}\langle w, Aw\rangle - \frac{|b|}{2} |Aw| \geqslant \gamma_A \quad \forall\, w \in \mathbb{K}^N,\ |w| = 1$,

(**5**) $\exists\, \delta_A > 1 : \frac{(2+b)}{|b|} \cdot \frac{\operatorname{Re}\langle w, Aw\rangle}{|w||Aw|} \geqslant \delta_A \quad \forall\, w \in \mathbb{K}^N,\ w \neq 0,\ Aw \neq 0$,

(**6**) $\exists\, \delta_A > 1 : \frac{(2+b)}{|b|} \cdot \mu_1(A) \geqslant \delta_A$,

(**7**) $\mu_1(A) > \frac{|b|}{(2+b)}$,

where $\mu_1(A)$ denotes the first antieigenvalue of A in the sense of Definition 5.17.

The scalar real case: Positivity. In the scalar real case $A=a\in\mathbb{R}$ (with $\mathbb{K}=\mathbb{R}$ and $N=1$) the statements (**1**) and (**2**) are equivalent, but they are in general not equivalent with (**3**). In particular, there exists a constant γ_a with (**2**) if and only if $(1+b)a>0$.

The scalar complex case: A cone condition. In the scalar complex case $A=\alpha\in\mathbb{C}$ (with $\mathbb{K}=\mathbb{C}$ and $N=1$) there exists a constant γ_α with (**3**) if and only if one of the following **cone conditions** hold

$$(\mathbf{8})\;\; |\operatorname{Im}\alpha| < \frac{2\sqrt{1+b}}{|b|}\operatorname{Re}\alpha,$$
$$(\mathbf{9})\;\; |\arg\alpha| < \arctan\left(\frac{2\sqrt{1+b}}{|b|}\right).$$

This conditions will be discussed below for normal matrices in more details.

First antieigenvalue and real angle. The statement (**7**) coincides with the L^p-antieigenvalue condition from (A4) and yields a lower p-dependent bound for the first antieigenvalue of the diffusion matrix A

$$\mu_1(A) > \frac{|p-2|}{p} \in [0,1[,\quad 1<p<\infty.$$

This implies an upper p-dependent bound for the (real) angle of A

$$\Phi_{\mathbb{R}}(A) := \cos^{-1}\left(\mu_1(A)\right) < \cos^{-1}\left(\frac{|p-2|}{p}\right) \in]0,\frac{\pi}{2}],\quad 1<p<\infty.$$

In general, one cannot derive an explicit expression for the first antieigenvalue $\mu_1(A)$ of a matrix A. However, for certain classes of matrices such as Hermitian and normal matrices it is possible to derive a closed formula for $\mu_1(A)$ as it is shown in the following two remarks.

$\boldsymbol{\mu_1(A)}$ for Hermitian matrices. If A is a Hermitian matrix, then $\mu_1(A)$ from (**7**) is given by

$$\mu_1(A) = \frac{\sqrt{\lambda_1^A\lambda_N^A}}{\frac{1}{2}\left(\lambda_1^A+\lambda_N^A\right)} = \frac{2\sqrt{\kappa_A}}{\kappa_A+1},$$

where $0<\lambda_1^A\leqslant\lambda_2^A\leqslant\cdots\leqslant\lambda_N^A$ denote the (real) eigenvalues of A and $\kappa_A:=\frac{\lambda_N^A}{\lambda_1^A}$ denotes the **spectral condition number of** A. In this case $\mu_1(A)$ is the quotient of the geometric mean $\sqrt{\lambda_1^A\lambda_N^A}$ and the arithmetic mean $\frac{1}{2}\left(\lambda_1^A+\lambda_N^A\right)$ of the smallest and largest eigenvalue of A. In particular, the equality $\mu_1(A)=\frac{\operatorname{Re}\langle w,Aw\rangle}{|Aw|}$ is satisfied for the antieigenvector $w=\left(\lambda_N^A\right)^{\frac{1}{2}}u_1+\left(\lambda_1^A\right)^{\frac{1}{2}}u_N$, where $u_1,u_N\in\mathbb{K}^N$ are orthogonal vectors with $Au_1=\lambda_1^Au_1$ and $Au_N=\lambda_N^Au_N$ such that $|w|=1$. This follows directly from the Greub-Rheinboldt inequality, [53, (7.4.12.11)], and can be found in [53, 7.4.P4].

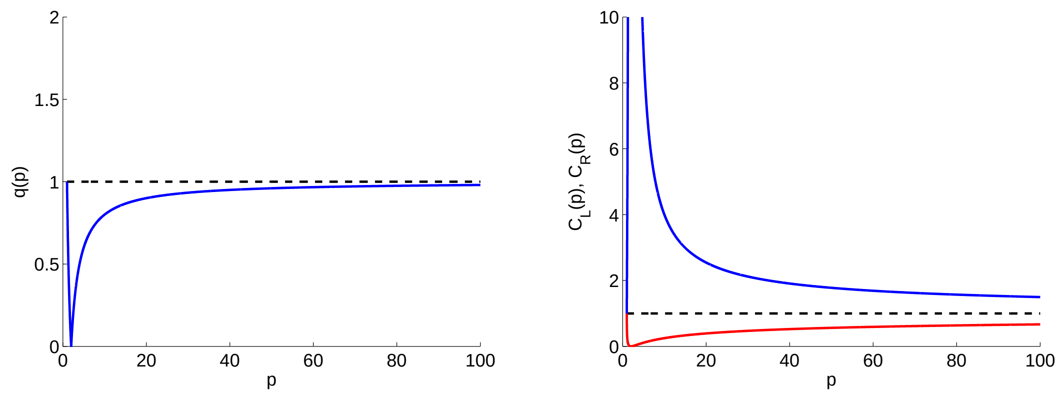
If we define $q := \frac{|p-2|}{p}$ for $1 < p < \infty$, see Figure 5.1(a), then $q \in [0, \infty[$ and the L^p-antieigenvalue condition (A4) is equivalent with

$$\frac{2-q^2-2\sqrt{1-q^2}}{q^2} < \kappa_A < \frac{2-q^2+2\sqrt{1-q^2}}{q^2}, \quad \text{for } 0 < q < 1.$$

Using the definition of q, this inequality implies

$$C_L(p) := \frac{p^2+4p-4-4p\sqrt{p-1}}{(p-2)^2} < \kappa_A < \frac{p^2+4p-4+4p\sqrt{p-1}}{(p-2)^2} =: C_R(p),$$

for $1 < p < \infty$ and $p \neq 2$, that is a lower and upper bound for the spectral condition number of A. Of course, since $0 < \lambda_1^A \leqslant \lambda_2^A \leqslant \cdots \leqslant \lambda_N^A$ not only $\kappa_A = \frac{\lambda_N^A}{\lambda_1^A}$ but also $\frac{\lambda_j^A}{\lambda_1^A}$ must be contained in the open interval $]C_L(p), C_R(p)[$ for every $1 \leqslant j \leqslant N$. The behavior of the constants $C_L(p)$ and $C_R(p)$ is depicted in Figure 5.1(b). In particular, to satisfy this condition for arbitrary very large p, i.e. p near ∞, the matrix A must be of the form $A = aI_N$ for some $0 < a \in \mathbb{R}$.



(a) (b)

Figure 5.1: q as a function on p (a) and constants C_L (red) and C_R (blue) in dependence on p (b)

$\mu_1(A)$ for normal matrices. If A is a normal matrix, then $\mu_1(A)$ from (**7**) is given by $\mu_1(A) = \min E \cup F$, where

$$E := \left\{ \frac{a_j}{\left|\lambda_j^A\right|} \,\middle|\, 1 \leqslant j \leqslant N \right\},$$

$$F := \left\{ \frac{2\sqrt{(a_j-a_i)\left(a_i\left|\lambda_j^A\right|^2 - a_j\left|\lambda_i^A\right|^2\right)}}{\left|\lambda_j^A\right|^2 - \left|\lambda_i^A\right|^2} \,\middle|\, 0 < \frac{a_j\left|\lambda_j^A\right|^2 - 2a_i\left|\lambda_j^A\right|^2 + a_j\left|\lambda_i^A\right|^2}{\left(\left|\lambda_i^A\right|^2 - \left|\lambda_j^A\right|^2\right)(a_i-a_j)} < 1, \right.$$
$$\left. 1 \leqslant i, j \leqslant N,\ \left|\lambda_i^A\right| \neq \left|\lambda_j^A\right| \right\},$$

and $\lambda_j^A = a_j + ib_j$ with $a_j, b_j \in \mathbb{R}$, $1 \leqslant j \leqslant N$, denote the eigenvalues of A. In particular, if

$$\mu_1(A) = \frac{a_j}{|\lambda_j^A|} \text{ for some } 1 \leqslant j \leqslant N, \tag{5.27}$$

then $\mu_1(A) = \frac{\operatorname{Re}\langle w,Aw\rangle}{|Aw|}$ for an antieigenvector $w \in \mathbb{K}^N$ with $|w_j| = 1$ and $|w_i| = 0$ for $1 \leqslant i \leqslant N$ with $i \neq j$. Conversely, if

$$\mu_1(A) = \frac{2\sqrt{(a_j - a_i)\left(a_i\left|\lambda_j^A\right|^2 - a_j\left|\lambda_i^A\right|^2\right)}}{\left|\lambda_j^A\right|^2 - \left|\lambda_i^A\right|^2} \text{ for some } 1 \leqslant i, j \leqslant N,\ i \neq j, \tag{5.28}$$

with $\left|\lambda_i^A\right| \neq \left|\lambda_j^A\right|$, then $\mu_1(A) = \frac{\operatorname{Re}\langle w,Aw\rangle}{|Aw|}$ for an antieigenvector $w \in \mathbb{K}^N$ with

$$\begin{aligned}
|w_i|^2 =& \frac{a_j\left|\lambda_j^A\right|^2 - 2a_i\left|\lambda_j^A\right|^2 + a_j\left|\lambda_i^A\right|^2}{\left(\left|\lambda_i^A\right|^2 - \left|\lambda_j^A\right|^2\right)(a_i - a_j)}, \\
|w_j|^2 =& \frac{a_i\left|\lambda_i^A\right|^2 - 2a_j\left|\lambda_i^A\right|^2 + a_i\left|\lambda_j^A\right|^2}{\left(\left|\lambda_i^A\right|^2 - \left|\lambda_j^A\right|^2\right)(a_i - a_j)}
\end{aligned}$$

and $|w_k| = 0$ for $1 \leqslant k \leqslant N$ with $k \neq i$ and $k \neq j$. This result can be found in [49, Theorem 5.1], [50, Theorem 3.1], [97, Theorem 1.1] and [95, Theorem 1]. The proof in [49, Theorem 5.1] is based on an application of the Lagrange multiplier method in order to solve a minimization problem. Furthermore, in [31] it was proved that the expression on the right hand side of (5.28) is an upper bound for $\mu_1(A)$. In [31] one can also find a geometrical interpretation of this equality by a semiellipse.

If $\mu_1(A)$ is given by (5.27) for some $1 \leqslant j \leqslant N$, then the L^p-antieigenvalue condition (A4) is equivalent with

$$\operatorname{Re}\lambda_j^A > \frac{|p-2|}{2\sqrt{p-1}}\left|\operatorname{Im}\lambda_j^A\right|,\ 1 < p < \infty.$$

This leads to a **cone condition** which postulates that every eigenvalues of A is even contained in a p-dependent sector Σ_p in the open right half-plane, called a **conic section**,

$$\begin{aligned}
\Sigma_p :=& \left\{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda|\,|p-2| < 2\sqrt{p-1}\operatorname{Re}\lambda\right\} \\
=& \left\{\lambda \in \mathbb{C} \mid |\arg\lambda| < \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right)\right\},\ 1 < p < \infty,
\end{aligned}$$

see Figure 5.2. The opening angle $|\varphi|$ is close to 0 for small and large p, i.e. p close to 1 or ∞, and it is $\frac{\pi}{2}$ for $p = 2$. Indeed, this is the same requirement as in the scalar complex case. In particular, to satisfy the cone condition for arbitrary very large p, the matrix A must be of the form $A = \operatorname{diag}(a_1, \ldots, a_N)$ for some positive $a_1, \ldots, a_N \in \mathbb{R}$.

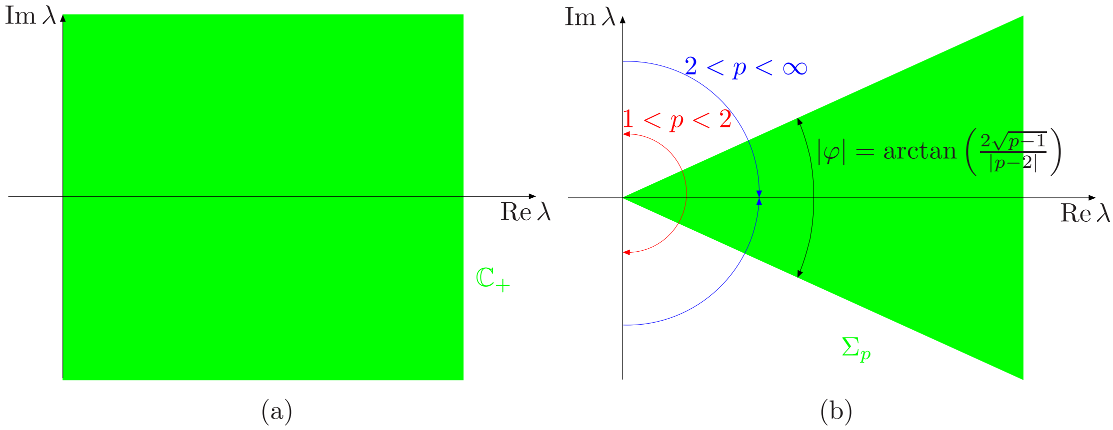


Figure 5.2: Sector for ellipticity assumption (A2) (a) and cone condition for antieigenvalue assumption (A4) for normal matrices A (b)

If $\mu_1(A)$ is given by (5.28) for some $1 \leqslant i,j \leqslant N$ with $i \neq j$, then the L^p-antieigenvalue condition (A4) is equivalent with

$$\frac{2\sqrt{(a_j - a_i)\left(a_i\left|\lambda_j^A\right|^2 - a_j\left|\lambda_i^A\right|^2\right)}}{\left|\lambda_j^A\right|^2 - \left|\lambda_i^A\right|^2} > \frac{|p-2|}{p},\ 1 < p < \infty.$$

that must be satisfied for every $1 \leqslant i,j \leqslant N$ with $|\lambda_j^A| \neq |\lambda_i^A|$. We emphasize the following equalities from [49, Section 6] and [31]

$$\begin{aligned}
&\frac{2\sqrt{(a_j - a_i)\left(a_i\left|\lambda_j^A\right|^2 - a_j\left|\lambda_i^A\right|^2\right)}}{\left|\lambda_j^A\right|^2 - \left|\lambda_i^A\right|^2} \\
=&\frac{2\sqrt{\frac{|\lambda_j^A|}{|\lambda_i^A|}\left[\left(\frac{a_i}{|\lambda_i^A|}\right)\left(\frac{|\lambda_j^A|}{|\lambda_i^A|}\right) - \frac{a_j}{|\lambda_j^A|}\right]\left[\left(\frac{a_j}{|\lambda_j^A|}\right)\left(\frac{|\lambda_j^A|}{|\lambda_i^A|}\right) - \frac{a_i}{|\lambda_i^A|}\right]}}{\left(\frac{|\lambda_j^A|}{|\lambda_i^A|}\right)^2 - 1} \\
=&\frac{2\sqrt{(r_i\rho_{ij} - r_j)(r_j\rho_{ij} - r_i)\rho_{ij}}}{\rho_{ij}^2 - 1},
\end{aligned}$$

where $\rho_{ij} := \frac{|\lambda_j^A|}{|\lambda_i^A|}$ and $r_k := \operatorname{Re}\frac{\lambda_k^A}{|\lambda_k^A|} = \frac{a_k}{|\lambda_k^A|}$ for $k = i, j$.

$\mu_1(A)$ for arbitrary matrices. If A is an arbitrary matrix, then for $\mu_1(A)$ from (**7**) there are only approximation results available. Such results are quite new and can be found in [96, Theorem 2].

Proof. The equivalence of (**1**), (**2**) and (**3**) is trivial for $b = 0$, so assume w.l.o.g. $b \neq 0$.
(**1**)$\Longleftrightarrow$(**2**): This follows directly by dividing both sides by $|z|^2|w|^2$.
(**2**)$\Longleftrightarrow$(**3**): We distinguish between the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Case 1: ($\mathbb{K} = \mathbb{R}$) Let $N \geqslant 2$. In this case we show the equivalence of

$$\langle w, Aw\rangle + b\,\langle w, z\rangle\,\langle z, Aw\rangle \geqslant \gamma_A \qquad \forall\, w, z \in \mathbb{R}^N,\ |z| = |w| = 1, \tag{5.29}$$

$$\left(1 + \frac{b}{2}\right)\langle w, Aw\rangle - \frac{|b|}{2}\,|Aw| \geqslant \gamma_A \qquad \forall\, w \in \mathbb{R}^N,\ |w| = 1, \tag{5.30}$$

for some $\gamma_A > 0$ by minimizing (5.29) with respect to z subject to $|z|^2 = 1$. Note that the minimum exists due to the boundedness of

$$|\langle z, Aw\rangle\,\langle w, z\rangle| \leqslant |z|^2|Aw||w| = |Aw|.$$

Subcase 1: (w, Aw linearly dependent) Let w and Aw be linearly dependent, then there exists $\lambda \in \mathbb{R}$ such that $Aw = \lambda w$. Since $|w| = 1$, we conclude $w \neq 0$ and therefore, $\lambda \in \sigma(A)$. Applying (5.29) with $z := w$

$$0 < \gamma_A \leqslant \langle w, Aw\rangle + b\,\langle w, w\rangle\,\langle w, Aw\rangle = (1 + b)\lambda$$

we deduce $\lambda > 0$, since $1 + b > 0$. In this case (5.29) and (5.30) reads as

$$\lambda\,|w|^2 + \lambda b\,\langle w, z\rangle^2 \geqslant \gamma_A \qquad \forall\, w, z \in \mathbb{R}^N,\ |z| = |w| = 1, \tag{5.31}$$

$$\left(1 + \frac{b}{2}\right)\lambda\,|w|^2 - \frac{|b|}{2}\,|\lambda|\,|w| \geqslant \gamma_A \qquad \forall\, w \in \mathbb{R}^N,\ |w| = 1. \tag{5.32}$$

The aim follows by minimization of $\lambda b\,\langle w, z\rangle^2$ with respect to z subject to $|z|^2 = 1$. If $b > 0$ then $\lambda b > 0$ and therefore, $\lambda b\,\langle w, z\rangle^2$ is minimal iff $\langle w, z\rangle^2$ is minimal. Choose $z \in w^\perp$ with $|z| = 1$ then the minimum is

$$\min_{\substack{z \in \mathbb{R}^N \\ |z|=1}} \lambda b\,\langle w, z\rangle^2 = \min_{\substack{z \in w^\perp \\ |z|=1}} \lambda b\,\langle w, z\rangle^2 = 0.$$

If $b < 0$ then $\lambda b < 0$ and therefore, $\lambda b\,\langle w, z\rangle^2$ is minimal iff $\langle w, z\rangle^2$ is maximal. Choose $z \in \{w, -w\}$ then the minimum is

$$\min_{\substack{z \in \mathbb{R}^N \\ |z|=1}} \lambda b\,\langle w, z\rangle^2 = \lambda b < 0.$$

Subcase 2: (w, Aw linearly independent) For this purpose we use the method of Lagrange multipliers for finding the local minima of (5.29) w.r.t. z. Consider the functions

$$\begin{aligned} f(w, z) &:= \langle w, Aw\rangle + b\,\langle w, z\rangle\,\langle z, Aw\rangle - \gamma_A, \\ g(z) &:= |z|^2 - 1 = 0 \end{aligned}$$

for every fixed $w \in \mathbb{R}^N$ with $|w| = 1$. The optimization problem is to minimize $f(w, z)$ w.r.t. $z \in \mathbb{R}^N$ subject to the constraint $g(z) = 0$.

1. We introduce a new variable $\mu \in \mathbb{R}$, called the Lagrange multiplier, and define the Lagrange function (Lagrangian)

$$\Lambda : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, \quad \Lambda(z,\mu) := f(z,w) + \mu g(z).$$

The solution of the minimization problem corresponds to a critical point of the Lagrange function. A necessary condition for critical point of Λ is that the Jacobian vanishes, i.e. $J_\Lambda(z,\mu) = 0$. This leads to the equations

$$b\,\langle z, Aw\rangle\, w + b\,\langle w, z\rangle\, Aw + 2\mu z = 0, \tag{5.33}$$

$$|z|^2 - 1 = 0, \tag{5.34}$$

i.e. every local minimizer z satisfies (5.33) and (5.34).
2. Multiplying (5.33) from the left by z^T we obtain

$$0 = 2b\,\langle z, Aw\rangle\,\langle w, z\rangle + 2\mu|z|^2 = 2b\alpha\beta + 2\mu,$$

and thus $\mu = -b\alpha\beta$, where $\alpha := \langle z, Aw\rangle$ and $\beta := \langle w, z\rangle$ are still to be determined. Now, inserting $\mu = -b\alpha\beta$ into (5.33) and dividing both sides by $b \neq 0$ we obtain

$$\alpha w + \beta Aw - 2\alpha\beta z = 0. \tag{5.35}$$

From (5.35) we deduce that if $\alpha = 0$ then $\beta = 0$ and vice versa. If $\alpha = \beta = 0$ then $z \in \{w, Aw\}^\perp$ and the minimum of $f(w,z)$ in z subject to $g(z) = 0$ is $\langle w, Aw\rangle - \gamma_A$.
In the following we consider the case $\alpha \neq 0$ and $\beta \neq 0$ and we show that in this case the minimum of $f(w,z)$ in z subject to $g(z) = 0$ is even smaller. Note that, assuming $\alpha \neq 0$ and $\beta \neq 0$, (5.35) yields the following representation for z

$$z = \frac{1}{2\alpha\beta}\left(\alpha w + \beta Aw\right) = \frac{1}{2\beta} w + \frac{1}{2\alpha} Aw, \tag{5.36}$$

We now look for possible solutions for α and β.
3. Multiplying (5.35) from the left by w^T and using $|w| = 1$ we obtain

$$0 = \alpha|w|^2 + \beta\,\langle w, Aw\rangle - 2\alpha\beta\,\langle w, z\rangle = \alpha + \beta q - 2\alpha\beta^2, \tag{5.37}$$

where $q := \langle w, Aw\rangle$. Multiplying (5.35) from the left by $(Aw)^T$ we obtain

$$0 = \alpha\beta\,\langle Aw, w\rangle + \beta\,\langle Aw, Aw\rangle - 2\alpha\beta\,\langle Aw, z\rangle = \alpha q + \beta r^2 - 2\alpha^2\beta, \tag{5.38}$$

where $r := |Aw|$. From (5.29) with $z := w$ we deduce that $q > 0$ since $1 + b > 0$. Moreover, we have $r > 0$: Assuming $r = |Aw| = 0$ yields $Aw = 0$ for some $|w| = 1$ which contradicts $\gamma_A > 0$, compare (5.29). Since $r > 0$, $q > 0$ and by assumption $\alpha \neq 0$ and $\beta \neq 0$, there exist four solutions of (5.37), (5.38) given by

$$(\alpha,\beta) \in \left\{ \left(\mp\sqrt{\frac{r(r-q)}{2}}, \pm\sqrt{\frac{r-q}{2r}} \right), \left(\pm\sqrt{\frac{r(r+q)}{2}}, \pm\sqrt{\frac{r+q}{2r}} \right) \right\}. \tag{5.39}$$

Note, that $r\pm q>0$ and therefore $(\alpha,\beta)\neq(0,0)$. This follows from the Cauchy-Schwarz inequality and $|w|=1$

$$\pm q\leqslant |q| = |\langle w, Aw\rangle| < |w||Aw| = r.$$

Note that we have indeed a strict inequality since w and Aw are linearly independent by our subcase.
4. Instead of investigating whether the Hessian of f at these points is positive definite or not, we evaluate the function f at the points (5.36) with (α,β) from (5.39) directly. First we observe that

$$f(w,z)=\langle w,Aw\rangle + b\langle w,z\rangle\langle z,Aw\rangle-\gamma_A = q+b\alpha\beta-\gamma_A. \tag{5.40}$$

We now distinguish between the two cases $b>0$ and $b<0$. If $b>0$ then the function $f(w,z)$ is minimal if $\operatorname{sgn}\alpha=-\operatorname{sgn}\beta$ and if $b<0$ then $f(w,z)$ is minimal if $\operatorname{sgn}\alpha=\operatorname{sgn}\beta$. Therefore, for the choice of

$$(\alpha,\beta)=\begin{cases}\left(\mp\sqrt{\frac{r(r-q)}{2}},\pm\sqrt{\frac{r-q}{2r}}\right) & ,\ b>0,\\ \left(\pm\sqrt{\frac{r(r+q)}{2}},\pm\sqrt{\frac{r+q}{2r}}\right) & ,\ b<0,\end{cases} \tag{5.41}$$

the term $b\alpha\beta$ is negative and we have found the global minimum. Thus, for $b>0$ we obtain

$$b\alpha\beta=-b\sqrt{\frac{r(r-q)}{2}}\sqrt{\frac{r-q}{2r}}=-\frac{b}{2}(r-q)=-\frac{|b|}{2}r+\frac{b}{2}q<0 \tag{5.42}$$

and similarly for $b<0$ we obtain

$$b\alpha\beta=b\sqrt{\frac{r(r+q)}{2}}\sqrt{\frac{r+q}{2r}}=\frac{b}{2}(r+q)=-\frac{|b|}{2}r+\frac{b}{2}q<0. \tag{5.43}$$

Therefore, using (5.40), (5.42) and (5.43), the global minimum of $f(w,z)$ in z subject to the constraint $g(z)=0$ is given by

$$\min_{\substack{z\in\mathbb{R}^N\\|z|=1}} f(w,z)=\min_{\substack{z\in\mathbb{R}^N\\|z|=1}}(q+b\alpha\beta-\gamma_A)=\left(1+\frac{b}{2}\right)q-\frac{|b|}{2}r-\gamma_A$$

for every fixed $w\in\mathbb{R}^N$ with $|w|=1$. In particular, defining

$$(z_\star,\mu_\star)=\left(\frac{1}{2\beta}w+\frac{1}{2\alpha}Aw,-b\alpha\beta\right)\ \text{with } \alpha,\ \beta \text{ from (5.41).} \tag{5.44}$$

the above minimum is attained at $z_\star$ from (5.44) since

$$f(w):=f(w,z_\star)=\left(1+\frac{b}{2}\right)q-\frac{|b|}{2}r-\gamma_A \tag{5.45}$$

for every fixed $w\in\mathbb{R}^N$ with $|w|=1$. Now, (5.45) must be nonnegative for every $w\in\mathbb{R}^N$ with $|w|=1$, which corresponds exactly (**3**).

Case 2: ($\mathbb{K}=\mathbb{C}$) In this case we apply Case 1 with $\mathbb{K}=\mathbb{R}$. For this purpose, we write

$$\mathbb{C}^N \ni w = w_1 + iw_2 \cong \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_{\mathbb{R}} \in \mathbb{R}^{2N},$$
$$\mathbb{C}^N \ni z = z_1 + iz_2 \cong \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_{\mathbb{R}} \in \mathbb{R}^{2N},$$
$$\mathbb{C}^{N,N} \ni A = A_1 + iA_2 \cong \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} = A_{\mathbb{R}} \in \mathbb{R}^{2N,2N}.$$

From

$$\langle w, z\rangle = \langle w_1, z_1\rangle + \langle w_2, z_2\rangle + i\left(\langle w_1, z_2\rangle - \langle w_2, z_1\rangle\right)$$

we deduce

$$\mathrm{Re}\,\langle w, z\rangle = \langle w_{\mathbb{R}}, z_{\mathbb{R}}\rangle, \quad \mathrm{Re}\,\langle w, Aw\rangle = \langle w_{\mathbb{R}}, A_{\mathbb{R}} w_{\mathbb{R}}\rangle, \quad |Aw| = |A_{\mathbb{R}} w_{\mathbb{R}}|.$$

Therefore,

$$\mathrm{Re}\,\langle w, Aw\rangle + b\mathrm{Re}\,\langle w, z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A \quad \forall\, w, z \in \mathbb{C}^N,\ |z| = |w| = 1,$$

translates into

$$\langle w_{\mathbb{R}}, A_{\mathbb{R}} w_{\mathbb{R}}\rangle + b\,\langle w_{\mathbb{R}}, z_{\mathbb{R}}\rangle\,\langle z_{\mathbb{R}}, A_{\mathbb{R}} w_{\mathbb{R}}\rangle \geqslant \gamma_A \quad \forall\, w_{\mathbb{R}}, z_{\mathbb{R}} \in \mathbb{R}^{2N},\ |z_{\mathbb{R}}| = |w_{\mathbb{R}}| = 1.$$

Due to Case 1 this is equivalent to

$$\left(1 + \frac{b}{2}\right)\langle w_{\mathbb{R}}, A_{\mathbb{R}} w_{\mathbb{R}}\rangle - \frac{|b|}{2}\,|A_{\mathbb{R}} w_{\mathbb{R}}| \geqslant \gamma_A \qquad \forall\, w_{\mathbb{R}} \in \mathbb{R}^{2N},\ |w_{\mathbb{R}}| = 1,$$

that translates back into

$$\left(1 + \frac{b}{2}\right)\mathrm{Re}\,\langle w, Aw\rangle - \frac{|b|}{2}\,|Aw| \geqslant \gamma_A \qquad \forall\, w \in \mathbb{C}^N,\ |w| = 1,$$

which proves the case $\mathbb{K}=\mathbb{C}$.

<u>(**4**)$\Longleftarrow$(**5**)</u>: Multiplying (**5**) from the left by $\frac{|b|}{2}|w||Aw|$ and using $|w|=1$ we obtain

$$\begin{aligned}\left(1 + \frac{b}{2}\right)\mathrm{Re}\,\langle w, Aw\rangle &\geqslant \frac{|b|}{2}|w||Aw|\delta_A = \frac{|b|}{2}|Aw| + \frac{|b|}{2}|Aw|\left(\delta_A - 1\right) \\ &\geqslant \frac{|b|}{2}|Aw| + \frac{|b|}{2}\frac{1}{|A^{-1}|}\left(\delta_A - 1\right) = \frac{|b|}{2}|Aw| + \gamma_A,\end{aligned}$$

for every $w \in \mathbb{K}^N$ with $|w| = 1$, where $\gamma_A := \frac{|b|}{2}\frac{1}{|A^{-1}|}\left(\delta_A - 1\right)$. Here we used $|w| = |A^{-1}Aw| \leqslant |A^{-1}||Aw|$.

<u>(**4**)$\Longrightarrow$(**5**)</u>: Dividing (**4**) by $\frac{|b|}{2}|Aw|$ we obtain

$$\frac{2\left(1 + \frac{b}{2}\right)\mathrm{Re}\,\langle w, Aw\rangle}{|b||Aw|} \geqslant \frac{2}{|b|}\frac{\gamma_A}{|Aw|} + 1 \quad \forall\, w \in \mathbb{K}^N,\ |w| = 1.$$

Now, let $w\in\mathbb{K}^N$ with $w\neq 0$, then $\left|\frac{w}{|w|}\right|=1$ and we further obtain

$$\frac{2\left(1+\frac{b}{2}\right)\operatorname{Re}\langle w,Aw\rangle}{|b||w||Aw|}\geqslant\frac{2}{|b|}\frac{\gamma_A}{|A|}+1=:\delta_A>1\quad\forall\, 0\neq w\in\mathbb{K}^N.$$

(**5**)$\Longleftrightarrow$(**6**): This follows by the definition of the first antieigenvalue of A.
(**6**)$\Longleftrightarrow$(**7**): trivial. □

5.6 The maximal domain (Part 1)

In this section we derive a characterization of the infinitesimal generator $A_p:\mathcal{D}(A_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ and of its maximal domain $\mathcal{D}(A_p)$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$. Problems of this type are also called **identification problems**.
The next theorem shows that the maximal domain $\mathcal{D}(A_p)$ coincide with $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and that the **formal** Ornstein-Uhlenbeck operator $\mathcal{L}_0$ coincide with the infinitesimal generator A_p on $\mathcal{D}(A_p)$, that can be considered as the **abstract** Ornstein Uhlenbeck operator. Therefore, A_p is called the **maximal realization** (or **maximal extension**) of the complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$ with maximal domain $\mathcal{D}(A_p)=\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$.
The main idea for the first part of the proof comes from [71, Proposition 2.2 and 3.2]. For the maximal domain of the scalar real-valued Ornstein-Uhlenbeck operator we refer to [73] and [83] for the L^p-spaces and to [75] for the L^p-spaces with invariant measure. In particular, we suggest that in the proof we apply Theorem 5.10(4), which states that $\mathcal{S}(\mathbb{R}^d,\mathbb{C}^N)$ is a core for $(A_p,\mathcal{D}(A_p))$, and Theorem 5.13, which yields unique solvability of the resolvent equation for $\mathcal{L}_0$.

Theorem 5.19 (Maximal domain, Part 1)**.** *Let the assumptions* (A1)–(A5) *be satisfied for* $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$*, then*

$$\mathcal{D}(A_p)=\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0),$$

where $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ *is given by*

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0):=\left\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p(\mathbb{R}^d,\mathbb{C}^N)\mid A\triangle v+\langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.$$

In particular,

$$A_pv=\mathcal{L}_0v \text{ for every } v\in\mathcal{D}(A_p),$$

i.e. A_p *is the maximal realization of* $\mathcal{L}_0$ *in* $L^p(\mathbb{R}^d,\mathbb{C}^N)$.

Proof. $\subseteq$: Let $v\in\mathcal{D}(A_p)$. Since $\mathcal{S}$ is dense in $\mathcal{D}(A_p)$ with respect to the graph norm $\|\cdot\|_{A_p}$ by Theorem 5.10(4), we have

$$\exists\ (v_n)_{n\in\mathbb{N}}\subset\mathcal{S}:\ \|v_n-v\|_{A_p}\to 0 \text{ as } n\to\infty.$$

This yields

$$\|v_n-v\|_{L^p}\to 0 \text{ as } n\to\infty.$$

and using Theorem 5.10(2)

$$\|\mathcal{L}_0 v_n - A_p v\|_{L^p} = \|A_p v_n - A_p v\|_{L^p} \to 0 \text{ as } n \to \infty,$$

where $A_p v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ because $v \in \mathcal{D}(A_p)$. Since obviously $\mathcal{S} \subset \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$, we have $(v_n)_{n\in\mathbb{N}} \subset \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and we deduce by the closedness of $\mathcal{L}_0 : \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ from Lemma 5.11 that $v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and $\mathcal{L}_0 v = A_p v$.
$\supseteq$: Let $v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. Choose $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and define $g := (\lambda I - \mathcal{L}_0)\, v$, thus $g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$. Then Corollary 5.7 yields a unique solution $v_\star \in \mathcal{D}(A_p)$ of $(\lambda I - A_p)\, v_\star = g$. Since $v_\star \in \mathcal{D}(A_p) \subseteq \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ we conclude $v_\star \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and $A_p v_\star = \mathcal{L}_0 v_\star$. Thus, we have

$$(\lambda I - \mathcal{L}_0)\, v_\star = g \quad \text{and} \quad (\lambda I - \mathcal{L}_0)\, v = g.$$

From the uniqueness of the resolvent equation for $\mathcal{L}_0$ from Theorem 5.13 we deduce $v = v_\star$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$. Recall $\mathcal{D}(A_p) \subseteq \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \subseteq L^p(\mathbb{R}^d, \mathbb{C}^N)$. Since $v, v_\star$ coincide in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$, $v, v_\star \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and $v_\star \in \mathcal{D}(A_p)$, we conclude $v \in \mathcal{D}(A_p)$ and $\mathcal{L}_0 v = A_p v$. □

A superset of the domain of $\mathcal{L}_0$. Combining Theorem 5.8, which yields $\mathcal{D}(A_p) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$, and Theorem 5.19, which yields $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ for $1 < p < \infty$, we even obtain that

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \mathcal{D}(A_p) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N),\ 1 < p < \infty,$$

and therefore

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \left\{ v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap W^{1,p}(\mathbb{R}^d, \mathbb{C}^N) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N) \right\},\ 1 < p < \infty.$$

Note that, in contrast to Theorem 5.8, Theorem 5.13 shows $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ only for $1 < p \leqslant 2$ but not for general $1 < p < \infty$.

Identification problem in $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$. As already mentioned after Theorem 5.3, the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ is strongly continuous on the exponentially weighted spaces $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$, that justifies to introduce the infinitesimal generator $A_{p,\theta}$. The identification of $\mathcal{D}(A_{p,\theta})$ is now much more complicated, since on the one hand one must check that the Schwartz space $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$ is a core for $(A_{p,\theta}, \mathcal{D}(A_{p,\theta}))$ and on the other hand one must prove resolvent estimates for $\mathcal{L}_0$ in exponentially weighted spaces $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ for $1 < p < \infty$ with domain

$$\mathcal{D}^p_{\mathrm{loc},\theta}(\mathcal{L}_0) := \left\{ v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_\theta(\mathbb{R}^d, \mathbb{C}^N) \mid \mathcal{L}_0 v \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N) \right\}.$$

The complete theory from Section 5.1–5.6 is also satisfied for $S = 0$. However, in this case we even have stronger results which are necessary for the next section.

Diffusion semigroup in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Consider the heat kernel

$$K(\psi,t) := (4\pi t A)^{-\frac{d}{2}} \exp\left(-(4tA)^{-1}|\psi|^2\right),$$

for the differential operator $\left[\mathcal{L}_0^{\mathrm{diff}} v\right](x) := A\triangle v(x)$, compare (4.24) with $B=0$. Then we define the (**d-dimensional**) **diffusion semigroup** (**Gaussian semigroup**, **heat semigroup**)

$$[G(t,0)v](x) := \begin{cases} \int_{\mathbb{R}^d} K(x-\xi,t)v(\xi)d\xi &, t>0 \\ v(x) &, t=0 \end{cases}, \; x\in\mathbb{R}^d, \tag{5.46}$$

Requiring assumption (A1) and (A2), Theorem 5.1–5.3 (with $S=0$) yield that $(G(t,0))_{t\geqslant 0}$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$. If we additionally require that the assumptions (A3) and (A4) are satisfied, then Theorem (5.19) (with $S=0$) states that the infinitesimal generator A_p^{diff} of $(G(t,0))_{t\geqslant 0}$ coincides with the diffusion operator $\mathcal{L}_0^{\mathrm{diff}}$ on its maximal domain given by

$$\mathcal{D}\left(\mathcal{L}_0^{\mathrm{diff}}\right) := \left\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p(\mathbb{R}^d,\mathbb{C}^N) \mid A\triangle v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}$$

for $1<p<\infty$. In particular, the graph norm of A_p^{diff} is given by

$$\|v\|_{A_p^{\mathrm{diff}}} = \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|A\triangle\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)},\; v\in\mathcal{D}(\mathcal{L}_0^{\mathrm{diff}}).$$

But in case $S=0$, we even have maximal L^p-regularity results, since the semigroup $(G(t,0))_{t\geqslant 0}$ is (in contrast to $(T_0(t))_{t\geqslant 0}$ for $S\neq 0$) analytic: Using the assumptions (A1)–(A4), we deduce from [67, Theorem 3.1.2 and 3.1.3] for the scalar complex-valued case that $\mathcal{L}_0^{\mathrm{diff}}$ is a sectorial operator in the sense of [67, Definition 2.0.1] and its maximal domain is even given by

$$\mathcal{D}\left(\mathcal{L}_0^{\mathrm{diff}}\right) := W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$$

for every $1<p<\infty$ with $N=1$. By our assumption (A1), this result extends also for $N>1$. We further conclude from [67, Lemma 6.1.1] for every $1<p<\infty$ that the graph norm is equivalent to $\|\cdot\|_{W^{2,p}}$, i.e. there exists some $C_{\mathrm{diff}}\geqslant 1$ such that

$$C_{\mathrm{diff}}^{-1}\|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|A\triangle v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C_{\mathrm{diff}}\|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} \tag{5.47}$$

for every $v\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$. Later, we still prove in Theorem 7.9 (with $S=Q=0$)

$$\left\{\lambda\in\mathbb{C}\mid \lambda\in\sigma(-\omega^2 A),\,\omega\in\mathbb{R}\right\}\subseteq\sigma_{\mathrm{ess}}(\mathcal{L}_0^{\mathrm{diff}}).$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$. Concluding we define the parabolic evolution family

$$[G(t,s)v](x) := \begin{cases} \int_{\mathbb{R}^d} K(x-\xi,t-s)v(\xi)d\xi &, t>s \\ v(x) &, t=s \end{cases}, \; x\in\mathbb{R}^d.$$

Rotation group in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. For any skew-symmetric matrix $S\in\mathbb{R}^{d,d}$ we define the **rotation group** by

$$[R(t)v](x) := v(e^{tS}x),\ t\in\mathbb{R},\ x\in\mathbb{R}^d.$$

Obviously, $(R(t))_{t\in\mathbb{R}}$ is a strongly continuous group in $W^{k,p}(\mathbb{R}^d,\mathbb{C}^N)$ for $k=0,1,2$ and $1\leqslant p<\infty$, compare proof of Theorem 5.3. The infinitesimal generator A_p^{drift} of $(R(t))_{t\in\mathbb{R}}$ coincides with the drift term $\left[\mathcal{L}_0^{\mathrm{drift}}v\right](x) := \langle Sx,\nabla v\rangle$ on its maximal domain given by

$$\mathcal{D}\left(\mathcal{L}_0^{\mathrm{drift}}\right) := \left\{v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\mid \langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}$$

for $1<p<\infty$. For a proof of these results we refer to [71, Proposition 2.2] and suggest that these results trivially extends to complex-valued systems.

We stress the following relations that follows directly from the definitions of T_0, R and G:

$$T_0(t) = R(t)G(t,0)\quad \forall\, t\geqslant 0, \tag{5.48}$$

$$G(t-s,0)R(s) = R(s)G(t,s)\quad \forall\, t\geqslant s. \tag{5.49}$$

5.7 Cauchy problems and exponential decay

In this section we study the **abstract Cauchy problem**

$$\begin{aligned} v_t(t) &= A_p v(t) + f(t),\ t\in]0,T],\\ v(0) &= v_0, \qquad\qquad\qquad t=0, \end{aligned} \tag{5.50}$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, where $A_p:\mathcal{D}(A_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ denotes the infinitesimal generator of the strongly continuous semigroup $(T_0(t))_{t\geqslant 0}$, $v_0\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ the initial data, $f:[0,T]\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ the inhomogeneity and $v:[0,T]\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ the solution of (5.50). Our aim in this section is to derive regularity results for the homogeneous and inhomogeneous initial value problem (5.50). For this purpose we introduce mild and classical solutions of (5.50), [34, Chapter VI.7].

Definition 5.20. Let the assumptions (A1), (A2), (A5) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Moreover, let $v_0\in L^p(\mathbb{R}^d,\mathbb{K}^N)$ and $f\in L^1([0,T],L^p(\mathbb{R}^d,\mathbb{K}^N))$ for some $T>0$. Then the function $v:[0,T]\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ given by

$$v(t) := T_0(t)v_0 + \int_0^t T_0(t-s)f(s)ds,\ t\in[0,T], \tag{5.51}$$

is called the **mild solution of** (5.50) **in** $[0,T]$. A function $v:[0,\infty[\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ is called the **mild solution of** (5.50) **in** $[0,\infty[$ if $v|_{[0,T]}$ is the mild solution of (5.50) in $[0,T]$ for every $T>0$.

Note that for $v_0\in L^p(\mathbb{R}^d,\mathbb{K}^N)$ and $f\in L^1([0,T],L^p(\mathbb{R}^d,\mathbb{K}^N))$ the mild solution of (5.50) is unique by its definition.

Definition 5.21. Let the assumptions (A1), (A2) and (A5) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Moreover, $v_0\in\mathcal{D}(A_p)$ and $f\in L^1([0,T],L^p(\mathbb{R}^d,\mathbb{K}^N))$ for some $T>0$. Then the function $v:[0,T]\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ is called a **classical solution of** (5.50) **in** $[0,T]$ if

$$v\in C([0,T],\mathcal{D}(A_p))\cap C^1(]0,T[,L^p(\mathbb{R}^d,\mathbb{K}^N))\quad\text{and}\quad(5.50)\text{ holds.}$$

A function $v:[0,\infty[\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ is called a **classical solution of** (5.50) **in** $[0,\infty[$ if $v|_{[0,T]}$ is a classical solution of (5.50) in $[0,T]$ for every $T>0$.

Assuming $v_0\in L^p(\mathbb{R}^d,\mathbb{K}^N)$ and $f\in L^1([0,T],L^p(\mathbb{R}^d,\mathbb{K}^N))$ one can show that every classical solution of (5.50) is also a mild solution of (5.50) and hence unique, [34, Chapter VI.7, 7.10 Exercise].

The following spatial L^p-regularity result for the mild solution of the homogeneous initial value problem (5.50), i.e. with $f=0$, is a direct consequence of Theorem 5.1.

Theorem 5.22 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1)*,* (A2) *and* (A5) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$*. Then for every radially nondecreasing weight function* $\theta\in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta\geqslant 0$ *and for every initial data* $v_0\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *we have* $v(t)\in W^{2,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *for every* $t>0$ *with*

$$\|v(t)\|_{L^p_\theta}\leqslant C_4(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad,\ t\geqslant 0,\tag{5.52}$$

$$\|D_iv(t)\|_{L^p_\theta}\leqslant C_5(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad,\ t>0,\ i=1,\ldots,d,\tag{5.53}$$

$$\|D_jD_iv(t)\|_{L^p_\theta}\leqslant C_6(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad,\ t>0,\ i,j=1,\ldots,d,\tag{5.54}$$

where $v:[0,\infty[\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ *given by* $v(t)=T_0(t)v_0$ *denotes the unique mild solution of* (5.50) *in* $[0,\infty[$ *with* $f=0$ *and the constants* $C_{4+|\beta|}(t)$ *are given by Theorem 5.1 for every* $|\beta|=0,1,2$.

Remark. In order to investigate the temporal regularity for mild solutions of (5.50) with $f=0$, one can show that

$$v\in C([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))\cap C(]0,T[,W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))\cap C^1(]0,T[,L^p_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)).$$

This statement was proved in [73, Theorem 3.3] for the scalar real-valued Ornstein-Uhlenbeck operator.

The following spatial L^p-regularity result for the mild solution of the inhomogeneous initial value problem (5.50) is an extension of Theorem 5.22 and follows again directly from Theorem 5.1. The major difference to Theorem 5.22 is that we are only able to prove that the mild solution of (5.50) belongs to $W^{1,p}(\mathbb{R}^d,\mathbb{C}^N)$ for $t>0$.

Theorem 5.23 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2) *and* (A5) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$, *for every initial data* $v_0 \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *and for every inhomogeneity* $f \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *we have* $v(t) \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *for every* $t>0$ *with*

$$
\begin{aligned}
\|v(t)\|_{L^p_\theta} &\leqslant C_4(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} + C_9(t)\,\|f\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} &&,\ t\geqslant 0,\\
\|D_i v(t)\|_{L^p_\theta} &\leqslant C_5(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} + C_{10}(t)\,\|f\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} &&,\ t>0,\ i=1,\ldots,d,
\end{aligned}
$$

where $v:[0,\infty[\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ *given by* (5.51) *denotes the unique mild solution of* (5.50) *in* $[0,\infty[$ *and the constants* $C_{4+|\beta|}(t)$ *and* $C_{9+|\beta|}(t)$ *are given by Theorem 5.1 and*

$$
C_{9+|\beta|}(t) := \int_0^t C_{4+|\beta|}(s)ds,
$$

respectively, for every $|\beta|=0,1$.

Concluding, we prove a time-space L^p-regularity result for the mild solution of (5.50). For this purpose, we mimic the proof of [73, Theorem 3.4]. Note, that one can identify $L^p([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))$ by $L^p(\mathbb{R}^d\times[0,T],\mathbb{C}^N)$. In the following we abbreviate $\Omega_T := \mathbb{R}^d\times]0,T[$. Moreover, we suggest that the theorem requires $f\in L^p(\Omega_T,\mathbb{C}^N)\cong L^p([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))$ that belongs to $L^1([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))$ on compact time intervals for every $1<p<\infty$.

Theorem 5.24 (Regularity for mild solution)**.** *Let the assumptions* (A1)–(A5) *be satisfied for* $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Moreover, let* v *given by* (5.51) *denote the unique mild solution of* (5.50) *in* $[0,T]$ *with* $v_0=0$ *and* $f\in L^p(\Omega_T,\mathbb{C}^N)$, *then*

$$
v \in W^{(2,1),p}_{\mathrm{loc}}(\Omega_T,\mathbb{C}^N)
$$

and satisfies

$$
v,\, v_t - \langle S\cdot,\nabla v\rangle,\, D_i v,\, D_j D_i v \in L^p(\Omega_T,\mathbb{C}^N).
$$

Remark. Note that Theorem 5.24 does neither say that $v_t\in L^p(\Omega_T,\mathbb{C}^N)$ not that $\langle S\cdot,\nabla v\rangle\in L^p(\Omega_T,\mathbb{C}^N)$. Only their difference $v_t-\langle S\cdot,\nabla v\rangle$ belongs to $L^p(\Omega_T,\mathbb{C}^N)$.

Proof. Let v be the unique mild solution of (5.50) with initial data $v_0=0$ and inhomogeneity $f\in L^p(\Omega_T,\mathbb{C}^N)\subseteq L^1([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))$. By Theorem 5.19 problem (5.50) can be written as

$$
\begin{aligned}
v_t(t) &= A\triangle v(t) + \langle S\cdot,\nabla v(t)\rangle + f(t),\ t\in]0,T],\\
v(0) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad ,\ t=0.
\end{aligned}
\tag{5.55}
$$

From (5.51), (5.48), (5.49) and $R(t)R(s)=R(t+s)$ for $t,s\in\mathbb{R}$ we obtain

$$
v(t) = \int_0^t T_0(t-s)f(s)ds = \int_0^t R(t-s)G(t-s,0)f(s)ds
$$

$$\begin{aligned}
&= \int_0^t R(t-s)G(t-s,0)R(s)R(-s)f(s)ds \\
&= \int_0^t R(t-s)R(s)G(t,s)R(-s)f(s)ds \\
&= \int_0^t R(t)G(t,s)R(-s)f(s)ds = R(t)\int_0^t G(t,s)R(-s)f(s)ds,\ t\in[0,T].
\end{aligned}$$

Defining $h(t) := R(-t)f(t)$ for $t\in[0,T]$ and

$$u(t) := \int_0^t G(t,s)h(s)ds,\ t\in[0,T],$$

we further obtain the relation $v(t) = R(t)u(t)$, i.e. $u(t) = R(-t)v(t)$ for every $t\in[0,T]$. $f\in L^p(\Omega_T,\mathbb{C}^N)$ implies $h\in L^p(\Omega_T,\mathbb{C}^N)\subseteq L^1(]0,T[,L^p(\mathbb{R}^d,\mathbb{C}^N))$ and hence u is the unique mild solution of

$$\begin{aligned}
u_t(t) =& A\triangle u(t) + h(t),\ t\in]0,T], \\
u(0) =& 0 \qquad\qquad\qquad\quad ,\ t=0.
\end{aligned} \tag{5.56}$$

Note that $v\in W^{(2,1),p}_{\mathrm{loc}}(\Omega_T,\mathbb{C}^N)$ is equivalent to $u\in W^{(2,1),p}_{\mathrm{loc}}(\Omega_T,\mathbb{C}^N)$.

In the following we even prove that $u\in W^{(2,1),p}(\Omega_T,\mathbb{C}^N)$: Since $h\in L^p(\Omega_T,\mathbb{C}^N)$ and since $C_{\mathrm{c}}^{\infty}(\Omega_T,\mathbb{C}^N)$ is dense in $L^p(\Omega_T,\mathbb{C}^N)$, there exists $h_n\in C_{\mathrm{c}}^{\infty}(\Omega_T,\mathbb{C}^N)$, $n\in\mathbb{N}$, such that $h_n\to h$ in $L^p(\Omega_T,\mathbb{C}^N)$ as $n\to\infty$. Let us define

$$u_n(t) := \int_0^t G(t,s)h_n(s)ds,\quad t\in[0,T],$$

then u_n is the unique mild solution of (5.56) with inhomogeneity h_n. Moreover, from $h_n\to h$ in $L^p(\Omega_T,\mathbb{C}^N)$ we deduce that $u_n\to u$ in $L^p(\Omega_T,\mathbb{C}^N)$ as $n\to\infty$ and hence $u\in L^p(\Omega_T,\mathbb{C}^N)$. Since $A\triangle : W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ is a sectorial operator in the sense of [67, Definition 2.0.1], an application of [67, Proposition 6.1.3] yields

$$u_n\in C([0,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))\cap C^1([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N)),\ n\in\mathbb{N},$$

and hence, u_n is even a classical solution of (5.56) with inhomogeneity h_n for every fixed $n\in\mathbb{N}$. Moreover, an application of [62, IV. Theorem 9.1] implies that there exists some n-independent constant $C = C(A,d,p,T) > 0$ such that

$$\left\|u_n\right\|_{W^{(2,1),p}(\Omega_T,\mathbb{C}^N)} \leqslant C\left\|h_n\right\|_{L^p(\Omega_T,\mathbb{C}^N)},\ n\in\mathbb{N}. \tag{5.57}$$

From (5.57) and $h_n\to h$ in $L^p(\Omega_T,\mathbb{C}^N)$ we deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{(2,1),p}(\Omega_T,\mathbb{C}^N)$. This follows from the fact that u_n-u_m is a classical solution of (5.56) with inhomogeneity h_n-h_m and thus using (5.57) we find for every $\varepsilon>0$ some $N_0\geqslant 0$ such that

$$\left\|u_n-u_m\right\|_{W^{(2,1),p}(\Omega_T,\mathbb{C}^N)} \leqslant C\left\|h_n-h_m\right\|_{L^p(\Omega_T,\mathbb{C}^N)} \leqslant \varepsilon\quad \forall\, n,m\geqslant N_0.$$

Consequently, u_n converges to some $\tilde{u} \in W^{(2,1),p}(\Omega_T, \mathbb{C}^N)$ as $n \to \infty$. Since we already know that $u_n \to u$ in $L^p(\Omega_T, \mathbb{C}^N)$ as $n \to \infty$, we conclude that $u = \tilde{u}$ in $L^p(\Omega_T, \mathbb{C}^N)$, thus $u \in W^{(2,1),p}(\Omega_T, \mathbb{C}^N)$. In particular, we deduce

$$\begin{aligned}\|u\|_{W^{(2,1),p}(\Omega_T,\mathbb{C}^N)} &\leqslant \|u - u_n\|_{W^{(2,1),p}(\Omega_T,\mathbb{C}^N)} + \|u_n\|_{W^{(2,1),p}(\Omega_T,\mathbb{C}^N)} \\ &\leqslant \|u - u_n\|_{W^{(2,1),p}(\Omega_T,\mathbb{C}^N)} + C\,\|h_n\|_{L^p(\Omega_T,\mathbb{C}^N)} \\ &\leqslant \|u - u_n\|_{W^{(2,1),p}(\Omega_T,\mathbb{C}^N)} + C\,\|h_n - h\|_{L^p(\Omega_T,\mathbb{C}^N)} \\ &\quad + C\,\|h\|_{L^p(\Omega_T,\mathbb{C}^N)}\end{aligned}$$

for every $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain

$$\|u\|_{W^{(2,1),p}(\Omega_T,\mathbb{C}^N)} \leqslant C\,\|h\|_{L^p(\Omega_T,\mathbb{C}^N)}\,.$$

Using $h(t) := R(-t)f(t)$ and $v(t) = R(t)u(t)$ we obtain

$$\begin{aligned}\Bigg(\|v\|^p_{L^p(\Omega_T,\mathbb{C}^N)} + \|v_t - \langle S\cdot, \nabla v\rangle\|^p_{L^p(\Omega_T,\mathbb{C}^N)} + \sum_{i=1}^{d} \|D_{x_i} v\|^p_{L^p(\Omega_T,\mathbb{C}^N)} \\ + \sum_{j=1}^{d}\sum_{i=1}^{d} \left\|D_{x_j} D_{x_i} v\right\|^p_{L^p(\Omega_T,\mathbb{K}^N)} \Bigg)^{\frac{1}{p}} \leqslant C\,\|f\|_{L^p(\Omega_T,\mathbb{C}^N)}\end{aligned}$$

This proves that $v \in W^{(2,1),p}_{\mathrm{loc}}(\Omega_T, \mathbb{C}^N)$, meaning that $v_t - \langle S\cdot, \nabla v\rangle \in L^p(\Omega_T, \mathbb{C}^N)$ but only $v_t, \langle S\cdot, \nabla v\rangle \in L^p_{\mathrm{loc}}(\Omega_T, \mathbb{C}^N)$. □

Remark. Note that [62, IV. Theorem 9.1] holds only for the scalar real-valued case, but it can be extended to the scalar complex-valued case, i.e. for $N = 1$. We do not outline the proof here. Using assumption (A1), this result also extends to complex-valued systems, i.e. with $N > 1$.

5.8 The maximal domain (Part 2)

In this section we prove a complete characterization for the maximal domain of the complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_0$, that is motivated by [73].

The next theorem states that the maximal domain of the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ coincides with the intersection of the domains of its diffusion $\mathcal{L}_0^{\mathrm{diff}}$ and drift part $\mathcal{L}_0^{\mathrm{drift}}$, i.e.

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0^{\mathrm{diff}} + \mathcal{L}_0^{\mathrm{drift}}) = \mathcal{D}^p_{\max}(\mathcal{L}_0^{\mathrm{diff}}) \cap \mathcal{D}^p_{\max}(\mathcal{L}_0^{\mathrm{drift}}).$$

This was proved in [73, Theorem 1] for the scalar real case.

Theorem 5.25 (Maximal domain, Part 2)**.** *Let the assumptions* (A1)–(A5) *be satisfied for* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*, then*

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \mathcal{D}^p_{\max}(\mathcal{L}_0),$$

where $\mathcal{D}^p_{\max}(\mathcal{L}_0)$ *is given by*

$$\mathcal{D}^p_{\max}(\mathcal{L}_0) := \left\{ v \in W^{2,p}(\mathbb{R}^d, \mathbb{C}^N) \mid \langle S\cdot, \nabla v\rangle \in L^p(\mathbb{R}^d, \mathbb{C}^N) \right\}.$$

Proof. $\supseteq$: Let $v\in\mathcal{D}^p_{\max}(\mathcal{L}_0)$, then we have $v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)$ and $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ since $v\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$. Moreover, $v\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$ implies $A\triangle v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$. Thus, using $\langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ we conclude $\mathcal{L}_0 v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$.
$\subseteq$: Let $v\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$, then $g:=\mathcal{L}_0 v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$. Then $w(t)=v$ is a classical solution of

$$\begin{aligned}\frac{d}{dt}w(t)&=\mathcal{L}_0 w(t)-g,\ t\in[0,T]\\ w(0)&=v.\end{aligned}$$

in the sense of Definition 5.21 and hence also a mild solution. On the other hand, since $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $g\in L^p([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))$ for every fixed $T>0$, the unique mild solution is given by

$$v=w(t)=T_0(t)v-\int_0^t T_0(t-s)g\,ds=:w_1(t)+w_2(t),\ t\in[0,T],$$

where w_1 is the mild solution of (5.50) in $[0,T]$ with initial data $v_0=v$ and inhomogeneity $f=0$. Theorem 5.22 states that $w_1(t)\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$ for every $t\in]0,T]$. Similarly, w_2 is the mild solution of (5.50) in $[0,T]$ with initial data $v_0=0$ and inhomogeneity $f=-g$. Because $g\in L^p([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))\cong L^p(\Omega_T,\mathbb{C}^N)$, Theorem 5.24 states that $w_2\in L^p(]0,T[,W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))$, i.e. $w_2(t)\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$ for almost every $t\in]0,T[$. If we consider such a $\bar t\in]0,T[$, we can deduce that

$$v=w(\bar t)=T_0(\bar t)v+\int_0^{\bar t}T_0(\bar t-s)g\,ds=w_1(\bar t)+w_2(\bar t)\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$$

and thus we have $A\triangle v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$. Consequently, using $\mathcal{L}_0 v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$, we conclude

$$\langle S\cdot,\nabla v\rangle=\mathcal{L}_0 v-A\triangle v\in L^p(\mathbb{R}^d,\mathbb{C}^N),$$

that means $v\in\mathcal{D}^p_{\max}(\mathcal{L}_0)$. This completes the proof. For the identification of the graph norm of A_p we need additionally an estimate for v in $W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$: Let $0<\varepsilon<T$ be arbitrary. By Theorem 5.1 (with $\theta\equiv 1$, $C_\theta=1$, $\kappa=0$) there exists a constant $C=C(A,d,p,\varepsilon,T)>0$ such that

$$\|w_1\|_{L^p([\varepsilon,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))}\leqslant C\,\|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.$$

Moreover, by the last inequality from the proof of Theorem 5.24 there exists a constant $C=C(A,d,p,T)>0$ such that

$$\begin{aligned}\|w_2\|_{L^p([\varepsilon,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))}\leqslant&\Big(\sum_{|\beta|\leqslant 2}\left\|D_x^\beta w_2\right\|^p_{L^p(\Omega_T,\mathbb{C}^N)}\Big)^{\frac1p}\leqslant C\,\|g\|_{L^p(\Omega_T,\mathbb{C}^N)}\\ =&CT\,\|\mathcal{L}_0 v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.\end{aligned}$$

Combining these estimates we deduce

$$\begin{aligned}(T-\varepsilon)^{\frac{1}{p}} \|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} &= \|v\|_{L^p([\varepsilon,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))} \\ &\leqslant \|w_1\|_{L^p([\varepsilon,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))} + \|w_2\|_{L^p([\varepsilon,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))} \\ &\leqslant C\left(\|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|\mathcal{L}_0 v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right).\end{aligned}$$

To make the constant independent on T and ε, we choose e.g. $T=1$ and $\varepsilon=\frac{T}{2}$ and conclude that there exists a constant $C=C(A,d,p)>0$ such that

$$\|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C\left(\|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|\mathcal{L}_0 v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right). \tag{5.58}$$

□

The following result yields an identification of the graph norm for A_p. The same result for the scalar real-valued case can also be found in [66, Proposition 9.4.2]. The techniques therein are based on bounded imaginary powers.

Corollary 5.26. *Let the assumptions* (A1)–(A5) *be satisfied for* $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$*, then the norms*

$$\begin{aligned}\|v\|_{A_p} &:= \|A_p v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} = \|\mathcal{L}_0 v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}, \\ \|v\|_{\mathcal{L}_0} &:= \|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} + \|\langle S\cdot,\nabla v\rangle\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)},\end{aligned}$$

are equivalent for $v\in\mathcal{D}^p_{\max}(\mathcal{L}_0)$*, i.e. there exist* $C_1,C_2\geqslant 1$ *such that*

$$C_1\|v\|_{\mathcal{L}_0} \leqslant \|v\|_{A_p} \leqslant C_2\|v\|_{\mathcal{L}_0}$$

for every $v\in\mathcal{D}^p_{\max}(\mathcal{L}_0)$.

Remark. Corollary 5.26 says that we can identify $(A_p,\|\cdot\|_{A_p})$ with $(\mathcal{L}_0,\|\cdot\|_{\mathcal{L}_0})$.

Proof. The second inequality in based on maximal L^p-regularity for $\mathcal{L}_0^{\mathrm{diff}}$: Using the triangle inequality, (5.47) and defining $C_2:=C_{\mathrm{diff}}\geqslant 1$, we obtain

$$\begin{aligned}\|v\|_{A_p} &= \|\mathcal{L}_0 v\|_{L^p} + \|v\|_{L^p} = \|A\triangle v + \langle S\cdot,\nabla v\rangle\|_{L^p} + \|v\|_{L^p} \\ &\leqslant \|A\triangle v\|_{L^p} + \|v\|_{L^p} + \|\langle S\cdot,\nabla v\rangle\|_{L^p} = \|v\|_{A_p^{\mathrm{diff}}} + \|\langle S\cdot,\nabla v\rangle\|_{L^p} \\ &\leqslant C_{\mathrm{diff}}\|v\|_{W^{2,p}} + \|\langle S\cdot,\nabla v\rangle\|_{L^p} \\ &\leqslant C_{\mathrm{diff}}\left(\|v\|_{W^{2,p}} + \|\langle S\cdot,\nabla v\rangle\|_{L^p}\right) = C_2\|v\|_{\mathcal{L}_0}.\end{aligned}$$

The first inequality follows from the characterization of the maximal domain from Theorem 5.25: Using (5.47) and the elliptic estimate (5.58) we obtain

$$\begin{aligned}\|v\|_{\mathcal{L}_0} &= \|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} + \|\langle S\cdot,\nabla v\rangle\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \\ &\leqslant \|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} + \|A\triangle v + \langle S\cdot,\nabla v\rangle\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|A\triangle v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \\ &\leqslant \|\mathcal{L}_0 v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + (1+C_{\mathrm{diff}})\|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)} \\ &\leqslant (C(1+C_{\mathrm{diff}})+1)\left(\|\mathcal{L}_0 v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \|v\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right) = C_1\|v\|_{A_p},\end{aligned}$$

where $C_1:=C(1+C_{\mathrm{diff}})+1>1$. □

As a consequence of Corollary 5.26 we deduce resolvent estimates for $v_\star$ in $W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$ and for $\langle S\cdot,\nabla v\rangle$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. This is an extension of Theorem 5.13 and Theorem 5.7, respectively.

Corollary 5.27 (Resolvent Estimates for $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ with $1<p<\infty$)**.** *Let the assumptions* (A1)–(A5) *be satisfied for* $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Moreover, let* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda>0$. *Then for every* $g\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ *the resolvent equation*

$$(\lambda I-\mathcal{L}_0)\,v=g$$

admits a unique solution $v_\star\in\mathcal{D}^p_{\max}(\mathcal{L}_0)$. *Moreover,* $v_\star$ *satisfies the resolvent estimates*

$$\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{1}{\operatorname{Re}\lambda}\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}, \tag{5.59}$$

$$\|v_\star\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{1}{C_1}\left(\frac{1+|\lambda|}{\operatorname{Re}\lambda}+1\right)\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}, \tag{5.60}$$

$$\|\langle S\cdot,\nabla v_\star\rangle\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{C_2\max\{1,|\lambda|\}}{C_1}\left(\frac{1+|\lambda|}{\operatorname{Re}\lambda}+1+\frac{C_1}{C_2}\right)\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}. \tag{5.61}$$

Proof. Inequality (5.59) was already proved in Theorem 5.13. Let us now prove inequality (5.60): Using (5.59), Corollary 5.26 and Theorem 5.13 we obtain

$$\begin{aligned}
&\|v_\star\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\|v_\star\|_{\mathcal{L}_0}\leqslant\frac{1}{C_1}\|v_\star\|_{A_p}\\
=&\frac{1}{C_1}\left(\|\mathcal{L}_0v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}+\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right)=\frac{1}{C_1}\left(\|\lambda v_\star-g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}+\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right)\\
\leqslant&\frac{1}{C_1}\left((1+|\lambda|)\,\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}+\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right)\leqslant\frac{1}{C_1}\left(\frac{1+|\lambda|}{\operatorname{Re}\lambda}+1\right)\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.
\end{aligned}$$

Finally, let us prove inequality (5.61): Using (5.47), (5.60) and $C_2=C_{\text{diff}}$ we obtain

$$\begin{aligned}
&\|\langle S\cdot,\nabla v_\star\rangle\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}=\|\lambda v_\star-A\triangle v_\star-g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\\
\leqslant&|\lambda|\,\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}+\|A\triangle v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}+\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\\
\leqslant&\max\{1,|\lambda|\}\left(\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}+\|A\triangle v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}+\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right)\\
\leqslant&\max\{1,|\lambda|\}\left(C_{\text{diff}}\,\|v_\star\|_{W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)}+\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\right)\\
\leqslant&\max\{1,|\lambda|\}\left(\frac{C_2}{C_1}\left(\frac{1+|\lambda|}{\operatorname{Re}\lambda}+1\right)+1\right)\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.
\end{aligned}$$

$\square$

6 Constant coefficient perturbations in $L^p(\mathbb{R}^d,\mathbb{C}^N)$

In this chapter we apply perturbation theory of semigroups to the operator

$$[\mathcal{L}_\infty v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x),\; x\in\mathbb{R}^d,\; d\geqslant 2,$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, where $v:\mathbb{R}^d\to\mathbb{C}^N$, $A,B\in\mathbb{C}^{N,N}$, $S\in\mathbb{R}^{d,d}$ skew-symmetric and $N\in\mathbb{N}$. Writing the operator as

$$[\mathcal{L}_\infty v](x) = [\mathcal{L}_0 v](x) - Bv(x),\; x\in\mathbb{R}^d,\; d\geqslant 2,$$

$\mathcal{L}_\infty$ can be seen as a constant coefficient perturbation of the complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_0$, that was analyzed in Chapter 5 before.

In Section 6.1 we investigate constant coefficient perturbations of A_p in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, where $(A_p,\mathcal{D}(A_p))$ denotes the infinitesimal generator of the complex-valued Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d,\mathbb{C}^N)$: Consider the bounded linear operator

$$E_p : L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N),\quad [E_p v](x) := -Bv(x) \tag{6.1}$$

on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for some $1\leqslant p<\infty$ and some matrix $B\in\mathbb{C}^{N,N}$. Then we analyze perturbations of the form

$$B_p : \mathcal{D}(B_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N),\quad [B_p v](x) := [A_p v](x) + [E_p v](x).$$

This means that the infinitesimal generator A_p is **perturbed** by the bounded operator E_p, which means that E_p is a bounded constant coefficient **perturbation** of A_p. Assuming (A1), (A2) and (A5) for $\mathbb{K}=\mathbb{C}$, we show in Theorem 6.1 that B_p with maximal domain $\mathcal{D}(B_p)=\mathcal{D}(A_p)$ is the infinitesimal generator of a strongly continuous semigroup $(T_\infty(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$. Moreover, this theorem yields a series representation for the semigroup $(T_\infty(t))_{t\geqslant 0}$, where the summands are defined recursively. This follows directly from an application of the bounded perturbation theorem, [34, III.1.3]. In Theorem 6.2 we require additionally assumption ($A8_B$) and prove that the semigroup is given by

$$[T_\infty(t)v](x) := \begin{cases} \int_{\mathbb{R}^d} H_\infty(x,\xi,t)v(\xi)d\xi & ,\, t>0\\ v(x) & ,\, t=0\end{cases}\quad ,x\in\mathbb{R}^d,$$

where $H_\infty(x,\xi,t)=H(x,\xi,t)$ denotes the heat kernel from Theorem 4.4. Note that for this result assumption ($A8_B$) is crucial and guarantees that H_0 and B commute.

Assuming (A1), (A2), (A5) and (A8$_B$) for $1 \leqslant p < \infty$ and $\mathbb{K}=\mathbb{C}$, we prove in Corollary 6.7 that the resolvent equation for B_p, which is given by

$$(\lambda I - B_p)\, v = g,$$

admits a unique solution $v_\star \in \mathcal{D}(A_p)$ for every $g \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -b_0$, where $-b_0 := s(-B)$ denotes the spectral bound of $-B$, cf. (1.18). This follows from some applications of abstract semigroup theory, [34, II.1]. In particular, if we require in addition the assumptions (A3) and (A4) for $1 < p < \infty$, then the identification problem for B_p is solved by Theorem 5.19 and 5.25, respectively, and we obtain $B_p = \mathcal{L}_\infty$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \mathcal{D}^p_{\max}(\mathcal{L}_0)$.

In Section 6.2 we derive a-priori estimates for the resolvent equation for B_p in exponentially weighted L^p-spaces. Assuming (A1), (A2), (A5) and (A8$_B$) for $\mathbb{K}=\mathbb{C}$, we prove in Theorem 6.8 that the solution $v_\star$ belongs to $W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ for every $g \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -b_0$. This is an extended version of Theorem 5.8.

For the sake of completeness note that, assuming (A1)–(A5) and (A8$_B$) for $1 < p < \infty$ and $\mathbb{K}=\mathbb{C}$, every $\lambda \in \mathbb{C}$ of the form

$$\lambda = -\lambda(\omega) - i\sum_{l=1}^{k} n_l\sigma_l, \quad n_l \in \mathbb{Z}, \quad \omega \in \mathbb{R}, \quad \lambda(\omega) \in \sigma(\omega^2 A + B),$$

belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L}_\infty)$ of $\mathcal{L}_\infty$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Hence, $\mathcal{L}_\infty$ is not sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $(T_\infty(t))_{t\geqslant 0}$ is not analytic on $L^p(\mathbb{R}^d,\mathbb{C}^N)$, whenever $S \neq 0$. These results will be proved later in Section 7.4 for more general perturbed Ornstein-Uhlenbeck operators. Their proofs combine and extend the results from [71] and [15].

6.1 Application of semigroup theory

Let the assumptions (A1), (A2) and (A5) be satisfied for $\mathbb{K}=\mathbb{C}$, then we denote by $(A_p,\mathcal{D}(A_p))$ the infinitesimal generator of the complex Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ from (5.3) on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 \leqslant p < \infty$. The semigroup $(T_0(t))_{t\geqslant 0}$ is strongly continuous in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1 \leqslant p < \infty$ and satisfies

$$\left\|T_0(t)\right\|_{\mathcal{L}(L^p,L^p)} \leqslant M_0 \quad \forall\, t \geqslant 0,$$

with $M_0 := \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}} \geqslant 1$. Moreover, if we additionally require the assumptions (A3) and (A4) for $1 < p < \infty$, then A_p is the maximal realization of $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ on its maximal domain, which is $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \mathcal{D}^p_{\max}(\mathcal{L}_0)$.

In this section we investigate constant coefficient perturbations of A_p in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 \leqslant p < \infty$. For this purpose, let E_p be given by (6.1). An application of [34, III.1.3 Bounded Perturbation Theorem, III.1.7 Corollary and III.1.10 Theorem] yields the following result.

Theorem 6.1 (Bounded Perturbation Theorem)**.** *Let the assumptions* (A1)*,* (A2)*,* (A5) *and* $B \in \mathbb{K}^{N,N}$ *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*. Then the operator*

$$B_p := A_p + E_p \quad \textit{with} \quad \mathcal{D}(B_p) := \mathcal{D}(A_p)$$

generates a strongly continuous semigroup $(T_\infty(t))_{t\geqslant 0}$ *on* $L^p(\mathbb{R}^d, \mathbb{C}^N)$ *satisfying*

$$\|T_\infty(t)\|_{\mathcal{L}(L^p,L^p)} \leqslant M_0 e^{M_0 t \|E_p\|_{\mathcal{L}(L^p,L^p)}} \quad \forall\, t \geqslant 0. \tag{6.2}$$

Moreover, for every $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *and* $t \geqslant 0$ *the semigroup* $(T_\infty(t))_{t\geqslant 0}$ *satisfies the integral equation (variation of parameters formula)*

$$T_\infty(t)v = T_0(t)v + \int_0^t T_0(t-s)E_p T_\infty(s)v ds,$$

is unique and can be obtained by

$$T_\infty(t) = \sum_{n=0}^{\infty} T_\infty^{(n)}(t) \tag{6.3}$$

where

$$T_\infty^{(0)}(t) := T_0(t), \quad T_\infty^{(n+1)}(t) := \int_0^t T_0(t-s)E_p T_\infty^{(n)}(s)ds. \tag{6.4}$$

Identification problem for B_p**.** Theorem 6.1 states that $(B_p, \mathcal{D}(A_p))$ is the infinitesimal generator of $(T_\infty(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$. If we additionally require the assumptions (A3) and (A4) for $1 < p < \infty$, then an application of Theorem 5.19 and Theorem 5.25 solves the identification problem for B_p in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ and we infer that

$$B_p = \mathcal{L}_\infty \quad \text{on} \quad \mathcal{D}(A_p) = \mathcal{D}^p_{\text{loc}}(\mathcal{L}_0) = \mathcal{D}^p_{\max}(\mathcal{L}_0).$$

To investigate the nonlinear problem for the Ornstein-Uhlenbeck operator, it is obligatory to have a more convenient representation for the semigroup of the perturbed operator B_p, since the estimate (6.2) shows that the semigroup $(T_\infty(t))_{t\geqslant 0}$ doesn't remain bounded as $t \to \infty$. The next theorem provides an explicit representation for the semigroup $(T_\infty(t))_{t\geqslant 0}$ for matrices $A, B \in \mathbb{C}^{N,N}$ that are simultaneously diagonalizable (over $\mathbb{C}$). This can easily be inferred from (6.3) using that the matrix B commutes with the Ornstein-Uhlenbeck kernel $H_0(x, \xi, t)$.

Theorem 6.2 (Semigroup representation)**.** *Let the assumptions* (A1)*,* (A2)*,* (A5) *and* (A8_B) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*. Then the semigroup* $(T_\infty(t))_{t\geqslant 0}$ *in* $L^p(\mathbb{R}^d, \mathbb{C}^N)$ *is given by*

$$[T_\infty(t)v](x) := \begin{cases} \int_{\mathbb{R}^d} H_\infty(x,\xi,t)v(\xi)d\xi & , t > 0 \\ v(x) & , t = 0 \end{cases} , x \in \mathbb{R}^d, \tag{6.5}$$

where $H_\infty(x, \xi, t) = H(x, \xi, t)$ *denotes the heat kernel from Theorem 4.4.*

Proof. From Theorem 6.1 we know that the semigroup $(T_\infty(t))_{t\geqslant 0}$ is given by (6.3) and (6.4). By induction over $n\in\mathbb{N}_0$ we show that

$$T_\infty^{(n)}(t)=T_0(t)\frac{(-tB)^n}{n!},\; n\in\mathbb{N}_0,\; t\geqslant 0. \tag{6.6}$$

The case $n=0$ is satisfied by (6.4). Let us consider the case $n\to n+1$: Using (6.4), (6.1), (6.6), (5.3), (A8$_B$) (that guarantees $BH_0(x,\xi,t)=H_0(x,\xi,t)B$) and Lemma 4.5 we obtain for every $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ with $1\leqslant p<\infty$ and every $t>0$

$$\begin{aligned}
T_\infty^{(n+1)}(t)v(x)=&\int_0^t T_0(t-s)E_pT_\infty^{(n)}(s)v(x)ds\\
=&-\int_0^t T_0(t-s)BT_0(s)\frac{(-sB)^n}{n!}v(x)ds\\
=&-\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}H_0(x,\psi,t-s)BH_0(\psi,\xi,s)\frac{(-sB)^n}{n!}v(\xi)d\psi d\xi ds\\
=&-\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}H_0(x,\psi,t-s)H_0(\psi,\xi,s)d\psi\frac{(-sB)^n}{n!}Bv(\xi)d\xi ds\\
=&\int_0^t s^n\int_{\mathbb{R}^d}H_0(x,\xi,t)\frac{(-B)^{n+1}}{n!}v(\xi)d\xi ds\\
=&\int_{\mathbb{R}^d}H_0(x,\xi,t)\frac{(-tB)^{n+1}}{(n+1)!}v(\xi)d\xi\\
=&T_0(t)\frac{(-tB)^{n+1}}{(n+1)!}v(x).
\end{aligned}$$

This proves (6.6). Now, (6.3), (5.3) and (A8$_B$) yield

$$\begin{aligned}
T_\infty(t)v(x)=&T_0(t)\left[\sum_{n=0}^\infty\frac{(-tB)^n}{n!}\right]v(x)=T_0(t)e^{-Bt}v(x)\\
=&\int_{\mathbb{R}^d}H_0(x,\xi,t)e^{-Bt}v(\xi)d\xi=\int_{\mathbb{R}^d}H_\infty(x,\xi,t)v(\xi)d\xi,\; t>0.
\end{aligned}$$

Finally, $T_\infty(0)v(x)=v(x)$ follows from (6.3) and (6.6), since $T_\infty^{(0)}(0)v(x)=v(x)$ and $T_\infty^{(n)}(0)v(x)=0$ for $n\geqslant 1$. $\square$

Simultaneous diagonalization of A and B. Note that if $A,B\in\mathbb{C}^{N,N}$ are diagonalizable matrices then A and B commute if and only if A and B are simultaneously diagonalizable (over $\mathbb{C}$), [53, Theorem 1.3.12]. Therefore, condition (A8$_B$) ensures that the matrices A and B commute. Moreover, also the inverse A^{-1}, that exists by assumption (A2), commutes with the matrix B. Furthermore, this yields that $A^{-\frac{d}{2}}$ and B commute and also that B and $\exp(A^{-1})$ commute by an application of the Baker-Campbell-Hausdorff formula. Combining these facts we obtain that

$$H_0(x,\xi,t)B=BH_0(x,\xi,t),\quad x,\xi\in\mathbb{R}^d,t>0,$$

i.e. $[B,H_0(x,\xi,t)]:=BH_0(x,\xi,t)-H_0(x,\xi,t)B=0$.

As already mentioned after Theorem 4.4, the situation changes dramatically if the assumption ($A8_B$) is not satisfied. In this case the kernel H_0 and B in general do not commute: Consider $X_1, X_2, Y \in \mathbb{C}^{N,N}$ then the Hadamard lemma states that

$$e^{X_1} Y e^{-X_1} = \sum_{m=0}^{\infty} \frac{1}{m!} [X_1, Y]_m = Y + [X_1, Y] + \frac{1}{2} [X_1, [X_1, Y]] + \cdots ,$$

where $[X,Y]_0 = Y$, $[X,Y]_1 = [X,Y] = XY - YX$ and $[X,Y]_m = \left[X, [X,Y]_{m-1}\right]$. This yields the following formula

$$X_2 e^{X_1} Y = Y X_2 e^{X_1} + [X_2, Y] e^{X_1} + \sum_{m=1}^{\infty} \frac{1}{m!} X_2 [X_1, Y]_m e^{X_1},$$

which we must apply for

$$X_1 = -(4tA)^{-1} \left| e^{tS} x - \xi \right|^2, \quad X_2 = (4\pi t A)^{-\frac{d}{2}}, \quad Y = B.$$

Theorem 6.3 (Boundedness on $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$). *Let the assumptions* (A1), (A2), (A5) *and* ($A8_B$) *be satisfied for* $1 \leqslant p \leqslant \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Then for every radial weight function* $\theta \in C(\mathbb{R}^d, \mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$ *and for every* $v \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$

(6.7) $$\|T_\infty(t)v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C_4(t) \|v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad , t \geqslant 0,$$

(6.8) $$\|D_i T_\infty(t)v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C_5(t) \|v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad , t > 0,\ i = 1, \ldots, d,$$

(6.9) $$\|D_j D_i T_\infty(t)v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant C_6(t) \|v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad , t > 0,\ i,j = 1, \ldots, d,$$

where the constants $C_{4+|\beta|}(t)$ *are from Section 4.3 for every* $|\beta| = 0, 1, 2$, *i.e.*

$$\begin{aligned}
C_4(t) =& C_\theta M^{\frac{d}{2}} e^{-b_0 t} \left[{}_1F_1\left(\frac{d}{2}; \frac{1}{2}; \kappa t\right) + 2 \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} (\kappa t)^{\frac{1}{2}} {}_1F_1\left(\frac{d+1}{2}; \frac{3}{2}; \kappa t\right) \right]^{\frac{1}{p}}, \\
C_5(t) =& C_\theta M^{\frac{d+1}{2}} e^{-b_0 t} (t a_{\min})^{-\frac{1}{2}} \left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} {}_1F_1\left(\frac{d+1}{2}; \frac{1}{2}; \kappa t\right) \right. \\
& \left. + 2 \frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} (\kappa t)^{\frac{1}{2}} {}_1F_1\left(\frac{d+2}{2}; \frac{3}{2}; \kappa t\right) \right]^{\frac{1}{p}}, \\
C_6(t) =& C_\theta M^{\frac{d+2}{2}} e^{-b_0 t} (t a_{\min})^{-1} \left[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} {}_1F_1\left(\frac{d+2}{2}; \frac{1}{2}; \kappa t\right) \right. \\
& + 2 \frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} (\kappa t)^{\frac{1}{2}} {}_1F_1\left(\frac{d+3}{2}; \frac{3}{2}; \kappa t\right) + \frac{\delta_{ij}}{2} M^{-1} {}_1F_1\left(\frac{d}{2}; \frac{1}{2}; \kappa t\right) \\
& \left. + \delta_{ij} M^{-1} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} (\kappa t)^{\frac{1}{2}} {}_1F_1\left(\frac{d+1}{2}; \frac{3}{2}; \kappa t\right) \right]^{\frac{1}{p}}.
\end{aligned}$$

In case $p = \infty$ *they are given by* $C_{4+|\beta|}(t)$ *with* $p = 1$, *where* $M := \frac{a_{\max}^2}{a_{\min} a_0} \geqslant 1$ *and* $\kappa := \frac{a_{\max}^2 \eta^2 p^2}{a_0} \geqslant 0$. *Note that* $C_{4+|\beta|}(t) \sim t^{\frac{-p|\beta|+d+|\beta|-1}{2p}} e^{-(b_0 - \frac{\kappa}{p})t}$ *as* $t \to \infty$ *and* $C_{4+|\beta|}(t) \sim t^{-\frac{|\beta|}{2}}$ *as* $t \to 0$ *for every* $|\beta| = 0, 1, 2$.

Proof. Using the semigroup representation from Theorem 6.2, the proof can be adopted from Theorem 5.1, where we have to replace T_0 and H_0 by T_∞ and $H_\infty = H$, respectively. □

For the next statement we refer to [34, II.1.3 Lemma, II.1.4 Theorem]:

Lemma 6.4. *Let the assumptions* (A1), (A2), (A5) *and* (A8$_B$) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$.
(**1**) *The generator* $B_p : \mathcal{D}(A_p) \subseteq L^p(\mathbb{R}^d,\mathbb{C}^N) \to L^p(\mathbb{R}^d,\mathbb{C}^N)$ *is a linear, closed and densely defined operator.*
(**2**) *For every* $v \in \mathcal{D}(A_p)$ *and* $t \geqslant 0$ *we have*

$$\begin{aligned}
&T_\infty(t)v \in \mathcal{D}(A_p)\\
&\frac{d}{dt}T_\infty(t)v = T_\infty(t)B_pv = B_pT_\infty(t)v
\end{aligned}$$

(**3**) *For every* $v \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ *and every* $t \geqslant 0$ *we have*

$$\int_0^t T_\infty(s)vds \in \mathcal{D}(A_p)$$

(**4**) *For every* $t \geqslant 0$ *we have*

$$\begin{aligned}
T_\infty(t)v - v =&B_p\int_0^t T_\infty(s)vds && \text{, for } v \in L^p(\mathbb{R}^d,\mathbb{C}^N),\\
=&\int_0^t T_\infty(s)B_pvds && \text{, for } v \in \mathcal{D}(A_p).
\end{aligned}$$

Since $(B_p,\mathcal{D}(A_p))$ is a closed operator on the Banach space $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1 \leqslant p < \infty$, we can introduce

$$\begin{aligned}
\sigma(B_p) :=&\{\lambda \in \mathbb{C} \mid \lambda I - B_p \text{ is not bijective}\} && \textbf{spectrum of } \boldsymbol{B_p},\\
\rho(B_p) :=&\mathbb{C}\backslash\sigma(B_p) && \textbf{resolvent set of } \boldsymbol{B_p},\\
R(\lambda,B_p) :=&(\lambda I - B_p)^{-1}\text{, for } \lambda \in \rho(B_p) && \textbf{resolvent of } \boldsymbol{B_p}.
\end{aligned}$$

The next identities follow from [34, II.1.9 Lemma].

Lemma 6.5. *Let the assumptions* (A1), (A2), (A5) *and* (A8$_B$) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Then for every* $\lambda \in \mathbb{C}$ *and* $t > 0$,

$$\begin{aligned}
e^{-\lambda t}T_\infty(t)v - v =&(B_p - \lambda I)\int_0^t e^{-\lambda s}T_\infty(s)vds && \text{, for } v \in L^p(\mathbb{R}^d,\mathbb{C}^N),\\
=&\int_0^t e^{-\lambda s}T_\infty(s)\left(B_p - \lambda I\right)vds && \text{, for } v \in \mathcal{D}(A_p).
\end{aligned}$$

By (6.7) from Theorem 6.3 (with $\theta \equiv 1$, $\eta = 0$ and $C_\theta = 1$) we have

$$\exists\,\omega_\infty \in \mathbb{R} \;\wedge\; \exists\,M_\infty \geqslant 1 :\; \|T_\infty(t)\|_{\mathcal{L}(L^p,L^p)} \leqslant M_\infty e^{\omega_\infty t} \quad \forall\,t \geqslant 0, \tag{6.10}$$

where $M_\infty := M_0 = \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}} \geqslant 1$ and $\omega_\infty := -b_0$, which gives a better estimate as in (6.2). For the next statement we refer to [34, II.1.10 Theorem].

Theorem 6.6. *Let the assumptions* (A1), (A2), (A5) *and* ($A8_B$) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$.
(**1**) *If* $\lambda \in \mathbb{C}$ *is such that* $R(\lambda)v := \int_0^\infty e^{-\lambda s} T_\infty(s) v ds$ *exists for every* $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$, *then*

$$\lambda \in \rho(B_p) \quad \textit{and} \quad R(\lambda, B_p) = R(\lambda).$$

(**2**) *If* $\lambda \in \mathbb{C}$ *satisfies* $\operatorname{Re}\lambda > \omega_\infty$, *then*

$$\lambda \in \rho(B_p), \quad R(\lambda, B_p) = R(\lambda)$$

and

$$\|R(\lambda, B_p)\|_{\mathcal{L}(L^p, L^p)} \leqslant \frac{M_\infty}{\operatorname{Re}\lambda - \omega_\infty}.$$

Theorem 6.6(2) states that the half–plane $\operatorname{Re}\lambda > \omega_\infty$ belongs to the resolvent set $\rho(B_p)$. Therefore, the spectrum $\sigma(B_p)$ is contained in the half–plane $\operatorname{Re}\lambda \leqslant \omega_\infty$. The **spectral bound** $s(B_p)$ **of** B_p, [34, II.1.12 Definition], defined by

$$-\infty \leqslant s(B_p) := \sup_{\lambda \in \sigma(B_p)} \operatorname{Re}\lambda \leqslant \omega_\infty = -b_0 < +\infty.$$

can be considered as the smallest value $\omega \in \mathbb{R}$ such that the spectrum is contained in the half–plane $\operatorname{Re}\lambda \leqslant \omega$.

A direct consequence of Theorem 6.6 is the following:

Corollary 6.7 (Solvability and uniqueness in $L^p(\mathbb{R}^d, \mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2), (A5) *and* ($A8_B$) *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Moreover, let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_\infty$. *Then for every* $g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *the resolvent equation*

$$(\lambda I - B_p)\, v = g$$

admits a unique solution $v_\star \in \mathcal{D}(A_p)$, *which is given by the integral expression*

$$\begin{aligned} v_\star = R(\lambda)g &= \int_0^\infty e^{-\lambda s} T_\infty(s) g ds \\ &= \int_0^\infty e^{-\lambda s} \int_{\mathbb{R}^d} H_\infty(\cdot, \xi, s) g(\xi) d\xi ds. \end{aligned} \tag{6.11}$$

Moreover, the following resolvent estimate holds

$$\|v_\star\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)} \leqslant \frac{M_\infty}{\operatorname{Re}\lambda - \omega_\infty} \|g\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)}.$$

Green's function of B_p. Let the assumptions (A1), (A2), (A5), (A8$_B$) and (A9$_B$) be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K}=\mathbb{C}$. By Corollary 6.7 (with $\lambda = 0$) we obtain for every $g \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ a unique solution $v_\star \in \mathcal{D}(A_p)$ of the resolvent equation

$$B_p v = g$$

which is given by (6.11). We believe one can apply Fubini's theorem in (6.11) to obtain

$$v_\star(x) = -\left[R(0)g\right](x) = \int_{\mathbb{R}^d} G(x,\xi)g(\xi)d\xi,$$

where

$$G(x,\xi) := -\int_0^\infty H_\infty(x,\xi,s)ds$$

denotes the Green's function of B_p. In particular, by Corollary 6.7 with $\lambda = 0$ the following resolvent estimate holds:

$$\left\|v_\star\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{M_\infty}{-\omega_\infty}\left\|g\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.$$

6.2 Exponential decay

In this section we prove a-priori estimates for the solution $v_\star$ of the resolvent equation $(\lambda I - B_p)\,v = g$ in exponentially weighted L^p-spaces. We show that the solution $v_\star \in \mathcal{D}(A_p)$ decays exponentially (at least) with the same rate as the inhomogeneity g. Note, that this result needs neither an explicit representation for the domain $\mathcal{D}(A_p)$ nor for the infinitesimal generator B_p. But the proof requires the integral expression for $v_\star$ from Corollary 6.7, that needs the explicit representation for the semigroup $(T_\infty(t))_{t\geqslant 0}$ from Theorem 6.2. The proof of the following result is similar to that one of Theorem 5.8.

Theorem 6.8 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1)*,* (A2)*,* (A5) *and* (A8$_B$) *be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $0<\vartheta<1$ and $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda > \omega_\infty$. Then for every radially nondecreasing weight function $\theta \in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ with $0 \leqslant \eta^2 \leqslant \vartheta\frac{a_0(\operatorname{Re}\lambda-\omega_\infty)}{a_{\max}^2 p^2}$ and for every $g \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ we have $v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ with*

$$\left\|v_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{C_7}{\operatorname{Re}\lambda-\omega_\infty}\left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}, \tag{6.12}$$

$$\left\|D_i v_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{C_8}{(\operatorname{Re}\lambda-\omega_\infty)^{\frac{1}{2}}}\left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)},\ i=1,\ldots,d, \tag{6.13}$$

where $v_\star \in \mathcal{D}(A_p)$ denotes the unique solution of $(\lambda I - B_p)v = g$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and the λ-independent constants C_7, C_8 are given by Lemma 4.8 (with $\omega=\omega_\infty$).

Proof. By Corollary 6.7 (with $H_\infty(x,\xi,t) = H(x,\xi,t)$) we have the representation

$$v_\star(x) = \int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d} H_\infty(x,\xi,t)g(\xi)d\xi dt, \tag{6.14}$$

Using this representation, the proof can be adopted from Theorem 5.8. In the last inequality, we must apply Lemma 4.8 with $\omega=\omega_\infty$. $\square$

7 Variable coefficient perturbations in $L^p(\mathbb{R}^d,\mathbb{C}^N)$

In this chapter we apply perturbation theory of semigroups to the operator

$$[\mathcal{L}_Q v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x) + Q(x)v(x),\ x\in\mathbb{R}^d,\ d\geqslant 2,$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, where $v:\mathbb{R}^d\to\mathbb{C}^N$, $A,B\in\mathbb{C}^{N,N}$, $S\in\mathbb{R}^{d,d}$ skew-symmetric, $Q\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ and $N\in\mathbb{N}$. Writing the operator as

$$[\mathcal{L}_Q v](x) = [\mathcal{L}_\infty v](x) + Q(x)v(x),\ x\in\mathbb{R}^d,\ d\geqslant 2,$$

$\mathcal{L}_Q$ can be seen as a variable coefficient perturbation of the perturbed complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_\infty$, that was analyzed in Chapter 6.

In Section 7.1 we investigate variable coefficient perturbations of B_p in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, where $(B_p,\mathcal{D}(A_p))$ denotes the infinitesimal generator of the semigroup $(T_\infty(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d,\mathbb{C}^N)$: Consider the bounded linear operator

$$F_p : L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N),\ [F_p v](x) := Q(x)v(x) \tag{7.1}$$

on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for some $1\leqslant p<\infty$ and some function $Q\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$. Then we analyze perturbations of the form

$$C_p:\mathcal{D}(C_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N),\quad [C_p v](x) := [B_p v](x) + [F_p v](x).$$

This signifies, similarly to Chapter 6, that the infinitesimal generator B_p is **perturbed** by the bounded operator F_p, which means that F_p is a bounded variable coefficient **perturbation** of B_p. Assuming (A1), (A2), (A5), (A8$_B$) and $Q\in L^\infty(\mathbb{R}^d,\mathbb{K}^{N,N})$ for $\mathbb{K}=\mathbb{C}$, we show in Theorem 7.1 that C_p with maximal domain $\mathcal{D}(C_p)=\mathcal{D}(A_p)$ is the infinitesimal generator of a strongly continuous semigroup $(T_Q(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$. This follows directly by the bounded perturbation theorem, [34, III.1.3]. In contrast to the constant coefficient perturbation from Chapter 6, we do not have an explicit representation for the semigroup $(T_Q(t))_{t\geqslant 0}$ in this case. Nevertheless, assuming (A1), (A2), (A5), (A8$_B$) and $Q\in L^\infty(\mathbb{R}^d,\mathbb{K}^{N,N})$ for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$, we show in Corollary 7.5 that the resolvent equation for C_p, which is given by

$$(\lambda I - C_p)\, v = g,$$

admits a unique solution $v_\star\in\mathcal{D}(A_p)$ for every $g\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda > -b_0 + M_\infty \|Q\|_{L^\infty}$, where $-b_0 := s(-B)$ and $M_\infty := \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}}$, cf. (1.18). This follows once more from some applications of abstract semigroup theory, [34, II.1]. In particular, if we require additonally the assumptions (A3) and (A4) for $1<p<\infty$, then the identification problem for C_p is again solved by Theorem 5.19 and 5.25, respectively, and we obtain $C_p=\mathcal{L}_Q$ on $\mathcal{D}(A_p)=\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)=\mathcal{D}^p_{\max}(\mathcal{L}_0)$.

In Section 7.2 we apply the results from Section 7.1 to small perturbations $Q=Q_\varepsilon\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ of B_p, meaning that Q_ε is small with respect to $\|\cdot\|_{L^\infty}$. Denoting by C_p^ε the infinitesimal generator of $(T_{Q_\varepsilon}(t))_{t\geqslant 0}$, we then derive some a-priori estimates for the resolvent equation $\left(\lambda I-C_p^\varepsilon\right)v=g$ in exponentially weighted L^p-spaces. Assuming (A1), (A2), (A5), (A8$_B$) and (A9$_B$) for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$ and $Q_\varepsilon\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ satisfying

$$\|Q_\varepsilon\|_{L^\infty}\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}=:K,$$

we prove in Theorem 7.6 that the solution $v_\star$ belongs to $W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ for every $g\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ and $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\geqslant-\frac{b_0}{3}$. Note that for small perturbations the bound $\operatorname{Re}\lambda\geqslant-\frac{b_0}{3}$ does not depend on the perturbation Q_ε. Moreover, the upper bound of the decay rate does not depend on λ any more.

In Section 7.3 we apply the results from Section 7.1 to perturbations $Q\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ of B_p, where Q falls below the constant K at infinity. Note that such perturbations Q can always be decomposed into the sum $Q=Q_\varepsilon+Q_c$ of a function $Q_\varepsilon\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, that is small with respect to $\|\cdot\|_{L^\infty}$, and a function $Q_c\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, that is compactly supported on $\mathbb{R}^d$. Denoting by C_p the infinitesimal generator of $(T_Q(t))_{t\geqslant 0}$, we then derive some a-priori estimates for the resolvent equation $(\lambda I-C_p)\,v=g$ in exponentially weighted L^p-spaces. Assuming (A1), (A2), (A5), (A8$_B$), (A9$_B$) for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$, $Q\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ satisfying

$$\operatorname*{ess\,sup}_{|x|\geqslant R_0}|Q(x)|_2\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\},\text{ for some } R_0>0,$$

$\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\geqslant-\frac{b_0}{3}$ and $g\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$, we prove in Theorem 7.7 that every solution $v_\star$ belongs to $W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$.

In Section 7.4 we investigate the essential spectrum of $\mathcal{L}_Q$. Assuming (A1)–(A5), (A8$_B$) and $Q\in L^\infty(\mathbb{R}^d,\mathbb{K}^{N,N})$ with

$$\operatorname*{ess\,sup}_{|x|\geqslant R}|Q(x)|_2\to 0\text{ as } R\to\infty$$

for $1<p<\infty$ and $\mathbb{K}=\mathbb{C}$, we show in Theorem 7.9 that every $\lambda\in\mathbb{C}$ satisfying the dispersion relation for $\mathcal{L}_Q$

$$\det\left(\lambda I_N+\omega^2A+B+i\sum_{l=1}^{k}n_l\sigma_lI_N\right)=0,\text{ for some }\omega\in\mathbb{R},\ n_l\in\mathbb{Z},$$

belongs to the essential spectrum of $\mathcal{L}_Q$. As a consequence we show in Corollary 7.10 that $\mathcal{L}_Q$ is not sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $(T_Q(t))_{t\geqslant 0}$ is not analytic on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$.

7.1 Application of semigroup theory

Let the assumptions (A1), (A2), (A5) and (A8$_B$) be satisfied for $\mathbb{K} = \mathbb{C}$, then we denote by $(B_p, \mathcal{D}(A_p))$ the infinitesimal generator of the semigroup $(T_\infty(t))_{t\geqslant 0}$ from (6.5) on $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$. The semigroup $(T_\infty(t))_{t\geqslant 0}$ is strongly continuous in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for every $1 \leqslant p < \infty$ and satisfies

$$\|T_\infty(t)\|_{\mathcal{L}(L^p,L^p)} \leqslant M_\infty e^{\omega_\infty t} \quad \forall\, t \geqslant 0,$$

with $M_0 := M_\infty := \left(\frac{a_{\max}^2}{a_{\min} a_0}\right)^{\frac{d}{2}} \geqslant 1$ and $\omega_\infty := -b_0 \in \mathbb{R}$. Moreover, if we additionally require the assumptions (A3) and (A4) for $1 < p < \infty$, then B_p is the maximal realization of $\mathcal{L}_\infty$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ on its maximal domain, which is $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \mathcal{D}^p_{\max}(\mathcal{L}_0)$.

In this section we investigate variable coefficient perturbations of B_p in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$. For this purpose, let F_p be given by (7.1). An application of [34, III.1.3 Bounded Perturbation Theorem, III.1.7 Corollary and III.1.10 Theorem] yields the following result.

Theorem 7.1 (Bounded Perturbation Theorem)**.** *Let the assumptions* (A1)*,* (A2)*,* (A5)*,* (A8$_B$) *and* $Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{N,N})$ *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*. Then the operator*

$$C_p := B_p + F_p \quad \textit{with} \quad \mathcal{D}(C_p) := \mathcal{D}(A_p)$$

generates a strongly continuous semigroup $(T_Q(t))_{t\geqslant 0}$ *on* $L^p(\mathbb{R}^d, \mathbb{C}^N)$*satisfying*

$$\|T_Q(t)\|_{\mathcal{L}(L^p,L^p)} \leqslant M_\infty e^{\left(\omega_\infty + M_\infty \|F_p\|_{\mathcal{L}(L^p,L^p)}\right)t} \quad \forall\, t \geqslant 0. \tag{7.2}$$

Moreover, for every $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *and* $t \geqslant 0$ *the semigroup* $(T_Q(t))_{t\geqslant 0}$ *satisfies the integral equation (variation of parameters formula)*

$$T_Q(t)v = T_\infty(t)v + \int_0^t T_\infty(t-s) F_p T_Q(s) v ds,$$

is unique and can be obtained by

$$T_Q(t) = \sum_{n=0}^{\infty} T_Q^{(n)}(t)$$

where

$$T_Q^{(0)}(t) := T_\infty(t), \quad T_Q^{(n+1)}(t) := \int_0^t T_\infty(t-s) F_p T_Q^{(n)}(s) ds.$$

Simultaneous diagonalization of A and B. Note that the statement from Theorem 7.1 remains true, if we omit the assumption (A8$_B$). In that case, we do not have a semigroup representation for $(T_\infty(t))_{t\geqslant 0}$ any more and the bound from (7.2) has accordingly to be modified.

Identification problem for C_p. Theorem 7.1 states that $(C_p,\mathcal{D}(A_p))$ is the infinitesimal generator of $(T_Q(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$. If we additionally assume the conditions (A3) and (A4) for $1<p<\infty$, then an application of Theorem 5.19 and Theorem 5.25 solves the identification problem for C_p in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and we infer that

$$C_p=\mathcal{L}_Q \quad \text{on} \quad \mathcal{D}(A_p)=\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)=\mathcal{D}^p_{\max}(\mathcal{L}_0).$$

Contrary to the case of constant coefficients, we cannot assume here, that $Q(x)$ commutes with both A and B for every $x\in\mathbb{R}^d$, since this is in general not satisfied in order to investigate the nonlinear problem of the Ornstein-Uhlenbeck operator. Thus, we are not able to derive a closed form for the representation of the semigroup $(T_Q(t))_{t\geqslant 0}$. In particular, it is not possible in this case to optimize the boundedness property of $\|T_Q(t)\|_{\mathcal{L}(L^p,L^p)}$ from Theorem 7.1.

An application of [34, II.1.3 Lemma, II.1.4 Theorem] yields the following result:

Lemma 7.2. *Let the assumptions* (A1), (A2), (A5), (A8$_B$) *and* $Q\in L^\infty(\mathbb{R}^d,\mathbb{K}^{N,N})$ *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$.
(**1**) *The generator* $C_p:\mathcal{D}(A_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ *is a linear, closed and densely defined operator.*
(**2**) *For every* $v\in\mathcal{D}(A_p)$ *and* $t\geqslant 0$ *we have*

$$\begin{aligned}&T_Q(t)v\in\mathcal{D}(A_p)\\ &\frac{d}{dt}T_Q(t)v=T_Q(t)C_pv=C_pT_Q(t)v\end{aligned}$$

(**3**) *For every* $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ *and every* $t\geqslant 0$ *we have*

$$\int_0^t T_Q(s)v\,ds\in\mathcal{D}(A_p)$$

(**4**) *For every* $t\geqslant 0$ *we have*

$$\begin{aligned}T_Q(t)v-v=&C_p\int_0^t T_Q(s)v\,ds && \text{, for } v\in L^p(\mathbb{R}^d,\mathbb{C}^N),\\ =&\int_0^t T_Q(s)C_pv\,ds && \text{, for } v\in\mathcal{D}(A_p).\end{aligned}$$

Since $(C_p,\mathcal{D}(A_p))$ is a closed operator on the Banach space $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$, we use the standard notion

$$\begin{aligned}\sigma(C_p):=&\{\lambda\in\mathbb{C}\mid \lambda I-C_p \text{ is not bijective}\} && \textbf{spectrum of } C_p,\\ \rho(C_p):=&\mathbb{C}\backslash\sigma(C_p) && \textbf{resolvent set of } C_p,\\ R(\lambda,C_p):=&(\lambda I-C_p)^{-1}\text{, for } \lambda\in\rho(C_p) && \textbf{resolvent of } C_p.\end{aligned}$$

The next identities follow from [34, II.1.9 Lemma].

Lemma 7.3. *Let the assumptions* (A1), (A2), (A5), (A8$_B$) *and* $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Then for every* $\lambda \in \mathbb{C}$ *and* $t > 0$,

$$\begin{aligned} e^{-\lambda t} T_Q(t)v - v &= (C_p - \lambda I) \int_0^t e^{-\lambda s} T_Q(s) v ds \qquad &&, \textit{for } v \in L^p(\mathbb{R}^d, \mathbb{C}^N), \\ &= \int_0^t e^{-\lambda s} T_Q(s) \left(C_p - \lambda I\right) v ds \qquad &&, \textit{for } v \in \mathcal{D}(A_p). \end{aligned}$$

The following statement comes from [34, II.1.10 Theorem].

Theorem 7.4. *Let the assumptions* (A1), (A2), (A5), (A8$_B$) *and* $Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{N,N})$ *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$.
(**1**) *If* $\lambda \in \mathbb{C}$ *is such that* $R(\lambda)v := \int_0^\infty e^{-\lambda s} T_Q(s) v ds$ *exists for every* $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$, *then*

$$\lambda \in \rho(C_p) \quad \textit{and} \quad R(\lambda, C_p) = R(\lambda).$$

(**2**) *If* $\lambda \in \mathbb{C}$ *satisfies* $\operatorname{Re}\lambda > \omega_\infty + M_\infty \|Q\|_{L^\infty}$, *then*

$$\lambda \in \rho(C_p), \quad R(\lambda, C_p) = R(\lambda)$$

and

$$\|R(\lambda, C_p)\|_{\mathcal{L}(L^p, L^p)} \leqslant \frac{M_\infty}{\operatorname{Re}\lambda - \left(\omega_\infty + M_\infty \|Q\|_{C_\mathrm{b}}\right)}.$$

Theorem 7.4(2) states that the half–plane $\operatorname{Re}\lambda > \omega_\infty + M_\infty \|Q\|_{L^\infty}$ belongs to the resolvent set $\rho(C_p)$. Therefore, the spectrum $\sigma(C_p)$ is contained in the half–plane $\operatorname{Re}\lambda \leqslant \omega_\infty + M_\infty \|Q\|_{L^\infty}$. The **spectral bound** $s(C_p)$ **of** C_p, [34, II.1.12 Definition], defined by

$$-\infty \leqslant s(C_p) := \sup_{\lambda \in \sigma(C_p)} \operatorname{Re}\lambda \leqslant \omega_\infty + M_\infty \|Q\|_{L^\infty} < +\infty$$

can be considered as the smallest value $\omega \in \mathbb{R}$ such that the spectrum is contained in the half–plane $\operatorname{Re}\lambda \leqslant \omega$.

A direct consequence of Theorem 7.4 is the following:

Corollary 7.5 (Solvability and uniqueness in $L^p(\mathbb{R}^d, \mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2), (A5), (A8$_B$) *and* $Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{N,N})$ *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Moreover, let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_\infty + M_\infty \|Q\|_{L^\infty}$. *Then for every* $g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *the resolvent equation*

$$(\lambda I - C_p)\, v = g$$

admits a unique solution $v_\star \in \mathcal{D}(A_p)$ *which satisfies the integral expression*

$$\begin{aligned} v_\star = R(\lambda) g &= \int_0^\infty e^{-\lambda s} T_Q(s) g ds \\ &= \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda s} H_\infty(\cdot, \xi, s) \left(g(\xi) + Q(\xi) v_\star(\xi)\right) d\xi ds. \end{aligned}$$

Moreover, it holds the resolvent estimate

$$\|v_\star\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)} \leqslant \frac{M_\infty}{\operatorname{Re}\lambda - (\omega_\infty + M_\infty \|Q\|_{L^\infty})} \|g\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)}.$$

7.2 Exponential decay for small perturbations

Let us consider the operator

$$[\mathcal{L}_{Q_\varepsilon}v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x) + Q_\varepsilon(x)v(x),\ x\in\mathbb{R}^d,\ d\geqslant 2,$$

for some sufficiently small $Q_\varepsilon \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$. Assuming (A1), (A2), (A5) and (A8$_B$) for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$, we can apply the results from Section 7.1. In the following we denote by $\left(C_p^\varepsilon, \mathcal{D}(A_p)\right)$ the infinitesimal generator of $(T_{Q_\varepsilon}(t))_{t\geqslant 0}$. We suggest that if we additionally require (A3) and (A4) for $1<p<\infty$, then we can replace the generator C_p^ε by $\mathcal{L}_{Q_\varepsilon}$ and its domain $\mathcal{D}(A_p)$ by $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and $\mathcal{D}^p_{\max}(\mathcal{L}_0)$, respectively.

The next theorem yields a-priori estimates for the solution of the resolvent equation $\left(\lambda I - C_p^\varepsilon\right)v = g$ in exponentially weighted L^p-spaces. This requires the additional assumption (A9$_B$). We show that for sufficiently small perturbations Q_ε the solution $v_\star\in\mathcal{D}(A_p)$ decays exponentially (at least) with the same rate as the inhomogeneity g.Note that the bound $\operatorname{Re}\lambda\geqslant -\frac{b_0}{3}$ does not depend on the perturbation Q_ε as in Corollary 7.5. Moreover, the upper bound for the decay rate does not depend on λ any more. The proof is based on an application of Corollary 7.5 and Theorem 6.8.

Theorem 7.6 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1)*,* (A2)*,* (A5)*,* (A8$_B$) *and* (A9$_B$) *be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Then for every $0<\vartheta<1$ and for every radially nondecreasing weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$ with $0\leqslant\eta^2\leqslant\vartheta\frac{2}{3}\frac{a_0b_0}{a_{\max}^2p^2}$, for every $Q_\varepsilon\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ with $\|Q_\varepsilon\|_{L^\infty}\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}$, for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\geqslant-\frac{b_0}{3}$ and for every $g\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ we have $v_\star\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ with*

$$\|v_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{C_7}{\operatorname{Re}\lambda+\frac{2b_0}{3}}\|g\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}, \tag{7.3}$$

$$\|D_iv_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{\sqrt{2}C_8}{\left(\operatorname{Re}\lambda+\frac{2b_0}{3}\right)^{\frac12}}\|g\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)},\ i=1,\ldots,d, \tag{7.4}$$

where $v_\star\in\mathcal{D}(A_p)$ denotes the unique solution of $(\lambda I-C_p^\varepsilon)v=g$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and the λ-independent constants C_7, C_8 are given by Lemma 4.8 (with $\omega=\omega_\infty$).

Proof. 1. Existence and uniqueness in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ (by Corollary 7.5): First we show that there exists a unique solution $v_\star\in\mathcal{D}(A_p)$ of $(\lambda I-C_p^\varepsilon)v=g$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Since θ is nondecreasing we have $g\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)$. Moreover, from $\operatorname{Re}\lambda\geqslant-\frac{b_0}{3}$, the choice of Q_ε and $b_0>0$, cp. (A9$_B$), we obtain

$$\operatorname{Re}\lambda\geqslant-\frac{b_0}{3}\geqslant-\frac{2}{3}b_0+M_\infty\|Q_\varepsilon\|_{L^\infty}>\omega_\infty+M_\infty\|Q_\varepsilon\|_{L^\infty},\quad \omega_\infty:=-b_0.$$

Thus, the statement follows directly from Corollary 7.5. In order to verify that this $v_\star$ belongs to $W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ and satisfies the inequalities (7.3) and (7.4) we must analyze $(\lambda I-C_p^\varepsilon)v=g$ in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$.

2. Existence in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ (by fixed point argument): Consider the fixed point equation

$$v = (\lambda I - B_p)^{-1} g + (\lambda I - B_p)^{-1} Q_\varepsilon v =: Fv$$

in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$. Since $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ and $b_0 > 0$ we obtain $\operatorname{Re}\lambda > -b_0$ and

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3}\frac{a_0 b_0}{a_{\max}^2 p^2} = \vartheta \frac{a_0 b_0}{a_{\max}^2 p^2} + \vartheta \frac{a_0}{a_{\max}^2 p^2}\left(-\frac{b_0}{3}\right) \leqslant \vartheta \frac{a_0(\operatorname{Re}\lambda + b_0)}{a_{\max}^2 p^2}.$$

Thus, for given $v \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ an application of Theorem 6.8 implies $Fv \in \mathcal{D}(A_p)$ and $Fv \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$. Moreover, the linear part of F is a contraction since (6.12) yields

$$\left\| (\lambda I - B_p)^{-1} Q_\varepsilon v \right\|_{L^p_\theta} \leqslant q \left\| v \right\|_{L^p_\theta} \quad \forall\, v \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$$

with Lipschitz constant

$$0 \leqslant q := \frac{C_7}{\operatorname{Re}\lambda + b_0} \left\| Q_\varepsilon \right\|_{L^\infty} \leqslant \frac{\frac{b_0}{3}}{\operatorname{Re}\lambda + b_0} \leqslant \frac{1}{2} < 1.$$

The last inequality follows from $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ and the choice of Q_ε. Hence, by the contraction mapping theorem F has a unique fixed point $u_\star \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ which even belongs to $\mathcal{D}(A_p)$ and satisfies $(\lambda I - C_p^\varepsilon)v = g$ in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$. Since $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N) \subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)$ both $v_\star$ and $u_\star$ solve $(\lambda I - C_p^\varepsilon)v = g$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and by uniqueness we deduce $v_\star = u_\star$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Moreover, we conclude that $v_\star = u_\star \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$.

3. L^p_θ- and $W^{1,p}_\theta$-estimates (by contraction mapping principle and bootstrapping): The L^p_θ-estimate (7.3) follows from the contraction mapping principle and

$$\begin{aligned}\left\| v_\star \right\|_{L^p_\theta} = \left\| u_\star \right\|_{L^p_\theta} \leqslant \frac{1}{1-q} \left\| F0 \right\|_{L^p_\theta} &\leqslant \frac{\operatorname{Re}\lambda + b_0}{\operatorname{Re}\lambda + b_0 - C_7 \left\| Q_\varepsilon \right\|_{L^\infty}} \frac{C_7}{\operatorname{Re}\lambda + b_0} \left\| g \right\|_{L^p_\theta} \\ &\leqslant \frac{C_7}{\operatorname{Re}\lambda + \frac{2b_0}{3}} \left\| g \right\|_{L^p_\theta}.\end{aligned}$$

Finally, the $W^{1,p}_\theta$-estimate (7.4) is proved by bootstrapping using the inequalities (6.13) and (7.3) for every $i = 1,\ldots,d$

$$\begin{aligned}\left\| D_i v_\star \right\|_{L^p_\theta} \leqslant & \frac{C_8}{(\operatorname{Re}\lambda + b_0)^{\frac{1}{2}}} \left\| g + Q_\varepsilon v_\star \right\|_{L^p_\theta} \\ \leqslant & \frac{C_8}{(\operatorname{Re}\lambda + b_0)^{\frac{1}{2}}} \left(\left\| g \right\|_{L^p_\theta} + \left\| Q_\varepsilon \right\|_{L^\infty} \left\| v_\star \right\|_{L^p_\theta} \right) \\ \leqslant & \frac{C_8}{(\operatorname{Re}\lambda + b_0)^{\frac{1}{2}}} \left(\left\| g \right\|_{L^p_\theta} + \frac{b_0}{3C_7} \frac{C_7}{\operatorname{Re}\lambda + \frac{2b_0}{3}} \left\| g \right\|_{L^p_\theta} \right) \\ = & \frac{C_8}{\left(\operatorname{Re}\lambda + \frac{2b_0}{3}\right)^{\frac{1}{2}}} \left(\frac{\operatorname{Re}\lambda + b_0}{\operatorname{Re}\lambda + \frac{2b_0}{3}} \right)^{\frac{1}{2}} \left\| g \right\|_{L^p_\theta} \leqslant \frac{\sqrt{2} C_8}{\left(\operatorname{Re}\lambda + \frac{2b_0}{3}\right)^{\frac{1}{2}}} \left\| g \right\|_{L^p_\theta}.\end{aligned}$$

For the last inequality we used $\operatorname{Re}\lambda \leqslant -\frac{b_0}{3}$ and

$$\frac{\operatorname{Re}\lambda + b_0}{\operatorname{Re}\lambda + \frac{2b_0}{3}} = 1 + \frac{\frac{b_0}{3}}{\operatorname{Re}\lambda + \frac{2b_0}{3}} \leqslant 2.$$

This shows that $v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$. $\square$

7.3 Exponential decay for compactly supported perturbations

Let us consider the operator

$$\left[\mathcal{L}_Q v\right](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x) + Q(x)v(x),\ x\in\mathbb{R}^d,\ d\geqslant 2,$$

for some $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ and let us assume for the moment that Q can decomposed into $Q = Q_\varepsilon + Q_c$, where $Q_\varepsilon \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ is small with respect to $\|\cdot\|_{L^\infty}$ and $Q_c \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ is compactly supported. Such a perturbation Q_c is also called a **compactly supported perturbation** of $\mathcal{L}_{Q_\varepsilon}$. Assuming (A1), (A2), (A5) and (A8$_B$) for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$, we can apply once more the results from Section 7.1. Let $(C_p,\mathcal{D}(A_p))$ denote the infinitesimal generator of $(T_Q(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$, then using $Q=Q_\varepsilon+Q_c$ we obtain for every $v\in\mathcal{D}(A_p)$

$$\left[C_p v\right](x) = \left[B_p v\right](x) + \left(Q_\varepsilon(x)+Q_c(x)\right)v(x) = \left[C_p^\varepsilon v\right](x) + Q_c(x)v(x),$$

where $\left(C_p^\varepsilon,\mathcal{D}(A_p)\right)$ denotes the infinitesimal generator of $\left(T_{Q_\varepsilon}(t)\right)_{t\geqslant 0}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. This means that the relatively compact perturbation Q of B_p is the same as a compact perturbation Q_c of C_p^ε. The relation will be of importance in the following proof.

In the next theorem we prove a-priori estimates for the solution of the resolvent equation $(\lambda I - C_p)\, v = g$ in exponentially weighted L^p-spaces. We show that for perturbations Q, that falls below a certain threshold in the far-field, the solution $v_\star\in\mathcal{D}(A_p)$ decays exponentially (at least) with the same rate as the inhomogeneity g. Similar to Theorem 7.6, the bound $\operatorname{Re}\lambda\geqslant -\frac{b_0}{3}$ does not depend on the perturbation Q_ε as in Corollary 7.5. The main idea of the proof is to combine the results from Corollary 7.5 and Theorem 7.6.

Theorem 7.7 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2), (A5), (A8$_B$) *and* (A9$_B$) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$*. Then for every* $0<\vartheta<1$*, for every radially nondecreasing weight function* $\theta\in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta\geqslant 0$ *with* $0\leqslant\eta^2\leqslant\vartheta\frac{2}{3}\frac{a_0 b_0}{a_{\max}^2 p^2}$*, for every* $Q\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ *with*

$$\operatorname*{ess\,sup}_{|x|\geqslant R_0} |Q(x)|_2 \leqslant \frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\},\ \textit{for some } R_0>0, \tag{7.5}$$

for every $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda\geqslant -\frac{b_0}{3}$ *and for every* $g\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *the following property is satisfies:*
Every solution $v_\star\in\mathcal{D}(A_p)$ *of the resolvent equation* $(\lambda I - C_p)\,v=g$ *in* $L^p(\mathbb{R}^d,\mathbb{C}^N)$ *satisfies* $v_\star\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$*.*

Proof. Let $0 < \vartheta < 1$ and let $\theta \in C(\mathbb{R}^d, \mathbb{R})$ be a radially nondecreasing weight function of exponential growth rate $\eta \geqslant 0$ with $0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$.
1. Decomposition of Q: For positive real R choose a C^∞ cut-off functio

$$\chi_R : [0,\infty[\to [0,1] \quad \text{with} \quad \chi_R(r) = \begin{cases} 0 & , r \leqslant R \\ \text{smooth} & , R \leqslant r \leqslant 2R \\ 1 & , r \geqslant 2R \end{cases}.$$

Then, we decompose Q as follows

$$Q(x) = \chi_{R_0}(|x|)Q(x) + (1 - \chi_{R_0}(|x|))\, Q(x) =: Q_\varepsilon(x) + Q_c(x),$$

where $R_0 > 0$ comes from (7.5). Note that $Q_\varepsilon, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ since $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ and $\chi_R(|\cdot|) \in C_{\mathrm{b}}(\mathbb{R}^d, [0,1])$. Moreover, Q_c is compactly supported because $Q_c(x) = 0$ for every $|x| \geqslant 2R_0$ and Q_ε is bounded by

$$\begin{aligned}
\|Q_\varepsilon\|_{L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})} &= \|\chi_{R_0}(|\cdot|)Q(\cdot)\|_{L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})} \\
&= \|\chi_{R_0}(|\cdot|)Q(\cdot)\|_{L^\infty(\mathbb{R}^d \backslash B_{R_0},\mathbb{C}^{N,N})} \\
&\leqslant \|\chi_{R_0}(|\cdot|)\|_{C_{\mathrm{b}}(\mathbb{R}^d \backslash B_{R_0},[0,1])} \|Q(\cdot)\|_{L^\infty(\mathbb{R}^d \backslash B_{R_0},\mathbb{C}^{N,N})} \\
&\leqslant \frac{b_0}{3} \min\left\{\frac{1}{C_7}, \frac{1}{M_\infty}\right\}.
\end{aligned}$$

Now, let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ and $g \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ and let $v_\star \in \mathcal{D}(A_p)$ a solution of $(\lambda I - C_p)\, v = g$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, i.e. $v_\star$ satisfies

$$\left(\lambda I - C_p^\varepsilon\right) v_\star = Q_c v_\star + g, \text{ in } L^p(\mathbb{R}^d, \mathbb{C}^N). \tag{7.6}$$

In the following, we consider the problem

$$\left(\lambda I - C_p^\varepsilon\right) u_\star = Q_c v_\star + g, \text{ in } L^p(\mathbb{R}^d, \mathbb{C}^N) \text{ and in } L^p_\theta(\mathbb{R}^d, \mathbb{C}^N). \tag{7.7}$$

Our aim is to show by Corollary 7.5 that $u_\star = v_\star$ (in $L^p(\mathbb{R}^d, \mathbb{C}^N)$) is the unique solution of (7.7) in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ and by Theorem 7.6 that $u_\star \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{C}^N)$.
2. Uniqueness in $L^p(\mathbb{R}^d, \mathbb{C}^N)$: Consider (7.7) in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. $Q_\varepsilon \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ and $\lambda \in \mathbb{C}$ satisfy

$$\operatorname{Re}\lambda \geqslant -\frac{b_0}{3} = -\frac{2}{3}b_0 + \frac{b_0}{3} \geqslant -\frac{2}{3}b_0 + M_\infty \|Q_\varepsilon\|_{L^\infty} > \omega_\infty + M_\infty \|Q_\varepsilon\|_{L^\infty}.$$

Hence, Corollary 7.5 (with $Q = Q_\varepsilon$ and inhomogeneity $Q_c v_\star + g$) implies a unique solution $u_\star \in \mathcal{D}(A_p)$ of (7.7) in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. Thus, we deduce that $w_\star := u_\star - v_\star \in \mathcal{D}(A_p)$ is a solution of

$$\left(\lambda I - C_p^\varepsilon\right) w_\star = 0, \text{ in } L^p(\mathbb{R}^d, \mathbb{C}^N). \tag{7.8}$$

Now, we apply Corollary 7.5 once more (with $Q = Q_\varepsilon$ and $g = 0$), which yields a unique solution $w_\star \in \mathcal{D}(A_p)$ of (7.8) in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, which satisfies $\|w_\star\|_{L^p} = 0$ by the resolvent estimate from Corollary 7.5. Hence, $u_\star = v_\star$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$.

3. Existence in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$: Consider (7.7) in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$. To apply Theorem 7.6, we only have to check that $Q_c v_\star + g \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$: From $v_\star \in \mathcal{D}(A_p)$, $g \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ and since Q_c is compactly supported in $B_{2R_0}(0)$ we obtain

$$\begin{aligned}
&\left\|Q_c v_\star + g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \left\|\theta Q_c v_\star\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\\
=&\left\|\theta Q_c v_\star\right\|_{L^p(B_{2R_0},\mathbb{C}^N)} + \left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\\
\leqslant&\left\|\theta\right\|_{C_{\mathrm{b}}(B_{2R_0},\mathbb{R})}\left\|(1-\chi_{R_0}(|\cdot|))Q\right\|_{L^\infty(B_{2R_0},\mathbb{C}^{N,N})}\left\|v_\star\right\|_{L^p(B_{2R_0},\mathbb{C}^N)} + \left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\\
\leqslant&\left\|\theta\right\|_{C_{\mathrm{b}}(B_{2R_0},\mathbb{R})}\left\|Q\right\|_{L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})}\left\|v_\star\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\\
=&C_{\theta,Q,R_0}\left\|v_\star\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}
\end{aligned}$$

i.e. $Q_c v_\star + g \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$. Therefore, Theorem 7.6 implies that the unique solution $u_\star \in \mathcal{D}(A_p)$ of (7.7) in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ satisfies $u_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$. Since $u_\star = v_\star$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and since $u_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N) \subseteq L^p_\theta(\mathbb{R}^d,\mathbb{C}^N) \subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)$, we conclude that $v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ as well. □

Remark. Since $v_\star \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ solves $\left(\lambda I - C^\varepsilon_p\right) v_\star = Q_c v_\star + g$ in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$, we deduce from (7.3) that

$$\begin{aligned}
\left\|v_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant&\frac{C_7}{\operatorname{Re}\lambda + \frac{2b_0}{3}}\left\|Q_c v_\star + g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\\
\leqslant&\frac{C_7}{\operatorname{Re}\lambda + \frac{2b_0}{3}}\left(C_{\theta,Q,R_0}\left\|v_\star\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\right).
\end{aligned}$$

Similarly, using (7.4) we obtain

$$\left\|D_i v_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant\frac{\sqrt{2}C_8}{\left(\operatorname{Re}\lambda + \frac{2b_0}{3}\right)^{\frac{1}{2}}}\left(C_{\theta,Q,R_0}\left\|v_\star\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} + \left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\right),$$

for $i = 1,\ldots,d$, where the constants C_7, C_8 are from Lemma 4.8 (with $\omega = \omega_\infty$).

7.4 Essential spectrum and analyticity

In this section we combine and extend the approaches from [15, Section 8.2, Theorem 8.1] and [71, Theorem 2.6] to compute the essential spectrum of $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $1 < p < \infty$. In order to transfer the results to the infinitesimal generator C_p, it is necessary to solve the identification problem for C_p. Therefore, we restrict $1 < p < \infty$ and require additionally the assumptions (A3) and (A4). In [15], Beyn and Lorenz have analyzed the case $p = d = 2$ for $\mathbb{K} = \mathbb{R}$. The spectrum of the Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^d,\mathbb{R})$, $1 < p < \infty$, without perturbation terms was studied by Metafune in [71].

First, let us introduce some notation about the spectrum of a closed and densely defined operator, [52]:

Definition 7.8. Let X be a (complex-valued) Banach space and let $A : \mathcal{D}(A) \subset X \to X$ be a closed, densely defined, linear operator. Moreover, let $\lambda \in \mathbb{C}$.
(**1**) $\lambda \in \rho(A)$ if and only if the following properties hold

- $(\lambda I - A) : \mathcal{D}(A) \to X$ is $1-1$ (injective) and onto (surjective),
- $(\lambda I - A)^{-1}$ is bounded on X.

The set $\rho(A)$ is called the **resolvent set of** A and $(\lambda I - A)^{-1}$ is called the **resolvent of** A. Moreover, the set $\sigma(A) := \mathbb{C}\backslash\rho(A)$ is called the **spectrum of** A and an element $\lambda \in \sigma(A)$ is called an **eigenvalue of** A.
(**2**) $\lambda_0 \in \sigma(A)$ is called **isolated** if and only if

$$\exists\, \varepsilon > 0 \,\forall\, \lambda \in \mathbb{C} \text{ with } 0 < |\lambda - \lambda_0| < \varepsilon : \lambda \in \rho(A).$$

(**3**) The **multiplicity** of $\lambda_0 \in \sigma(A)$ is defined as the dimension of the algebraic eigenspace

$$\left\{ v \in X \mid (\lambda_0 I - A)^k = 0 \text{ for some } k \in \mathbb{N} \right\}.$$

(**4**) $\lambda \in \mathbb{C}$ is called a **normal point of** A if and only if one of the following properties hold

- $\lambda \in \rho(A)$,
- $\lambda \in \sigma_{\mathrm{point}}(A) := \{\lambda \in \sigma(A) \mid \lambda \text{ is isolated with finite multiplicity}\}$.

The set $\sigma_{\mathrm{point}}(A)$ is called the **point spectrum of** A.
(**5**) The set

$$\sigma_{\mathrm{ess}}(A) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is not a normal point of } A\}$$

is called the **essential spectrum of** A.

By definition it holds

$$\mathbb{C} = \rho(A) \,\dot{\cup}\, \sigma(A) = \rho(A) \,\dot{\cup}\, \left(\sigma_{\mathrm{ess}}(A) \,\dot{\cup}\, \sigma_{\mathrm{point}}(A)\right).$$

We first give a short motivation, how we can determine the essential spectrum of $\mathcal{L}_Q$, see [15, Section 8.2] for the case $d = p = 2$ and see [71, Theorem 2.6] for the essential spectrum of the drift term in $L^p(\mathbb{R}^d, \mathbb{R})$, $1 < p < \infty$:
The main idea for detecting the essential spectrum is to look for solutions of the initial value problem

$$\begin{aligned} v_t(x,t) &= \left[\mathcal{L}_Q v\right](x,t), \; x \in \mathbb{R}^d, \; t > 0, \\ v(x,0) &= v_0(x), \qquad\quad x \in \mathbb{R}^d, \; t = 0. \end{aligned}$$

1. Orthogonal transformation. For space dimensions $d \geqslant 3$ the axis of rotation is in general not orthogonal to some plane (x_l, x_k), $1 \leqslant l, k \leqslant d$. Furthermore, in space dimensions $d \geqslant 4$ the pattern can also rotate rigidly around several axes of rotation simultaneously. The first step is to separate the axes of rotation in such a way that they are orthogonal to (completely) different planes. For this purpose we perform an orthogonal transformation: Since $S \in \mathbb{R}^{d,d}$ with $S^T = -S$ we have $\sigma(S) \subset i\mathbb{R}$. Let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S, $1 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$, then

$$\exists\, P \in \mathbb{R}^{d,d} \text{ orthogonal matrix}: \; S = P\Lambda^S_{\text{block}}P^T,$$

where

$$\Lambda^S_{\text{block}} = \begin{pmatrix} \Lambda^S_1 & & & & & 0 \\ & \ddots & & & & \\ & & \Lambda^S_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix} \in \mathbb{R}^{d,d}, \quad \Lambda^S_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \in \mathbb{R}^{2,2},$$

for every $j = 1, \ldots, k$. The orthogonal transformation of coordinates

$$x = T_1(y) := Py, \; y \in \mathbb{R}^d$$

and $\tilde{v}(y,t) := v(T_1(y),t)$ yield the transformed equation

$$\tilde{v}_t(y,t) = \left[\mathcal{L}_{Q,T_1}\tilde{v}\right](y,t), \; y \in \mathbb{R}^d, \; t > 0,$$

where the transformed operator is given by

$$\left[\mathcal{L}_{Q,T_1}\tilde{v}\right](y,t) = A\triangle v(y,t) + \left\langle \Lambda^S_{\text{block}}y, \nabla\tilde{v}(y,t)\right\rangle - B\tilde{v}(y,t) + Q(T_1(y))\tilde{v}(y,t),$$

with

$$\left\langle \Lambda^S_{\text{block}}y, \nabla\tilde{v}(y,t)\right\rangle = \sum_{l=1}^{k} \sigma_l \left(y_{2l}D_{2l-1} - y_{2l-1}D_{2l}\right)\tilde{v}(y,t).$$

2. Transformation into several planar polar coordinates. Since now we have k angular derivatives in k different planes it is advisable to transform the coordinates of every such a plane into planar polar coordinates via

$$\begin{pmatrix} y_{2l-1} \\ y_{2l} \end{pmatrix} = T(r_l, \phi_l) := \begin{pmatrix} r_l\cos\phi_l \\ r_l\sin\phi_l \end{pmatrix}, \; l = 1, \ldots, k, \; \phi_l \in]-\pi, \pi], \; r_l > 0.$$

All further coordinates, i.e. $y_{2k+1}, \ldots, y_d$, remain fixed. Denoting the total transformation by $T_2(r_1, \phi_1, \ldots, r_k, \phi_k, y_{2k+1}, \ldots, y_d)$ we obtain from $\hat{v}(\xi,t) := \tilde{v}(T_2(\xi),t)$

$$\hat{v}_t(\xi,t) = \left[\mathcal{L}_{Q,T_2}\hat{v}\right](\xi,t), \; \xi \in (]0,\infty[\times]-\pi,\pi])^k \times \mathbb{R}^{d-2k}, \; t > 0,$$

where $\xi := (r_1, \phi_1, \ldots, r_k, \phi_k, y_{2k+1}, \ldots, y_d)$ and the transformed operator is given by

$$\begin{aligned}\left[\mathcal{L}_{Q,T_2}\hat{v}\right](\xi,t) =& A\left[\sum_{l=1}^{k}\left(\partial_{r_l}^2 + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right) + \sum_{l=2k+1}^{d}\partial_{y_l}^2\right]\hat{v}(\xi,t)\\ &- \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v}(\xi,t) - B\hat{v}(\xi,t) + Q(T_1(T_2(\xi)))\hat{v}(\xi,t).\end{aligned}$$

3. Simplified operator (limit operator, far-field operator). Since the essential spectrum depends only on the limiting equation for $|x| \to \infty$, we let formally $r_l \to \infty$ for every $1 \leqslant l \leqslant k$ and obtain

$$\hat{v}_t(\xi,t) = \left[\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\hat{v}\right](\xi,t),\ \xi \in (]0,\infty[\times]-\pi,\pi])^k \times \mathbb{R}^{d-2k},\ t > 0,$$

with the simplified operator

$$\left[\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\hat{v}\right](\xi,t) = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{y_l}^2\right]\hat{v}(\xi,t) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v}(\xi,t) - B\hat{v}(\xi,t).$$

where we assumed that $\sup_{|x|\geqslant R}|Q(x)| \to 0$ as $R \to \infty$.

4. Temporal Fourier transform. In order to eliminate the time derivative we perform a Fourier transform with respect to the time variable t. From $\check{v}(\xi) := e^{-\lambda t}\hat{v}(\xi,t)$, $\lambda \in \mathbb{C}$, we obtain

$$\left[\left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)\check{v}\right](\xi) = 0,\ \xi \in (]0,\infty[\times]-\pi,\pi])^k \times \mathbb{R}^{d-2k}.$$

5. Angular Fourier transform. Using a Fourier transform with respect to the space variable ξ, we eliminate all spatial derivatives, including radial and angular derivatives. For this purpose, we choose

$$\begin{aligned}\check{v}(\xi) := \exp\left(\frac{i\omega}{k}\sum_{l=1}^{k}r_l\right)\exp\left(i\sum_{l=1}^{k}n_l\phi_l\right)\underline{v},\ n_l \in \mathbb{Z},\ \omega \in \mathbb{R},\ \underline{v} \in \mathbb{C}^N,\ |\underline{v}| = 1,\\ \phi_l \in]-\pi,\pi],\ r_l > 0,\ l = 1,\ldots,k,\end{aligned}$$

that is sometimes called the **angular Fourier decomposition**. This yields

$$\left[\left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)\check{v}\right](\xi) = \left(\lambda I_N + \omega^2 A + i\sum_{l=1}^{k}n_l\sigma_l I_N + B\right)\check{v}(\xi).$$

The angular Fourier decomposition is a well-known tool for investigating essential spectrum, also in case of spiral waves, see [38].

6. Finite-dimensional eigenvalue problem. Hence, $\left[\left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)\check{v}\right](\xi) = 0$ for every ξ if and only if $\lambda \in \mathbb{C}$ satisfies the following finite-dimensional eigenvalue problem

$$\left(\omega^2 A + B\right)\underline{v} = -\left(\lambda + i\sum_{l=1}^{k}n_l\sigma_l\right)\underline{v}.$$

Theorem 7.9 (Essential spectrum of $\mathcal{L}_Q$)**.** *Let the assumptions* (A1)–(A5), (A8$_B$) *and $Q\in L^\infty(\mathbb{R}^d,\mathbb{K}^{N,N})$ with*

$$\eta_R := \operatorname*{ess\,sup}_{|x|\geqslant R} |Q(x)|_2 \to 0 \text{ as } R\to\infty$$

be satisfied for $1<p<\infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $\lambda_j(\omega)$ denote the eigenvalues of $\omega^2 A+B$ for $j=1,\ldots,N$. Then every number $\lambda\in\mathbb{C}$ with

$$\lambda = -\lambda_j(\omega) - i\sum_{l=1}^{k} n_l\sigma_l,\; n_l\in\mathbb{Z},\; \omega\in\mathbb{R},\; j=1,\ldots,N, \tag{7.9}$$

belongs to the essential spectrum of $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$, i.e. $\lambda\in\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$.

Dispersion relation for $\mathcal{L}_Q$. The dispersion relation for $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ with $1<p<\infty$ states that every $\lambda\in\mathbb{C}$ satisfying

$$\det\left(\lambda I_N + \omega^2 A + B + i\sum_{l=1}^{k} n_l\sigma_l I_N\right) = 0, \text{ for some } \omega\in\mathbb{R},\; n_l\in\mathbb{Z},$$

belongs to the essential spectrum of $\mathcal{L}_Q$, i.e. $\lambda\in\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$. This condition is necessary for the localization (and the existence) of the essential spectrum. Note that we have not proved that every $\lambda\in\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$ can be represented as in (7.9).

Essential spectrum at localized rotating waves. Later on, in Theorem 9.10 we apply Theorem 7.9 to the linearized operator

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx,\nabla v(x)\rangle + Df(v_\star(x))v(x)$$

with

$$-B = Df(v_\infty),\qquad Q(x) = Df(v_\star(x)) - Df(v_\infty),\; x\in\mathbb{R}^d,$$

where $v_\star(x)$ denotes the profile of a localized rotating pattern. This is motivated by the fact, that the essential spectrum of $\mathcal{L}$ provides informations about the linear (or spectral) stability of $\mathcal{L}$ at localized rotating waves $v_\star$.

Density of essential spectrum in a half-plane. Consider the set

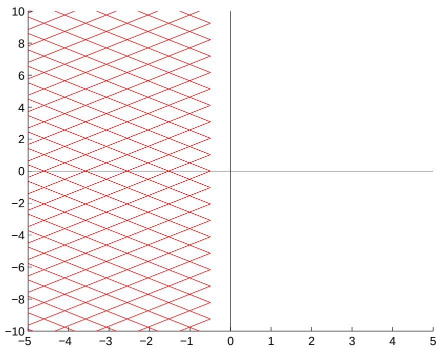
$$\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}_Q) := \left\{\lambda\in\mathbb{C}\mid \det\left(\lambda I_N + \omega^2 A + B + i\sum_{l=1}^{k} n_l\sigma_l I_N\right) = 0,\; \omega\in\mathbb{R},\; n_l\in\mathbb{Z}\right\}.$$

Theorem 7.9 shows that $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}_Q)\subseteq\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$. Moreover, we have the inclusion $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}_Q)\subseteq\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda\leqslant -b_0\}$, where $b_0=-s(-B)$. If there exists σ_n,σ_m such that $\sigma_n\sigma_m^{-1}\notin\mathbb{Q}$ then $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}_Q)$ is dense in the half-plane $\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda\leqslant -b_0\}$, i.e. $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}_Q) = \{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda\leqslant -b_0\}$. Otherwise $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}_Q)$ is a discrete subgroup of $\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda\leqslant -b_0\}$ (independently of p). The reason for this conclusion is given by Metafune in [71, Theorem 2.6]: There, it is proved that the essential spectrum of the drift term is dense in $i\mathbb{R}$, i.e. $\sigma_{\mathrm{ess}}(\langle Sx,\nabla v(x)\rangle) = i\mathbb{R}$, if and only if there exists

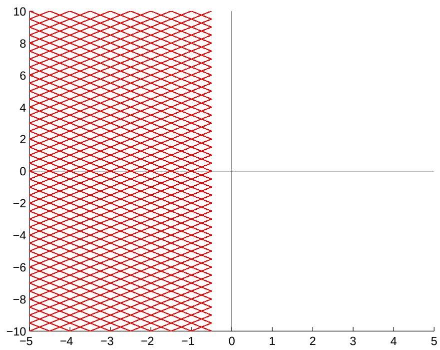
σ_m, σ_m such that $\sigma_n \sigma_m \notin \mathbb{Q}$. Otherwise, $\sigma_{\text{ess}}(\langle Sx, \nabla v(x) \rangle)$ is a discrete subgroup of $i\mathbb{R}$ (independently of p).

Effect of assumption $(\mathrm{A9}_B)$ **on the location of essential spectrum.** If we require in addition the stability condition $(\mathrm{A9}_B)$, then $b_0 = -s(-B) > 0$. Hence, $\operatorname{Re} \lambda_j(\omega) > 0$ and thus $\sigma_{\text{ess}}^{\text{part}}(\mathcal{L}_Q) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leqslant -b_0\} \subset \mathbb{C}_-$, where $\mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < 0\}$.

Figure 7.1 illustrates the set $\sigma_{\text{ess}}^{\text{part}}(\mathcal{L}_Q)$ in the scalar case for $A = \frac{1}{2}(1+i)$, $B = \frac{1}{2}$ and $Q = 0$. Figure 7.1(a) shows the part of the essential spectrum of $\mathcal{L}_Q$ for $\sigma_1 = 1.027$ and space dimension $d = 2$ and $d = 3$. In this case $\sigma_{\text{ess}}^{\text{part}}(\mathcal{L}_Q)$ forms a zig-zag curve, see [15] for $d = 2$, and is not dense in $\left\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leqslant -\frac{1}{2}\right\}$. Note that $\sigma_{\text{ess}}^{\text{part}}(\mathcal{L}_Q)$ can only be dense in $\left\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leqslant -\frac{1}{2}\right\}$ for space dimensions $d \geqslant 4$. Figure 7.1(b) shows the part of essential spectrum of $\mathcal{L}_Q$ for $\sigma_1 = 1$, $\sigma_2 = 1.5$ and $d = 4$. The eigenvalues σ_1, σ_2 satisfy $\sigma_1 \sigma_2^{-1} \in \mathbb{Q}$ and $\sigma_2 \sigma_1^{-1} \in \mathbb{Q}$. Thus, the set $\sigma_{\text{ess}}^{\text{part}}(\mathcal{L}_Q)$ is not dense in $\left\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leqslant -\frac{1}{2}\right\}$. Figure 7.1(c) shows the part of essential spectrum of $\mathcal{L}_Q$ for $\sigma_1 = 1$, $\sigma_2 = \frac{1}{2}\exp(1)$ and $d = 4$. In this case, the eigenvalues σ_1, σ_2 satisfy $\sigma_1 \sigma_2^{-1} \notin \mathbb{Q}$ and $\sigma_2 \sigma_1^{-1} \notin \mathbb{Q}$. Thus, the set $\sigma_{\text{ess}}^{\text{part}}(\mathcal{L}_Q)$ is dense in $\left\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leqslant -\frac{1}{2}\right\}$. This shows, that also the essential spectrum $\sigma_{\text{ess}}(\mathcal{L}_Q)$ changes dramatically depending on the eigenvalues of S.



(a) $d \in \{2, 3\}$, not dense



(b) $d = 4$, not dense

(c) $d = 4$, dense

Figure 7.1: Essential spectrum of $\mathcal{L}_Q$ for parameters $A = \frac{1}{2}(1+i)$, $B = \frac{1}{2}$ and $Q = 0$

Proof. 1. Let $R\geqslant 2$ be large and let $\chi_R:[0,\infty[\to[0,1]$ be a cut-off function such that $\chi_R\in C^2([0,\infty[,[0,1])$ with bounded derivatives independently of R and

$$\chi_R(r)=\begin{cases}0 & ,\ r\in[0,R-1],\\ \in[0,1] & ,\ r\in[R-1,R],\\ 1 & ,\ r\in[R,2R],\\ \in[0,1] & ,\ r\in[2R,2R+1],\\ 0 & ,\ r\in[2R+1,\infty[.\end{cases}$$

2. Define

$$\begin{aligned}v_R(\xi)&:=\left(\prod_{l=1}^{k}\chi_R(r_l)\right)\chi_R(|\tilde{x}|)v(\xi)\\ &=\left(\prod_{l=1}^{k}\chi_R(r_l)\right)\chi_R(|\tilde{x}|)\exp\left(i\omega\sum_{l=1}^{k}r_l\right)\exp\left(i\sum_{l=1}^{k}n_l\phi_l\right)\hat{v},\end{aligned}$$

where $\tilde{x}:=(x_{2k+1},\ldots,x_d)$, $\xi:=(r_1,\phi_1,\ldots,r_k,\phi_k,\tilde{x})$, $n_l\in\mathbb{Z}$, $\omega\in\mathbb{R}$, $\hat{v}\in\mathbb{C}^N$ with $|\hat{v}|=1$, $\phi_l\in]-\pi,\pi]$, $r_l>0$ and $l=1,\ldots,k$. By definition of χ_R we have

$$\left(\lambda I-\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)v_R(\xi)=0, \tag{7.10}$$

whenever $|\tilde{x}|\in[0,R-1]\cup[2R+1,\infty[$ or $r_l\in[0,R-1]\cup[2R+1,\infty[$ for some $1\leqslant l\leqslant k$. Moreover, by the choice of λ and by definition of χ_R we have

$$\left(\lambda I-\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)v_R(\xi)=0, \tag{7.11}$$

if $|\tilde{x}|,r_l\in[R,2R]$ for every $l=1,\ldots,k$.
3. By the choice of λ,

$$\begin{aligned}&\partial_{r_l}^2\left(\chi_R(r_l)e^{i\omega r_l}\right)=\chi_R''(r_l)e^{i\omega r_l}+2i\omega\chi_R'(r_l)e^{i\omega r_l}+\chi_R(r_l)\partial_{r_l}^2e^{i\omega r_l},\ l=1,\ldots,k,\\ &\partial_{x_l}^2\left(\chi_R(|\tilde{x}|)\right)=\frac{|\tilde{x}|^2-x_l^2}{|\tilde{x}|^3}\chi_R'(|\tilde{x}|)+\frac{x_l^2}{|\tilde{x}|^2}\chi_R''(|\tilde{x}|),\ l=2k+1,\ldots,d,\end{aligned}$$

the triangle inequality, $|\chi_R(r)|\leqslant 1$, $\chi_R'(r)\leqslant\|\chi_R\|_{C_{\mathrm{b}}^2}$, $\chi_R''(r)\leqslant\|\chi_R\|_{C_{\mathrm{b}}^2}$, $|v(\xi)|=1$ and $\frac{1}{|\tilde{x}|}\leqslant\frac{1}{R-1}\leqslant 1$, since $R\geqslant 2$, we have

$$\begin{aligned}&\left|\left(\lambda I-\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)v_R(\xi)\right|\\ =&\left|\left(\lambda I-A\left[\sum_{l=1}^{k}\partial_{r_l}^2+\sum_{l=2k+1}^{d}\partial_{x_l}^2\right]+\sum_{l=1}^{k}\sigma_l\partial_{\phi_l}+B\right)\left(\prod_{l=1}^{k}\chi_R(r_l)\right)\chi_R(|\tilde{x}|)v(\xi)\right|\\ =&\left|\left(\prod_{l=1}^{k}\chi_R(r_l)\right)\chi_R(|\tilde{x}|)\underbrace{\left(\lambda I-A\left[\sum_{l=1}^{k}\partial_{r_l}^2+\sum_{l=2k+1}^{d}\partial_{x_l}^2\right]+\sum_{l=1}^{k}n_l\partial_{\phi_l}+B\right)v(\xi)}_{=0\text{ (by the choice of }\lambda)}\right.\\ &-A\sum_{l=1}^{k}\left(\chi_R''(r_l)+2i\omega\chi_R'(r_l)\right)\left(\prod_{\substack{j=1\\ j\neq l}}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi)\end{aligned}$$

$$
\begin{aligned}
&- A\sum_{l=2k+1}^{d}\left(\frac{|\tilde{x}|^2-x_l^2}{|\tilde{x}|^3}\chi_R'(|\tilde{x}|)+\frac{x_l^2}{|\tilde{x}|^2}\chi_R''(|\tilde{x}|)\right)\left(\prod_{j=1}^{k}\chi_R(r_j)\right)v(\xi)\Bigg|\\
\leqslant{}& |A|_2\sum_{l=1}^{k}\left(|\chi_R''(r_l)|+2|\omega|\,|\chi_R'(r_l)|\right)\left(\prod_{\substack{j=1\\ j\neq l}}^{k}|\chi_R(r_j)|\right)|\chi_R(|\tilde{x}|)|\,|v(\xi)|\\
&+|A|_2\left(\frac{|d-2k-1|}{|\tilde{x}|}|\chi_R'(|\tilde{x}|)|+|\chi_R''(|\tilde{x}|)|\right)\left(\prod_{j=1}^{k}|\chi_R(r_j)|\right)|v(\xi)|\\
\leqslant{}& |A|_2\left(k(1+2|\omega|)+|d-2k-1|+1\right)\|\chi_R\|_{C_{\mathrm{b}}^2}=:C,
\end{aligned}
$$

for every $|\tilde{x}|, r_l\in[R-1,R]\cup[R,2R]\cup[2R,2R+1]$ and $1\leqslant l\leqslant k$.
4. Furthermore, we have by the definition of χ_R, $|v(\xi)|^p=1$ and by the transformation theorem

$$
\begin{aligned}
&\|v_R\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}^p=\int_{\mathbb{R}^d}|v_R(x)|^p\,dx\\
=&\int_0^\infty\int_{-\pi}^{\pi}\cdots\int_0^\infty\int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-2k}}\left(\prod_{l=1}^{k}r_l\right)|v_R(\xi)|^p\,d\tilde{x}d\phi_kdr_k\cdots d\phi_1dr_1\\
=&\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}\cdots\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}\int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1}\left(\prod_{l=1}^{k}r_l\right)|v_R(\xi)|^p\,d\xi\\
=&\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}\cdots\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}\int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1}\left(\prod_{l=1}^{k}r_l\right)\left(\prod_{l=1}^{k}\chi_R^p(r_l)\right)\chi_R^p(|\tilde{x}|)d\xi\\
=&\int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1}\chi_R^p(|\tilde{x}|)d\tilde{x}\prod_{l=1}^{k}\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}r_l\chi_R^p(r_l)d\phi_ldr_l\\
=&\Bigg(\int_{R-1\leqslant|\tilde{x}|\leqslant R}\underbrace{\chi_R^p(|\tilde{x}|)}_{\geqslant 0}d\tilde{x}+\int_{R\leqslant|\tilde{x}|\leqslant 2R}\underbrace{\chi_R^p(|\tilde{x}|)}_{=1}d\tilde{x}+\int_{2R\leqslant|\tilde{x}|\leqslant 2R+1}\underbrace{\chi_R^p(|\tilde{x}|)}_{\geqslant 0}d\tilde{x}\Bigg)\\
&\cdot\prod_{l=1}^{k}2\pi\Bigg(\int_{R-1}^{R}r_l\underbrace{\chi_R^p(r_l)}_{\geqslant 0}dr_l+\int_R^{2R}r_l\underbrace{\chi_R^p(r_l)}_{=1}dr_l+\int_{2R}^{2R+1}r_l\underbrace{\chi_R^p(r_l)}_{\geqslant 0}dr_l\Bigg)\\
\geqslant&\int_{R\leqslant|\tilde{x}|\leqslant 2R}1d\tilde{x}\cdot\prod_{l=1}^{k}2\pi\int_R^{2R}r_ldr_l=CR^{\tilde{d}}\prod_{l=1}^{k}3\pi R^2=(3\pi)^kCR^{2k+\tilde{d}}=CR^d,
\end{aligned}
$$

where $d\xi:=d\tilde{x}d\phi_kdr_k\cdots d\phi_1dr_1$ and $\tilde{d}:=d-2k$ denotes the dimension of the $\tilde{x}$-integral. Moreover, we used the following formula with $a=R$ and $b=2R$

$$
\int_{a\leqslant|\tilde{x}|\leqslant b}1d\tilde{x}=\begin{cases}1 & ,\ \tilde{d}=0,\\ b-a & ,\ \tilde{d}=1,\\ 2\pi\frac{\pi^{\frac{\tilde{d}-2}{2}}}{\Gamma\left(\frac{\tilde{d}}{2}\right)}\frac{\left(b^{\tilde{d}}-a^{\tilde{d}}\right)}{\tilde{d}} & ,\ \tilde{d}\geqslant 2.\end{cases}\tag{7.12}
$$

5. Furthermore, we have by (7.10)

$$
\left\|\left(\lambda I-\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)v_R\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}^p=\int_{\mathbb{R}^d}\left|\left(\lambda I-\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)v_R(x)\right|^p dx
$$

$$
\begin{aligned}
&= \int_0^\infty \int_{-\pi}^{\pi} \cdots \int_0^\infty \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-2k}} \left(\prod_{l=1}^k r_l\right) \left|\left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right) v_R(\xi)\right|^p d\tilde{x} d\phi_k dr_k \cdots d\phi_1 dr_1 \\
&= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1} \left(\prod_{l=1}^k r_l\right) \left|\left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right) v_R(\xi)\right|^p d\xi
\end{aligned}
$$

Defining $\tilde{d} := d - 2k$ we distinguish between the following cases:
Case 1: ($\tilde{d} = 0$). From step 3, (7.11), the multinomial theorem,

$$
(7.13) \qquad \int_{R-1}^{R} r_l dr_l = \frac{1}{2}(2R-1), \; \int_{R}^{2R} r_l dr_l = \frac{1}{2}3R^2, \; \int_{2R}^{2R+1} r_l dr_l = \frac{1}{2}(4R+1),
$$

$k = \frac{d}{2}$ and

$$
(2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3} \leqslant CR^{j_1+2j_2+j_3} = CR^{k+j_2} \leqslant CR^{k+k-1} = CR^{d-1}
$$

we further obtain

$$
\begin{aligned}
&= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \left(\prod_{l=1}^k r_l\right) \left|\left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right) v_R(\xi)\right|^p d\tilde{x} d\phi_k dr_k \cdots d\phi_1 dr_1 \\
&\leqslant \sum_{\substack{j_1+j_2+j_3=k \\ j_2\neq k}} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} C^p \left(\prod_{l=1}^k r_l\right) (2\pi)^k dr_1 \cdots dr_k \\
&= \sum_{\substack{j_1+j_2+j_3=k \\ j_2\neq k}} \binom{k}{j_1, j_2, j_3} C^p (2\pi)^k \frac{1}{2^k} (2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3} \leqslant CR^{d-1}.
\end{aligned}
$$

Case 2: ($\tilde{d} \geqslant 1$). Again from step 3, (7.11), the multinomial theorem, (7.12), $(2R-1)^{j_1} \leqslant CR^{j_1}$, $(3R^2)^{j_2} \leqslant CR^{2j_2}$ and $(4R+1)^{j_3} \leqslant CR^{j_3}$ we further obtain

$$
\begin{aligned}
\leqslant & \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} \\
& \quad \int_{R-1\leqslant|\tilde{x}|\leqslant R} C^p \left(\prod_{l=1}^k r_l\right) (2\pi)^k d\tilde{x} dr_1 \cdots dr_k \\
& + \sum_{\substack{j_1+j_2+j_3=k \\ j_2\neq k}} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} \\
& \quad \int_{R\leqslant|\tilde{x}|\leqslant 2R} C^p \left(\prod_{l=1}^k r_l\right) (2\pi)^k d\tilde{x} dr_1 \cdots dr_k \\
& + \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} \\
& \quad \int_{2R\leqslant|\tilde{x}|\leqslant 2R+1} C^p \left(\prod_{l=1}^k r_l\right) (2\pi)^k d\tilde{x} dr_1 \cdots dr_k
\end{aligned}
$$

$$= \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} \frac{1}{2^k} (2R-1)^{j_1} (3R^2)^{j_2} (4R+1)^{j_3}$$

$$\left\{ \begin{array}{ll} 1 & , \tilde{d} = 1 \\ 2\pi \frac{\pi^{\frac{\tilde{d}-2}{2}}}{\Gamma\left(\frac{\tilde{d}}{2}\right)} \frac{\left(R^{\tilde{d}} - (R-1)^{\tilde{d}}\right)}{\tilde{d}} & , \tilde{d} \geqslant 2 \end{array} \right\} C^p (2\pi)^k$$

$$+ \sum_{\substack{j_1+j_2+j_3=k \\ j_2 \neq k}} \binom{k}{j_1, j_2, j_3} \frac{1}{2^k} (2R-1)^{j_1} (3R^2)^{j_2} (4R+1)^{j_3}$$

$$\left\{ \begin{array}{ll} R & , \tilde{d} = 1 \\ 2\pi \frac{\pi^{\frac{\tilde{d}-2}{2}}}{\Gamma\left(\frac{\tilde{d}}{2}\right)} \frac{\left((2R)^{\tilde{d}} - R^{\tilde{d}}\right)}{\tilde{d}} & , \tilde{d} \geqslant 2 \end{array} \right\} C^p (2\pi)^k$$

$$+ \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} \frac{1}{2^k} (2R-1)^{j_1} (3R^2)^{j_2} (4R+1)^{j_3}$$

$$\left\{ \begin{array}{ll} 1 & , \tilde{d} = 1 \\ 2\pi \frac{\pi^{\frac{\tilde{d}-2}{2}}}{\Gamma\left(\frac{\tilde{d}}{2}\right)} \frac{\left((2R)^{\tilde{d}} - R^{\tilde{d}}\right)}{\tilde{d}} & , \tilde{d} \geqslant 2 \end{array} \right\} C^p (2\pi)^k$$

$$\leqslant \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} C R^{j_1+2j_2+j_3+\tilde{d}-1} + \sum_{\substack{j_1+j_2+j_3=k \\ j_2 \neq k}} \binom{k}{j_1, j_2, j_3} C R^{j_1+2j_2+j_3+\tilde{d}}$$

$$+ \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} C R^{j_1+2j_2+j_3+\tilde{d}-1} \leqslant C R^{d-1}.$$

6. Now, let us consider the operator $\mathcal{L}_{Q,T_2}$ instead of $\mathcal{L}_{Q,T_2}^{\mathrm{sim}}$. By definition of χ_R we have

$$\left[(\lambda I - \mathcal{L}_{Q,T_2}) \, v_R \right] (\xi) = 0,$$

whenever $|\tilde{x}| \in [0, R-1] \cup [2R+1, \infty[$ or $r_l \in [0, R-1] \cup [2R+1, \infty[$ for some $1 \leqslant l \leqslant k$. Moreover, we have by the choice of λ, by definition of χ_R and since $R \geqslant 1$

$$\begin{aligned}
&|(\lambda I - \mathcal{L}_{Q,T_2}) \, v_R(\xi)| \\
&= \left| \left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right) v_R(\xi) - A \sum_{l=1}^{k} \left(\frac{1}{r_l} \partial_{r_l} + \frac{1}{r_l^2} \partial_{\phi_l} \right) v_R(\xi) - Q(\xi) v_R(\xi) \right| \\
&= \left| A \sum_{l=1}^{k} \left(\frac{1}{r_l} \partial_{r_l} + \frac{1}{r_l^2} \partial_{\phi_l} \right) v_R(\xi) + Q(\xi) v_R(\xi) \right| \\
&\leqslant |A|_2 \sum_{j=1}^{k} \left(\frac{|i\omega|}{r_l} + \frac{|in_l|^2}{r_l^2} \right) + |Q(\xi)| \\
&\leqslant |A|_2 \sum_{j=1}^{k} \left(|\omega| + |n_l|^2 \right) \frac{1}{r_l} + \eta_R \\
&\leqslant \left(|A|_2 \sum_{j=1}^{k} \left(|\omega| + |n_l|^2 \right) \frac{1}{r_l} + \eta_R \right)^{\frac{1}{p}}, \; 1 < p < \infty
\end{aligned}$$

if $|\tilde{x}|, r_l \in [R,2R]$ for every $l=1,\ldots,k$.
7. From the choice of λ, step 3, $\frac{1}{r_l}\leqslant\frac{1}{R-1}\leqslant 1$ (since $R\geqslant 2$), $\frac{1}{r_l^2}\leqslant 1$, $|\chi_R(y)|\leqslant 1$, $|\chi_R'(y)|\leqslant\|\chi_R\|_{C_{\mathrm{b}}^2}$ and $|v(\xi)|=1$ we obtain

$$\begin{aligned}
&\left|\left(\lambda I-\mathcal{L}_{Q,T_2}\right)v_R(\xi)\right|\\
=&\left|\left(\lambda I-\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)v_R(\xi)-A\sum_{l=1}^{k}\left(\frac{1}{r_l}\partial_{r_l}+\frac{1}{r_l^2}\partial_{\phi_l}\right)v_R(\xi)-Q(\xi)v_R(\xi)\right|\\
=&\Bigg|\left(\lambda I-\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)v_R(\xi)-A\sum_{l=1}^{k}\frac{1}{r_l}\chi_R'(r_l)\left(\prod_{\substack{j=1\\j\neq l}}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi)\\
&-A\sum_{l=1}^{k}\frac{1}{r_l}i\omega\left(\prod_{j=1}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi)-A\sum_{l=1}^{k}\frac{1}{r_l^2}in_l\left(\prod_{j=1}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi)\\
&-Q(\xi)\left(\prod_{j=1}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi)\Bigg|\\
\leqslant&\left|\left(\lambda I-\mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right)v_R(\xi)\right|+|A|_2\sum_{l=1}^{k}\frac{1}{r_l}\left|\chi_R'(r_l)\right|\left(\prod_{\substack{j=1\\j\neq l}}^{k}\left|\chi_R(r_j)\right|\right)\left|\chi_R(|\tilde{x}|)\right|\left|v(\xi)\right|\\
&+|A|_2\sum_{l=1}^{k}\frac{1}{r_l}|\omega|\left(\prod_{j=1}^{k}\left|\chi_R(r_j)\right|\right)\left|\chi_R(|\tilde{x}|)\right|\left|v(\xi)\right|\\
&+|A|_2\sum_{l=1}^{k}\frac{1}{r_l^2}|n_l|\left(\prod_{j=1}^{k}\left|\chi_R(r_j)\right|\right)\left|\chi_R(|\tilde{x}|)\right|\left|v(\xi)\right|\\
&+\|Q\|_{L^\infty}\left(\prod_{j=1}^{k}\left|\chi_R(r_j)\right|\right)\left|\chi_R(|\tilde{x}|)\right|\left|v(\xi)\right|\\
\leqslant& C+|A|_2\left(k\,\|\chi_R\|_{C_{\mathrm{b}}^2}+k\,|\omega|+\sum_{l=1}^{k}|n_l|+\|Q\|_{L^\infty}\right)=C,
\end{aligned}$$

for every $|\tilde{x}|, r_l\in[R-1,R]\cup[R,2R]\cup[2R,2R+1]$ and $1\leqslant l\leqslant k$.
8. Hence, we obtain from the transformation theorem and step 6

$$\begin{aligned}
&\left\|\left(\lambda I-\mathcal{L}_Q\right)v_R\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}^p\\
&=\int_{\mathbb{R}^d}\left|\left(\lambda I-\mathcal{L}_Q\right)v_R(x)\right|^p dx=\int_{\mathbb{R}^d}\left|\left(\lambda I-\mathcal{L}_{Q,T_1}\right)v_R(x)\right|^p dx\\
&=\int_0^\infty\int_{-\pi}^{\pi}\cdots\int_0^\infty\int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-2k}}\left(\prod_{l=1}^{k}r_l\right)\left|\left(\lambda I-\mathcal{L}_{Q,T_2}\right)v_R(\xi)\right|^p d\xi\\
&=\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}\cdots\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}\int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1}\left(\prod_{l=1}^{k}r_l\right)\left|\left(\lambda I-\mathcal{L}_{Q,T_2}\right)v_R(\xi)\right|^p d\xi
\end{aligned}$$

Using the abbreviation $\tilde{d}:=d-2k$ we distinguish again between the following cases:

Case 1: $(\tilde{d}=0)$. From step 6, step 7 and (7.13) we deduce

$$
\begin{aligned}
=&\int_R^{2R}\int_{-\pi}^{\pi}\cdots\int_R^{2R}\int_{-\pi}^{\pi}\left(\prod_{l=1}^{k}r_l\right)\left[\sum_{l=1}^{k}\frac{|A|_2\left(|\omega|+|n_l|^2\right)}{r_l}+\eta_R\right]d\phi_k dr_k\cdots d\phi_1 dr_1\\
&+\sum_{\substack{j_1+j_2+j_3\\ j_2\neq k}}\binom{k}{j_1,j_2,j_3}\left(\int_{R-1}^{R}\right)^{j_1}\left(\int_{R}^{2R}\right)^{j_2}\left(\int_{2R}^{2R+1}\right)^{j_3}C^p\left(\prod_{l=1}^{k}r_l\right)(2\pi)^k dr_1\cdots dr_k\\
\leqslant&\int_R^{2R}\cdots\int_R^{2R}(2\pi)^k\left[\sum_{l=1}^{k}\left(\prod_{\substack{j=1\\ j\neq l}}^{k}r_j\right)|A|_2\left(|\omega|+|n_l|^2\right)\right]\\
&+(2\pi)^k\left(\prod_{l=1}^{k}r_l\right)\eta_R dr_k\cdots dr_1+CR^{d-1}\\
=&\sum_{l=1}^{k}|A|_2\left(|\omega|+|n_l|^2\right)\int_R^{2R}\cdots\int_R^{2R}\prod_{\substack{j=1\\ j\neq l}}^{k}r_j dr_1\cdots dr_k\\
&+(2\pi)^k\eta_R\int_R^{2R}\cdots\int_R^{2R}\left(\prod_{l=1}^{k}r_l\right)dr_1\cdots dr_k+CR^{d-1}\\
=&\sum_{l=1}^{k}(2\pi)^k|A|_2\left(|\omega|+|n_l|^2\right)\left(\prod_{\substack{j=1\\ j\neq l}}^{k}\int_R^{2R}r_j dr_j\right)\int_R^{2R}dr_l\\
&+(2\pi)^k\eta_R\prod_{j=1}^{k}\int_R^{2R}r_j dr_j+CR^{d-1}\\
=&\left(\sum_{l=1}^{k}(2\pi)^k|A|_2\left(|\omega|+|n_l|^2\right)\left(\frac{3}{2}\right)^{k-1}R^{2k-1}\right)+(2\pi)^k\eta_R\left(\frac{3}{2}\right)^{k}R^{2k}+CR^{d-1}\\
\leqslant&CR^{d-1}+CR^d\eta_R.
\end{aligned}
$$

Here we refer to case 1 from step 5 for an estimate of the sum.
Case 2: $(\tilde{d}\geqslant 1)$. From the procedure used in case 2 from step 5 and in case 1 and (7.12) we obtain

$$
\begin{aligned}
&\leqslant\int_R^{2R}\int_{-\pi}^{\pi}\cdots\int_R^{2R}\int_{-\pi}^{\pi}\int_{R\leqslant|\tilde{x}|\leqslant 2R}\left(\prod_{l=1}^{k}r_l\right)\left[\sum_{l=1}^{k}\frac{|A|_2\left(|\omega|+|n_l|^2\right)}{r_l}+\eta_R\right]d\xi\\
&\quad+CR^{d-1}\\
&\leqslant\left(CR^{2k-1}+CR^{2k}\eta_R\right)\int_{R\leqslant|\tilde{x}|\leqslant 2R}d\tilde{x}+CR^{d-1}\\
&\leqslant CR^{2k-1+\tilde{d}}+CR^{d-1}+CR^{2k+\tilde{d}}\eta_R=CR^{d-1}+CR^d\eta_R.
\end{aligned}
$$

The constant CR^{d-1} in the first inequality comes from an estimate of three sums, compare case 2 from step 5. For the second inequality compare case 1.
9. Define

$$
w_R:=\frac{v_R}{\|v_R\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}}\in L^p(\mathbb{R}^d,\mathbb{C}^N),
$$

which belongs to $L^p(\mathbb{R}^d,\mathbb{C}^N)$ by step 4, then we obtain from step 4 and step 8

$$\begin{aligned}\|(\lambda I-\mathcal{L}_Q)\,w_R\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)} =& \frac{\|(\lambda I-\mathcal{L}_Q)\,v_R\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)}}{\|v_R\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)}}\\ \leqslant& \frac{CR^{d-1}+CR^d\eta_R}{CR^d}=\frac{C}{R}+\eta_R\to 0 \text{ as } R\to\infty.\end{aligned}$$

10. Hence, we must have

$$\lambda\in\sigma(\mathcal{L}_Q) \text{ or } (\lambda I-\mathcal{L}_Q)^{-1} \text{ is unbounded on } L^p(\mathbb{R}^d,\mathbb{C}^N).$$

If $\lambda=-\lambda_j(\omega)-i\sum_{l=1}^{k}n_l\sigma_l\in\sigma(\mathcal{L}_Q)$, i.e. λ is an eigenvalue of $\mathcal{L}_Q$, then varying $\omega\in\mathbb{R}$ shows that λ cannot be isolated, i.e. λ is not a normal point of $\mathcal{L}_Q$. Therefore, all such numbers λ belongs to the essential spectrum of $\mathcal{L}_Q$, i.e. $\lambda\in\sigma_{\text{ess}}(\mathcal{L}_Q)$. □

The next Corollary states that for every $1<p<\infty$ the semigroup $(T_Q(t))_{t\geqslant 0}$ is not analytic on $L^p(\mathbb{R}^d,\mathbb{C}^N)$, $1<p<\infty$, whenever $S\neq 0$. We refer to [108] and also to [71] and [83] for the scalar real-valued case. We refer to [34, II.4.5], for a definition of an analytic semigroup, and to [34, II.4.1], for the definition of a sectorial operator.

Corollary 7.10 (Analyticity of $(T_Q(t))_{t\geqslant 0}$)**.** *Let the assumptions* (A1)–(A5), (A8$_B$) *with* $S\neq 0$ *and* $Q\in L^\infty(\mathbb{R}^d,\mathbb{K}^{N,N})$ *with*

$$\eta_R:=\operatorname*{ess\,sup}_{|x|\geqslant R}|Q(x)|_2\to 0 \text{ as } R\to\infty$$

be satisfied for $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then the operator* $\mathcal{L}_Q$ *is not sectorial in* $L^p(\mathbb{R}^d,\mathbb{C}^N)$ *and, consequently, the corresponding semigroup* $(T_Q(t))_{t\geqslant 0}$ *is not analytic on* $L^p(\mathbb{R}^d,\mathbb{C}^N)$.

Proof. We show that $\mathcal{L}_Q$ is not sectorial: For this purpose, we verify that

$$\forall\,\delta\in]0,\tfrac{\pi}{2}]\;\exists\,\lambda\in\Sigma_{\frac{\pi}{2}+\delta}:=\left\{\lambda\in\mathbb{C}\mid|\arg(\lambda)|<\frac{\pi}{2}+\delta\right\}\backslash\{0\}:\;\lambda\in\sigma_{\text{ess}}(\mathcal{L}_Q),$$

i.e. $\lambda\notin\rho(\mathcal{L}_Q)$. Let $\delta\in]0,\frac{\pi}{2}]$ and let $\lambda\in\mathbb{C}$ be chosen as in (7.9). Let us fix w.l.o.g. $j\in\{1,\ldots,N\}$, $\omega=0$ and $n_2,\ldots,n_k=0$. Then λ has the form $\lambda=-\mu-in_1\sigma_1$ for some $\mu\in\sigma(B)$. Choose $n_1\in\mathbb{Z}$ so large, that $\lambda\in\Sigma_{\frac{\pi}{2}+\delta}$. Now, Theorem 7.9 implies $\lambda\in\sigma_{\text{ess}}(\mathcal{L}_Q)$, hence $\lambda\notin\rho(\mathcal{L}_Q)$. Thus, $\mathcal{L}_Q$ cannot be sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and, consequently, the semigroup $(T_Q(t))_{t\geqslant 0}$ cannot be analytic on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ by [34, Theorem II.4.6]. Note, that the proof needs that $\sigma_l\neq 0$ for at least one such l, that is guaranteed by the assumption $S\neq 0$. □

The case $S=0$**.** A crucial part in the proof of Corollary 7.10 plays the fact that $S\neq 0$. The assertion is in general not true for $S=0$: Consider for example the simplest case $S=B=Q=0$ with $A=I_N$, then it is well known that the corresponding diffusion semigroup is analytic on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$. This result remains valid for arbitrary diffusion matrices $A\in\mathbb{C}^N$ satisfying the assumptions (A1)–(A4) for $1<p<\infty$.

8 Nonlinear problems and complex Ornstein-Uhlenbeck operators

In this chapter we investigate the nonlinear problem

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2, \tag{8.1}$$

for the complex Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle,\ x \in \mathbb{R}^d,$$

in $L^p(\mathbb{R}^d, \mathbb{K}^N)$ with $A \in \mathbb{K}^{N,N}$, skew-symmetric matrix $S \in \mathbb{R}^{d,d}$, nonlinearity $f : \mathbb{K}^N \to \mathbb{K}^N$ and $v : \mathbb{R}^d \to \mathbb{K}^N$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $N \in \mathbb{N}$.

In Section 8.1 we consider the nonlinear problem (8.1) for $\mathbb{K} = \mathbb{R}$ and prove the main result from Theorem 1.8. Assuming (A4)–(A9) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{R}$, we prove in Theorem 1.8 that $v_\star - v_\infty$ and its derivatives up to order 1 decay exponentially in space at a certain rate, whenever $v_\star$ is a classical solution of (8.1) such that $v_\star \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and $v_\star - v_\infty$ falls below a certain threshold in the far-field. The proof is based on an application of Theorem 7.7, that requires the identification of the generator C_p and its maximal domain from Theorem 5.19. For a detailed treatment of this result we refer to Section 1.2. For an outline of the proof see Section 1.3.

In Section 8.2 we consider the nonlinear problem (8.1) for $\mathbb{K} = \mathbb{C}$ whose nonlinearities are of the form

$$f : \mathbb{C}^N \to \mathbb{C}^N, \quad f(u) = g\left(|u|^2\right) u,$$

where $g : \mathbb{R} \to \mathbb{C}^{N,N}$ is a sufficiently smooth function. Assuming (A4) and (A5) for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$ and assuming some additional properties for the function g, we prove in Corollary 8.1 an extension of Theorem 1.8 to complex systems. The proof is based on an application of Theorem 1.8. For this purpose we transform the N-dimensional complex-valued system (8.1) into a coupled $2N$-dimensional real-valued system.

8.1 Proof of main theorem

We are now able to prove our main result from Theorem 1.8:

Proof. Let $0 < \vartheta < 1$ be fixed, $1 < p < \infty$ and $\theta \in C(\mathbb{R}^d, \mathbb{R})$ be a radially nondecreasing weight function of exponential growth rate $\eta \geqslant 0$ with $0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$, where $a_{\max}$, a_0 and b_0 are from (1.18).

1. Let $v_\star$ denote a classical solution of (1.20) satisfying $v_\star - v_\infty \in L^p(\mathbb{R}^d,\mathbb{R}^N)$ and (1.19). From Taylor's theorem, (A6) and (A7) we obtain

$$\begin{aligned} f(v_\star(x)) =& \underbrace{f(v_\infty)}_{=0} + \underbrace{Df(v_\infty)}_{=:-B}(v_\star(x)-v_\infty) \\ &+ \underbrace{\int_0^1 \left(Df(v_\infty + t(v_\star(x)-v_\infty)) - Df(v_\infty)\right)dt\,(v_\star(x)-v_\infty)}_{=:Q(x)} \\ =& -B\,(v_\star(x)-v_\infty) + Q(x)\,(v_\star(x)-v_\infty)\,. \end{aligned}$$

2. Defining $w_\star := v_\star - v_\infty$ then $w_\star \in C^2(\mathbb{R}^d,\mathbb{R}^N)\cap C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}^N)\cap L^p(\mathbb{R}^d,\mathbb{R}^N)$ since $v_\star$ is a classical solution of (1.11) and $v_\star - v_\infty \in L^p(\mathbb{R}^d,\mathbb{R}^N)$, and we obtain

$$\begin{aligned} 0 =& A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) \\ =& A\triangle\,(v_\star(x)-v_\infty) + \langle Sx, \nabla\,(v_\star(x)-v_\infty)\rangle \\ &- B\,(v_\star(x)-v_\infty) + Q(x)\,(v_\star(x)-v_\infty) \\ =& A\triangle w_\star(x) + \langle Sx,\nabla w_\star(x)\rangle - Bw_\star(x) + Q(x)w_\star(x) = \left[\mathcal{L}_Q w_\star\right](x). \end{aligned}$$

3. In order to apply Theorem 7.7 (with $C_p = \mathcal{L}_Q$ and $\lambda = 0$) we have to verify, that the assumptions are satisfied. Note that the application of Theorem 7.7 with $C_p = \mathcal{L}_Q$ requires additionally that the assumptions (A3) and (A4) are fulfilled, which are necessary to solve the identification problem for C_p. Let us check the assumptions: Assumption (A1) follows from (A8). The assumption (A4) and (A5) are directly satisfied and assumption (A4) implies (A3) and (A2). Using the definition of B, the assumptions (A8$_B$) and (A9$_B$) follow from (A8) and (A9), respectively. It remains to verify $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, (7.5), $w_\star \in \mathcal{D}(A_p)$ and $\mathcal{L}_Q w_\star = 0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$.
4. First we show that $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$. From $w_\star \in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}^N)$ we obtain

$$|v_\infty + tw_\star(x)| \leqslant |v_\infty| + t\,|w_\star(x)| \leqslant |v_\infty| + \|w_\star\|_\infty =: R_1$$

for every $x \in \mathbb{R}^d$ and $0 \leqslant t \leqslant 1$. Using (A6) this implies

$$\begin{aligned} |Q(x)|_2 &\leqslant \int_0^1 |Df(v_\infty + tw_\star(x))|_2 + |Df(v_\infty)|_2\,dt \\ &\leqslant \sup_{z\in B_{R_1}(0)} |Df(z)|_2 + |Df(v_\infty)|_2 \end{aligned}$$

for every $x \in \mathbb{R}^d$, which is of course finite by the continuity of Df on compact sets. Taking the suprema over $x \in \mathbb{R}^d$ we obtain $Q \in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}^{N,N})$, thus $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$.
5. We next verify (7.5): Let us choose $K_1 = K_1(A,f,v_\infty,d,p,\theta,\vartheta) > 0$ such that

$$K_1\left(\sup_{z\in B_{K_1}(v_\infty)} \left|D^2f(z)\right|_2\right) \leqslant \frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\} \tag{8.2}$$

is satisfied, where $C_7 = C_7(A,d,p,\theta,\vartheta)$ is from Lemma 4.8, $M_\infty = M_\infty(A,d)$ from (6.10), $b_0 = b_0(f,v_\infty)$ from (1.18) and

$$\left|D^2f(z)\right|_2 := \left\|D^2f(z)\right\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^{N,N})} := \sup_{\substack{v\in\mathbb{R}^N\\|v|=1}} \left|D^2f(z)v\right|_2.$$

The fundamental theorem of calculus, (A6), (1.20) and the choice of K_1 yield

$$\begin{aligned}
&|Q(x)|_2\\
=&\left|\int_0^1 Df(v_\infty+tw_\star(x))-Df(v_\infty)dt\right|_2\\
=&\left|\int_0^1\int_0^1 D^2f(v_\infty+s(v_\infty+tw_\star(x)-v_\infty))ds(v_\infty+tw_\star(x)-v_\infty)dt\right|_2\\
=&\left|\int_0^1\int_0^1 D^2f(v_\infty+stw_\star(x))ds\cdot tw_\star(x)dt\right|_2\\
\leqslant&\int_0^1\int_0^1 \sup_{|x|\geqslant R_0}\left|D^2f(v_\infty+st(v_\star(x)-v_\infty))\right|_2 ds\cdot t\left|v_\star(x)-v_\infty\right|dt\\
\leqslant&K_1\left(\sup_{z\in B_{K_1}(v_\infty)}\left|D^2f(z)\right|_2\right)\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}
\end{aligned}$$

for every $|x|\geqslant R_0$. Taking the suprema over $|x|\geqslant R_0$ yields

$$\sup_{|x|\geqslant R_0}|Q(x)|_2\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}.$$

6. Now we verify that $w_\star\in\mathcal{D}(A_p)$: An application of Theorem 5.19 shows that $\mathcal{D}(A_p)=\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. Therefore, it suffices to show that $w_\star\in L^p(\mathbb{R}^d,\mathbb{C}^N)$, $w_\star\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)$ and $\mathcal{L}_0w_\star\in L^p(\mathbb{R}^d,\mathbb{C}^N)$. By assumption we know that $w_\star=v_\star-v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^N)$. Moreover, we deduce $w_\star\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^N)$ from $w_\star\in C^2(\mathbb{R}^d,\mathbb{R}^N)$. It remains to prove that $\mathcal{L}_0w_\star\in L^p(\mathbb{R}^d,\mathbb{R}^N)$: Since $v_\star\in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}^N)$ there exists a constant $R_1>0$ such that $|v_\star(x)-v_\infty|\leqslant R_1$ for every $x\in\mathbb{R}^d$. From (A6) we deduce that f is locally Lipschitz continuous, i.e. there exists $L=L(R_1)\geqslant 0$ such that

$$|f(v_\star(x))-f(v_\infty)|\leqslant L\,|v_\star(x)-v_\infty|$$

for every $x\in\mathbb{R}^d$. Now, we obtain from (A7) and (1.11)

$$\begin{aligned}
\|\mathcal{L}_0w_\star\|^p_{L^p}=&\int_{\mathbb{R}^d}\left|\left[\mathcal{L}_0w_\star\right](x)\right|^p dx=\int_{\mathbb{R}^d}\left|\left[\mathcal{L}_0v_\star\right](x)\right|^p dx\\
=&\int_{\mathbb{R}^d}|f(v_\star(x))|^p dx=\int_{\mathbb{R}^d}|f(v_\star(x))-f(v_\infty)|^p dx\leqslant L^p\int_{\mathbb{R}^d}|v_\star(x)-v_\infty|^p dx\\
=&L^p\,\|v_\star-v_\infty\|^p_{L^p}=L^p\,\|w_\star\|^p_{L^p}
\end{aligned}$$

This yields $\mathcal{L}_0w_\star\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and thus $w_\star\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$.
7. Finally, we verify $\mathcal{L}_Qw_\star=0$ in $L^p(\mathbb{R}^d,\mathbb{R}^N)$: From $w_\star\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and $Q\in L^\infty(\mathbb{R}^d,\mathbb{R}^{N,N})$ we deduce from Hölders inequality that $\mathcal{L}_Qw_\star\in L^p(\mathbb{R}^d,\mathbb{R}^N)$. Further, since $w_\star\in C^2(\mathbb{R}^d,\mathbb{R}^N)$ satisfies $\left[\mathcal{L}_Qw_\star\right](x)=0$ pointwise for every $x\in\mathbb{R}^d$, we deduce from $\mathcal{L}_Qw_\star\in L^p(\mathbb{R}^d,\mathbb{R}^N)$ that $\mathcal{L}_Qw_\star=0$ in $L^p(\mathbb{R}^d,\mathbb{R}^N)$.
Now, we can apply Theorem 7.7 that yields $w_\star=v_\star-v_\infty\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{R}^N)$. $\square$

8.2 Application to complex-valued systems

In this section we extend the result from Theorem 1.8 to complex systems (8.1). The proof is based on an application of Theorem 1.8

Corollary 8.1. *Let the assumptions* (A4) *and* (A5) *be satisfied for* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Moreover, let* $g \in C^2(\mathbb{R}, \mathbb{C}^{N,N})$ *such that* A *and* $g(0)$ *are simultaneously diagonalizable (over* $\mathbb{C}$*),* $\sigma(g(0)) \subset \mathbb{C}_-$ *and define*

$$f : \mathbb{C}^N \to \mathbb{C}^N, \quad f(u) = g\left(|u|^2\right) u. \tag{8.3}$$

Then for every $0 < \vartheta < 1$ *and for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d, \mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$ *with*

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$$

$a_{\max}$, a_0 *from* (1.18), $b_0 = -s(g(0))$, *there is a constant* $K_1 = K_1(A, g, d, p, \theta, \vartheta) > 0$ *with the following property:*
Every classical solution $v_\star$ *of*

$$A \triangle v(x) + \langle Sx, \nabla v(x) \rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d, \tag{8.4}$$

such that $v_\star \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *and*

$$\sup_{|x| \geqslant R_0} |v_\star(x)| \leqslant K_1 \text{ for some } R_0 > 0 \tag{8.5}$$

satisfies

$$v_\star \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{C}^N).$$

Proof. 1. We transform the N-dimensional complex-valued system (8.4) into the coupled $2N$-dimensional real-valued system

$$A_{\mathbb{R}} \triangle v_{\mathbb{R}}(x) + \langle Sx, \nabla v_{\mathbb{R}}(x) \rangle + f_{\mathbb{R}}(v_{\mathbb{R}}(x)) = 0,\ x \in \mathbb{R}^d, \tag{8.6}$$

For this purpose, we decompose $A = A_1 + iA_2$ with $A_1, A_2 \in \mathbb{R}^{N,N}$, $v = v_1 + iv_2$ with $v_1, v_2 : \mathbb{R}^d \to \mathbb{R}^N$, $f_1, f_2 : \mathbb{R}^{2N} \to \mathbb{R}^N$ with $f_1(u_1, u_2) = \operatorname{Re} f(u_1 + iu_2)$, $f_2(u_1, u_2) = \operatorname{Im} f(u_1 + iu_2)$, $g = g_1 + ig_2$ with $g_1, g_2 : \mathbb{R} \to \mathbb{R}^{N,N}$ and define

$$A_{\mathbb{R}} := \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}, \quad v_{\mathbb{R}} := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and

$$f_{\mathbb{R}}(v_{\mathbb{R}}) := \begin{pmatrix} f_1(v_{\mathbb{R}}) \\ f_2(v_{\mathbb{R}}) \end{pmatrix} = \begin{pmatrix} g_1(|v_{\mathbb{R}}|) & -g_2(|v_{\mathbb{R}}|) \\ g_2(|v_{\mathbb{R}}|) & g_1(|v_{\mathbb{R}}|) \end{pmatrix} v_{\mathbb{R}},$$

where $A_{\mathbb{R}} \in \mathbb{R}^{2N,2N}$, $v_{\mathbb{R}} \in \mathbb{R}^{2N}$ and $f_{\mathbb{R}} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$.

2. In order to apply Theorem 1.8 to the $2N$-dimensional problem (8.6), we have to verify, that the assumptions (A4)–(A9) are satisfied for $\mathbb{K}=\mathbb{R}$. For this purpose, we collect some relations of A and $A_{\mathbb{R}}$:

$$\text{(8.7)} \qquad \lambda\in\sigma(A) \quad\Longleftrightarrow\quad \lambda,\overline{\lambda}\in\sigma(A_{\mathbb{R}}),$$

$$\text{(8.8)} \qquad Y^{-1}AY=\Lambda_A \quad\Longleftrightarrow\quad \begin{pmatrix} iY & \overline{Y} \\ Y & -i\overline{Y}\end{pmatrix} A_{\mathbb{R}} \begin{pmatrix} iY & \overline{Y} \\ Y & -i\overline{Y}\end{pmatrix}^{-1} = \begin{pmatrix} \Lambda_A & 0 \\ 0 & \overline{\Lambda_A}\end{pmatrix},$$

$$\text{(8.9)} \qquad \operatorname{Re}\langle v,Av\rangle = \langle v_{\mathbb{R}},A_{\mathbb{R}}v_{\mathbb{R}}\rangle\,,\ |v|=|v_{\mathbb{R}}|\,,\ |Av|=|A_{\mathbb{R}}v_{\mathbb{R}}|\,.$$

Since A satisfies (A4) for some $1<p<\infty$ and $\mathbb{K}=\mathbb{C}$, we deduce from (8.9), that $A_{\mathbb{R}}$ satisfies (A4) for the same $1<p<\infty$ and $\mathbb{K}=\mathbb{R}$. We casually note that if A satisfies (A1), (A2), (A3) for some $1<p<\infty$ and $\mathbb{K}=\mathbb{C}$ then $A_{\mathbb{R}}$ satisfies (A1), (A2), (A3) for the same $1<p<\infty$ and $\mathbb{K}=\mathbb{R}$, that follows from (8.8), (8.7), (8.9), respectively. Assumption (A5) is directly satisfied. Since $g\in C^2(\mathbb{R},\mathbb{C}^{N,N})$ we deduce that $f_{\mathbb{R}}\in C^2(\mathbb{R}^{2N},\mathbb{R}^{2N})$, meaning that assumption (A6) is satisfied for $\mathbb{K}=\mathbb{R}$. Choosing $v_\infty=0\in\mathbb{R}^{2N}$, then $f_{\mathbb{R}}(v_\infty)=0$ and condition (A7) is satisfied. Since A and $g(0)$ are simultaneously diagonalizable (over $\mathbb{C}$), we deduce from (8.8) that $A_{\mathbb{R}}$ and

$$Df_{\mathbb{R}}(0)=\begin{pmatrix} g_1(0) & -g_2(0) \\ g_2(0) & g_1(0)\end{pmatrix}$$

are simultaneously diagonalizable (over $\mathbb{C}$), meaning that assumption (A8) is satisfied for $\mathbb{K}=\mathbb{R}$. Finally, since $\sigma(g(0))\subset\mathbb{C}_-$ we deduce from (8.7) that $\sigma(Df_{\mathbb{R}}(0))\subset\mathbb{C}_-$. Thus, assumption (A9) is also satisfied.

3. Let $0<\vartheta<1$ be fixed and $\theta\in C(\mathbb{R}^d,\mathbb{R})$ be a radially nondecreasing weight function of exponential growth rate $\eta\geqslant 0$ with $0\leqslant\eta^2\leqslant\vartheta\frac{2}{3}\frac{a_0b_0}{a_{\max}^2p^2}$, where $a_{\max}$, a_0 and b_0 are from (1.18) with $A_{\mathbb{R}}$ and $Df_{\mathbb{R}}(0)$ instead of A and $Df(v_\infty)$. Moreover, let $v_\star$ be a classical solution of (8.4) satisfying $v_\star\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and (8.5). Then the function

$$v_{\mathbb{R},\star}:=\begin{pmatrix}\operatorname{Re}v_\star \\ \operatorname{Im}v_\star\end{pmatrix}$$

is a classical solution of (8.6), which also satisfies $v_{\mathbb{R},\star}\in L^p(\mathbb{R}^d,\mathbb{R}^{2N})$ and (8.5) since $|v_\star(x)|=|v_{\mathbb{R},\star}(x)|$. Now, an application of Theorem 1.8 yields $v_{\mathbb{R},\star}\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{R}^{2N})$ and thus $v_\star\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$. $\square$

Exponential decay for holomorphic nonlinearities. For the exponential decay in the complex-valued case, Corollary 8.1 requires that the nonlinearity $f:\mathbb{C}^N\to\mathbb{C}^N$ has the special form (8.3). The form (8.3) often arises in applications, for example in complex Ginzburg-Landau equations but also in Schrödinger equations and Gross-Pitaevskii equations. Note that under the more restrictive assumption that $f:\mathbb{C}^N\to\mathbb{C}^N$ is holomorphic, we can directly adopt the proof of Theorem 1.8. But in applications the nonlinearity is often not holomorphic. For instance, the nonlinearity of the cubic-quintic complex Ginzburg-Landau equation is not holomorphic at the origin.

9 Eigenvalue problems for the linearized differential operator

In this chapter we analyze the eigenvalue problem

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\; x \in \mathbb{R}^d,\; d \geqslant 2, \tag{9.1}$$

for the linearized differential operator

$$\left[\mathcal{L}v\right](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\; x \in \mathbb{R}^d \tag{9.2}$$

where $v : \mathbb{R}^d \to \mathbb{C}^N$, $A \in \mathbb{R}^{N,N}$, $S \in \mathbb{R}^{d,d}$ skew-symmetric, $f : \mathbb{R}^N \to \mathbb{R}^N$ sufficiently smooth, $\lambda \in \mathbb{C}$ and $v_\star : \mathbb{R}^d \to \mathbb{R}^N$ denotes a classical solution of the nonlinear problem (1.11). Related to the co-rotating frame (1.16), the operator $\mathcal{L}$ describes the linearization at the profile $v_\star$ of the rotating wave solution $u_\star$, cf. Definition 1.1. Investigations of the corresponding eigenvalue problem (9.1) are motivated by the stability theory of rotating patterns, [15]. In order to investigate the eigenvalue problem (9.1) in the complex case with $\mathbb{K} = \mathbb{C}$, we stress that N-dimensional complex-valued systems must generally be transformed to $2N$-dimensional real-valued systems as performed in the proof of Corollary 8.1. In this chapter we are mainly interested in finding classical solutions (λ, v) of the eigenvalue problem (9.1). Such a solution consists of an eigenfunction $v \in C^2(\mathbb{R}^d, \mathbb{C}^N)$ and an associated eigenvalue $\lambda \in \mathbb{C}$. Moreover, we are also interested in the exponential decay of these eigenfunctions.

In Section 9.1 we introduce some basic definitions for classical solutions of (9.1) and for the spectral stability of rotating waves, [38]. Decomposing the spectrum $\sigma(\mathcal{L})$ into the union of the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ and the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ gives rise to investigate both parts of $\sigma(\mathcal{L})$ in the following sections.

In Section 9.2 we analyze the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ of $\mathcal{L}$ and the shape of the correspondig eigenfunctions. Assuming $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ to be a classical solution of (1.11), we show in Theorem 9.4 that every

$$\lambda \in \sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S),\; \lambda_1 \neq \lambda_2\}$$

belongs to the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ of $\mathcal{L}$ and that their corresponding eigenfunction v has the form

$$v(x) = \left\langle C^{\mathrm{rot}}x + C^{\mathrm{tra}}, \nabla v_\star(x)\right\rangle,\; x \in \mathbb{R}^d,$$

for explicitly given $C^{\mathrm{tra}} \in \mathbb{C}^d$ and skew-symmetric $C^{\mathrm{rot}} \in \mathbb{C}^{d,d}$. This part of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ contains $\frac{d(d+1)}{2}$ eigenvalues and is caused by the rotational and translational symmetries from the $\mathrm{SE}(d)$-action. Note that by the skew-symmetry of S, all these isolated eigenvalues are located on the imaginary axis. In particular, we conclude that $0 \in \sigma_{\mathrm{point}}(\mathcal{L})$ and that the rotational term $v(x) = \langle Sx, \nabla v_\star(x)\rangle$ is an eigenfunction associated to the eigenvalue 0. The point spectrum contains in general further isolated eigenvalues, but a complete characterization of the point spectrum is often very delicate and will not be performed here. The results for the point spectrum do not depend upon whether the rotating pattern is localized or nonlocalized. This is in strict contrast to the essential spectrum, that depends strongly on the asymptotic behavior of the rotating wave at infinity. We conclude with some examples for the two and three dimensional case.

In Section 9.3 we investigate the exponential decay of the eigenfunctions v, which strongly depends on the asymptotic behavior of the pattern $v_\star$ and on the real part of its corresponding eigenvalue λ. Assuming the assumptions of our main result from Theorem 1.8, we prove in Theorem 9.8 that v and its first order derivatives decay exponentially in space at the same rate as the pattern $v_\star$, whenever v is a classical solution of (9.1) such that $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \frac{s(Df(v_\infty))}{3}$. The proof is based on an application of Theorem 7.7, that requires once more the identification of the maximal domain from Theorem 5.19. We deduce that the rotational term $v(x) = \langle Sx, \nabla v_\star(x)\rangle$ belongs to $W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$, whenever $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$.

In Section 9.4 we study the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$ for exponentially localized patterns $v_\star$. Assuming the assumptions of our main result from Theorem 1.8, we prove in Theorem 9.10 that every $\lambda \in \mathbb{C}$ satisfying

$$\det\left(\lambda I_N + \omega^2 A - Df(v_\infty) + i\sum_{l=1}^{k} n_l \sigma_l I_N\right) = 0, \text{ for some } \omega \in \mathbb{R},\ n_l \in \mathbb{Z},$$

belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 < p < \infty$. The result follows directly from an application of Theorem 7.9.

In Section 9.5 we analyze the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$ for Archimedean spiral patterns $v_\star$, a special kind of a nonlocalized rotating wave with $d = 2$, and formulate a dispersion relation for Archimedean spiral waves. Most of the results from this section are not completely new and we refer to [92] and [38], but also to [93].

9.1 Classical solutions and spectral stability

We are interested in solutions (λ, v) of (9.1) in the following sense:

Definition 9.1. A function $v : \mathbb{R}^d \to \mathbb{C}^N$ is called a **classical solution of** (9.1) for some $\lambda \in \mathbb{C}$, if

$$v \in C^2(\mathbb{R}^d, \mathbb{C}^N) \tag{9.3}$$

and v solves (9.1) pointwise.

We transfer the definition for spectral stability of traveling waves to rotating waves, see [38, Section 3.1.2], and introduce the strong spectral stability.

Definition 9.2. A rotating wave solution $u_\star : \mathbb{R}^d \times [0,\infty[\to \mathbb{K}^N$ of (1.14) given by

$$u_\star(x,t) = v_\star(e^{-tS}(x - x_\star)$$

is called **spectrally stable**, if $\sigma(\mathcal{L}) \subset \mathbb{C}_- \cup i\mathbb{R} = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant 0\}$. Moreover, a rotating wave solution $u_\star : \mathbb{R}^d \times [0,\infty[\to \mathbb{K}^N$ of (1.14) is called **strongly spectrally stable**, if it is spectrally stable and every $\lambda \in \sigma(\mathcal{L})$ with Re $\lambda = 0$ is caused by the SE(d)-group action.

The eigenvalues caused by the SE(d)-group action are described in detail in Theorem 9.4.

For rotating waves on unbounded domains it is well known that one usually derive nonlinear stability from strong spectral stability. Note that nonlinear stability only implies spectral stability, but in general not strong spectral stability. However, both definitions motivate investigations of the spectrum $\sigma(\mathcal{L})$ of the linearization $\mathcal{L}$. For this purpose, we decompose $\sigma(\mathcal{L})$ into

$$\sigma(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L}) \,\dot{\cup}\, \sigma_{\mathrm{point}}(\mathcal{L}),$$

where $\sigma_{\mathrm{ess}}(\mathcal{L})$ and $\sigma_{\mathrm{point}}(\mathcal{L})$ denote the essential and the point spectrum of $\mathcal{L}$, respectively, cf. Definition 7.8. In the following sections we analyze these two sets in more detail.

9.2 Point spectrum and the shape of eigenfunctions

In this section we analyze the isolated points in the spectrum of the operator $\mathcal{L}$ which are caused by the group action on SE(d). It is convenient to work in the **Euclidean Sobolev space**

$$\begin{aligned} W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N) &:= \left\{v \in W^{2,p}(\mathbb{R}^d,\mathbb{K}^N) \mid \langle S\cdot, \nabla v\rangle \in L^p(\mathbb{R}^d,\mathbb{K}^N)\ \forall\, S \in \mathfrak{so}(d)\right\}, \\ \|v\|_{W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)} &:= \|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{K}^N)} + \sup_{S\in\mathfrak{so}(d)} \|\langle Sx, \nabla v\rangle\|_{L^p(\mathbb{R}^d,\mathbb{K}^N)}\,,\ 1<p<\infty, \end{aligned}$$

which is the intersection of the spaces $\mathcal{D}^p_{\max}(\mathcal{L}_0)$ for every $S \in \mathfrak{so}(d)$, see Example 10.6. To investigate the point spectrum of $\mathcal{L}$ we need the following Lemma.

Lemma 9.3 (Group action on SE(d))**.** *Let $v \in L^p(\mathbb{R}^d,\mathbb{K}^N)$ for $1<p<\infty$ and $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Moreover, let the group action*

$$a(\cdot)v : \mathrm{SE}(d) \to L^p(\mathbb{R}^d,\mathbb{K}^N), \quad g := (R,\tau) \mapsto a(R,\tau)v$$

be given by

$$[a(R,\tau)v]\,(x) := v(R^{-1}(x-\tau))$$

for $g = (R,\tau) \in \mathrm{SO}(d) \ltimes \mathbb{R}^d = \mathrm{SE}(d) =: G$. Then, for $v \in W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)$ the derivative of $a(\cdot)v$ with respect to g evaluated at $g = \mathbb{1}$ is the mapping

$$d\,[a(\mathbb{1})v] : T_{\mathbb{1}}\mathrm{SE}(d) \to L^p(\mathbb{R}^d,\mathbb{K}^N), \quad \mu \mapsto d\,[a(\mathbb{1})v]\,\mu$$

given by

$$d\,[a(\mathbb{1})v(x)]\,(S,\lambda) = -\left\langle Sx + I_d\lambda, \nabla v(x)\right\rangle,$$

where $T_{\mathbb{1}}SE(d) = \mathfrak{se}(d)$ and $\mu = (S,\lambda) \in \mathfrak{so}(d) \times \mathbb{R}^d = \mathfrak{se}(d)$.

Proof. We prove the result pointwise and note that all derivatives converge in $L^p(\mathbb{R}^d, \mathbb{K}^N)$. For $v \in W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ the derivative of $a(\cdot)v$ with respect to $g = (R, \tau)$ is given by the mapping

$$d\,[a(g)v] : T_g\mathrm{SE}(d) \to L^p(\mathbb{R}^d, \mathbb{K}^N), \quad \mu \mapsto d\,[a(g)v]\,\mu$$

where $T_g\mathrm{SE}(d)$ denotes the tangential space of $\mathrm{SE}(d)$ at g. The right hand side $d\,[a(g)v]$ can be computed in a formal way as follows:

$$\begin{aligned}
&d\,[a(g)v(x)] := \frac{d}{dg}\,[a(g)v(x)] = \frac{d}{d(R,\tau)}\,[a(R,\tau)v]\,(x)\\
=&\frac{d}{d(R,\tau)}v(R^{-1}(x-\tau)) = \left[\frac{\partial}{\partial R}v(R^{-1}(x-\tau)), \frac{\partial}{\partial \tau}v(R^{-1}(x-\tau))\right]\\
=&-\left[\left(Dv(R^{-1}(x-\tau))\cdot\int_0^1 e^{(1-\alpha)X(S)}(I_{ij}-I_{ji})e^{\alpha X(S)}d\alpha\cdot(x-\tau)\right)_{\substack{i=1,\ldots,d-1\\ j=i+1,\ldots,d}},\right.\\
&\left.\left(Dv(R^{-1}(x-\tau))R^{-1}e_l\right)_{l=1,\ldots,d}\right]
\end{aligned}$$

where $S \in \mathfrak{so}(d)$ is chosen such that $\exp(S) = R$ for R given by the group element and X is defined by $X(S) := -\sum_{l=1}^{d-1}\sum_{k=l+1}^{d} S_{lk}(I_{lk} - I_{kl})$. To prove the last equality, recall the informations about the special Euclidean group $\mathrm{SE}(d)$ from Section 3. For the first term we use the definition of $X(S)$ from above and apply (3.1) to obtain

$$\begin{aligned}
&\frac{\partial}{\partial S_{ij}}v\left(\exp\left(-\sum_{l=1}^{d-1}\sum_{k=l+1}^{d} S_{lk}(I_{lk}-I_{kl})\right)(x-\tau)\right)\\
=&Dv(R^{-1}(x-\tau))\cdot\frac{\partial}{\partial S_{ij}}\left(e^{X(S)}(x-\tau)\right)\\
=&Dv(R^{-1}(x-\tau))\cdot\int_0^1 e^{(1-\alpha)X(S)}\left[\frac{\partial}{\partial S_{ij}}X(S)\right]e^{\alpha X(S)}d\alpha\cdot(x-\tau)\\
=&-Dv(R^{-1}(x-\tau))\cdot\int_0^1 e^{(1-\alpha)X(S)}(I_{ij}-I_{ji})e^{\alpha X(S)}d\alpha\cdot(x-\tau)
\end{aligned}$$

for every $i = 1, \ldots, d-1$ and $j = i+1, \ldots, d$. This yields

$$\begin{aligned}
&\frac{\partial}{\partial R}v(R^{-1}(x-\tau)) = \left(\frac{\partial}{\partial S_{ij}}v\left(\exp\left(-\sum_{l=1}^{d-1}\sum_{k=l+1}^{d} S_{lk}(I_{lk}-I_{kl})\right)(x-\tau)\right)\right)_{\substack{i=1,\ldots,d-1\\ j=i+1,\ldots,d}}\\
=&\left(-Dv(R^{-1}(x-\tau))\cdot\int_0^1 e^{(1-\alpha)X(S)}(I_{ij}-I_{ji})e^{\alpha X(S)}d\alpha\cdot(x-\tau)\right)_{\substack{i=1,\ldots,d-1\\ j=i+1,\ldots,d}}.
\end{aligned}$$

The second term is even simpler: From

$$\begin{aligned}\frac{\partial}{\partial\tau_l}v(R^{-1}(x-\tau)) =&Dv(R^{-1}(x-\tau))\cdot\frac{\partial}{\partial\tau_l}R^{-1}(x-\tau)\\ =&-Dv(R^{-1}(x-\tau))\cdot R^{-1}e_l\end{aligned}$$

for $l=1,\ldots,d$ we deduce

$$\begin{aligned}\frac{\partial}{\partial\tau}v(R^{-1}(x-\tau)) &= \left[\left(\frac{\partial}{\partial\tau_l}v(R^{-1}(x-\tau))\right)_{l=1,\ldots,d}\right]\\ &= \left[\left(-Dv(R^{-1}(x-\tau))R^{-1}e_l\right)_{l=1,\ldots,d}\right].\end{aligned}$$

The derivative $d[a(g)v]$ at $g=\mathbb{1}$ leads to the simple expression

$$\begin{aligned}d\left[a(\mathbb{1})v(x)\right] &= -\left[\left(Dv(x)(I_{ij}-I_{ji})x\right)_{\substack{i=1,\ldots,d-1\\ j=i+1,\ldots,d}},\left(Dv(x)e_l\right)_{l=1,\ldots,d}\right]\\ &= -\left[\left((x_jD_i-x_iD_j)v(x)\right)_{\substack{i=1,\ldots,d-1\\ j=i+1,\ldots,d}},\left(D_lv(x)\right)_{l=1,\ldots,d}\right].\end{aligned}$$

The unit element is $\mathbb{1}=(R,\tau)=(I_d,0)\in\mathrm{SO}(d)\ltimes\mathbb{R}^d$. To guarantee the relation $\exp(S)=R=I_d$ for some $S\in\mathfrak{so}(d)$ we choose $S=0$. Therefore, we have $X(S)=X(0)=0$, thus $e^{(1-\alpha)X(S)}=e^{\alpha X(S)}=I_d$ and the integral equals $(I_{ij}-I_{ji})$. In order to evaluate $d\left[a(\mathbb{1})v\right]$ at $\mu=(S,\lambda)\in\mathfrak{so}(d)\times\mathbb{R}^d$ we use the basis of $\mathfrak{so}(d)$ from Section 3 once more and obtain for every $(S,\lambda)\in\mathfrak{se}(d)$

$$\begin{aligned}&d\left[a(\mathbb{1})v(x)\right](S,\lambda)\\ &= -\left[\left((x_jD_i-x_iD_j)v(x)\right)_{\substack{i=1,\ldots,d-1\\ j=i+1,\ldots,d}},\left(D_lv(x)\right)_{l=1,\ldots,d}\right]\cdot\begin{pmatrix}(S_{ij})_{\substack{i=1,\ldots,d-1\\ j=i+1,\ldots,d}}\\ (\lambda_l)_{l=1,\ldots,d}\end{pmatrix}\\ &= -\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}S_{ij}\left(x_jD_i-x_iD_j\right)v(x)-\sum_{l=1}^{d}\lambda_lD_lv(x)\\ &= -\sum_{i=1}^{d}\sum_{j=1}^{d}S_{ij}x_jD_iv(x)-\sum_{l=1}^{d}\lambda_lD_lv(x) = -\sum_{i=1}^{d}\left((Sx)_i+\lambda_i\right)D_iv(x)\\ &= -\sum_{i=1}^{d}(Sx+I_d\lambda)_iD_iv(x) = -\left\langle Sx+I_d\lambda,\nabla v(x)\right\rangle.\end{aligned}$$

$\square$

The following theorem gives informations about the point spectrum and the shape of the eigenfunctions of $\mathcal{L}$. As mentioned before we are not able to determine the whole point spectrum of $\mathcal{L}$, but all eigenvalues on the imaginary axis that are due to the SE(d)-action and their associated eigenfunctions. The procedure for the proof is already well known in the literature, for instance for traveling waves and spiral waves, and is based on the following algebraical observation: Considering the nonlinear equation (1.19), one applies the group action $a(g)$ to both sides and takes the derivative with respect to g at $g = \mathbb{1}$. This leads to a number of $\frac{d(d+1)}{2}$ equations, which equals the dimension of SE(d).

Theorem 9.4 (Point spectrum of $\mathcal{L}$ on the imaginary axis)**.** *Let $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ be a classical solution of* (1.19)*. Moreover, let $U \in \mathbb{C}^{d,d}$ be the unitary matrix from* (3.2)*, then the function $v : \mathbb{R}^d \to \mathbb{C}^N$ given by*

$$\begin{aligned} v(x) &= \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} C_{ij}^{\mathrm{rot}}(x_j D_i - x_i D_j) v_\star(x) + \sum_{l=1}^{d} C_l^{\mathrm{tra}} D_l v_\star(x) \\ &= -d\left[a(\mathbb{1})v_\star(x)\right] \begin{pmatrix} C^{\mathrm{rot}} \\ C^{\mathrm{tra}} \end{pmatrix} = \left\langle C^{\mathrm{rot}} x + I_d C^{\mathrm{tra}}, \nabla v_\star(x) \right\rangle \end{aligned}$$

is a classical solution of the eigenvalue problem $\mathcal{L}v = \lambda v$ for every $C^{\mathrm{rot}} \in \mathbb{C}^{d,d}$ and $C^{\mathrm{tra}} \in \mathbb{C}^d$ satisfying

$$(\lambda, (C^{\mathrm{rot}}, C^{\mathrm{tra}})) = (-\lambda_l^S, (0, Ue_l)),$$

for some $l = 1, \ldots, d$, or

$$(\lambda, (C^{\mathrm{rot}}, C^{\mathrm{tra}})) = (-(\lambda_n^S + \lambda_m^S), (U(I_{nm} - I_{mn})U^T, 0)),$$

for some $n = 1, \ldots, d-1$ and $m = n+1, \ldots, d$. Thus, $v \in C^2(\mathbb{R}^d, \mathbb{C}^N)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda \in i\mathbb{R}$.

Remark. Later on we will show that the eigenfunctions v decay exponentially, see Theorem 9.8. Moreover, it is possible to deduce from Theorem 7.7 that the eigenvalues λ are actually in the L^p-point spectrum of $\mathcal{L}$, i.e. $\lambda \in \sigma_{\mathrm{point}}(\mathcal{L})$, meaning that they are isolated and have finite multiplicity. This will be proved elsewhere by using Fredholm theory.

Proof. 1. Let $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ be a classical solution of (1.19), i.e. $v_\star$ satisfies the nonlinear problem

$$0 = A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)),\ x \in \mathbb{R}^d.$$

Applying the group action $a(g)$ from Lemma 9.3 on both hand sides yields

$$0 = a(g)\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right),\ x \in \mathbb{R}^d.$$

Taking the derivative $\frac{d}{dg}$ at $g = \mathbb{1}$, we obtain by Lemma 9.3

$$\begin{aligned} 0 &= d\left[a(\mathbb{1})\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)\right] \\ &= -\left[\left((x_j D_i - x_i D_j)\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)\right)_{\substack{i=1,\ldots,d-1 \\ j=i+1,\ldots,d}},\right. \end{aligned}$$

$$\left.\left(D_l\left(A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))\right)\right)_{l=1,\ldots,d}\right].$$

This leads to a total of $\frac{d(d+1)}{2}$ equations

$$0=(x_jD_i-x_iD_j)\left(A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))\right), \tag{9.4}$$
$$0=D_l\left(A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))\right). \tag{9.5}$$

for $i=1,\ldots,d-1$, $j=i+1,\ldots,d$ and $l=1,\ldots,d$, where $\frac{d(d+1)}{2}=\dim\mathrm{SE}(d)$.
2. We apply the differential expressions $(x_jD_i-x_iD_j)$ and D_l to the nonlinear equation and transfer them directly in front of $v_\star(x)$. For this purpose we have to investigate commutator relations between this two terms and the differential expressions. In order to transform (9.5), we observe that

$$\bullet D_l\left(A\triangle v_\star(x)\right)=A\triangle D_lv_\star(x),$$
$$\bullet D_l\left(f(v_\star(x))\right)=Df(v_\star(x))D_lv_\star(x),$$
$$\bullet D_l\left\langle Sx,\nabla v_\star(x)\right\rangle=D_l\sum_{i=1}^d(Sx)_i\,D_iv_\star(x)=\sum_{i=1}^d\left[(Sx)_iD_l+(Se_l)_i\right]D_iv_\star(x)$$
$$=\sum_{i=1}^d(Sx)_iD_iD_lv_\star(x)+\sum_{i=1}^d(Se_l)_i\,D_iv_\star(x)=\langle Sx,\nabla D_lv_\star(x)\rangle+\langle Se_l,\nabla v_\star(x)\rangle,$$

for every $l=1,\ldots,d$, where $Se_l=S_{\cdot l}$. Similarly, for equation (9.4) we obtain

$$\begin{aligned}
&\bullet A\triangle\left((x_jD_i-x_iD_j)v_\star(x)\right)=A\sum_{k=1}^dD_k^2\left((x_jD_i-x_iD_j)v_\star(x)\right)\\
&=A\sum_{k=1}^d\left(2\delta_{kj}D_iD_k+x_jD_iD_k^2-2\delta_{ki}D_jD_k-x_iD_jD_k^2\right)v_\star(x)\\
&=A(x_jD_i-x_iD_j)\triangle v_\star(x)+2D_iD_jv_\star(x)-2D_jD_iv_\star(x)\\
&=(x_jD_i-x_iD_j)A\triangle v_\star(x),\\
&\bullet(x_jD_i-x_iD_j)f(v_\star(x))=Df(v_\star(x))(x_jD_i-x_iD_j)v_\star(x),
\end{aligned}$$

for every $i=1,\ldots,d-1$ and $j=i+1,\ldots,d$. The transformation of the rotational term $\langle Sx,\nabla v_\star(x)\rangle$ is much more involved: Using

$$x_kD_mD_lv_\star(x)=D_m(x_kD_lv_\star(x))-\delta_{mk}D_lv_\star(x)$$

and

$$\begin{aligned}
&x_k\left\langle Sx,\nabla D_lv_\star(x)\right\rangle=\sum_{m=1}^d(Sx)_m\,x_kD_mD_lv_\star(x)\\
&=\sum_{m=1}^d(Sx)_m\,D_m(x_kD_lv_\star(x))-\sum_{m=1}^d(Sx)_k\delta_{mk}D_lv_\star(x)\\
&=\langle Sx,\nabla(x_kD_lv_\star(x))\rangle-(Sx)_kD_lv_\star(x),
\end{aligned}$$

we deduce

$$
\begin{aligned}
&(x_jD_i - x_iD_j)\left\langle Sx, \nabla v_\star(x)\right\rangle \\
=&x_j\left(\left\langle Sx, \nabla D_i v_\star(x)\right\rangle + \left\langle Se_i, \nabla v_\star(x)\right\rangle\right) - x_i\left(\left\langle Sx, \nabla D_j v_\star(x)\right\rangle + \left\langle Se_j, \nabla v_\star(x)\right\rangle\right) \\
=&\left\langle Sx, \nabla((x_jD_i - x_iD_j)v_\star(x))\right\rangle + x_j\left\langle Se_i, \nabla v_\star(x)\right\rangle - x_i\left\langle Se_j, \nabla v_\star(x)\right\rangle \\
&- (Sx)_j D_i v_\star(x) + (Sx)_i D_j v_\star(x).
\end{aligned}
$$

We now simplify the remaining four terms in the last equation. Using

$$
\begin{aligned}
&x_j\left\langle Se_i, \nabla v_\star(x)\right\rangle + (Sx)_i D_j v_\star(x) = x_j\sum_{n=1}^{d} S_{ni}D_n v_\star(x) + \sum_{n=1}^{d} S_{in}x_n D_j v_\star(x) \\
=&-\sum_{\substack{n=1\\n\neq j}}^{d} S_{in}x_jD_nv_\star(x) + \sum_{\substack{n=1\\n\neq j}}^{d} S_{in}x_nD_jv_\star(x) - S_{ij}x_jD_jv_\star(x) + S_{ij}x_jD_jv_\star(x) \\
=&\sum_{\substack{n=1\\n\neq j}}^{d} S_{in}\left(x_nD_j - x_jD_n\right)v_\star(x),
\end{aligned}
$$

and analogously

$$
-\left[x_i\left\langle Se_j, \nabla v_\star(x)\right\rangle + (Sx)_j D_i v_\star(x)\right] = \sum_{\substack{n=1\\n\neq i}}^{d} S_{jn}\left(x_nD_i - x_iD_n\right)v_\star(x)
$$

as well as

$$
\left\langle Se_l, \nabla v_\star(x)\right\rangle = \sum_{n=1}^{d} S_{nl}D_n v_\star(x) = -\sum_{n=1}^{d} S_{ln}D_n v_\star(x)
$$

and taking all into account, (9.4) and (9.5) can be rewritten as

$$
\begin{aligned}
(9.6)\qquad 0 =&\mathcal{L}\left((x_jD_i - x_iD_j)v_\star(x)\right) + \sum_{\substack{n=1\\n\neq j}}^{d} S_{in}(x_nD_j - x_jD_n)v_\star(x) \\
&- \sum_{\substack{n=1\\n\neq i}}^{d} S_{jn}(x_nD_i - x_iD_n)v_\star(x),
\end{aligned}
$$

$$
(9.7)\qquad 0 =\mathcal{L}\left(D_l v_\star(x)\right) - \sum_{n=1}^{d} S_{ln}D_n v_\star(x).
$$

for $i = 1, \ldots, d-1$, $j = i+1, \ldots, d$ and $l = 1, \ldots, d$.
3. We now reduce $\mathcal{L}v = \lambda v$ to a finite dimensional eigenvalue problem. For this purpose, we put the ansatz

$$
v(x) := \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} C_{ij}^{\mathrm{rot}}(x_jD_i - x_iD_j)v_\star(x) + \sum_{l=1}^{d} C_l^{\mathrm{tra}} D_l v_\star(x),\ C_{ij}^{\mathrm{rot}}, C_l^{\mathrm{tra}} \in \mathbb{C}
$$

into the original eigenvalue problem $\mathcal{L}v = \lambda v$ and transform the equation in such a way, that we are able to require equality for every summand.

$$\begin{aligned}
&\sum_{i=1}^{d-1}\sum_{j=i+1}^{d} \lambda C_{ij}^{\mathrm{rot}}(x_jD_i - x_iD_j)v_\star(x) + \sum_{l=1}^{d} \lambda C_l^{\mathrm{tra}} D_l v_\star(x) = \lambda v(x) = \mathcal{L}v(x)\\
=&\sum_{i=1}^{d-1}\sum_{j=i+1}^{d} C_{ij}^{\mathrm{rot}} \mathcal{L}\left((x_jD_i - x_iD_j)v_\star(x)\right) + \sum_{l=1}^{d} C_l^{\mathrm{tra}} \mathcal{L}\left(D_l v_\star(x)\right)\\
=&-\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\sum_{\substack{n=1\\n\neq j}}^{d} C_{ij}^{\mathrm{rot}} S_{in}(x_nD_j - x_jD_n)v_\star(x)\\
&+\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\sum_{\substack{n=1\\n\neq i}}^{d} C_{ij}^{\mathrm{rot}} S_{jn}(x_nD_i - x_iD_n)v_\star(x) + \sum_{l=1}^{d}\sum_{n=1}^{d} C_l^{\mathrm{tra}} S_{ln} D_n v_\star(x)
\end{aligned}$$

We next modify each of these three terms. For this purpose we use the skew-symmetry of $S \in \mathbb{R}^{d,d}$ and the abbreviation $D_{(i,j)}^x := x_jD_i - x_iD_j$. The first term can be simplified by

$$\begin{aligned}
&-\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\sum_{\substack{n=1\\n\neq j}}^{d} C_{ij}^{\mathrm{rot}} S_{in}(x_nD_j - x_jD_n)v_\star(x)\\
=&-\sum_{n=1}^{d-1}\sum_{i=n+1}^{d}\sum_{\substack{j=1\\j\neq i}}^{d} C_{ni}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x) = -\sum_{i=2}^{d}\sum_{\substack{j=1\\j\neq i}}^{d}\sum_{n=1}^{i-1} C_{ni}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x)\\
=&-\sum_{i=2}^{d}\sum_{j=i+1}^{d}\sum_{n=1}^{i-1} C_{ni}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x) - \sum_{i=2}^{d}\sum_{j=1}^{i-1}\sum_{n=1}^{i-1} C_{ni}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x)\\
=&-\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\sum_{n=1}^{i-1} C_{ni}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x) - \sum_{j=1}^{d-1}\sum_{i=j+1}^{d}\sum_{n=1}^{i-1} C_{ni}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x)\\
=&-\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\sum_{n=1}^{i-1} C_{ni}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x) + \sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\sum_{n=1}^{j-1} C_{nj}^{\mathrm{rot}} S_{ni} D_{(i,j)}^x v_\star(x)\\
=&\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\left[-\sum_{n=1}^{j-1} S_{in}C_{nj}^{\mathrm{rot}} + \sum_{n=1}^{i-1} S_{jn}C_{ni}^{\mathrm{rot}}\right](x_jD_i - x_iD_j)v_\star(x),
\end{aligned}$$

the second term by

$$\begin{aligned}
&\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\sum_{\substack{n=1\\n\neq i}}^{d} C_{ij}^{\mathrm{rot}} S_{jn}(x_nD_i - x_iD_n)v_\star(x)\\
=&\sum_{i=1}^{d-1}\sum_{n=i+1}^{d}\sum_{\substack{j=1\\j\neq i}}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x) = \sum_{i=1}^{d-1}\sum_{\substack{j=1\\j\neq i}}^{d}\sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^x v_\star(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^{x} v_\star(x) + \sum_{i=1}^{d-1} \sum_{j=1}^{i-1} \sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^{x} v_\star(x) \\
&= \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^{x} v_\star(x) + \sum_{j=1}^{d-2} \sum_{i=j+1}^{d-1} \sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^{x} v_\star(x) \\
&= \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^{x} v_\star(x) + \sum_{j=1}^{d-1} \sum_{i=j+1}^{d} \sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^{x} v_\star(x) \\
&= \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj} D_{(i,j)}^{x} v_\star(x) - \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \sum_{n=j+1}^{d} C_{jn}^{\mathrm{rot}} S_{ni} D_{(i,j)}^{x} v_\star(x) \\
&= \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \left[- \sum_{n=i+1}^{d} S_{jn} C_{in}^{\mathrm{rot}} + \sum_{n=j+1}^{d} S_{in} C_{jn}^{\mathrm{rot}} \right] (x_j D_i - x_i D_j) v_\star(x)
\end{aligned}
$$

and the third term by

$$
\sum_{l=1}^{d} \sum_{n=1}^{d} C_{l}^{\mathrm{tra}} S_{ln} D_n v_\star(x) = \sum_{n=1}^{d} \sum_{l=1}^{d} C_{n}^{\mathrm{tra}} S_{nl} D_l v_\star(x) = \sum_{l=1}^{d} \left[- \sum_{n=1}^{d} S_{ln} C_{n}^{\mathrm{tra}} \right] D_l v_\star(x).
$$

Thus, we conclude

$$
\begin{aligned}
&\sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \lambda C_{ij}^{\mathrm{rot}} (x_j D_i - x_i D_j) v_\star(x) + \sum_{l=1}^{d} \lambda C_{l}^{\mathrm{tra}} D_l v_\star(x) \\
=& \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \left[- \sum_{n=1}^{j-1} S_{in} C_{nj}^{\mathrm{rot}} + \sum_{n=j+1}^{d} S_{in} C_{jn}^{\mathrm{rot}} + \sum_{n=1}^{i-1} S_{jn} C_{ni}^{\mathrm{rot}} - \sum_{n=i+1}^{d} S_{jn} C_{in}^{\mathrm{rot}} \right] \\
&\cdot (x_j D_i - x_i D_j) v_\star(x) + \sum_{l=1}^{d} \left[- \sum_{n=1}^{d} S_{ln} C_{n}^{\mathrm{tra}} \right] D_l v_\star(x).
\end{aligned}
$$

Requiring equality of summands yields for $i = 1, \ldots, d-1$, $j = i+1, \ldots, d$ and $l = 1, \ldots d$

$$
\lambda C_{ij}^{\mathrm{rot}} = \sum_{n=1}^{j-1} C_{nj}^{\mathrm{rot}} S_{ni} - \sum_{n=j+1}^{d} C_{jn}^{\mathrm{rot}} S_{ni} - \sum_{n=1}^{i-1} C_{ni}^{\mathrm{rot}} S_{nj} + \sum_{n=i+1}^{d} C_{in}^{\mathrm{rot}} S_{nj}, \tag{9.8}
$$

$$
\lambda C_{l}^{\mathrm{tra}} = - \sum_{n=1}^{d} C_{n}^{\mathrm{tra}} S_{ln}. \tag{9.9}
$$

In order to determine the solutions $(\lambda, (C^{\mathrm{rot}}, C^{\mathrm{tra}}))$ of (9.8)–(9.9) we postulate $C_{ij}^{\mathrm{rot}} = -C_{ji}^{\mathrm{rot}}$ for every $i, j = 1, \ldots, d$, i.e. in the following we consider C^{rot} as a (complex-valued) matrix $C^{\mathrm{rot}} \in \mathbb{C}^{d,d}$ that is assumed to be skew-symmetric. Note that $C_{ii}^{\mathrm{rot}} = 0$ by equation (9.8). Thus, the equation (9.8)–(9.9) leads to a $\frac{d(d+1)}{2}$-dimensional eigenvalue problem

$$\lambda C^{\mathrm{rot}} = -SC^{\mathrm{rot}} + \left(SC^{\mathrm{rot}}\right)^T, \tag{9.10}$$

$$\lambda C^{\mathrm{tra}} = -SC^{\mathrm{tra}}. \tag{9.11}$$

Since (9.10)–(9.11) arise in block diagonal form, it is sufficient to solve the eigenvalue problems separatly, i.e. $(\lambda, C^{\mathrm{rot}})$ solves (9.10) if and only if $(\lambda, (C^{\mathrm{rot}}, 0))$ solves (9.8)–(9.9). Similarly, $(\lambda, C^{\mathrm{tra}})$ solves (9.11) if and only if $(\lambda, (0, C^{\mathrm{tra}}))$ solves (9.8)–(9.9). Equation (9.10) is a matrix eigenvalue problem that admits exactly $\frac{d(d-1)}{2}$ solutions. This number equals the dimension $\dim \mathrm{SO}(d)$. The eigenvalue problem (9.11) has exactly d solutions. Altogether, we have $\frac{d(d+1)}{2}$ solutions, which coincide with the dimension of $\mathrm{SE}(d)$.
4. We start to solve (9.11): From (3.2) we deduce $S = U\Lambda_S\bar{U}^T$ with a unitary matrix $U \in \mathbb{C}^{d,d}$, $\Lambda_S = \mathrm{diag}(\lambda_1^S, \ldots, \lambda_d^S)$ and $\lambda_i \in \sigma(S)$ pairwise different. Multiplying (9.11) from left by $\bar{U}^T$ and defining $w := \bar{U}^T C^{\mathrm{tra}} \in \mathbb{C}^d$ we obtain

$$\lambda w = \lambda \bar{U}^T C^{\mathrm{tra}} = -\bar{U}^T S C^{\mathrm{tra}} = -\bar{U}^T U \Lambda_S \bar{U}^T C^{\mathrm{tra}} = -\Lambda_S w. \tag{9.12}$$

The solutions of (9.12) are given by $(\lambda, w) = (-\lambda_l^S, e_l)$ for $l = 1, \ldots, d$. Thus, the transformation $C^{\mathrm{tra}} = Uw$ yields the solutions $(\lambda, C^{\mathrm{tra}}) = (-\lambda_l^S, Ue_l)$ of (9.11) for $l = 1, \ldots, d$. Consequently, $(\lambda, (C^{\mathrm{rot}}, C^{\mathrm{tra}})) = (-\lambda_l, (0, Ue_l))$ for $l = 1, \ldots, d$ are d solutions of (9.8)–(9.9).
5. Finally, we solve (9.10): We use the decomposition $S = U\Lambda_S\bar{U}^T$ once more. Multiplying (9.10) from left by $\bar{U}^T$, from right by $\bar{U}$ and defining $W := \bar{U}^T C^{\mathrm{rot}} \bar{U} \in \mathbb{C}^{d,d}$ we obtain

$$\begin{aligned} \lambda W &= \lambda \bar{U}^T C^{\mathrm{rot}} \bar{U} = -\bar{U}^T S C^{\mathrm{rot}} \bar{U} + \bar{U}^T \left(SC^{\mathrm{rot}}\right)^T \bar{U} \\ &= -\bar{U}^T \bar{U} \Lambda_S \bar{U}^T C^{\mathrm{rot}} \bar{U} + \left(\bar{U}^T U \Lambda_S \bar{U}^T C^{\mathrm{rot}} \bar{U}\right)^T = -\Lambda_S W + W^T \Lambda_S. \end{aligned} \tag{9.13}$$

The solutions of (9.13) are given by $(\lambda, W) = (-(\lambda_n^S + \lambda_m^S), I_{nm} - I_{mn})$ for $n = 1, \ldots, d-1$ and $m = n+1, \ldots, d$. Thus, the transformation $C^{\mathrm{rot}} = UWU^T$ yields the solutions $(\lambda, C^{\mathrm{rot}}) = (-(\lambda_n^S + \lambda_m^S), U(I_{nm} - I_{mn})U^T)$ of (9.10) for $n = 1, \ldots, d-1$ and $m = n+1, \ldots, d$, where $C^{\mathrm{rot}} = U(I_{nm} - I_{mn})U^T \in \mathbb{C}^{d,d}$ is of course skew-symmetric. Consequently, $(\lambda, (C^{\mathrm{rot}}, C^{\mathrm{tra}})) = (-(\lambda_n^S + \lambda_m^S), (U(I_{nm} - I_{mn})U^T, 0))$ for $n = 1, \ldots, d-1$ and $m = n+1, \ldots, d$ are $\frac{d(d-1)}{2}$ solutions of (9.8)–(9.9). $\square$

The previous Theorem 9.4 and the subsequent remark prove that the point spectrum of the linearization $\mathcal{L}$ contains the spectrum of S and the direct sum of its different eigenvalues. We summarize this result in the following:

Corollary 9.5. *Let the assumptions of Corollary 9.9 be satisfied, then the inclusion*

$$\sigma_{\mathrm{point}}^{\mathrm{part}}(\mathcal{L}) := \sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S),\ \lambda_1 \neq \lambda_2\} \subseteq \sigma_{\mathrm{point}}(\mathcal{L})$$

holds for the L^p-spectrum of $\mathcal{L}$, in particular, $0 \in \sigma_{\mathrm{point}}(\mathcal{L})$.

Rotational term. If $v_\star \in C^3(\mathbb{R}^d,\mathbb{R}^N)$, then for space dimensions $d \geqslant 2$

$$v(x) = \langle Sx, \nabla v_\star(x)\rangle \in C^2(\mathbb{R}^d,\mathbb{R}^N)$$

is a classical solution of $\mathcal{L}v = 0$, i.e. v is an eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda = 0$. This can be shown directly and follows also from Theorem 9.4 with $\lambda = 0$, $C^{\mathrm{tra}} = 0 \in \mathbb{R}^d$, $C^{\mathrm{rot}} = S \in \mathbb{R}^{d,d}$. This was also observed in [15] for $d = 2$.

Multiplicities of isolated eigenvalues. Theorem 9.4 gives also information about the multiplicity of the isolated eigenvalues of $\mathcal{L}$. More precisely, for any fixed skew-symmetric $S \in \mathbb{R}^{d,d}$, Theorem 9.4 yields an lower bound for the multiplicities. But note that multiplicities depends on $S \in \mathbb{R}^{d,d}$, i.e. varying $S \in \mathbb{R}^{d,d}$ leads to different eigenvalues with multiplicities.

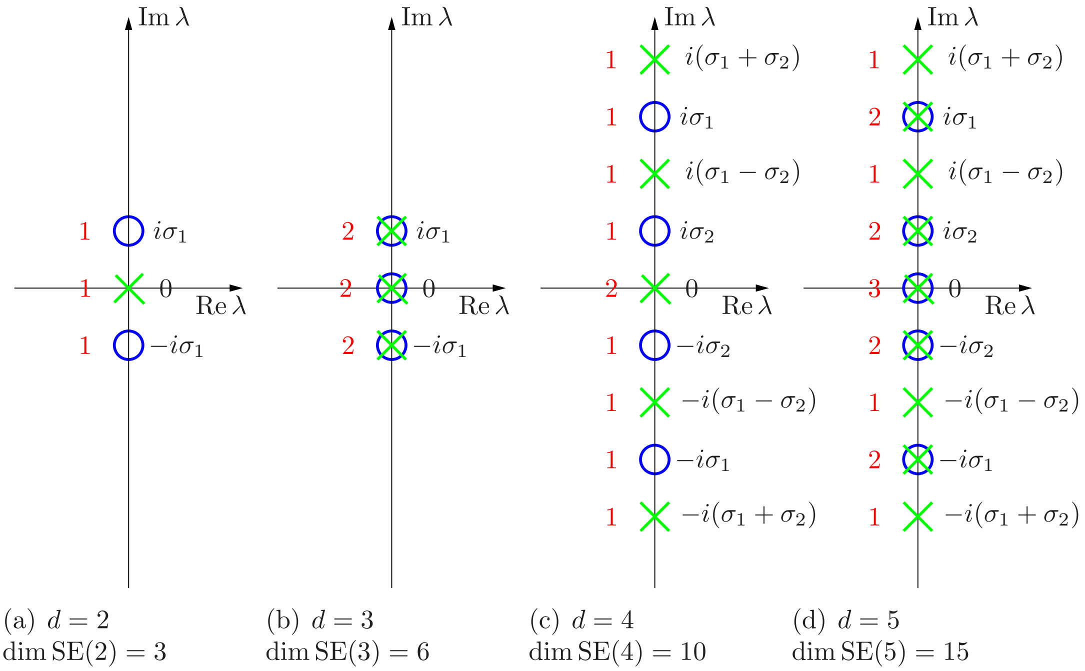


Figure 9.1: Point spectrum of the linearization $\mathcal{L}$ on the imaginary axis $i\mathbb{R}$ for space dimension $d = 2, 3, 4, 5$ given by Theorem 9.4.

Figure 9.1 shows the eigenvalues $\lambda \in \sigma_{\mathrm{point}}(\mathcal{L})$ from Corollary 9.5 and their corresponding multiplicities for the different space dimensions $d = 2, 3, 4, 5$. The eigenvalues $\lambda \in \sigma(S)$ are illustrated by the blue circles, the eigenvalues $\lambda \in \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S),\ \lambda_1 \neq \lambda_2\}$ are illustrated by the green crosses. The imaginary values to the right of the symbols denote the precise values of eigenvalues and the numbers to the left their corresponding multiplicities. We observe that for space dimension d there are $\dim \mathrm{SE}(d) = \frac{d(d+1)}{2}$ eigenvalues on the imaginary axis.

Example 9.6 (Point spectrum of $\mathcal{L}$ for $d = 2$)**.** In the two dimensional case the skew-symmetric matrix $S \in \mathbb{R}^{2,2}$ has the form

$$S = \begin{pmatrix} 0 & S_{12} \\ -S_{12} & 0 \end{pmatrix},$$

where $\sigma_1 = S_{12}$ and $k = 1$. The matrix S is unitary diagonalizable with diagonal matrix $\Lambda_S \in \mathbb{C}^{2,2}$ and unitary matrix $U \in \mathbb{C}^{2,2}$ given by

$$\Lambda_S = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \; U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

where $\lambda_1^S = i\sigma_1$ and $\lambda_2^S = -i\sigma_1$. Note that the j-th column of the matrix U contains the normalized eigenvector $v_j^S := Ue_j \in \mathbb{C}^2$ for the j-th eigenvalue λ_j^S. Using $U(I_{12} - I_{21})U^T = -i(I_{12} - I_{21})$, Theorem 9.4 yields the following eigenfunctions, compare also [15, Lemma 2.3],

$$\begin{aligned}
\lambda_1 &= i\sigma_1, & v_1(x) &= D_1 v_\star(x) + iD_2 v_\star(x), \\
\lambda_2 &= -i\sigma_1, & v_2(x) &= D_1 v_\star(x) - iD_2 v_\star(x), \\
\lambda_3 &= 0, & v_3(x) &= (x_2 D_1 - x_1 D_2) v_\star(x).
\end{aligned}$$

Example 9.7 (Point spectrum of $\mathcal{L}$ for $d = 3$)**.** In the three dimensional case the skew symmetric matrix $S \in \mathbb{R}^{3,3}$ has the form

$$S = \begin{pmatrix} 0 & S_{12} & S_{13} \\ -S_{12} & 0 & S_{23} \\ -S_{13} & -S_{23} & 0 \end{pmatrix},$$

where $\sigma_1 = \sqrt{S_{12}^2 + S_{13}^2 + S_{23}^2}$ and $k = 1$. The matrix S is unitary diagonalizable with diagonal matrix $\Lambda_S \in \mathbb{C}^{3,3}$ and unitary matrix $U \in \mathbb{C}^{3,3}$ given by

$$\Lambda_S = \begin{pmatrix} i\sigma_1 & 0 & 0 \\ 0 & -i\sigma_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; U = \begin{pmatrix} \frac{\sigma_1 S_{13} - iS_{12}S_{23}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & \frac{\sigma_1 S_{13} + iS_{12}S_{23}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & \frac{S_{23}}{\sigma_1} \\ \frac{\sigma_1 S_{23} + iS_{12}S_{13}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & \frac{\sigma_1 S_{23} - iS_{12}S_{13}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & -\frac{S_{13}}{\sigma_1} \\ \frac{i(S_{13}^2+S_{23}^2)}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & \frac{-i(S_{13}^2+S_{23}^2)}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & \frac{S_{12}}{\sigma_1} \end{pmatrix},$$

where $\lambda_1^S = i\sigma_1$, $\lambda_2^S = -i\sigma_1$ and $\lambda_3^S = 0$. Note once more that the j-th column of the matrix U contains the normalized eigenvector $v_j^S := Ue_j \in \mathbb{C}^3$ for the j-th eigenvalue λ_j^S. Using

$$\begin{aligned}
U(I_{12} - I_{21})U^T &= \frac{i}{\sigma_1} S, \\
U(I_{13} - I_{31})U^T &= \begin{pmatrix} 0 & -\frac{\sqrt{2(S_{13}^2+S_{23}^2)}}{2\sigma_1} & \frac{S_{12}S_{13}+i\sigma_1 S_{23}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} \\ \frac{\sqrt{2(S_{13}^2+S_{23}^2)}}{2\sigma_1} & 0 & -\frac{-S_{12}S_{23}+i\sigma_1 S_{13}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} \\ -\frac{S_{12}S_{13}+i\sigma_1 S_{23}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & \frac{-S_{12}S_{23}+i\sigma_1 S_{13}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & 0 \end{pmatrix}, \\
U(I_{23} - I_{32})U^T &= \begin{pmatrix} 0 & -\frac{\sqrt{2(S_{13}^2+S_{23}^2)}}{2\sigma_1} & -\frac{-S_{12}S_{13}+i\sigma_1 S_{23}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} \\ \frac{\sqrt{2(S_{13}^2+S_{23}^2)}}{2\sigma_1} & 0 & \frac{S_{12}S_{23}+i\sigma_1 S_{13}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} \\ \frac{-S_{12}S_{13}+i\sigma_1 S_{23}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & -\frac{S_{12}S_{23}+i\sigma_1 S_{13}}{\sigma_1\sqrt{2(S_{13}^2+S_{23}^2)}} & 0 \end{pmatrix},
\end{aligned}$$

Theorem 9.4 yields the following eigenfunctions

$$
\begin{aligned}
\lambda_1 &= i\sigma_1, & v_1(x) &= (\sigma_1 S_{13} - iS_{12}S_{23})D_1 v_\star(x) + (\sigma_1 S_{23} + iS_{12}S_{13})D_2 v_\star(x)\\
&& &\quad + i(S_{13}^2 + S_{23}^2)D_3 v_\star(x),\\
\lambda_2 &= -i\sigma_1, & v_2(x) &= (\sigma_1 S_{13} + iS_{12}S_{23})D_1 v_\star(x) + (\sigma_1 S_{23} - iS_{12}S_{13})D_2 v_\star(x)\\
&& &\quad - i(S_{13}^2 + S_{23}^2)D_3 v_\star(x),\\
\lambda_3 &= 0, & v_3(x) &= S_{23}D_1 v_\star(x) - S_{13}D_2 v_\star(x) + S_{12}D_3 v_\star(x),\\
\lambda_4 &= 0, & v_4(x) &= S_{12}(x_2 D_1 - x_1 D_2)v_\star(x) + S_{13}(x_3 D_1 - x_1 D_3)v_\star(x)\\
&& &\quad + S_{23}(x_3 D_2 - x_2 D_3)v_\star(x),\\
\lambda_5 &= -i\sigma_1, & v_5(x) &= -(S_{13}^2 + S_{23}^2)(x_2 D_1 - x_1 D_2)v_\star(x)\\
&& &\quad + (S_{12}S_{13} + i\sigma_1 S_{23})(x_3 D_1 - x_1 D_3)v_\star(x)\\
&& &\quad + (S_{12}S_{23} - i\sigma_1 S_{13})(x_3 D_2 - x_2 D_3)v_\star(x),\\
\lambda_6 &= i\sigma_1, & v_6(x) &= -(S_{13}^2 + S_{23}^2)(x_2 D_1 - x_1 D_2)v_\star(x)\\
&& &\quad + (S_{12}S_{13} - i\sigma_1 S_{23})(x_3 D_1 - x_1 D_3)v_\star(x)\\
&& &\quad + (S_{12}S_{23} + i\sigma_1 S_{13})(x_3 D_2 - x_2 D_3)v_\star(x).
\end{aligned}
$$

9.3 Exponential decay of eigenfunctions and of the rotational term

The following theorem proves that eigenfunctions of $\mathcal{L}$ decay exponentially in space, whenever their associated (isolated) eigenvalues are sufficiently close to the imaginary axis.

Theorem 9.8 (Exponential decay of eigenfunctions)**.** *Let the assumptions* (A4)–(A9) *be satisfied for* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{R}$. *Then for every* $0 < \vartheta < 1$ *and for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d, \mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$ *with*

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$$

and $a_{\max}$, a_0, b_0 *from* (1.18), *there exists a constant* $K_1 = K_1(A, f, v_\infty, d, p, \theta, \vartheta) > 0$ *with the following property:*
Given a classical solution $v_\star$ *of* (1.19) *such that* $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ *and* (1.20) *hold. Then every classical solution* $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *of the eigenvalue problem*

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\ x \in \mathbb{R}^d, \tag{9.14}$$

with $\lambda \in \mathbb{C}$ *and* $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ *satisfies*

$$v \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{C}^N).$$

Proof. Let $0 < \vartheta < 1$ be fixed, $1 < p < \infty$ and $\theta \in C(\mathbb{R}^d, \mathbb{R})$ be a radially nondecreasing weight function of exponential growth rate $\eta \geqslant 0$ with

$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$, where $a_{\max}$, a_0 and b_0 are from (1.18).
1. Let v denote a classical solution of (9.1) satisfying $v \in L^p(\mathbb{R}^d, \mathbb{R}^N)$, then

$$\begin{aligned} 0 &= \lambda v(x) - [\mathcal{L}v](x) = \lambda v(x) - (A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x) + Q(x)v(x)) \\ &= (\lambda I - \mathcal{L}_Q)\, v(x), \end{aligned}$$

where $B := -Df(v_\infty)$ and $Q(x) := Df(v_\star(x)) - Df(v_\infty)$.
2. In order to apply Theorem 7.7 (with $C_p = \mathcal{L}_Q$) we have to verify, that the assumptions are satisfied. Note that the application of Theorem 7.7 with $C_p = \mathcal{L}_Q$ requires additionally that the assumptions (A3) and (A4) are fulfilled, which are necessary to solve the identification problem for C_p. Let us check the assumptions: Assumption (A1) follows from (A8). The assumption (A4) and (A5) are directly satisfied and assumption (A4) implies (A3) and (A2). Using the definition of B, the assumptions (A8$_B$) and (A9$_B$) follow from (A8) and (A9), respectively. It remains to verify $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$, (7.5), $v \in \mathcal{D}(A_p)$ and $\mathcal{L}_Q v = \lambda v$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$.
3. First we show that $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$: Since $v_\star - v_\infty \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^N)$ we obtain

$$|v_\star(x) - v_\infty| \leqslant \|v_\star - v_\infty\|_\infty =: R_1$$

for every $x \in \mathbb{R}^d$. Using (A6) this implies

$$|Q(x)|_2 = |Df(v_\star(x)) - Df(v_\infty)|_2 \leqslant \sup_{z \in B_{R_1}(0)} |Df(z)|_2 + |Df(v_\infty)|_2\,,$$

for every $x \in \mathbb{R}^d$, which is of course finite by the continuity of Df on compact sets. Taking the suprema over $x \in \mathbb{R}^d$ we obtain $Q \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$, thus $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$.
4. We next verify (7.5): Let us choose $K_1 = K_1(A, f, v_\infty, d, p, \theta, \vartheta) > 0$ (as in Theorem 1.8) such that

$$K_1 \left(\sup_{z \in B_{K_1}(v_\infty)} \left|D^2 f(z)\right|_2 \right) \leqslant \frac{b_0}{3} \min\left\{ \frac{1}{C_7}, \frac{1}{M_\infty} \right\}$$

is satisfied, where $C_7 = C_7(A, d, p, \theta, \vartheta)$ is from Lemma 4.8, $M_\infty = M_\infty(A, d)$ from (6.10), $b_0 = b_0(f, v_\infty)$ from (1.18). The fundamental theorem of calculus, (A6), (1.20) and the choice of K_1 yield

$$\begin{aligned} &|Q(x)|_2 = |Df(v_\star(x)) - Df(v_\infty)|_2 \\ \leqslant &\int_0^1 \left|D^2 f(v_\infty + s(v_\star(x) - v_\infty))\right|_2 ds\, |v_\star(x) - v_\infty| \\ \leqslant &\int_0^1 \left(\sup_{|x| \geqslant R_0} \left|D^2 f(v_\infty + s(v_\star(x) - v_\infty))\right|_2 \right) ds \left(\sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \right) \\ \leqslant & K_1 \left(\sup_{z \in B_{K_1}(v_\infty)} \left|D^2 f(z)\right|_2 \right) \leqslant \frac{b_0}{3} \min\left\{ \frac{1}{C_7}, \frac{1}{M_\infty} \right\} \end{aligned}$$

for every $|x| \geqslant R_0$. Taking the suprema over $|x| \geqslant R_0$ yields

$$\sup_{|x| \geqslant R_0} |Q(x)|_2 \leqslant \frac{b_0}{3} \min\left\{ \frac{1}{C_7}, \frac{1}{M_\infty} \right\}.$$

5. Next we verify that $v \in \mathcal{D}(A_p)$: An application of Theorem 5.19 shows that $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. Therefore, it suffices to show that $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$, $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N)$ and $\mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$. By assumption we know that $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$. Moreover, we deduce $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N)$ from $v \in C^2(\mathbb{R}^d, \mathbb{C}^N)$. It remains to prove that $\mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$: From (A6) we deduce that Df is locally Lipschitz continuous, i.e. there exists $L = L(R_1) \geqslant 0$ such that

$$|Df(v_\star(x)) - Df(v_\infty)| \leqslant L\, |v_\star(x) - v_\infty|$$

for every $x \in \mathbb{R}^d$. Now, we obtain from (9.14) and Hölder's inequality

$$\begin{aligned} \|\mathcal{L}_0 v\|_{L^p} &\leqslant |\lambda|\, \|v\|_{L^p} + |Df(v_\infty)|\, \|v\|_{L^p} + \|Df(v_\star(x)) - Df(v_\infty)\|_{L^\infty} \|v\|_{L^p} \\ &\leqslant (|\lambda| + |Df(v_\infty)| + LR_1)\, \|v\|_{L^p}. \end{aligned}$$

This yields $\mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$, thus $v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$.
6. Finally, we verify that $\mathcal{L}_Q v = \lambda v$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$: From $v \in \mathcal{D}(A_p)$ we deduce that both $\mathcal{L}_Q v$ and $\lambda v - \mathcal{L}_Q v$ belong to $L^p(\mathbb{R}^d, \mathbb{C}^N)$. Since $[\mathcal{L}_Q v](x) = \lambda v(x)$ for every $x \in \mathbb{R}^d$, we obtain $\mathcal{L}_Q v = \lambda v$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. Now, we can apply Theorem 7.7 that yields $v \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{C}^N)$. □

Exponential decay for $\mathrm{Re}\,\lambda > -b_0$**.** Usually one expects that every eigenfunction associated to an eigenvalue $\lambda \in \sigma_{\mathrm{point}}(\mathcal{L})$ decays exponentially in space, i.e. for λ satisfying satisfies $\mathrm{Re}\,\lambda > -b_0$. However, Theorem 9.8 provides that $\mathrm{Re}\,\lambda \geqslant -\frac{b_0}{3}$ must be satisfied. The lower bound $-\frac{b_0}{3}$, that is larger than $-b_0$, is necessary to obtain a uniform decay rate, i.e. a decay rate that does not depend on $\mathrm{Re}\,\lambda$ any more. Note that one can also prove the exponential decay of eigenfunctions associated to eigenvalues with $\mathrm{Re}\,\lambda > -b_0$, but one usually obtains different decay rates. An eigenvalue that is located near $-b_0$ implies a small decay rate for the eigenfunction.

A direct consequence of Theorem 9.8 is the following:

Corollary 9.9 (Exponential decay of the rotational term)**.** *Let the assumptions* (A4)–(A9) *be satisfied for* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{R}$. *Then for every* $0 < \vartheta < 1$ *and for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d, \mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$ *with*

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$$

with $a_{\max}$, a_0, b_0 *from* (1.18), *there exists a constant* $K_1 = K_1(A, f, v_\infty, d, p, \theta, \vartheta) > 0$ *with the following property:*
Given a classical solution $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ *of* (1.19) *such that* $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ *and* (1.20) *hold. Then the classical solution*

$$v(x) = \langle Sx, \nabla v_\star(x)\rangle$$

of the eigenvalue problem (9.14) *with* $\lambda = 0$ *satisfies*

$$v \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N).$$

Proof. In order to apply Theorem 9.8 for $v(x) = \langle Sx, \nabla v_\star(x)\rangle$, we have to check that $\langle Sx, \nabla v_\star(x)\rangle \in L^p(\mathbb{R}^d, \mathbb{R}^N)$. Our main result from Theorem 1.8 states that $v_\star \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$. This directly implies that $\langle Sx, \nabla v_\star(x)\rangle \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and the application of Theorem 9.8 completes the proof. A further possibility to verify $\langle Sx, \nabla v_\star(x)\rangle \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ works as follows: In the proof of Theorem 1.8 we showed that $v_\star \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. By Theorem 5.25 we have $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \mathcal{D}^p_{\max}(\mathcal{L}_0)$, thus we deduce $\langle Sx, \nabla v_\star(x)\rangle \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ by the definition of $\mathcal{D}^p_{\max}(\mathcal{L}_0)$. ☐

Exponential decay of $A\triangle v_\star$. A crucial implication of Corollary 9.9 is that $A\triangle(v_\star - v_\infty) \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)$: For this purpose, we consider the steady state equation

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d.$$

Following the proof of Theorem 1.8, we decompose the last term into

$$f(v_\star(x)) = -B(v_\star(x) - v_\infty) + Q(x)\,(v_\star(x) - v_\infty)\,.$$

Defining $w_\star := v_\star - v_\infty$ we obtain

$$0 = A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle - Bw_\star(x) + Q(x)w_\star(x),\ x \in \mathbb{R}^d,$$

with $Q \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$. Theorem 1.8 shows that $w_\star \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$ and thus, the last two terms belong to $L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)$. Assuming additional smoothness on $v_\star$, i.e. $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$, we deduce from Corollary 9.9 that $\langle S\cdot, \nabla v_\star\rangle$ and therefore also $\langle S\cdot, \nabla w_\star\rangle$ belong to $W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$. This implies that $A\triangle w_\star \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)$ for $1 < p < \infty$. Note that for $Q \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^{N,N})$ this procedure even implies $A\triangle w_\star \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$.

9.4 Essential spectrum and dispersion relation of localized rotating waves

In this section we investigate the essential spectrum of the linearization about a localized rotating wave. The following Theorem is an application of Theorem 7.9 with

$$-B = Df(v_\infty), \qquad Q(x) = Df(v_\star(x)) - Df(v_\infty),\ x \in \mathbb{R}^d.$$

Note that Theorem 1.8 implies that $Q \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$ satisfies . For a detailed treatment of the essential spectrum we refer to Section 7.4

Theorem 9.10 (Essential spectrum of $\mathcal{L}$)**.** *Let the assumptions of Theorem 1.8 be satisfied. Moreover, let $\lambda_j(\omega)$ denote the eigenvalues of $\omega^2 A - Df(v_\infty)$ for $j = 1, \ldots, N$ and let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S. Then every number $\lambda \in \mathbb{C}$ with*

$$\lambda = -\lambda_j(\omega) - i\sum_{l=1}^{k} n_l\sigma_l,\ n_l \in \mathbb{Z},\ \omega \in \mathbb{R},\ j = 1, \ldots, N,$$

belongs to the essential spectrum of $\mathcal{L}$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, i.e. $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L})$.

Dispersion relation for $\mathcal{L}$ at localized rotating waves. The dispersion relation for the linearization $\mathcal{L}$ at a localized rotating wave in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ with $1<p<\infty$ states that every $\lambda\in\mathbb{C}$ satisfying

$$\det\left(\lambda I_N+\omega^2 A-Df(v_\infty)+i\sum_{l=1}^{k}n_l\sigma_l I_N\right)=0,\text{ for some }\omega\in\mathbb{R},\ n_l\in\mathbb{Z},\tag{9.15}$$

belongs to the essential spectrum of $\mathcal{L}$, i.e. $\lambda\in\sigma_{\mathrm{ess}}(\mathcal{L})$. The dispersion relation for $d=2$ can be found in [15].

Location and density of the essential spectrum. Let us define the set

$$\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}):=\{\lambda\in\mathbb{C}\mid\lambda\text{ satisfies (9.15)}\}.$$

Theorem 9.10 shows that $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})\subseteq\sigma_{\mathrm{ess}}(\mathcal{L})$. Moreover, (9.15) for $\omega=0$ yields the inclusion $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})\subseteq\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\leqslant s(Df(v_\infty))\}$. If there exists σ_n,σ_m such that $\sigma_n\sigma_m^{-1}\notin\mathbb{Q}$ then $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})$ is dense in the half-plane $\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\leqslant s(Df(v_\infty))\}$, i.e. $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})=\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\leqslant s(Df(v_\infty))\}$. Otherwise $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})$ is a discrete subgroup of $\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\leqslant s(Df(v_\infty))\}$ (independently of p). The density observations come originally from [71, Theorem 2.6] and they are illustrated in Figure 7.1. Note that the spectral condition (A9), stating that the matrix $Df(v_\infty)$ is stable, implies that $s(Df(v_\infty))<0$, hence $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})\subseteq\mathbb{C}_-$, which is necessary to guarantee spectral stability, cf. Definition 9.2.

9.5 Essential spectrum and dispersion relation of nonlocalized rotating waves

In this section we investigate the essential spectrum of the linearization about a rigidly rotating spiral wave. This serves as an backward material for the numerical examples in Section 10.4. A rotating spiral wave is a special type of nonlocalized rotating waves for $d=2$. Most of the results come from [92] and [38], but we also refer to [93].

Recall the reaction diffusion equation (1.14)

$$\begin{aligned}u_t(x,t)=&A\triangle u(x,t)+f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 1,\\ u(x,0)=&u_0(x)\qquad\qquad\qquad\quad,\ t=0,\ x\in\mathbb{R}^d.\end{aligned}$$

Definition 9.11. A function $u_\infty:\mathbb{R}\times[0,\infty[\to\mathbb{K}^N$ is called a **traveling wave** solution of (1.14) if it has the form

$$u_\infty(x,t)=v_\infty(x-c_\infty t)$$

with **profile** (or **pattern**) $v_\infty:\mathbb{R}\to\mathbb{K}^N$ and **translational velocity** $c_\infty\in\mathbb{R}$ with $c_\infty\neq 0$. A traveling wave u_∞ of (1.14) is called an **one-dimensional periodic wavetrain with period** $T>0$ if v_∞ is a T-periodic function, i.e. $v_\infty(\xi+T)=v_\infty(\xi)$ for every $x\in\mathbb{R}$.

Note that v_∞ is T-periodic if and only if u_∞ is T-periodic w.r.t. x. In particular, v_∞ satisfies the one-dimensional traveling wave equation

$$Av''(\xi)+cv'(\xi)+f(v(\xi))=0.$$

Assumption 9.12. *Let $v_\infty:\mathbb{R}\to\mathbb{K}^N$ be the profile of an one-dimensional T-periodic wavetrain u_∞ of (1.14) for $d=1$ that travels at speed c_∞, i.e. v_∞ satisfies the* ***compatibility conditions***

(A10) $\qquad v_\infty$ *is T-periodic,*

(A11) $\qquad Av''_\infty(\xi)+c_\infty v'_\infty(\xi)+f(v_\infty(\xi))=0,\ \xi\in\mathbb{R}.$

We assume a rotating Archimedean spiral wave solution $u_\star$ of (1.14) in the following sense:

Definition 9.13. A rotating wave solution $u_\star:\mathbb{R}^2\times[0,\infty[\to\mathbb{K}^N$ of (1.14) in the sense of Definition 1.1 (with $d=2$) is called a **(rigidly) rotating Archimedean spiral wave** if the complex version $v_\star^{\mathrm{pol}}(r,\phi):=v_\star(T_1(T_2(r,\phi)))$ of the pattern $v_\star(x)$ satisfies $v_\star^{\mathrm{pol}}\in C_b(]0,\infty[\times\mathbb{R},\mathbb{K}^N)$ and

$$\lim_{r\to\infty}\left|v_\star^{\mathrm{pol}}(r,\phi)-v_\infty\left(r-\frac{\phi}{k_\infty}\right)\right|=0 \text{ uniformly for } \phi\in\mathbb{R} \tag{9.16}$$

for some $k_\infty\in\mathbb{R}$ with $k_\infty\neq 0$ and for some function $v_\infty:\mathbb{R}\to\mathbb{K}^N$ satisfying (A10) and (A11) with $T=\frac{2\pi}{k_\infty}$ and $c_\infty=\frac{\sigma}{k_\infty}$, where $\sigma=S_{12}$ denotes the angular velocity of the Archimedean spiral wave.

The condition (9.16) states that the pattern $v_\star^{\mathrm{pol}}$ is **Archimedean far away from the center of rotation** and describes in a certain sense a counterpart or an extension of (1.20). v_∞ is called the **asymptotic wavetrain (of the spiral wave solution)** or the **asymptotic profile (of the spiral wave)** and k_∞ is called the **wavenumber (of the periodic wavetrain)**. In particular, we advise the following important relation

$$c_\infty=\frac{\sigma}{k_\infty}.$$

This relation states that the translational speed c_∞ of the periodic wavetrain coincide with the quotient of the rotational speed σ of the spiral wave and the wavenumber k_∞ of the periodic wavetrain.

We now start to investigate the essential spectrum of the linearization at the spiral wave. For this purpose we assume $\mathbb{K}=\mathbb{R}$. In case $\mathbb{K}=\mathbb{C}$ we must transform the N-dimensional complex-valued equation into a $2N$-dimensional real-valued system, c.f. proof of Corollary 8.1. For the moment let $d\geqslant 2$ and consider

$$\left[\mathcal{L}_0 v\right](x)+f(v(x)):=A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+f(v(x))=0,\ x\in\mathbb{R}^d,$$

where $A\in\mathbb{R}^{N,N}$ with $\sigma(A)\subset\mathbb{C}_+$, $S\in\mathbb{R}^{d,d}$ with $-S=S^T$ and $f\in C^1(\mathbb{R}^N,\mathbb{R}^N)$. The linearization at the profile $v_\star$ of the rotating wave $u_\star$ is given by

$$\left[\mathcal{L}v\right](x)=A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+Df(v_\star(x))v(x).$$

In the following we look for solutions of the initial value problem

$$\begin{aligned} v_t(x,t) &= [\mathcal{L}v](x,t),\ x\in\mathbb{R}^d,\ t>0,\\ v(x,0) &= v_0(x), \qquad x\in\mathbb{R}^d,\ t=0. \end{aligned}$$

Similarly to Section 7.4, we now derive a dispersion relation for the linearization at the profile of an Archimedean spiral wave:

1. Orthogonal transformation. The transformation $T_1(y) = Py$ and $\tilde{v}(y,t) = v(T_1(y),t)$ yield

$$\tilde{v}_t(y,t) = [\mathcal{L}_{T_1}\tilde{v}](y,t) = A\triangle\tilde{v}(y,t) + \left\langle \Lambda^S_{\mathrm{block}} y, \nabla\tilde{v}(y,t)\right\rangle + Df(v_\star(T_1(y)))\tilde{v}(y,t)$$

with

$$\left\langle \Lambda^S_{\mathrm{block}} y, \nabla\tilde{v}(y,t)\right\rangle = \sum_{l=1}^{k} \sigma_l\left(y_{2l}D_{2l-1} - y_{2l-1}D_{2l}\right)\tilde{v}(y,t).$$

2. Several planar polar coordinates. Choosing $T_2(\xi)$ as in Section 7.4 for $\xi = (r_1,\phi_1,\ldots,r_k,\phi_k,y_{2k+1},\ldots,y_d)$, then $\hat{v}(\xi,t) = \tilde{v}(T_2(\xi),t)$ yields

$$\begin{aligned} \hat{v}_t(\xi,t) = [\mathcal{L}_{T_2}\hat{v}](\xi,t) =& A\left[\sum_{l=1}^{k}\left(\partial^2_{r_l} + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial^2_{\phi_l}\right) + \sum_{l=2k+1}^{d}\partial^2_{y_l}\right]\hat{v}(\xi,t)\\ &-\sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v}(\xi,t) + Df(v_\star(T_1(T_2(\xi))))\hat{v}(\xi,t), \end{aligned}$$

where we define $v_\star^{\mathrm{pol}}(\xi) = v_\star(T_1(T_2(\xi)))$.

3. Rigidly rotating spiral waves for $d=2$. From now on we consider rigidly rotating spiral wave solutions $u_\star$. For this purpose let $d=2$, thus $\xi=(r,\phi)$ with $r:=r_1$, $\phi:=\phi_1$ and $\sigma:=\sigma_1$. Adding and subtracting the $\frac{2\pi}{k_\infty}$-periodic function $Df(v_\infty(r-\frac{\phi}{k_\infty}))\hat{v}(r,\phi,t)$ we obtain

$$\begin{aligned} \hat{v}_t(r,\phi,t) = [\mathcal{L}_{T_2}\hat{v}](r,\phi,t) =& A\left(\partial^2_r + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_\phi\right)\hat{v}(r,\phi,t) - \sigma\partial_\phi\hat{v}(r,\phi,t)\\ &+\left(Df(v_\star^{\mathrm{pol}}(r,\phi)) - Df(v_\infty(r-\frac{\phi}{k_\infty}))\right)\hat{v}(r,\phi,t)\\ &+Df(v_\infty(r-\frac{\phi}{k_\infty}))\hat{v}(r,\phi,t). \end{aligned}$$

Note that $Df(v_\star^{\mathrm{pol}}(r,\phi)) - Df(v_\infty(r-\frac{\phi}{k_\infty})) \to 0$ for $r\to\infty$ uniformly for $\phi\in\mathbb{R}$. Moreover, $\hat{v}$ is a 2π-periodic function w.r.t. ϕ.

4. Simplified operator (limit operator, far-field operator). Neglecting the terms of order $\mathcal{O}\left(\frac{1}{r}\right)$ we obtain the far-field linearization

$$\hat{v}_t(r,\phi,t) = A\partial^2_r\hat{v}(r,\phi,t) - \sigma\partial_\phi\hat{v}(r,\phi,t) + Df(v_\infty(r-\frac{\phi}{k_\infty}))\hat{v}(r,\phi,t).$$

5. Angular transformation. $\hat{v}(r,\phi,t) = \bar{v}(r, r - \frac{\phi}{k_\infty}, t)$ and $\xi = r - \frac{\phi}{k_\infty}$ yield

$$\bar{v}_t(r,\xi,t) = A\left(\partial_r^2 + 2\partial_{r\xi} + \partial_\xi^2\right)\bar{v}(r,\xi,t) + \frac{\sigma}{k_\infty}\partial_\xi \bar{v}(r,\xi,t) + Df(v_\infty(\xi))\bar{v}(r,\xi,t)$$

where $Df(v_\infty(\xi))$ and $\bar{v}(r,\xi,t)$ are $\frac{2\pi}{k_\infty}$-periodic functions w.r.t. ξ.

6. Temporal Fourier transform. $\bar{v}(r,\xi,t) = e^{\lambda t}\check{v}(r,\xi)$, $\lambda \in \mathbb{C}$, and multiply from left by $e^{-\lambda t}$ yields

$$\lambda \check{v}(r,\xi) = A\underbrace{\left(\partial_r^2 + 2\partial_{r\xi} + \partial_\xi^2\right)}_{=\left(\partial_r + \partial_\xi\right)^2}\check{v}(r,\xi) + \frac{\sigma}{k_\infty}\partial_\xi \check{v}(r,\xi) + Df(v_\infty(\xi))\check{v}(r,\xi),$$

where $\check{v}(r,\xi)$ is $\frac{2\pi}{k_\infty}$-periodic w.r.t. ξ.

7. Radial Fourier transform. $\check{v}(r,\xi) = e^{\nu r}\underline{v}(\xi)$, $\nu \in \mathbb{C}$, and multiply from left by $e^{-\nu r}$ yields

$$\lambda \underline{v}(\xi) = A\underbrace{\left(\nu^2 + 2\nu\partial_\xi + \partial_\xi^2\right)}_{=\left(\nu + \partial_\xi\right)^2}\underline{v}(\xi) + \frac{\sigma}{k_\infty}\partial_\xi \underline{v}(\xi) + Df(v_\infty(\xi))\underline{v}(\xi),$$

where $\underline{v}(\xi)$ is $\frac{2\pi}{k_\infty}$-periodic.

8. Bloch wave transformation w.r.t. ξ. $\underline{v}(\xi) = e^{-\nu\xi}v(\xi)$, $\nu \in \mathbb{C}$ from step 7, and multiply from left by $e^{\nu\xi}$ yield

$$\lambda v(\xi) = A\partial_\xi^2 v(\xi) - \frac{\sigma\nu}{k_\infty}v(\xi) + \frac{\sigma}{k_\infty}\partial_\xi v(\xi) + Df(v_\infty(\xi))v(\xi),$$

with $\frac{2\pi}{k_\infty}$-periodic function $Df(v_\infty(\xi))$. Add $\frac{\sigma\nu}{k_\infty}v(\xi)$ and define $\tilde{\lambda} := \lambda + \frac{\sigma\nu}{k_\infty}$ we obtain

$$\tilde{\lambda} v(\xi) = A\partial_\xi^2 v(\xi) + \frac{\sigma}{k_\infty}\partial_\xi v(\xi) + Df(v_\infty(\xi))v(\xi).$$

The $\frac{2\pi}{k_\infty}$-periodicity of $\underline{v}(\xi)$ and the Bloch wave transformation $\underline{v}(\xi) = e^{-\nu\xi}v(\xi)$ leads to the Floquet boundary condition

$$e^{\nu\frac{2\pi}{k_\infty}}v(\xi) = v(\xi + \frac{2\pi}{k_\infty}), \quad e^{\nu\frac{2\pi}{k_\infty}}v_\xi(\xi) = v_\xi(\xi + \frac{2\pi}{k_\infty})$$

Note that for a 2nd order problem we have to impose boundary conditions for v and v_ξ to make the problem well-posed. This is expressed by the periodicity conditions for v and v_ξ. Indeed, if v is sufficiently smooth, the second equality can directly be deduced by differentiating the first one. Altogether, we obtain a Floquet boundary value problem

$$\tilde{\lambda} v(\xi) = A\partial_\xi^2 v(\xi) + c_\infty\partial_\xi v(\xi) + Df(v_\infty(\xi))v(\xi),\ \xi \in [0,T], \tag{9.17}$$

$$e^{\nu T}v(0) = v(T), \quad e^{\nu T}v_\xi(0) = v_\xi(T). \tag{9.18}$$

where $Df(v_\infty(\xi)) \in \mathbb{R}^{N,N}$ is a continuous, $\frac{2\pi}{k_\infty}$-periodic function, $c_\infty := \frac{\sigma}{k_\infty} \in \mathbb{R}$ and period $T := \frac{2\pi}{k_\infty}$. We now look for solutions (λ, ν, v) satisfying (9.17)–(9.18).

9. Application of Floquet theory. Using $u := v_\xi$, we transform the N-dimensional 2nd order boundary value problem (9.17)–(9.18) into a $2N$-dimensional $1st$ order boundary value problem

$$\begin{aligned}\begin{pmatrix} v \\ u \end{pmatrix}_\xi (\xi) &= \begin{pmatrix} 0 & I_N \\ A^{-1}(\tilde{\lambda} I_N - Df(v_\infty(\xi))) & -c_\infty A^{-1} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} (\xi),\ \xi \in [0,T], \\ e^{\nu T} \begin{pmatrix} v \\ u \end{pmatrix} (0) &= \begin{pmatrix} v \\ u \end{pmatrix} (T).\end{aligned}$$

Note that A^{-1} exists due to the fact that $\operatorname{Re}\sigma(A) > 0$. Abbreviating $w := \begin{pmatrix} v \\ u \end{pmatrix}$ with $w : \mathbb{R} \to \mathbb{R}^{2N}$ we obtain

$$w_\xi(\xi) = \tilde{A}(\xi) w(\xi),\ \xi \in [0,T], \tag{9.19}$$

$$e^{\nu T} w(0) = w(T) \tag{9.20}$$

with continuous, T-periodic matrix-valued function $\tilde{A} : \mathbb{R} \to \mathbb{R}^{2N,2N}$ given by

$$\tilde{A}(\xi) := \begin{pmatrix} 0 & I_N \\ A^{-1}(\tilde{\lambda} I_N - Df(v_\infty(\xi))) & -c_\infty A^{-1} \end{pmatrix}$$

It is well known that in general one cannot derive an explicit solution representation for (9.19)–(9.20), but one can apply Floquet theory, see [51] and [24]: For this purpose, let $W(\xi)$ with $W : \mathbb{R} \to \mathbb{R}^{2N,2N}$ be a fundamental matrix solution of (9.19), where we assume without loss of generality that $W(0) = I_{2N}$. An application of Floquet's theorem, see [51, Theorem 7.1] or [24, Theorem 2.83], yields that W has the form

$$W(\xi) = P(\xi) e^{B\xi} \quad \text{(Floquet normal form)}$$

with T-periodic function $P : \mathbb{R} \to \mathbb{C}^{2N,2N}$ and constant matrix $B \in \mathbb{C}^{2N,2N}$. The proof of Floquet's theorem shows that the matrix $P(\xi)$ is invertible for all $\xi \in \mathbb{R}$. Moreover, P satisfies $P(0) = I_{2N}$, since $W(0) = I_{2N}$, and W satisfies

$$W(T) = P(T) e^{BT} = P(0) e^{BT} = W(0) e^{BT} = e^{BT}$$

by the T-periodicity of P. The invertible matrix e^{BT} is called the **monodromy matrix** of (9.19). Its eigenvalues $\rho \in \sigma(e^{TB})$ are called the **characteristic multipliers** (or **Floquet multipliers**) **of** (9.19). Furthermore, an element $\mu \in \mathbb{C}$ satisfying $\rho = e^{\mu T}$ for some $\rho \in \sigma(e^{TB})$ is called a **characteristic exponent** (or **Floquet exponent**) **of** (9.19). The Floquet multipliers ρ of (9.19) are unique. The Floquet exponents μ of (9.19) are not unique but their real parts $\operatorname{Re}\mu$. This can be accepted by the fact that if μ is a Floquet exponent of (9.19) then also $\mu + 2\pi i \frac{k}{T}$ is a Floquet exponent of (9.19) for every $k \in \mathbb{Z}$. For a detailed treatment of Floquet theory for homogeneous linear periodic systems we refer to [51] and [24].

10. Evolution operator. Defining

$$\Phi_{\tilde{\lambda}}(\xi, \psi) := W(\xi) W(\psi)^{-1} = P(\xi) e^{B(\xi - \psi)} P(\psi)^{-1} \in \mathbb{C}^{2N,2N}$$

the evolution operator of (9.19) is by the periodic map

$$\Phi_{\tilde{\lambda}}(T,0) := W(T)W(0)^{-1} = P(T)e^{B(T-0)}P(0)^{-1} = e^{BT} \in \mathbb{C}^{2N,2N},$$

where we used $P(T) = P(0) = I_{2N}$, since $W(0) = I_{2N}$. Since $W(T) = W(0)e^{BT}$ and $W(0) = I_{2N}$ we deduce

$$\Phi_{\tilde{\lambda}}(T,0) = W(T)W(0)^{-1} = W(0)e^{BT}W(0)^{-1} = e^{tB}.$$

We now look for solutions w of (9.19)–(9.20), that have the special form

$$w(\xi) = W(\xi)w_\nu, \quad \text{for } \xi \in \mathbb{R} \text{ and for some } w_\nu \in \mathbb{R}^{2N}.$$

Due to the Floquet boundary conditions (9.20) we have

$$e^{\nu T}w(0) = w(T) = W(T)w_\nu = e^{BT}w_\nu = \Phi_{\tilde{\lambda}}(T,0)w(0).$$

Subtracting $e^{\nu T}w(0)$ we obtain

$$\left(\Phi_{\tilde{\lambda}}(T,0) - e^{\nu T}I_{2N}\right)w(0) = 0.$$

11. Dispersion relation for Archimedean spiral waves. Using $\tilde{\lambda} = \lambda + \frac{\sigma\nu}{k_\infty}$ we define

$$d(\lambda,\nu) := \det\left(\Phi_{\lambda+\frac{\sigma\nu}{k_\infty}}(T,0) - e^{\nu T}I_{2N}\right).$$

Since we are interested only in bounded solutions, we choose $\nu = ik$, $k \in \mathbb{R}$, where k is called the **Bloch wavenumber**. This and $T = \frac{2\pi}{k_\infty}$ yield

$$d(\lambda,ik) = \det\left(\Phi_{\lambda+i\sigma\frac{k}{k_\infty}}(\frac{2\pi}{k_\infty},0) - e^{2\pi i\frac{k}{k_\infty}}I_{2N}\right) = 0.$$

The dispersion relation for Archimedean spiral waves states: If $d(\lambda,ik) = 0$ for some $k \in \mathbb{R}$, then $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L})$, i.e.

$$d(\lambda,ik) = 0 \text{ for some } k \in \mathbb{R} \quad \Rightarrow \quad \lambda \in \sigma_{\mathrm{ess}}(\mathcal{L}).$$

Therefore, we have the inclusion

$$\{\lambda \in \mathbb{C} \mid d(\lambda,ik) = 0 \text{ for some } k \in \mathbb{R}\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

We suggest that the eigenvalue λ and the Bloch wave number k are related by the dispersion relation. In particular, we cannot derive an explicit expression for the dispersion relation due to the fact that equation (9.19)–(9.20) cannot be solved explicitly. But, we are able to give a better characterization of the structure of the essential spectrum. For this purpose, we first note that if (λ,ik) solves $d(\lambda,ik) = 0$ then also $(\tilde{\lambda},\tilde{k}) := (\lambda - i\sigma n, i(k+nk_\infty))$ solves $d(\tilde{\lambda},\tilde{k}) = 0$ for every $n \in \mathbb{Z}$, i.e.

$$d(\lambda,ik) = 0 \text{ for some } k \in \mathbb{R} \quad \Rightarrow \quad d(\lambda - i\sigma n, i(k+nk_\infty)) = 0 \text{ for every } n \in \mathbb{Z}.$$

Using $e^{2\pi i n} = 1$ for every $n \in \mathbb{Z}$ this follows easily from

$$\begin{aligned} d(\lambda - i\sigma n, i(k + nk_\infty)) &= \det\left(\Phi_{\lambda - i\sigma n + i\sigma\frac{k+nk_\infty}{k_\infty}}(\frac{2\pi}{k_\infty}, 0) - e^{2\pi i\frac{k+nk_\infty}{k_\infty}} I_{2N}\right) \\ = \det\left(\Phi_{\lambda + i\sigma\frac{k}{k_\infty}}(\frac{2\pi}{k_\infty}, 0) - e^{2\pi i\frac{k}{k_\infty}} I_{2N}\right) &= d(\lambda, ik) = 0. \end{aligned}$$

Therefore, we can consider w.l.o.g. $k \in [0, k_\infty[$. Moreover, if (λ, ik) solves $d(\lambda, ik) = 0$ then also $(\bar{\lambda}, -ik)$ solves $d(\bar{\lambda}, -ik) = 0$, i.e.

$$d(\lambda, ik) = 0 \text{ for some } k \in \mathbb{R} \quad \Rightarrow \quad d(\bar{\lambda}, -ik) = 0.$$

Thus, for $k \in [0, k_\infty[$ there are $2N$ solutions of $d(\lambda, ik) = 0$, that we denote by $\lambda^l(\nu) = \lambda^l(ik)$, $l = 1, \ldots, 2N$. We deduce the inclusion

$$\left\{\lambda^l(ik) + i\sigma\mathbb{Z} \mid k \in \mathbb{R},\, l = 1, \ldots, 2N\right\} \subseteq \sigma_{\text{ess}}(\mathcal{L}). \tag{9.21}$$

12. Shape of the spectral curves. We finally discuss the shape of the spectral curves $\Gamma^l := \left\{\lambda^l(ik) \mid k \in \mathbb{R}\right\}$ for fixed $l = 1, \ldots, 2N$: In order to apply the implicit function theorem we consider the continuously differentiable function

$$\tilde{d} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2,\ (\lambda, k) \mapsto \tilde{d}\left(\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, k\right) := d(\lambda_1 + i\lambda_2, ik) = d(\lambda, ik).$$

Then the mapping $\tilde{d}$ satisfies

$$\tilde{d}\left(\begin{pmatrix}0\\0\end{pmatrix}, 0\right) = \begin{pmatrix}0\\0\end{pmatrix} \quad \text{and} \quad \tilde{d}_\lambda\left(\begin{pmatrix}0\\0\end{pmatrix}, 0\right) \in \mathbb{R}^{2,2} \text{ is invertible.}$$

at the point $(\lambda, k) = ((0,0)^T, 0)$. An application of the implicit function theorem yields open sets $U = U((0,0)^T) \subset \mathbb{R}^2$ and $V = V(0) \subset \mathbb{R}$ as well as a continuously differentiable function $g : V \to U$ with $g(0) = (0,0)^T$ such that $\tilde{d}(g(k), k) = 0$ for every $k \in V$. Considering λ as a function of k, i.e. $\lambda(k) := g(k)$, this means that

$$d(\lambda(ik), ik) = \tilde{d}(\lambda(k), k) = 0 \text{ for every } k \in V.$$

Further, the function $\tilde{d}$ satisfies

$$\tilde{d}_k(\lambda(0), 0) = \begin{pmatrix}0\\0\end{pmatrix}.$$

We deduce, that the function $\lambda(k)$ and therefore also the spectral curves Γ^l have locally the shape of a parabola near $k = 0$. Thus, we can think about $\{\lambda^l(ik) + i\sigma\mathbb{Z} \mid k \in \mathbb{R}\}$ from (9.21) as infinitely many copies of parabolas along the imaginary axis for every $l = 1, \ldots, 2N$. The parabolas in the complex plane are opened to the left and touch the imaginary axis at $i\sigma\mathbb{Z}$. In particular, the distance on the imaginary axis of two neighboring parabolas equals the rotational velocity σ of the Archimedean spiral wave.

We conclude with extensions and corresponding open problems:

Essential spectrum of scroll waves and scroll rings. We believe that this approach extends to scroll wave and scroll ring solutions for $d = 3$. In this case, there exists a further period in z-direction, that has to take into account in the computation of the essential spectrum. This question seems to be still an open problem.

Hyperbolic-parabolic systems. Furthermore, we believe that the approach also extends to matrices A that do have a zero eigenvalue. This is motivated by the existence of rigidly rotating spiral waves in Barkley's model for $D = 0$ and seems also to be an open problem.

10 Freezing approach and numerical results

In this chapter we recall some basic ideas and results from the field of equivariant evolution equations. We then introduce the freezing method, that is an approach for the approximation of relative equilibria. An application to reaction-diffusion equations shows that rotating waves are a special kind of relative equilibria. Solving numerically the resulting freezing system for the Examples from Section 2.1 yields approximations for the profiles of rotating waves and their group velocities. Moreover, we investigate numerically the spectrum of the linearization at localized and nonlocalized rotating waves. Afterwards, we introduce the decompose and freeze method for equivariant evolution equations. We apply the general theory for reaction-diffusion systems and extend this approach to higher space dimensions in order to investigate interactions of multi-solitons. At the end of the chapter we numerically investigate the interaction of several spinning solitons in the two-dimensional Ginzburg-Landau equation.

In Section 10.1 we consider abstract evolution equations

$$u_t(t) = F(u(t)),\ 0 < t < T \qquad \text{with} \qquad u(0) = u_0,\ t = 0, \tag{10.1}$$

for some densely defined function $F : X \supset Y \to X$ on a $\mathbb{K}$-valued Banach space $(X, \|\cdot\|)$. Assuming (10.1) to be equivariant with respect to a group action $a(\cdot)u : G \to X$ of a finite-dimensional (not necessarily compact) Lie group G on X,

$$F(a(\gamma)u) = a(\gamma)F(u),\ \gamma \in G,\ u \in Y,$$

we investigate so called relative equilibria of (10.1)

$$u_\star(t) = a(\gamma_\star(t))v_\star,\ \gamma_\star(t) \in G,\ v_\star \in Y. \tag{10.2}$$

For some general theory of equivariant evolution equations we refer to [25], [39] and [43]. At the end of this section, in Example 10.6, we apply the theory for

$$G = \mathrm{SE}(d), \quad F(u) = A\triangle u + f(u), \quad X = L^p(\mathbb{R}^d, \mathbb{K}^N), \quad 1 < p < \infty, \tag{10.3}$$

This leads us to general reaction-diffusion systems on $\mathbb{R}^d$. Their solutions, which are of the from (10.2), are called rotating waves.

In Section 10.2 we introduce the freezing method in an abstract setting, [18]. The freezing method is an approach for approximating relative equilibria such as traveling and rotating waves. The idea of the freezing method is to decompose the solution of (10.1) into a group motion and a profile, not only at or near relative equilibria as in (10.2), but also for the general Cauchy Problem (10.1). Introducing new unknowns $\gamma(t) \in G$ and $v(t) \in Y$ such that the solution u of (10.1) is of the form

$$u(t) = a(\gamma(t))v(t),\ 0 \leqslant t < T,$$

we transform (10.1) into a differential algebraic evolution equation

$$\begin{aligned} &(10.4a) && v_t(t) = F(v(t)) - d\left[a(\mathbb{1})v(t)\right]\mu(t), && v(0) = u_0, \\ &(10.4b) && 0 = \Psi(v(t),\mu(t)), && \\ &(10.4c) && \gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu(t), && \gamma(0) = \mathbb{1}, \end{aligned}$$

where $d\left[a(\mathbb{1})v(t)\right]\mu(t)$ denotes the derivative of the group action $a(\cdot)u$ and $\Psi : Y \times \mathfrak{g} \to \mathfrak{g}^*$ denotes a so-called phase condition. Well-known phase conditions are the fixed and the orthogonal phase condition, that we briefly discuss. The equation (10.4c) is known as the reconstruction equation. In Example 10.7 we apply the presented theory to reaction-diffusion systems, compare (10.3). This leads to a partial differential algebraic evolution equation (10.5), which we analyze numerically in the next section. In Example 10.8 we analytically solve the reconstruction equation for reaction-diffusion systems with $G = \mathrm{SE}(d)$. The freezing method was independently proposed in [18] and [89]. For a more detailed treatment about the freezing method of single structures we also refer to [103], [19], [105], [20] and [16]. The freezing approach for hyperbolic-parabolic systems is analyzed in [85], [86], [87], [88] and partially in [16]. Results for the freezing method of stochastic traveling waves can be found for instance in [65].

In Section 10.3 we analyze the freezing system

$$\begin{aligned} &(10.5a) && \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + f(v(x,t)) + \langle S(t)x + I_d\lambda(t), \nabla v(x,t)\rangle, \\ v(x,0) &= u_0(x), \end{aligned} \\ &(10.5b) && 0 = \Psi(v(\cdot,t),\mu(t)), \\ &(10.5c) && \begin{pmatrix} R_t(t) \\ \tau_t(t) \end{pmatrix} = \begin{pmatrix} R(t)S(t) \\ R(t)\lambda(t) \end{pmatrix}, \quad \begin{pmatrix} R(0) \\ \tau(0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix}. \end{aligned}$$

for the Examples from Section 2.1. To solve (10.5) numerically, we truncate (10.5) to a bounded domain and postulate homogeneous Neumann boundary conditions. Solving this resulting system until a certain end time T yields a profile $v_\star(x) := v(x,T)$ and velocities $(S_\star, \lambda_\star) := (S(T), \lambda(T))$. The definition is justified by the fact that we expect $v(t) \to v_\star$ and $(S(t), \lambda(t)) \to (S_\star, \lambda_\star)$ as $t \to \infty$. Hence, the solution of (10.5) provides an approximation of the rotating wave $u_\star$, compare (1.15) and Example 10.8.

In Section 10.4 we investigate the spectral properties of the linearizations about rotating waves for the Examples from Section 2.1 and Section 10.3. For this purpose we consider the eigenvalue problem

$$(10.6) \qquad \left[\lambda I - \mathcal{L}\right] v(x) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2,$$

where $\mathcal{L}$ denotes the linearization about a rotating wave profile $v_\star$

$$\left[\mathcal{L}v\right](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x \in \mathbb{R}^d.$$

To solve (10.6) numerically, we restrict the equation (10.6) to a bounded domain and impose homogeneous Neumann boundary conditions. Solving the resulting system yields a prescribed number of eigenvalues and their associated eigenfunctions.

In Section 10.5 we introduce the decompose and freeze method in an abstract setting, [17]. Introducing new unknowns $\gamma_j(t) \in G$ and $v_j(t) \in Y$ for $i = 1, \ldots, m$ such that the solution u of (10.1) is of the form

$$u(t) = \sum_{j=1}^{m} a(\gamma_j(t))v_j(t),\; 0 \leqslant t < T,$$

we transform (10.1) into a nonlinearly coupled system of differential algebraic evolution equations

$$\begin{aligned}
&(10.7a) \quad v_{j,t}(t) = F(v_j(t)) - d\left[a(\mathbb{1})v_j(t)\right]\mu_j(t) + \frac{\varphi}{\sum_{k=1}^{m} b(\gamma_j^k(t))\varphi}, \qquad v_j(0) = v_j^0,\\
&\qquad\qquad \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t))v_k(t)\right) - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t))v_k(t)\right)\right]\\
&(10.7b) \quad 0 = \Psi(v_j(t), \mu_j(t)),\\
&(10.7c) \quad \gamma_{j,t}(t) = dL_{\gamma_j(t)}(\mathbb{1})\mu_j(t), \qquad \gamma_j(0) = \gamma_j^0,
\end{aligned}$$

with abbreviation $\gamma_j^k(t) := \gamma_j^{-1}(t) \circ \gamma_k(t)$. Finally, in Example 10.15 we apply the presented theory to reaction-diffusion systems. This leads to a nonlinearly coupled system of partial differential algebraic evolution equations, which we analyze numerically in the next section. The decompose and freeze method comes originally from [17]. Since the approach is quite new, there doesn't exist many results concerning the nonlinear stability theory of multi-structures. So far, there is only a nonlinear stability result for multifronts and multipulses in one space dimension, [99]. An extension of this method to multi-solitons in higher space dimensions can be found in [16].

In Section 10.6 we are mainly interested into the interaction of multi-solitons. For this purpose we numerically solve the decompose and freeze system

$$\begin{aligned}
v_{j,t}(x,t) =& A\triangle v_j(x,t) + f(v_j(x,t)) + \langle S_j(t)x + I_d\lambda_j(t), \nabla v_j(x,t)\rangle\\
(10.8a) \quad & + \frac{\varphi(x)}{\sum_{k=1}^{m} b(\gamma_j^k(t))\varphi(x)}\left[f\left(\sum_{k=1}^{m} a(\gamma_j^k(t))v_k(x,t)\right)\right.\\
& \left. - \sum_{k=1}^{m} f\left(a(\gamma_j^k(t))v_k(x,t)\right)\right], \quad v_j(x,0) = v_j^0(x)\\
(10.8b) \quad 0 =& \Psi(v_j(\cdot,t), \mu_j(t)),\\
(10.8c) \quad & \begin{pmatrix} R_{j,t}(t) \\ \tau_{j,t}(t) \end{pmatrix} = \begin{pmatrix} R_j(t)S_j(t) \\ R_j(t)\lambda_j(t) \end{pmatrix}, \quad \begin{pmatrix} R_j(0) \\ \tau_j(0) \end{pmatrix} = \begin{pmatrix} R_j^0 \\ \tau_j^0 \end{pmatrix},
\end{aligned}$$

for the cubic-quintic complex Ginzburg-Landau equation from Example 2.1. We observe different situations, in which the solitons repel from each other (weak interaction), collide with each other into a single soliton (strong interaction) or permanent collide with each other (phase shift interaction). Using the decompose and freeze method, we analyze the temporal change of the profiles, their velocities and their positions for these interaction processes.

10.1 Equivariant evolution equations

In this section we briefly introduce some basic theory of equivariant evolution equations, [25], [39], [43].

Consider a **general abstract evolution equation**

$$
\begin{aligned}
u_t(t) =& F(u(t)),\ 0 < t < T,\\
u(0) =& u_0\ ,\ t = 0,
\end{aligned} \tag{10.9}
$$

on a $\mathbb{K}$-valued Banach space $(X, \|\cdot\|)$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We assume that the operator F, given by

$$F : X \supset Y = \mathcal{D}(F) \to X, \quad u \mapsto F(u), \quad \overline{Y} = X.$$

is defined on a dense subspace Y of X. The whole approach can also be generalized to Banach manifolds rather than Banach spaces, [19], [105], [85].

Definition 10.1. A function $u \in C^1(]0, T[, X) \cap C([0, T[, Y)$ is called a **solution of** (10.9) **on** $[0, T[$ if u solves (10.9) pointwise.

Let $(G, \circ)$ denote a finite-dimensional (not necessarily compact) Lie group with group operation

$$\circ : G \times G \to G, \quad (\gamma_1, \gamma_2) \mapsto \gamma_1 \circ \gamma_2,$$

unit element $\mathbb{1}$ and dimension $\dim G = q < \infty$. By $\gamma^{-1} \in G$ we denote the inverse of $\gamma \in G$, i.e. $\gamma^{-1} \circ \gamma = \gamma \circ \gamma^{-1} = \mathbb{1}$. Moreover, let $\mathfrak{g} = T_{\mathbb{1}}G$ denote the Lie algebra associated with G, that is the tangent space of G at $\mathbb{1}$ and has the same dimension as G, i.e. $\dim \mathfrak{g} = q < \infty$.

The left multiplication by $\gamma \in G$ on G is now defined via

$$L_\gamma : G \to G, \quad g \mapsto L_\gamma(g) := \gamma \circ g$$

with derivative denoted by

$$dL_\gamma(g) : T_gG \to T_{\gamma \circ g}G, \quad \mu \mapsto dL_\gamma(g)\mu.$$

Here, T_gG denotes the tangent space of G at $g \in G$. Thus, for $g = \mathbb{1}$ we have

$$dL_\gamma(\mathbb{1}) : \mathfrak{g} = T_{\mathbb{1}}G \to T_\gamma G, \quad \mu \mapsto dL_\gamma(\mathbb{1})\mu.$$

Furthermore, let

$$a : G \to GL(X), \quad \gamma \mapsto a(\gamma)$$

denote the action of the Lie group G on X via a representation in $GL(X)$, meaning that a is a homomorphism satisfying the following properties

$$a(\mathbb{1}) = I, \quad a(\gamma_1 \circ \gamma_2) = a(\gamma_1)a(\gamma_2),$$

where I denotes the unit element of $GL(X)$. Moreover, we assume that the mapping

$$a(\cdot)u : G \to X, \quad \gamma \mapsto a(\gamma)u$$

is continuous for every fixed $u \in X$ and continuously differentiable for every fixed $u \in Y$, where the derivative is given by

$$d\,[a(\gamma)u] : T_\gamma G \to X, \quad \mu \mapsto d\,[a(\gamma)u]\,\mu.$$

In the special case $\gamma = \mathbb{1}$ we have

$$d\,[a(\mathbb{1})u] : \mathfrak{g} = T_{\mathbb{1}}G \to X, \quad \mu \mapsto d\,[a(\mathbb{1})u]\,\mu$$

for every fixed $u \in Y$. Furthermore, we assume that

$$a(\gamma)Y = Y \quad \forall\, \gamma \in G.$$

Indeed, it is sufficient to require only $a(\gamma)Y \subseteq Y$, because of

$$Y = a(\gamma)a(\gamma^{-1})Y \subseteq a(\gamma)Y.$$

In the literature there exists a general principle for the construction of the function space Y such that all our assumptions are satisfied. But note that this result does not provide an explicit representation for Y. The following proposition can be found in [18, Proposition 2.4], that is an extension of [94, Theorem 4.5].

Proposition 10.2. *Let $(X_0, \|\cdot\|_0)$ be a Banach space and let $a : G \to GL(X_0)$ be a homomorphism. Then*

$$X_1 := \left\{ u \in X_0 \mid \|u\|_1 := \sup_{\gamma \in G} \|a(\gamma)u\|_0 < \infty \right\}$$

is a Banach space with respect to the norm $\|\cdot\|_1$ and the operators $a(\gamma)|_{X_1}$ are isometries in $GL(X_1)$. Further, the space

$$X_2 := \{u \in X_1 \mid\ a(\cdot)u \text{ is continuous in } G\}$$

is a closed subspace of $(X_1, \|\cdot\|_1)$ such that $a(\gamma)|_{X_2} \in GL(X_2)$ acts continuously. Finally,

$$X_3 := \{u \in X_2 \mid\ a(\cdot)u \text{ is continuously differentiable in } G\}$$

is a dense subspace of X_2 and can be written as

$$X_3 = \bigcap_{\mu \in \mathfrak{g}} \mathcal{D}(\mu),$$

where $\mathcal{D}(\mu)$ is the domain of the infinitesimal generator of the C^0-semigroup $(a(\exp(\mu t)))_{t \geqslant 0}$.

Note that all assumptions above are satisfied for the choice

$$X := X_2 \quad \text{and} \quad Y := X_3.$$

Finally, we require that the evolution equation (10.9) is equivariant in the following sense:

Definition 10.3. The evolution equation (10.9) is called **equivariant under the group action** a **of** G **on** X if

$$F\left(a(\gamma)u\right) = a(\gamma)F(u) \quad \forall\, u \in Y\ \forall\, \gamma \in G.$$

We are interested in a special kind of solution of the equivariant evolution equation (10.9), so called relative equilibria, [19, Definition 4.1].

Definition 10.4. (**1**) A solution $u_\star$ of (10.9) on $[0,\infty[$ is called a **relative equilibrium** (**with respect to the action** a **of** G **on** X) if it has the form

$$u_\star(t) = a(\gamma_\star(t))v_\star,\ t \geqslant 0$$

for some $v_\star \in Y$ and $\gamma_\star \in C^1(]0,\infty[,G) \cap C([0,\infty[,G)$.
(**2**) A solution $u_\star$ of (10.9) on $[0,\infty[$ is called a **relative periodic orbit** (**with respect to the action** a **of** G **on** X) if it has the form

$$u_\star(t) = a(\gamma_\star(t))v_\star(t),\ t \geqslant 0$$

for some periodic $v_\star \in C^1(]0,\infty[,Y) \cap C([0,\infty[,Y)$ with nontrivial period $T > 0$ and $\gamma_\star \in C^1(]0,\infty[,G) \cap C([0,\infty[,G)$.

Consider the **group orbit of an element** $v \in Y$

$$\mathcal{O}_G(v) := \left\{a(\gamma)v \mid \gamma \in G\right\}.$$

We point out that a relative equilibrium $u_\star(t) = a(\gamma_\star(t))v_\star$ lies for all times in a single group orbit, i.e. $u_\star(t) \in \mathcal{O}_G(v_\star)$ for every $t \geqslant 0$. An essential feature of relative equilibria is that they never come alone: If $u_\star(t) = a(\gamma_\star(t))v_\star$ is a relative equilibrium of (10.9) then also $a(\gamma \circ \gamma_\star(t))v_\star$ is a relative equilibrium of (10.9) for every $\gamma \in G$:

$$\begin{aligned}&\frac{d}{dt}\left(a(\gamma\circ\gamma_\star(t))v_\star\right) = \frac{d}{dt}\left(a(\gamma)a(\gamma_\star(t))v_\star\right) = \frac{d}{dt}a(\gamma)u_\star(t) = a(\gamma)\frac{d}{dt}u_\star(t)\\ =&a(\gamma)F(u_\star(t)) = a(\gamma)F(a(\gamma_\star(t))v_\star) = F(a(\gamma)a(\gamma_\star(t))v_\star) = F(a(\gamma\circ\gamma_\star(t))v_\star).\end{aligned}$$

This means that relative equilibria always come in families.

Nonlinear Stability of relative equilibria: One main issue for the investigation of relative equilibria is nonlinear stability of relative equilibria (also called stability of relative equilibria with asymptotic phase). Since relative equilibria always appears in families, we have to modify the classical Lyapunov stability in order to investigate the stability of relative equilibria, [16, Definition 2], [105].

Definition 10.5. A relative equilibrium $u_\star$ of (10.9) with $u_\star(t) = a(\gamma_\star(t))v_\star$ for $t \in [0,\infty[$ is called **orbitally stable** (with respect to given norms $\|\cdot\|_1$, $\|\cdot\|_2$ on Y) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any initial value $u_0 \in Y$ with $\|u_0 - v_\star\|_1 \leqslant \delta$ the following property hold:
The Cauchy problem (10.9) has a unique solution $u \in C^1(]0,\infty[,X) \cap C([0,\infty[,Y)$ and the solution u satisfies

$$\inf_{\gamma\in G} \|u(t) - a(\gamma)v_\star\|_2 \leqslant \varepsilon \quad \forall\, t \geqslant 0.$$

Moreover, $u_\star$ is called **stable with asymptotic phase** if in addition there exists a $\delta_0 > 0$ such that for any initial value $u_0 \in Y$ with $\|u_0 - v_\star\|_1 \leqslant \delta$ there exists some $\gamma_\infty \in G$ (depending on u_0) such that

$$\|u(t) - a(\gamma_\infty \circ \gamma_\star(t))v_\star\|_2 \to 0 \quad \text{as} \quad t \to \infty.$$

The value γ_∞ is called the **asymptotic phase** and depends in general on the initial data u_0.

In the remaining part of this section we classify general reaction-diffusion systems into the general framework of abstract equivariant evolution equations. As we will see below, rotating waves are a special kind of relative equilibria in reaction-diffusion systems.

Example 10.6 (Reaction diffusion systems, Part 1)**.** Let us consider a system of reaction diffusion equations

$$\begin{aligned} u_t(x,t) =& A\triangle u(x,t) + f(u(x,t)),\, t > 0,\, x \in \mathbb{R}^d,\, d \geqslant 2, \\ u(x,0) =& u_0(x) \qquad\qquad\qquad\quad ,\, t = 0,\, x \in \mathbb{R}^d, \end{aligned} \tag{10.10}$$

on the Banach space $(X,\|\cdot\|)$ given by $\left(L^p(\mathbb{R}^d,\mathbb{K}^N),\|\cdot\|_{L^p}\right)$ with $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, $1 < p < \infty$, diffusion matrix $A \in \mathbb{K}^{N,N}$ and nonlinearity $f : \mathbb{K}^N \to \mathbb{K}^N$. The operator

$$F : X \supset Y = \mathcal{D}(F) \to X, \quad u \mapsto F(u) := A\triangle u + f(u)$$

is defined on the dense subspaces Y, that will be characterized below.
Let $d \in \mathbb{N}$ with $d \geqslant 2$ and let

$$G = \mathrm{SE}(d) = \mathrm{SO}(d) \ltimes \mathbb{R}^d$$

denote the special Euclidean group of dimension $q = \dim \mathrm{SE}(d) = \frac{d(d+1)}{2}$, that is the semidirect product of the special orthogonal group

$$\mathrm{SO}(d) = \{R \in \mathbb{R}^{d,d} \mid R^T = R^{-1} \text{ and } \det(R) = 1\}$$

of dimension $\dim \mathrm{SO}(d) = \frac{d(d-1)}{2}$ with the Abelian translation group $\mathbb{R}^d$ of dimension d. $\mathrm{SE}(d)$ consists of all pairs

$$\gamma = (R,\tau) \in \mathrm{SE}(d),\, R \in \mathrm{SO}(d),\, \tau \in \mathbb{R}^d,$$

is equipped with the group operation $\circ : \mathrm{SE}(d) \times \mathrm{SE}(d) \to \mathrm{SE}(d)$ defined by

$$\gamma_1 \circ \gamma_2 = (R_1, \tau_1) \circ (R_2, \tau_2) = (R_1 R_2, \tau_1 + R_1 \tau_2),$$

and has the unit element $\mathbb{1} = (I_d, 0)$. The inverse of $\gamma = (R, \tau) \in \mathrm{SE}(d)$ is

$$\gamma^{-1} = (R, \tau)^{-1} = (R^{-1}, -R^{-1}\tau).$$

Moreover, the Lie algebra of $\mathrm{SE}(d)$ is given by

$$\mathfrak{g} = T_{\mathbb{1}} SE(d) = \mathfrak{se}(d) = \mathfrak{so}(d) \times \mathbb{R}^d,$$

that is the direct product of the space of skew-symmetric matrices

$$\mathfrak{so}(d) = \{S \in \mathbb{R}^{d,d} \mid S^T = -S\},$$

of dimension $\dim \mathfrak{so}(d) = \frac{d(d-1)}{2}$ with the Abelian translation group $\mathbb{R}^d$ of dimension d. The Lie algebra $\mathfrak{se}(d)$ has also the dimension $q = \dim \mathfrak{se}(d) = \frac{d(d+1)}{2}$.

The left multiplication by some $\gamma = (R, \tau) \in \mathrm{SE}(d)$ on $\mathrm{SE}(d)$ is defined by

$$L_\gamma : \mathrm{SE}(d) \to \mathrm{SE}(d), \quad g = (\tilde{R}, \tilde{\tau}) \mapsto L_\gamma(g) := \gamma \circ g = (R\tilde{R}, \tau + R\tilde{\tau})$$

with derivative at $g = \mathbb{1}$

$$dL_\gamma(\mathbb{1}) : \mathfrak{g} = \mathfrak{se}(d) \to T_\gamma \mathrm{SE}(d), \quad \mu \mapsto dL_\gamma(\mathbb{1})\mu := (RS, R\lambda),$$

where $\gamma = (R, \tau) \in \mathrm{SE}(d)$ and $\mu = (S, \lambda) \in \mathfrak{se}(d)$.

Let $(X, \|\cdot\|)$ still be chosen as above, then we define the $\mathrm{SE}(d)$-action on X via

$$a(\cdot)u : \mathrm{SE}(d) \to X, \quad \gamma = (R, \tau) \mapsto [a(\gamma)u](\cdot) := u(R^{-1}(\cdot - \tau)).$$

A short computation shows that the action $a : \mathrm{SE}(d) \to GL(X)$ is indeed a homomorphism, since we have

$$[a(\mathbb{1})u](x) = [a(I_d, 0)u](x) = u(I_d^{-1}(x - 0)) = u(x),\ x \in \mathbb{R}^d,$$

and

$$\begin{aligned} &[a(\gamma_1 \circ \gamma_2)u](x) = [a(R_1R_2, \tau_1 + R_1\tau_2)u](x) \\ =&u\left((R_1R_2)^{-1}(x - (\tau_1 + R_1\tau_2))\right) = u\left(R_2^{-1}\left(R_1^{-1}(x - \tau_1) - \tau_2\right)\right) \\ =&\left[a(\gamma_1)u(R_2^{-1}(\cdot - \tau_2))\right](x) = [a(\gamma_1)a(\gamma_2)u](x),\ x \in \mathbb{R}^d. \end{aligned}$$

For $u \in Y$ the derivative of $a(\gamma)u$ with respect to $\gamma \in G$ at $\gamma = \mathbb{1}$

$$d[a(\mathbb{1})u] : \mathfrak{g} = \mathfrak{se}(d) \to X, \quad \mu \mapsto d[a(\mathbb{1})u]\mu$$

is given by, cf. Lemma 9.3,

$$d[a(\mathbb{1})u(x)]\mu = -\sum_{i=1}^{d} (Sx + I_d\lambda)_i D_i u(x) = -\langle Sx + I_d\lambda, \nabla u(x)\rangle \tag{10.11}$$

$$= -\sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_j D_i - x_i D_j\right) u(x) - \sum_{l=1}^{d}\lambda_l D_l u(x),$$

where $\mu = (S,\lambda) \in \mathfrak{so}(d)\times\mathbb{R}^d = \mathfrak{se}(d)$.

Let us now discuss the choice of the function space Y, that is obtained from Proposition 10.2: For $(X_0, \|\cdot\|_0) = \left(L^p(\mathbb{R}^d,\mathbb{K}^N), \|\cdot\|_{L^p}\right)$ with $1<p<\infty$ we have

$$X := X_2 = X_1 = X_0 := L^p(\mathbb{R}^d,\mathbb{K}^N), \quad \|\cdot\|_1 = \|\cdot\|_0 := \|\cdot\|_{L^p}.$$

A difficult task in general is to derive a full characterization of X_3. But thanks to our extensive investigations in Chapter 5 we are now able to present a connection between the abstract semigroup theory for the Ornstein-Uhlenbeck operator from Chapter 5 and the maximal domain of F: Let $A\in\mathbb{K}^{N,N}$ satisfy the assumptions (A1) and (A4), then an application of Proposition 10.2 and Theorem 5.25 yield the following characterization for the domain $Y=\mathcal{D}(F)$ of F

$$\begin{aligned} Y := X_3 =& \{u\in L^p \mid \ a(\cdot)u \text{ is continuously differentiable in SE}(d)\} \\ =& \bigcap_{(S,\lambda)\in\mathfrak{se}(d)} \mathcal{D}^p_{\max}(\mathcal{L}_0) \\ =& \bigcap_{(S,\lambda)\in\mathfrak{se}(d)} \left\{v\in W^{2,p} \mid \langle S\cdot + I_d\lambda, \nabla v\rangle \in L^p\right\} \\ =& \left\{v\in W^{2,p} \mid \langle (I_{ij}-I_{ji})\cdot, \nabla v\rangle \in L^p,\ \langle e_l,\nabla v\rangle\in L^p\right\} \\ =& \left\{v\in W^{2,p} \mid \langle (I_{ij}-I_{ji})\cdot, \nabla v\rangle \in L^p\right\} \end{aligned}$$

for every $i=1,\ldots,d-1$, $j=i,\ldots,d$ and $l=1,\ldots,d$. Therefore, we define the **Euclidean Sobolev space** (**of order** 2 **with exponent** p)

$$W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N) := \left\{v\in W^{2,p}(\mathbb{R}^d,\mathbb{K}^N) \mid \langle S\cdot,\nabla v\rangle \in L^p(\mathbb{R}^d,\mathbb{K}^N)\ \forall\, S\in\mathfrak{so}(d)\right\}$$

for every $1<p<\infty$, which is equipped with the norm, compare Corollary 5.26,

$$\|v\|_{W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)} := \|v\|_{W^{2,p}(\mathbb{R}^d,\mathbb{K}^N)} + \sup_{S\in\mathfrak{so}(d)} \|\langle Sx,\nabla v\rangle\|_{L^p(\mathbb{R}^d,\mathbb{K}^N)}.$$

Note that a first step in order to obtain an explicit representation for Y was done in Theorem 5.19 together with our a-priori estimates from Theorem 5.8, that yields a local version for the domain

$$\begin{aligned} Y = \big\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{K}^N)\cap W^{1,p}(\mathbb{R}^d,\mathbb{K}^N) \mid A\triangle v + \langle S\cdot + \lambda I_d, \nabla v\rangle \in L^p(\mathbb{R}^d,\mathbb{K}^N) \\ \forall\,(S,\lambda)\in\mathfrak{se}(d)\big\}, \end{aligned}$$

which is only equipped with the graph norm of $\mathcal{L}_0$. The domain $W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)$ is used in [15] for $p=d=2$ and $\mathbb{K}=\mathbb{R}$ with $H^2_{\mathrm{Eucl}}(\mathbb{R}^2,\mathbb{R}^N) := W^{2,2}_{\mathrm{Eucl}}(\mathbb{R}^2,\mathbb{R}^N)$. We notice that for the space of bounded uniformly continuous functions a full characterization for the domain Y is still an open problem. Only in the scalar real-valued case with $K=\mathbb{R}$ and $N=1$, an application of [29, Proposition 3.5] yields a local version for the domain Y. The abstract representation for the domain Y in

$C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{K}^N)$ was used in [113, Section 2.3.1] for the first time with $d=2$, $\mathbb{K}=\mathbb{R}$ and $N=1$.

A straightforward computation shows that the reaction diffusion system (10.10) is equivariant under the SE(d)-action on $W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)$: On the one hand it is trivial to see that

$$\begin{aligned}a(\gamma)F(u(x)) =& a(R,\tau)\left(A\triangle u(x)+f(u(x))\right)\\ =& A[\triangle u](R^{-1}(x-\tau))+f(u(R^{-1}(x-\tau)))\end{aligned}$$

and on the other hand a short computation shows that

$$\begin{aligned}
&F(a(\gamma)u(x)) = A\triangle\left[u(R^{-1}(x-\tau))\right]+f(u(R^{-1}(x-\tau)))\\
=&A\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}\left[u(R^{-1}(x-\tau))\right]+f(u(R^{-1}(x-\tau)))\\
=&A\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\sum_{k=1}^{d}\left(\frac{\partial}{\partial\xi_k}u(R^{-1}(x-\tau))\right)\left(\frac{\partial}{\partial x_i}\left(R^{-1}(x-\tau)\right)_k\right)+f(u(R^{-1}(x-\tau)))\\
=&A\sum_{i=1}^{d}\sum_{k=1}^{d}\sum_{l=1}^{d}\left(\frac{\partial}{\partial\xi_l}\frac{\partial}{\partial\xi_k}u(R^{-1}(x-\tau))\right)\left(\frac{\partial}{\partial x_i}\left(R^{-1}(x-\tau)\right)_l\right)\left(\frac{\partial}{\partial x_i}\left(R^{-1}(x-\tau)\right)_k\right)\\
&+A\sum_{i=1}^{d}\sum_{k=1}^{d}\left(\frac{\partial}{\partial\xi_k}u(R^{-1}(x-\tau))\right)\left(\frac{\partial^2}{\partial x_i^2}\left(R^{-1}(x-\tau)\right)_k\right)+f(u(R^{-1}(x-\tau)))\\
=&A\sum_{k=1}^{d}\sum_{l=1}^{d}\left(\frac{\partial}{\partial\xi_l}\frac{\partial}{\partial\xi_k}u(R^{-1}(x-\tau))\right)\delta_{lk}+f(u(R^{-1}(x-\tau)))\\
=&A\sum_{k=1}^{d}\left[\frac{\partial^2}{\partial\xi_k^2}u\right](R^{-1}(x-\tau))+f(u(R^{-1}(x-\tau)))\\
=&A\left[\triangle u\right](R^{-1}(x-\tau))+f(u(R^{-1}(x-\tau))).
\end{aligned}$$

Here we used that the equalities

$$\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}\left(R^{-1}(x-\tau)\right)_k=0,\ \sum_{i=1}^{d}\left(\frac{\partial}{\partial x_i}\left(R^{-1}(x-\tau)\right)_l\right)\left(\frac{\partial}{\partial x_i}\left(R^{-1}(x-\tau)\right)_k\right)=\delta_{lk}$$

are satisfied for every $k,l\in\{1,\dots,d\}$ and $(R,\tau)\in\mathrm{SE}(d)$. Moreover, one shows that $a(\gamma)W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)=W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)$ for every $\gamma\in\mathrm{SE}(d)$, but we omit the details.

Relative equilibria of (10.10) on $[0,\infty[$ are now of the form

$$u_\star(x,t)=v_\star\left(R_\star^{-1}(t)\cdot(x-\tau_\star(t))\right),\ v_\star\in W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N),\ \gamma_\star=(R_\star,\tau_\star)\in\mathrm{SE}(d).$$

Examples for relative equilibria of the reaction-diffusion systems (10.10) are rotating waves with $(R_\star(t),\tau_\star(t))=(\exp(tS_\star),x_\star)$ for some $(S_\star,x_\star)\in\mathfrak{se}(d)$, but also traveling waves with $(R_\star(t),\tau_\star(t))=(I_d,\lambda_\star t)$. We are mainly interested in rotating waves. As we know from the abstract theory above, they always come in families: If $u_\star(x,t)=v_\star(e^{-tS_\star}(x-x_\star))$ is a rotating wave of (10.10), then also $v_\star(e^{-tS_\star}R^{-1}(x-(\tau+Rx_\star)))$ is a rotating wave of (10.10) for every $(R,\tau)\in\mathrm{SE}(d)$.

Nonlinear Stability of rotating waves: An important issue is to investigate nonlinear stability of rotating waves, also known as stability with asymptotic phase. Let $u_\star(x,t) = v_\star(e^{-tS_\star}(x - x_\star))$ be a rotating wave of (10.10) with $\gamma_\star(t) = (e^{tS_\star}, x_\star)$ and let two possibly different norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ be given. Then we are interested in nonlinear stability of rotating waves, cp. Definition 10.5:

Problem 1. *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any initial value $u_0 \in W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ with $\|u_0 - v_\star\|_1 \leqslant \delta$ the following property hold: The reaction diffusion system* (10.10) *has a unique solution $u \in C^1(]0,\infty[, L^p(\mathbb{R}^d, \mathbb{K}^N)) \cap C([0,\infty[, W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N))$ and the solution u satisfies*

$$\inf_{\gamma \in \mathrm{SE}(d)} \|u(t) - a(\gamma)v_\star\|_2 \leqslant \varepsilon \quad \forall\, t \geqslant 0.$$

Moreover, there exists a $\delta_0 > 0$ such that for any initial value $u_0 \in W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ with $\|u_0 - v_\star\|_1 \leqslant \delta$ there exists some asymptotic phase $\gamma_\infty \in \mathrm{SE}(d)$ such that the solution u satisfies

$$\|u(t) - a(\gamma_\infty \circ \gamma_\star(t))v_\star\|_2 \to 0 \quad \textit{as} \quad t \to \infty.$$

A nonlinear stability result of 2-dimensional localized rotating waves in parabolic reaction diffusion systems was proved in [15, Theorem 1.1] for $d = p = 2$, $\mathbb{K} = \mathbb{R}$ and $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|_{H^2}$. But so far, there are no further nonlinear stability results for higher dimensional rotating waves. Nonlinear stability results for traveling waves in parabolic reaction diffusion equations are well known in the literature and can be found in the monographs [52], [110] and in the survey article [91]. For a nonlinear stability result of traveling waves in hyperbolic and mixed hyperbolic-parabolic equations we refer to [85], [86], [87] and [88]. A nonlinear stability result for multi-fronts and and multi-pulses in parabolic reaction diffusion equations, that we discuss below in Section 10.5, can be found in [99].

An essential feature of all stability results is to derive nonlinear stability from linear stability (also called strong spectral stability). The proof for the nonlinear stability of 2-dimensional localized rotating waves from [15, Theorem 1.1] requires three essential assumptions: The profile $v_\star$ of the rotating wave and their partial derivatives up to order 2 are localized in the sense of Definition 1.1. The matrix $Df(v_\infty)$ is stable, i.e. $\mathrm{Re}\, Df(v_\infty) < 0$ meaning that all eigenvalues have a negative real part. And finally, the eigenvalues of the linearized operator

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x).$$

satisfy suitable eigenvalue conditions. In order to extend the nonlinear stability result from [15, Theorem 1.1] to rotating waves in several space dimensions, it is useful to analyze the decay of rotating waves and the spectrum of the linearization. In particular, Problem 1 shows that the characterization of the maximal domain $Y = \mathcal{D}(F)$ plays also a fundamental role. The exponential decay of rotating waves we have investigated in our main result from Theorem 1.8. Moreover, we have analyzed the eigenvalue problem for the linearized differential operator both analytically in Chapter 9 and numerically in Section 10.4 below. As we have seen above, the characterization for the domain $W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ is based on the characterization of $\mathcal{D}^p_{\max}(\mathcal{L}_0)$ as described in Theorem 5.25.

10.2 Freezing method for single-structures

In the following we briefly recall the main concept of the **freezing method** for single-structures. The main idea of this approach is to approximate relative equilibria of equivariant evolution equations, which is based on a decomposition of the solution u of (10.9) into a group motion and a profile. For a more general and detailed treatment we refer to [18] and [89], but also to [103], [19], [105], [20], [16].

Consider a general equivariant evolution equation (10.9). We introduce new functions $\gamma(t) \in G$ and $v(t) \in Y$ such that the solution u of (10.9) is of the form

$$u(t) = a(\gamma(t))v(t),\ 0 \leqslant t < T. \tag{10.12}$$

Inserting the ansatz (10.12) into (10.9)

$$\begin{aligned} a(\gamma(t))v_t(t) + d\left[a(\gamma(t))v(t)\right]\gamma_t(t) = \frac{d}{dt}\left[a(\gamma(t))v(t)\right] = u_t(t) \\ = F(u(t)) = F(a(\gamma(t))v(t)) = a(\gamma(t))F(v(t)) \end{aligned}$$

and applying $a(\gamma^{-1}(t))$ to both sides we obtain

$$v_t(t) = F(v(t)) - a(\gamma^{-1}(t))d\left[a(\gamma(t))v(t)\right]\gamma_t(t),\ 0 < t < T. \tag{10.13}$$

At this point, it is convenient to introduce $\mu(t) \in \mathfrak{g} = T_{\mathbb{1}}G$ via

$$\gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu(t),\ 0 < t < T. \tag{10.14}$$

Then, differentiating $a(\gamma(t))a(g)v(t) = a(\gamma(t)\circ g)v(t) = a(L_{\gamma(t)}(g))v(t)$ with respect to g at $g = \mathbb{1}$ yields

$$a(\gamma(t))d\left[a(\mathbb{1})v(t)\right]\mu(t) = d\left[a(\gamma(t))v(t)\right]dL_{\gamma(t)}(\mathbb{1})\mu(t),\ 0 < t < T. \tag{10.15}$$

Thus, requiring (10.14), equation (10.13) can be written as

$$v_t(t) = F(v(t)) - d\left[a(\mathbb{1})v(t)\right]\mu(t),\ 0 < t < T. \tag{10.16}$$

To compensate the extra variable $\mu(t)$, we finally impose $q = \dim\mathfrak{g}$ phase conditions $\Psi(v(t),\mu(t)) = 0$, that are defined by a functional

$$\Psi : Y \times \mathfrak{g} \to \mathfrak{g}^*, \quad (v,\mu) \mapsto \Psi(v,\mu), \tag{10.17}$$

where $\mathfrak{g}^*$ denotes the dual space of the Lie algebra $\mathfrak{g}$, which is isomorphic to $\mathbb{R}^q$, i.e. $\mathfrak{g}^* \cong \mathbb{R}^q$. To take the initial data from equation (10.9) into account, we equip the γ-equation (10.14) with the initial condition $\gamma(0) = \mathbb{1}$. Thus, using (10.12) at $t = 0$ the initial condition for the v-equation (10.16) is given by $v(0) = u_0$.
This leads to the abstract formulation of the freezing method as **differential algebraic evolution equation (DAE)**

$$v_t(t) = F(v(t)) - d\left[a(\mathbb{1})v(t)\right]\mu(t), \qquad v(0) = u_0, \tag{10.18a}$$

$$0 = \Psi(v(t),\mu(t)), \tag{10.18b}$$

$$\gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu(t), \qquad \gamma(0) = \mathbb{1}. \tag{10.18c}$$

The equations (10.18a) and (10.18c) must be satisfied for $t > 0$ and the equation (10.18b) for $t \geqslant 0$. In applications (10.18a) is a PDE, (10.18b) an algebraic constraint and (10.18c) an ODE. The ODE from (10.18c) is called the **reconstruction equation** in [89], is decoupled from the first two equations (10.18a) and (10.18b) and can be solved in a post-processing step.

We now explain a well-studied possibility for the choice of the algebraic constraint (10.18b), that is called the **phase condition**. In the sequel, we will distinguish between the fixed phase condition and the orthogonal phase condition. For this purpose, let us assume that

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}, \quad (u, v) \mapsto \langle u, v \rangle$$

is a continuous inner product on the $\mathbb{K}$-valued Banachspace $(X, \|\cdot\|)$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, i.e. $|u| := \sqrt{\langle u, u \rangle} \leqslant C \|u\|$. If $(X, \|\cdot\|, (\cdot, \cdot))$ is a Hilbert space, we can choose e.g. $\langle \cdot, \cdot \rangle := (\cdot, \cdot)$ and hence $|\cdot| = \|\cdot\|$. But in general we do not assume this.

Type 1: (fixed phase condition). Choose a template function $\hat{v} \in Y$. The **fixed phase condition** is a minimization condition that requires $\hat{v}$ to be the closest point to $v(t)$ on the group orbit of $\hat{v}$ given by $\mathcal{O}(\hat{v}) := \{a(\gamma)\hat{v} \mid \gamma \in G\}$, i.e.

$$\min_{\gamma \in G} |v(t) - a(\gamma)\hat{v}| = |v(t) - \hat{v}| \,.$$

The necessary condition is

$$\begin{aligned}
0 &= \left[\frac{d}{d\gamma} |v(t) - a(\gamma)\hat{v}|^2\right]_{\gamma = \mathbb{1}} = \left[\frac{d}{d\gamma} \langle v(t) - a(\gamma)\hat{v}, v(t) - a(\gamma)\hat{v}\rangle\right]_{\gamma=\mathbb{1}} \\
&= \left[\langle -d\,[a(\gamma)\hat{v}]\,, v(t) - a(\gamma)\hat{v}\rangle + \langle v(t) - a(\gamma)\hat{v}, -d\,[a(\gamma)\hat{v}]\rangle\right]_{\gamma=\mathbb{1}} \\
&= \langle -d\,[a(\mathbb{1})\hat{v}]\,, v(t) - a(\mathbb{1})\hat{v}\rangle + \langle v(t) - a(\mathbb{1})\hat{v}, -d\,[a(\mathbb{1})\hat{v}]\rangle \\
&= -\langle d\,[a(\mathbb{1})\hat{v}]\,, v(t) - \hat{v}\rangle - \langle v(t) - \hat{v}, d\,[a(\mathbb{1})\hat{v}]\rangle \\
&= -2\mathrm{Re}\,\langle v(t) - \hat{v}, d\,[a(\mathbb{1})\hat{v}]\rangle\,,
\end{aligned}$$

that is a mapping from $T_{\mathbb{1}}G$ into $\mathbb{R}^q \cong \mathfrak{g}^*$. In the numerical computations we will replace the phase condition (10.18b) by

$$0 = -2\mathrm{Re}\,\langle v(t) - \hat{v}, d\,[a(\mathbb{1})\hat{v}]\,\omega\rangle \quad \forall\,\omega \in \mathfrak{g}, \tag{10.19}$$

that leads to a DAE of index 2. To reduce the index we choose a basis $e^1, \ldots, e^q$ of $\mathfrak{g} = T_{\mathbb{1}}G$, evaluate (10.19) at $\omega = e^j$ for every $j = 1, \ldots, q$, multiply by $-\frac{1}{2}$, differentiate with respect to t and insert the differential equation (10.18a) to obtain

$$\begin{aligned}
0 &= \frac{d}{dt}\mathrm{Re}\,\left\langle v(t) - \hat{v}, d\,[a(\mathbb{1})\hat{v}]\,e^j\right\rangle = \mathrm{Re}\,\left\langle v_t(t), d\,[a(\mathbb{1})\hat{v}]\,e^j\right\rangle \\
&= \mathrm{Re}\,\left\langle F(v(t)), d\,[a(\mathbb{1})\hat{v}]\,e^j\right\rangle - \mathrm{Re}\,\left\langle d\,[a(\mathbb{1})v(t)]\,\mu(t), d\,[a(\mathbb{1})\hat{v}]\,e^j\right\rangle \\
&=: \Psi_{\mathrm{fix}}^{(j)}(v(t), \mu(t)),
\end{aligned} \tag{10.20}$$

that is the j-th component of the phase operator Ψ_{fix}. This leads to the fixed phase condition $\Psi_{\mathrm{fix}}(v(t), \mu(t)) = 0 \in \mathbb{R}^q \cong \mathfrak{g}^*$. If we replace (10.18b) by (10.20) we end up with a DAE of index 1, provided that $v(t)$ and $\hat{v}$ are sufficiently close for every $t \geqslant 0$.

Type 2: (orthogonal phase condition). The **orthogonal phase condition** is also a minimization condition and requires that the temporal change $|v_t|$ is minimal at each time instance, i.e.

$$\min_{\mu\in\mathfrak{g}} |v_t(t)| = \min_{\mu\in\mathfrak{g}} |F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu| = |F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu(t)|\,.$$

The necessary condition is

$$\begin{aligned}
0 &= \left[\frac{d}{d\mu}\,|v_t(t)|^2\right]_{\mu=\mu(t)} = \left[\frac{d}{d\mu}\,|F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu|^2\right]_{\mu=\mu(t)} \\
&= \left[\frac{d}{d\mu}\,\langle F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu, F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu\rangle\right]_{\mu=\mu(t)} \\
&= [\langle -d\,[a(\mathbb{1})v(t)]\,, F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu\rangle \\
&\quad + \langle F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu, -d\,[a(\mathbb{1})v(t)]\rangle]_{\mu=\mu(t)} \\
&= -\,\langle d\,[a(\mathbb{1})v(t)]\,, F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu(t)\rangle \\
&\quad -\,\langle F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu(t), d\,[a(\mathbb{1})v(t)]\rangle \\
&= -\,2\mathrm{Re}\,\langle F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu(t), d\,[a(\mathbb{1})v(t)]\rangle \\
&= -\,2\mathrm{Re}\,\langle v_t(t), d\,[a(\mathbb{1})v(t)]\rangle\,,
\end{aligned}$$

that is a mapping from $\mathfrak{g} = T_{\mathbb{1}}G$ into $\mathbb{R}^q \cong \mathfrak{g}^*$, i.e. the condition reads as

$$0 = -2\mathrm{Re}\,\langle v_t(t), d\,[a(\mathbb{1})v(t)]\,\omega\rangle \quad \forall\,\omega\in\mathfrak{g}. \tag{10.21}$$

For the numerical computations one usually uses condition (10.21) instead of (10.18b). Next, we choose a basis $e^1,\ldots,e^q$ of $T_{\mathbb{1}}G$, evaluate (10.21) at e^j for every $j=1,\ldots,q$, multiply by $-\frac{1}{2}$ and insert the differential equation (10.18a) to obtain

(10.22)
$$\begin{aligned}
0 &= \mathrm{Re}\,\left\langle v_t(t), d\,[a(\mathbb{1})v(t)]\,e^j\right\rangle \\
&= \mathrm{Re}\,\left\langle F(v(t)) - d\,[a(\mathbb{1})v(t)]\,\mu(t), d\,[a(\mathbb{1})v(t)]\,e^j\right\rangle \\
&= \mathrm{Re}\,\left\langle F(v(t)), d\,[a(\mathbb{1})v(t)]\,e^j\right\rangle - \mathrm{Re}\,\left\langle d\,[a(\mathbb{1})v(t)]\,\mu(t), d\,[a(\mathbb{1})v(t)]\,e^j\right\rangle \\
&=: \Psi^{(j)}_{\mathrm{orth}}(v(t),\mu(t)),
\end{aligned}$$

that is the j-th component of the phase operator Ψ_{orth}. This leads to the orthogonal phase condition $\Psi_{\mathrm{orth}}(v(t),\mu(t)) = 0 \in \mathbb{R}^q \cong \mathfrak{g}^*$. If we replace (10.18b) by (10.22) we end up with a DAE of index 1, provided that the isotropy group of $v(t)$, that is given by $H_{v(t)} := \{\gamma\in G \mid a(\gamma)v(t) = v(t)\}$, is trivial for every $t\geqslant 0$.

Example 10.7 (Reaction diffusion systems, Part 2)**.** We continue with Example 10.6 and assume that the solution u of (10.10) can be written as

$$u(x,t) = a(\gamma(t))v(x,t) = v(R(t)^{-1}(x-\tau(t)),t),\ t\geqslant 0,\ x\in\mathbb{R}^d,\ d\geqslant 2, \tag{10.23}$$

where $\gamma(t) = (R(t), \tau(t)) \in \mathrm{SE}(d)$ and $v(\cdot,t) \in Y$. Inserting the freezing ansatz (10.23) into (10.10) and applying $a(\gamma^{-1}(t))$ to both sides yields

$$v_t(x,t) = A\triangle v(x,t) + f(v(x,t)) - a(\gamma^{-1}(t))d\left[a(\gamma(t))v(x,t)\right]\gamma_t(t), \tag{10.24}$$

for $t > 0$ and $x \in \mathbb{R}^d$. Introducing $\mu(t) \in \mathfrak{se}(d) = T_{\mathbb{1}}\mathrm{SE}(d)$ via (10.14), the v-equation (10.24) can be transformed into (10.16), where $d[a(\mathbb{1})v(x,t)]\mu(t)$ is given by (10.11). To compensate the extra variable $\mu(t)$ we additionally require $\dim \mathfrak{se}(d) = \frac{d(d+1)}{2}$ phase conditions, given by the functional

$$\Psi : Y \times \mathfrak{se}(d) \to (\mathfrak{se}(d))^*, \quad (v,\mu) \mapsto \Psi(v,\mu), \tag{10.25}$$

where the dual space $(\mathfrak{se}(d))^*$ of the Lie algebra $\mathfrak{se}(d)$ is isomorphic to $\mathbb{R}^{\dim \mathfrak{se}(d)}$, i.e. $(\mathfrak{se}(d))^* \cong \mathbb{R}^{\frac{d(d-1)}{2}}$. The reconstruction equation is given by

$$\begin{pmatrix} R_t(t) \\ \tau_t(t) \end{pmatrix} = \gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu(t) = \begin{pmatrix} R(t)S(t) \\ R(t)\lambda(t) \end{pmatrix}, \quad \begin{pmatrix} R(0) \\ \tau(0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix},$$

where $\gamma(t) = (R(t), \tau(t)) \in \mathrm{SE}(d)$ and $\mu(t) = (S(t), \lambda(t)) \in \mathfrak{se}(d)$. Thus, the freezing method yields a **partial differential algebraic evolution equation (PDAE)**

$$\begin{aligned} v_t(x,t) =& A\triangle v(x,t) + f(v(x,t)) + \langle S(t)x + I_d\lambda(t), \nabla v(x,t)\rangle, \\ v(x,0) =& u_0(x), \end{aligned} \tag{10.26a}$$

$$0 = \Psi(v(\cdot,t), \mu(t)), \tag{10.26b}$$

$$\begin{pmatrix} R_t(t) \\ \tau_t(t) \end{pmatrix} = \begin{pmatrix} R(t)S(t) \\ R(t)\lambda(t) \end{pmatrix}, \quad \begin{pmatrix} R(0) \\ \tau(0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix}. \tag{10.26c}$$

Let us discuss the two mentioned possibilities for the choice of the phase condition (10.26b). For this purpose let us consider the Hilbert space $X = L^2(\mathbb{R}^d, \mathbb{K}^N)$ equipped with the inner product

$$(\cdot,\cdot)_{L^2} : L^2(\mathbb{R}^d, \mathbb{K}^N) \times L^2(\mathbb{R}^d, \mathbb{K}^N) \to \mathbb{K}, \quad (u,v) \mapsto (u,v)_{L^2} := \int_{\mathbb{R}^d} \overline{u(x)}^T v(x)dx$$

and $\langle \cdot,\cdot \rangle := (\cdot,\cdot)_{L^2}$. Moreover, let $\hat{v} \in W^{2,2}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ denote a template function.

The necessary condition for the fixed phase condition from (10.19) yields

$$0 = \mathrm{Re}\left(v(t) - \hat{v}, \left\langle \tilde{S}x + I_d\tilde{\lambda}, \nabla\hat{v} \right\rangle\right)_{L^2} \quad \forall\, (\tilde{S}, \tilde{\lambda}) \in \mathfrak{se}(d).$$

Plugging $(I_{ij} - I_{ji}, 0)$ and $(0, e_l)$, that is indeed a basis of $\mathfrak{se}(d)$, into this equation we obtain

$$\begin{aligned} 0 =& \mathrm{Re}\,\left(v(t) - \hat{v}, (x_j D_i - x_i D_j)\hat{v}\right)_{L^2}, \quad i = 1,\ldots,d-1,\ j = i+1,\ldots,d, \\ 0 =& \mathrm{Re}\,\left(v(t) - \hat{v}, D_l\hat{v}\right)_{L^2}, \quad l = 1,\ldots,d. \end{aligned}$$

To guarantee that the index of the PDAE equals 1 we have to require that

$$\mathrm{Re}\,\left((x_j D_i - x_i D_j)v(t), (x_l D_k - x_k D_l)\hat{v}\right)_{L^2} \neq 0,$$

$$\begin{aligned}
&\mathrm{Re}\,\left((x_jD_i - x_iD_j)v(t), D_m\hat{v}\right)_{L^2} \neq 0,\\
&\mathrm{Re}\,\left(D_nv(t), (x_lD_k - x_kD_l)\hat{v}\right)_{L^2} \neq 0,\\
&\mathrm{Re}\,\left(D_nv(t), D_m\hat{v}\right)_{L^2} \neq 0.
\end{aligned}$$

is satisfied for every $i,k = 1,\ldots,d-1$, $j = i+1,\ldots,d$, $l = k+1,\ldots,d$ and $n,m = 1,\ldots,d$.

The necessary condition for the orthogonal phase condition from (10.21) yields

$$0 = -2\mathrm{Re}\,\left(v_t(t), \left\langle \tilde{S}x + I_d\tilde{\lambda}, \nabla v(t)\right\rangle\right)_{L^2} \quad \forall\,(\tilde{S},\tilde{\lambda}) \in \mathfrak{se}(d).$$

Plugging $(I_{ij} - I_{ji}, 0)$ and $(0, e_l)$ once more, we obtain

$$\begin{aligned}
0 =&\mathrm{Re}\,\left(v_t(t), (x_jD_i - x_iD_j)v(t)\right)_{L^2}, \quad i = 1,\ldots,d-1,\ j = i+1,\ldots,d\\
0 =&\mathrm{Re}\,\left(v_t(t), D_lv(t)\right)_{L^2}, \quad l = 1,\ldots,d.
\end{aligned}$$

To guarantee that the index of the PDAE equals 1 we require that

$$\begin{aligned}
&\mathrm{Re}\,\left((x_jD_i - x_iD_j)v(t), (x_lD_k - x_kD_l)v(t)\right)_{L^2} \neq 0,\\
&\mathrm{Re}\,\left((x_jD_i - x_iD_j)v(t), D_mv(t)\right)_{L^2} \neq 0,\\
&\mathrm{Re}\,\left(D_nv(t), D_mv(t)\right)_{L^2} \neq 0.
\end{aligned}$$

Approximation of localized rotating waves on bounded domains: An important issue is to investigate approximations of rotating waves to bounded domains. We formulate such a result for the choice $p = 2$.

Let $(v_\star, (S_\star, \lambda_\star)) \in W^{2,2}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N) \times \mathfrak{se}(d)$ be a solution of

$$\begin{aligned}
0 =&A\triangle v_\star(x) + \langle S_\star x + \lambda_\star, \nabla v_\star(x)\rangle + f(v_\star(x)) &&, x \in \mathbb{R}^d,\\
0 =&\mathrm{Re}\,\langle v_\star - \hat{v}, (x_jD_i - x_iD_j)\hat{v}\rangle_{L^2(\mathbb{R}^d,\mathbb{K}^N)} &&, i = 1,\ldots,d-1,\ j = i+1,\ldots,d\\
0 =&\mathrm{Re}\,\langle v_\star - \hat{v}, D_l\hat{v}\rangle_{L^2(\mathbb{R}^d,\mathbb{K}^N)} &&, l = 1,\ldots,d
\end{aligned}$$

where $\hat{v} \in W^{2,2}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ denotes an appropriate reference function. Then we are interested in solving the following problem:

Problem 2. *There exist some $\rho > 0$ and $R_0 > 0$ such that for every radius $R > R_0$ the boundary value problem*

$$\begin{aligned}
0 =&A\triangle v_R(x) + \langle S_R x + \lambda_R, \nabla v_R(x)\rangle + f(v_R(x)) &&, x \in B_R(0),\\
0 =&v_R(x) &&, x \in \partial B_R(0),\\
0 =&\mathrm{Re}\,\langle v_R - \hat{v}, (x_jD_i - x_iD_j)\hat{v}\rangle_{L^2(B_R(0),\mathbb{K}^N)} &&, i = 1,\ldots,d-1,\ j = i+1,\ldots,d,\\
0 =&\mathrm{Re}\,\langle v_R - \hat{v}, D_l\hat{v}\rangle_{L^2(B_R(0),\mathbb{K}^N)} &&, l = 1,\ldots,d,
\end{aligned}$$

has a unique solution $(v_R, (S_R, \lambda_R))$ in a neighborhood of

$$\begin{aligned}
B_\rho(v_\star|_{B_R(0)}, (S_\star, \lambda_\star)) = \Big\{&(v, (S,\lambda)) \in W^{2,2}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N) \times \mathfrak{se}(d) \mid \\
&\left\|v_\star|_{B_R(0)} - v\right\|_{W^{2,2}_{\mathrm{Eucl}}(B_R(0),\mathbb{K}^N)} + d((S_\star,\lambda_\star),(S,\lambda)) \leqslant \rho\Big\}.
\end{aligned}$$

Moreover, there exist some $C > 0$ and $\eta > 0$ such that

$$\|v_R - v_\star\|_{W^{2,2}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)} + d((S_R,\lambda_R),(S_\star,\lambda_\star)) \leqslant Ce^{-\eta R}.$$

An approximation theorem for relative equilibria on bounded intervals in one space dimension can be found in [98] and [104]. But so far, there seems to be no approximation results for localized rotating waves. This is still an open problem.

In order to extend the result from [98] to an approximation theorem for localized rotating waves, as formulated in Problem 2, we must analyze solvability and uniqueness of the inhomogeneous Cauchy problem for the Ornstein-Uhlenbeck operator on bounded domains, compare Section 1.6. Moreover, we must study the truncation error. Therefore, we plug the solution of the $v_\star$-equation into the v_R-equation. If every term is exponentially small, then we can conclude that the truncation error is exponentially small. Of course, $v_\star$ satisfies the PDE from the v_R-equation on the whole $B_R(0)$. But both the boundary condition and the phase conditions possess a defect. To show that these defects are exponentially small, we need pointwise exponential decay of the rotational term and of the derivatives of $v_\star$ up to order 2. Our main Theorem 1.8 implies exponential decay of $v_\star$ in L^p-spaces, but does not yet provide us with pointwise estimates. For this purpose, we must extend Theorem 1.8 to spaces of bounded uniformly continuous functions or to spaces of Hölder continuous functions, compare Section 1.6.

Example 10.8 (Reaction diffusion systems, Part 3)**.** We continue with Example 10.7 and compute the motion $\gamma(t) = (R(t), \tau(t)) \in \mathrm{SE}(d)$ in the special Euclidean group when the solution $v(t)$ has reached its relative equilibrium $v_\star$, i.e. we compute $\gamma(t)$ for a given $\mu_\star = (S_\star, \lambda_\star) \in \mathfrak{se}(d)$ from the reconstruction equation

$$\begin{pmatrix} R_t(t) \\ \tau_t(t) \end{pmatrix} = \gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu_\star = \begin{pmatrix} R(t)S_\star \\ R(t)\lambda_\star \end{pmatrix}, \quad \begin{pmatrix} R(0) \\ \tau(0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix}, \tag{10.27}$$

The R-equation is decoupled from the τ-equation and admits the solution

$$R(t) = \exp(tS_\star). \tag{10.28}$$

Inserting the solution for R from (10.28) into the τ-equation yields

$$\tau_t(t) = \exp(tS_\star)\lambda_\star, \quad \tau(0) = 0$$

with solution

$$\tau(t) = E(tS_\star)t\lambda_\star, \quad E(X) := \sum_{n=0}^{\infty} \frac{X^n}{(n+1)!}, \quad X \in \mathbb{R}^{d,d}. \tag{10.29}$$

Note that

$$\begin{aligned} \tau_t(t) &= \frac{d}{dt}\left[E(tS_\star)t\lambda_\star\right] = \frac{d}{dt}\sum_{n=0}^{\infty} \frac{t^{n+1}S_\star^n}{(n+1)!}\lambda_\star = \sum_{n=0}^{\infty} \frac{(tS_\star)^n}{n!}\lambda_\star = \exp(tS_\star)\lambda_\star, \\ \tau(0) &= E(0 \cdot S_\star) \cdot 0 \cdot \lambda_\star = I_d \cdot 0 \cdot \lambda_\star = 0. \end{aligned}$$

If $X \in \mathbb{R}^{d,d}$ is invertible, we can represent $E(X)$ by

$$E(X) = X^{-1}\sum_{n=0}^{\infty} \frac{X^{n+1}}{(n+1)!} = X^{-1}\left(\exp(X) - I_d\right) = \left(\exp(X) - I_d\right) X^{-1}.$$

Thus, we deduce from (10.28) and (10.29) that the solution for the reconstruction equation (10.27) is given by

$$\begin{pmatrix} R_\star(t) \\ \tau_\star(t) \end{pmatrix} := \begin{pmatrix} R(t) \\ \tau(t) \end{pmatrix} = \begin{pmatrix} \exp(tS_\star) \\ E(tS_\star)t\lambda_\star \end{pmatrix},\ 0 \leqslant t \leqslant T. \tag{10.30}$$

Consider the relative equilibrium

$$u_\star(x,t) = a(R_\star(t),\tau_\star(t))v_\star(x) = v_\star\left(R_\star(t)^{-1}(x-\tau_\star(t))\right).$$

If a special point $x_\star \in \mathbb{R}^d$ of the profile $v_\star$ is of interest, e.g. the tip of a spiral as in [21], then this point will be visible at position $x(t)$ with

$$x_\star = R_\star(t)^{-1}(x(t)-\tau_\star(t)).$$

The solution (10.30) of the reconstruction equation (10.27) yields

$$\begin{aligned} x(t) =& R_\star(t)x_\star + \tau_\star(t) = \exp(tS_\star)x_\star + E(tS_\star)t\lambda_\star \\ =& \sum_{n=0}^{\infty}\frac{(tS_\star)^n}{n!}x_\star + \sum_{n=0}^{\infty}\frac{(tS_\star)^n}{(n+1)!}t\lambda_\star = I_d x_\star + \sum_{n=1}^{\infty}\frac{(tS_\star)^n}{n!}x_\star + \sum_{n=0}^{\infty}\frac{(tS_\star)^n}{(n+1)!}t\lambda_\star \\ =& x_\star + \sum_{n=0}^{\infty}\frac{t^{n+1}S_\star^n\left(S_\star x_\star + \lambda_\star\right)}{(n+1)!}. \end{aligned}$$

For $x(t) \in \mathbb{R}^d$, that remain fixed with respect to the time evolution, the point $x_\star \in \mathbb{R}^d$ must satisfy the equation

$$S_\star x_\star + \lambda_\star = 0 \tag{10.31}$$

for a given $\mu_\star = (S_\star,\lambda_\star) \in \mathfrak{se}(d)$. In the following we discuss about the solutions of (10.31). For this purpose, we define the **kernel** (or **null space**) and the **range of** $S_\star$ by

$$\begin{aligned} \mathcal{N}(S_\star) &:= \left\{x_\star \in \mathbb{R}^d \mid S_\star x_\star = 0\right\}, \\ \mathcal{R}(S_\star) &:= \left\{y_\star \in \mathbb{R}^d \mid \exists\, x_\star \in \mathbb{R}^d :\ S_\star x_\star = y_\star\right\}. \end{aligned}$$

Case 1: ($\mathrm{rank}(S_\star) = d$). In this case the matrix $S_\star$ has full rank, i.e. $\mathcal{N}(S_\star) = \{0\}$ is trivial and $\mathcal{R}(S_\star) = \mathbb{R}^d$, hence $S_\star$ is invertible and the only point that remains fixed in time is given by

$$c_{\mathrm{rot}} := -S_\star^{-1}\lambda_\star \in \mathbb{R}^d.$$

The point $c_{\mathrm{rot}} \in \mathbb{R}^d$ is called the **center of rotation in** $\mathbb{R}^d$, since the relative equilibrium satisfies

$$\begin{aligned} u_\star(x,t) =& v_\star\left(R_\star^{-1}(t)\left(x-\tau_\star(t)\right)\right) = v_\star\left(\exp(-tS_\star)(x-E(tS_\star)t\lambda_\star)\right) \\ =& v_\star\left(\exp(-tS_\star)(x-(\exp(tS_\star)-I_d)(tS_\star)^{-1}t\lambda_\star)\right) \\ =& v_\star\left(\exp(-tS_\star)x - S_\star^{-1}\lambda_\star + \exp(-tS_\star)S_\star^{-1}\lambda_\star\right) \\ =& v_\star\left(\exp(-tS_\star)(x+S_\star^{-1}\lambda_\star) - S_\star^{-1}\lambda_\star\right) \\ =& v_\star\left(\exp(-tS_\star)(x-c_{\mathrm{rot}}) + c_{\mathrm{rot}}\right). \end{aligned}$$

Defining the shifted profile $v_\star^{c_{\mathrm{rot}}}(x) := v_\star(x+c_{\mathrm{rot}})$ we obtain the rotating wave

$$u_\star(x,t) = v_\star^{c_{\mathrm{rot}}}\left(\exp(-tS_\star)(x-c_{\mathrm{rot}})\right).$$

Case 2: ($\mathrm{rank}(S_\star) < d$). In order to solve the equation (10.31), let us first consider the corresponding homogeneous equation $S_\star x_\star = 0$. Using the orthogonal transformation $S_\star = P\Lambda^{S_\star}_{\mathrm{block}}P^T$ from Section 3 with block diagonal matrix $\Lambda^{S_\star}_{\mathrm{block}}$ yields

$$S_\star x_\star = 0 \quad\Longleftrightarrow\quad x_\star \in \mathcal{N}(S_\star) = \mathrm{span}\left\{Pe_{k+1},\ldots,Pe_d\right\} \cup \{0\}.$$

Let us next consider the inhomogeneous equation (10.31). To guarantee that equation (10.31) admits at least one solution $x_\star$ we must require $-\lambda_\star \in \mathcal{R}(S_\star)$, otherwise there exists no solution. If $-\lambda_\star \in \mathcal{R}(S_\star)$, then we deduce

$$S_\star x_\star = -\lambda_\star \quad\Longleftrightarrow\quad x_\star \in \left\{x_{\mathrm{sv}} + x_{\mathrm{dv}} \mid x_{\mathrm{dv}} \in \mathcal{N}(S_\star)\right\}, \tag{10.32}$$

where x_{sv} is any solution of (10.31). Here, x_{sv} and x_{dv} are called the **support vector** and the **direction vector in $\mathbb{R}^d$**, respectively. Using that x_{sv} is a solution of (10.31), a formal computation

$$\begin{aligned}
&- R_\star^{-1}(t)\tau_\star(t) = -\exp(-tS_\star)E(tS_\star)t\lambda_\star \\
&= -\left(\sum_{n=0}^{\infty}\frac{(-tS_\star)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(tS_\star)^n}{(n+1)!}\right)t\lambda_\star = -\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-tS_\star)^k(tS_\star)^{n-k}}{k!(n-k+1)!}\right)t\lambda_\star \\
&= -\left(\sum_{n=0}^{\infty}(tS_\star)^n\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k+1)!}\right)t\lambda_\star = -\left(\sum_{n=0}^{\infty}\frac{(-tS_\star)^n}{(n+1)!}\right)t\lambda_\star \\
&= \left(\sum_{n=0}^{\infty}\frac{(-tS_\star)^n}{(n+1)!}\right)tS_\star x_{\mathrm{sv}} = -\left(\sum_{n=0}^{\infty}\frac{(-tS_\star)^{n+1}}{(n+1)!}\right)x_{\mathrm{sv}} = -\left(\sum_{n=1}^{\infty}\frac{(-tS_\star)^n}{n!}\right)x_{\mathrm{sv}} \\
&= \left(I_d - \sum_{n=0}^{\infty}\frac{(-tS_\star)^n}{n!}\right)x_{\mathrm{sv}} = \left(I_d - \exp(-tS_\star)\right)x_{\mathrm{sv}} = \left(I_d - R_\star^{-1}(t)\right)x_{\mathrm{sv}}
\end{aligned}$$

shows that the relative equilibrium satisfies

$$\begin{aligned}
u_\star(x,t) =& v_\star\left(R_\star^{-1}(t)\left(x - \tau_\star(t)\right)\right) = v_\star\left(R_\star^{-1}(t)\left(x - x_{\mathrm{sv}}\right) + x_{\mathrm{sv}}\right) \\
=& v_\star\left(\exp\left(-tS_\star\right)\left(x - x_{\mathrm{sv}}\right) + x_{\mathrm{sv}}\right).
\end{aligned}$$

Defining the shifted profile $v_\star^{x_{\mathrm{sv}}}(x) := v_\star(x + x_{\mathrm{sv}})$ we obtain the rotating wave

$$u_\star(x,t) = v_\star^{x_{\mathrm{sv}}}\left(\exp(-tS_\star)(x - x_{\mathrm{sv}})\right).$$

Obviously, the choice of the support vector x_{sv} is still arbitrary. To make the choice unique, we solve a rank-deficient least squares problem, [42, Section 5.5]: Let $(S_\star,\lambda_\star) \in \mathfrak{se}(d)$ with $\mathrm{rank}(S_\star) = 2k < d$ and let the singular value decomposition $S_\star = U\Sigma V^T$ of $S_\star$ from (3.3) be given. An application of [42, Theorem 5.5.1] shows that

$$x_{\mathrm{sv}} = -\sum_{i=1}^{2k}\frac{U_{\cdot,i}^T\lambda_\star}{\Sigma_{i,i}}V_{\cdot,i} \tag{10.33}$$

minimizes $\left\|S_\star x + \lambda_\star\right\|_2$ with respect to the Euclidean norm $\left\|\cdot\right\|_2$ and has the smallest 2-norm of all **minimizers** x belonging to the **set of minimizers** $\chi = \left\{x \in \mathbb{R}^d \mid \left\|S_\star x + \lambda_\star\right\|_2 = \min\right\}$. Here, $U_{\cdot,i}$ and $V_{\cdot,i}$ denote the i-th column of U and V, respectively.

Note that the solution set from (10.32) one obtains from the freezing method for free. This set yields a center of rotation for $d = 2$, an axis of rotation for $d = 3$, a center or a plane of rotation for $d = 4$ and several axis of rotation as well as hyperplanes of rotation for $d \geqslant 5$.

Moreover, note that the calculations above extends also to

$$\begin{pmatrix} R_t(t) \\ \tau_t(t) \end{pmatrix} = \gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu_\star = \begin{pmatrix} R(t)S_\star \\ R(t)\lambda_\star \end{pmatrix}, \quad \begin{pmatrix} R(0) \\ \tau(0) \end{pmatrix} = \begin{pmatrix} R_0 \\ \tau_0 \end{pmatrix},$$

for some initial value $(R_0, \tau_0) \in \mathrm{SE}(d)$. This equation admits the solution

$$\begin{pmatrix} R_\star(t) \\ \tau_\star(t) \end{pmatrix} := \begin{pmatrix} R(t) \\ \tau(t) \end{pmatrix} = \begin{pmatrix} R_0 \exp(tS_\star) \\ R_0 E(tS_\star)t\lambda_\star + \tau_0 \end{pmatrix}, \; 0 \leqslant t \leqslant T.$$

But in the numerical examples below we only consider the case $(R_0, \tau_0) = (I_d, 0)$.

We next consider the special cases $d = 2$ and $d = 3$ in more detail.

Special Case 1: $(d = 2)$. From the assumption $0 \neq S_\star \in \mathfrak{so}(2)$ we can deduce that the matrix $S_\star$ is invertible, i.e. $\mathcal{N}(S_\star) = \{0\}$ and $\mathcal{R}(S_\star) = \mathbb{R}^2$. This yields the **center of rotation in** $\mathbb{R}^2$

$$\begin{aligned} c_{\mathrm{rot}}^{2D} &:= -\, S_\star^{-1}\lambda_\star = -\begin{pmatrix} 0 & S_{12} \\ -S_{12} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{pmatrix} \\ &= -\frac{1}{S_{12}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{pmatrix} = \frac{1}{S_{12}} \begin{pmatrix} \lambda^{(2)} \\ -\lambda^{(1)} \end{pmatrix} \in \mathbb{R}^2. \end{aligned} \tag{10.34}$$

and the relative equilibrium satsifies

$$u_\star(x,t) = v_\star\left(\exp(-tS_\star)(x - c_{\mathrm{rot}}^{2D}) + c_{\mathrm{rot}}^{2D}\right) = v_\star^{c_{\mathrm{rot}^{2D}}}\left(exp(-tS_\star)(x - c_{\mathrm{rot}}^{2D})\right).$$

In particular, the time, that the pattern needs for exact one rotation about c_{rot}^{2D}, is given by the **temporal period of rotation for** $d = 2$

$$T^{2D} = \frac{2\pi}{|\sigma_1|} = \frac{2\pi}{|S_{12}|},$$

that can be determined by $T := \min\{t > 0 \mid \exp(-tS_\star) = I_d\}$ for arbitrary space dimensions $d \geqslant 2$.

Special Case 2: $(d = 3)$. For space dimension $d = 3$ the matrix $0 \neq S_\star \in \mathfrak{so}(3)$ is indeed not invertible. More precisely, the null space of $S_\star$ contains the **direction vector** of the axis of rotation, i.e.

$$\mathcal{N}(S_\star) = \left\{ r \begin{pmatrix} S_{23} \\ -S_{13} \\ S_{12} \end{pmatrix} \mid r \in \mathbb{R} \right\}.$$

Thus, the axis of rotation has the form

$$x_\star = x_{\mathrm{sv}}^{3D} + r \begin{pmatrix} S_{23} \\ -S_{13} \\ S_{12} \end{pmatrix}, \; r \in \mathbb{R}.$$

Using the singular value decomposition $S = U\Sigma V^T$ from (3.3) with

$$S = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \ \Lambda_{\text{block}}^{S_\star} = \begin{pmatrix} 0 & \sigma_1 & 0 \\ -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P = \frac{1}{\sigma_1}\begin{pmatrix} \frac{b\sigma_1}{\sqrt{b^2+c^2}} & -\frac{ac}{\sqrt{b^2+c^2}} & c \\ \frac{c\sigma_1}{\sqrt{b^2+c^2}} & \frac{ab}{\sqrt{b^2+c^2}} & -b \\ 0 & \sqrt{b^2+c^2} & a \end{pmatrix}, \ \Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U = \frac{1}{\sigma_1}\begin{pmatrix} -\frac{ac}{\sqrt{b^2+c^2}} & \frac{b\sigma_1}{\sqrt{b^2+c^2}} & c \\ \frac{ab}{\sqrt{b^2+c^2}} & \frac{c\sigma_1}{\sqrt{b^2+c^2}} & -b \\ \sqrt{b^2+c^2} & 0 & a \end{pmatrix}, \ V = \frac{1}{\sigma_1}\begin{pmatrix} -\frac{b\sigma_1}{\sqrt{b^2+c^2}} & -\frac{ac}{\sqrt{b^2+c^2}} & c \\ -\frac{c\sigma_1}{\sqrt{b^2+c^2}} & -\frac{ab}{\sqrt{b^2+c^2}} & -b \\ 0 & \sqrt{b^2+c^2} & a \end{pmatrix},$$

$\sigma_1 := \sqrt{a^2+b^2+c^2} > 0$ and $a := S_{12}$, $b := S_{13}$, $c := S_{23}$ and solving the rank-deficient least squares problem leads to the **support vector**

$$x_{\text{sv}}^{3D} = \frac{1}{S_{12}^2+S_{13}^2+S_{23}^2}\begin{pmatrix} S_{12}\lambda_2 + S_{13}\lambda_3 \\ -S_{12}\lambda_1 + S_{23}\lambda_3 \\ -S_{13}\lambda_1 - S_{23}\lambda_2 \end{pmatrix},$$

cf. (10.33). This yields a very simple formula for the **axis of rotation**

(10.35)
$$a_{\text{rot}}^{3D}(r) := \frac{1}{S_{12}^2+S_{13}^2+S_{23}^2}\begin{pmatrix} S_{12}\lambda_2 + S_{13}\lambda_3 \\ -S_{12}\lambda_1 + S_{23}\lambda_3 \\ -S_{13}\lambda_1 - S_{23}\lambda_2 \end{pmatrix} + r\begin{pmatrix} S_{23} \\ -S_{13} \\ S_{12} \end{pmatrix}, \ r \in \mathbb{R}.$$

In this case, the relative equilibrium satisfies

$$u_\star(x,t) = v_\star\left(\exp(-tS_\star)(x - x_{\text{sv}}^{3D}) + x_{\text{sv}}^{3D}\right) = v_\star^{x_{\text{sv}}^{3D}}\left(exp(-tS_\star)(x - x_{\text{sv}}^{3D})\right)$$

In particular, the time, that the pattern needs for exact one rotation about $a_{\text{rot}}(r)$, is given by the **temporal period of rotation for** $d = 3$

$$T^{3D} = \frac{2\pi}{|\sigma_1|} = \frac{2\pi}{|\sqrt{S_{12}^2+S_{13}^2+S_{23}^2}|}.$$

10.3 Numerical examples of single-structures

In this section we investigate numerically the freezing system

$$\begin{aligned} v_t(x,t) =& A\triangle v(x,t) + f(v(x,t)) + \langle S(t)x + I_d\lambda(t), \nabla v(x,t)\rangle, \\ v(x,0) =& u_0(x), \end{aligned} \quad (10.36a)$$

$$0 = \Psi(v(\cdot,t),\mu(t)), \quad (10.36b)$$

$$\begin{pmatrix} R_t(t) \\ \tau_t(t) \end{pmatrix} = \begin{pmatrix} R(t)S(t) \\ R(t)\lambda(t) \end{pmatrix}, \quad \begin{pmatrix} R(0) \\ \tau(0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix}. \quad (10.36c)$$

from Example 10.7, cf. (10.26). Our aim is to compute an approximation of the rotating wave $u_\star$, in the sense that we approximate the profile $v_\star$ and the velocities $(S_\star, \lambda_\star)$, separately.

To solve the partial differential algebraic system (10.36) numerically, we first truncate (10.36) from the original domain $\mathbb{R}^d$ to a bounded domain $\Omega \subseteq \mathbb{R}^d$. Since the truncation requires additional boundary conditions and since we don't want to affect the asymptotic behavior of the wave near the boundary, we purpose homogeneous Neumann boundary conditions, also known as no-flux boundary conditions. This leads to truncated versions of (10.36) and their solutions can be considered as approximations of the original rotating wave.

In the examples below, we numerically solve the truncated version of the PDAE from (10.36) including the additional boundary conditions. The finite domain $\Omega \subseteq \mathbb{R}^d$ will be a circular disk if $d = 2$ or a cube if $d = 3$. We use continuous piecewise linear finite elements in space and the BDF method of order 2 in time. The computations require suitable initial data v_0 and reference functions $\hat{v}$ that both come actually from a simulation and are chosen as the solution u at the end time from Example 2.1, 2.2 and 2.3, respectively. For the numerical computations we use Comsol Multiphysics$^{\mathrm{TM}}$, [1].

Example 10.9 (Ginzburg-Landau equation)**.** Consider the freezing system for the cubic-quintic complex Ginzburg-Landau equation (QCGL) from Example 2.1

$$\begin{aligned}
&(10.37\text{a}) && v_t = \alpha\triangle v + v\left(\mu + \beta\,|v|^2 + \gamma\,|v|^4\right) \\
& && \qquad + \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}(x_j D_i - x_i D_j)v + \sum_{l=1}^{d}\lambda_l D_l v, \quad v(\cdot,t_0) = v_0 \\
&(10.37\text{b}) && 0 = \mathrm{Re}\,(v-\hat{v}, (x_j D_i - x_i D_j)\hat{v})_{L^2}, \quad i = 1,\ldots,d-1,\ j = i+1,\ldots,d \\
&(10.37\text{c}) && 0 = \mathrm{Re}\,(v-\hat{v}, D_l\hat{v})_{L^2}, \quad l = 1,\ldots,d \\
&(10.37\text{d}) && \begin{pmatrix} R_t \\ \tau_t \end{pmatrix} = \begin{pmatrix} RS \\ R\lambda \end{pmatrix}, \quad \begin{pmatrix} R(t_0) \\ \tau(t_0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix},
\end{aligned}$$

with $v : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$, $\alpha, \beta, \gamma, \mu \in \mathbb{C}$ and $\mathrm{Re}\,\alpha > 0$.

(**1**): For the parameter values (2.4) we know from Example 2.1 that the QCGL exhibits spinning soliton solutions $u(x,t)$ for space dimensions $d = 2$ and $d = 3$, cf. Figure 2.1. In the examples below we approximate the patterns $v_\star$ as well as the rotational and translational velocities, that are contained in S and λ, respectively. Further, using the reconstruction equation (10.37d) we determine the centers of rotation $c_{\mathrm{rot}}^{2D} \in \mathbb{R}^2$ for $d = 2$ and the axis of rotation $a_{\mathrm{rot}}^{3D} \in C(\mathbb{R},\mathbb{R}^3)$ for $d = 3$.

Figure 10.1(b)–10.1(d) shows the real part (b), imaginary part (c) and the absolute value (d) for the approximation of the profile $v_\star$ of the spinning soliton in $\mathbb{R}^2$ as the solution of (10.37) on a circular disk of radius $R = 20$ centered in the origin at time $t = 400$. Figure 10.1(e) shows the translational and rotational velocities. $\mu^{(1)} = \lambda^{(1)}$ and $\mu^{(2)} = \lambda^{(2)}$ denotes the translational velocity in x_1- and x_2-direction, respectively, and $\mu^{(3)} = S_{12}$ denotes the rotational velocity in the (x_1,x_2)-plane. Their values at time $t = 400$ are

$$(10.38) \qquad \mu^{(1)} = 0.002926, \quad \mu^{(2)} = -0.01691, \quad \mu^{(3)} = 1.0270.$$

Recall that we have a clockwise rotation, if $S_{12} > 0$, and a counter clockwise rotation, if $S_{12} < 0$. Thus, the spinning soliton rotates clockwise. The reconstruction equation and the velocities yield the center of rotation, cf. (10.34),

$$c_{\mathrm{rot}}^{2D} = \frac{1}{\mu^{(3)}} \begin{pmatrix} \mu^{(2)} \\ -\mu^{(1)} \end{pmatrix} = \begin{pmatrix} -\frac{0.01691}{1.0270} \\ -\frac{0.002926}{1.0270} \end{pmatrix} = \begin{pmatrix} -0.016465 \\ -0.002849 \end{pmatrix}.$$

The temporal period, that the spinning soliton in $\mathbb{R}^2$ needs for exact one rotation, is given by

$$T^{2D} = \frac{2\pi}{|\mu^{(3)}|} = 6.118.$$

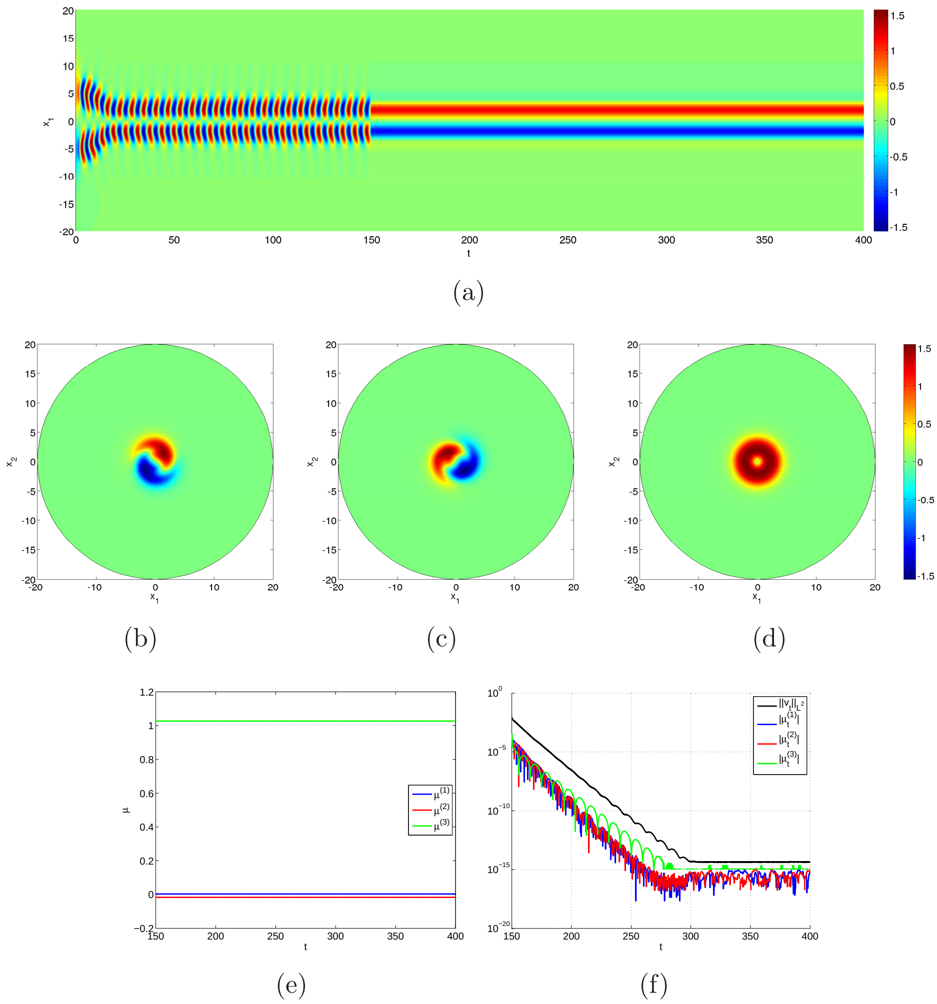
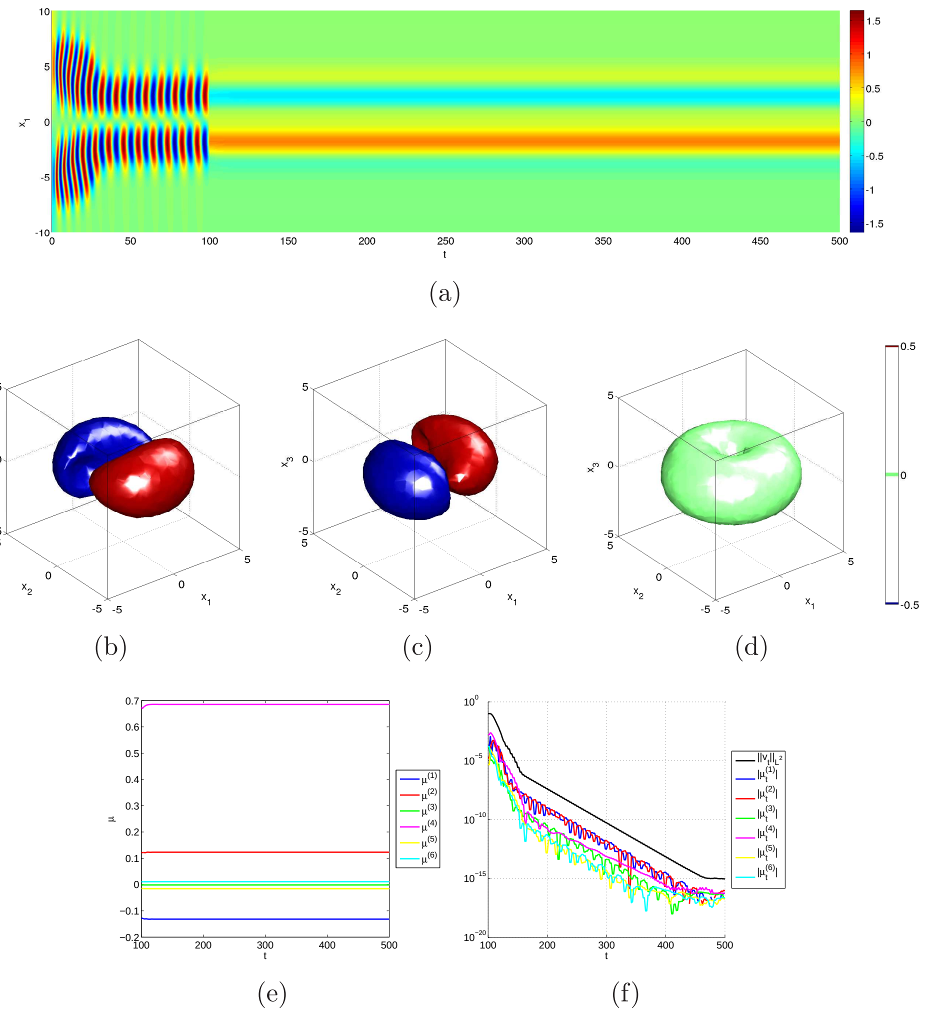


Figure 10.1: Frozen solitons of QCGL for $d = 2$

Figure 10.1(f) shows that neither the approximation v of the profile $v_\star$ nor the velocities $\mu^{(1)}$, $\mu^{(2)}$ and $\mu^{(3)}$ vary in time any more, i.e. both v and $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$ are stationary at time $t = 400$. For the computation of (10.37) with $d = 2$ we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.25$, the BDF method of order 2 with absolute tolerance atol $= 10^{-7}$, relative tolerance rtol $= 10^{-2}$ and maximal stepsize $\triangle t = 0.2$, homogeneous Neumann boundary conditions and fixed phase conditions. The initial data and the reference function come from a simulation: First we solved the nonfrozen system (2.1) until time $t = 150$, as explained in Example 2.1, then we solved the freezing system (10.37) from $t_0 = 150$ to $T = 400$, where the initial data and the reference function is chosen as the solution of the nonfrozen equation (2.1) at time $t = 150$, cf. Figure 2.1(a)–2.1(c). This general procedure is also displayed in Figure 10.1(a), that shows a space-time diagram on the line $x_1 \in [-20, 20]$ for $x_2 = 0$ and $0 \leqslant t \leqslant 400$.



(a) (b) (c) (d) (e) (f)

Figure 10.2: Frozen solitons of QCGL for $d = 3$

Figure 10.2(b)–10.2(d) shows the real part (b), imaginary part (c) and the absolute value (d) for the approximation of the profile $v_\star$ of the spinning soliton in $\mathbb{R}^3$ as the solution of (10.37) on a cube with edge length $L = 20$ centered in the origin at time $t = 500$. Figure 10.2(e) shows the translational and rotational velocities. $\mu^{(1)} = \lambda^{(1)}$, $\mu^{(2)} = \lambda^{(2)}$ and $\mu^{(3)} = \lambda^{(3)}$ denotes the translational velocity in x_1-, x_2- and x_3-direction, respectively, and $\mu^{(4)} = S_{12}$, $\mu^{(5)} = S_{13}$ and $\mu^{(6)} = S_{23}$ denote the rotational velocities in the (x_1, x_2)-, (x_1, x_3)- and (x_2, x_3)-plane. Their values at time $t = 500$ are

$$\begin{aligned} &\mu^{(1)} = -0.1315, \quad \mu^{(2)} = 0.1231, \quad \mu^{(3)} = -0.001496, \\ &\mu^{(4)} = 0.6855, \quad \mu^{(5)} = -0.01558, \quad \mu^{(6)} = 0.01086. \end{aligned} \tag{10.39}$$

The reconstruction equation, the rank-deficient least squares problem and the velocities yield the axis of rotation, cf. (10.35),

$$a_{\mathrm{rot}}^{3D}(r) = \begin{pmatrix} 0.179489 \\ 0.191649 \\ -0.007199 \end{pmatrix} + r \begin{pmatrix} 0.01086 \\ 0.01558 \\ 0.6855 \end{pmatrix}, \quad r \in \mathbb{R}.$$

The temporal period, that the spinning soliton in $\mathbb{R}^3$ needs for exact one rotation, is given by

$$T^{3D} = \frac{2\pi}{\left|\sqrt{(\mu^{(4)})^2 + (\mu^{(5)})^2 + (\mu^{(6)})^2}\right|} = 9.162327.$$

Figure 10.2(f) shows once more that neither the approximation v of the profile $v_\star$ nor the velocities $\mu^{(1)}, \ldots, \mu^{(6)}$ vary in time any more. For the computation of (10.37) with $d = 3$ we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.8$, the BDF method of order 2 with absolute tolerance atol $= 10^{-5}$, relative tolerance rtol $= 10^{-2}$ and maximal stepsize $\triangle t = 2.0$, homogeneous Neumann boundary conditions on all faces and fixed phase conditions. The initial data and the reference function come again from a simulation: First we solved the nonfrozen system (2.1) until time $t = 100$, as explained in Example 2.1, then we solved the freezing system (10.37) from $t_0 = 100$ to $T = 500$, where the initial data and the reference function is chosen as the solution of the nonfrozen equation (2.1) at time $t = 100$, cf. Figure 2.1(d)–2.1(e). This general procedure is also displayed in Figure 10.2(a), that shows a space-time diagram on the line $x_1 \in [-20, 20]$ for $x_2 = x_3 = 0$ and $0 \leqslant t \leqslant 500$.

(**2**): For the parameter values (2.5) we know from Example 2.1 that the QCGL exhibits rigidly rotating spiral wave solutions $u(x,t)$ for space dimension $d = 2$, cf. Figure 2.2.

Figure 10.3(b)–10.3(d) shows the real part (b), imaginary part (c) and the absolute value (d) for the approximation of the profile $v_\star$ of the spiral wave in $\mathbb{R}^2$ as the solution of (10.37) on a circular disk of radius $R = 20$ centered in the origin at time $t = 500$. Figure 10.3(e) shows the translational and rotational velocities. $\mu^{(1)} = \lambda^{(1)}$ and $\mu^{(2)} = \lambda^{(2)}$ denotes again the translational velocity in x_1- and x_2-direction, respectively, and $\mu^{(3)} = S_{12}$ denotes the rotational velocity. Their values at time $t = 500$ are

$$\mu^{(1)} = 0.02616, \quad \mu^{(2)} = -0.01027, \quad \mu^{(3)} = 1.323. \tag{10.40}$$

Since $S_{12} > 0$, the spiral wave rotates clockwise. In particular, the spiral wave rotates faster than the $2D$-spinning solitons. The reconstruction equation and the velocities yield the center of rotation, cf. (10.34),

$$c_{\mathrm{rot}}^{2D} = \frac{1}{\mu^{(3)}} \begin{pmatrix} \mu^{(2)} \\ -\mu^{(1)} \end{pmatrix} = \begin{pmatrix} -\frac{0.01027}{1.323} \\ -\frac{0.02616}{1.323} \end{pmatrix} = \begin{pmatrix} -0.007763 \\ -0.019773 \end{pmatrix}.$$

The temporal period, that the spiral wave needs for exact one rotation, is given by

$$T^{2D} = \frac{2\pi}{|\mu^{(3)}|} = 4.7492.$$

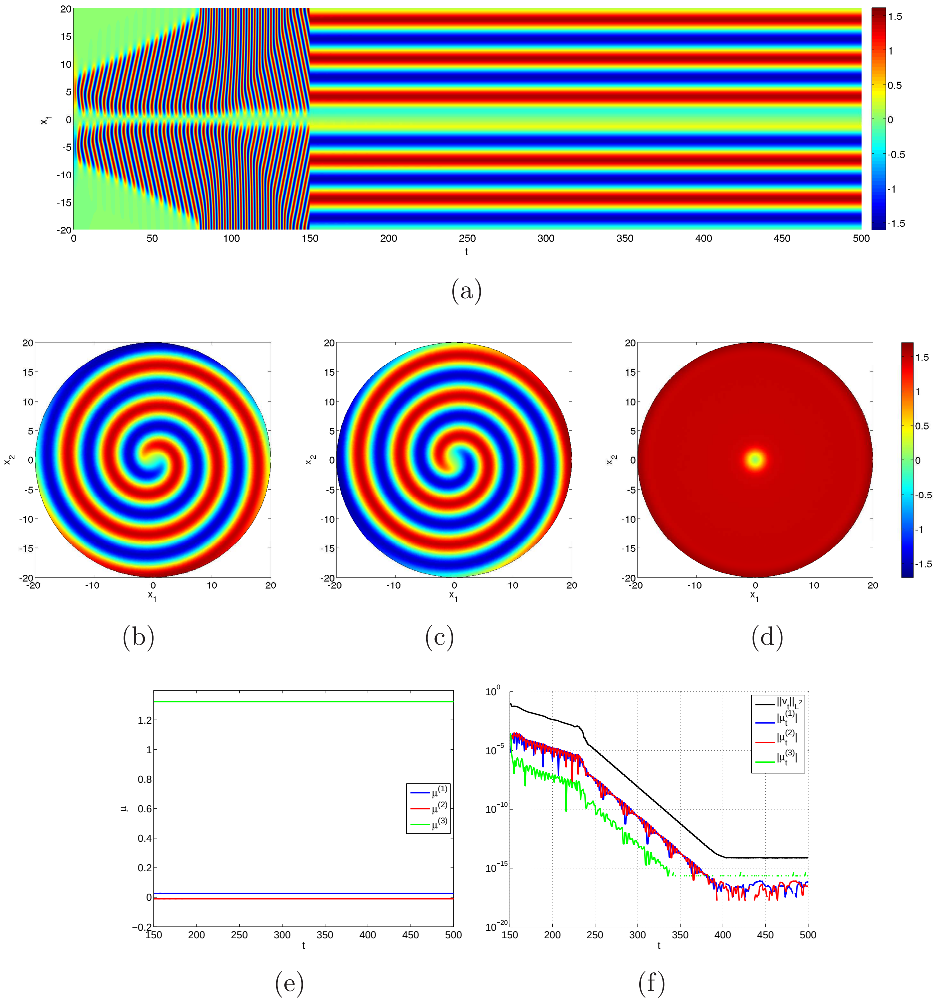


Figure 10.3: Frozen spiral of QCGL for $d = 2$

Figure 10.3(f) shows that neither the approximation v of the profile $v_\star$ nor the velocities μ_1, μ_2 and μ_3 vary in time any more. For the computation of (10.37) with $d = 2$ we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.25$, the BDF method of order 2 with absolute tolerance $\mathrm{atol} = 10^{-6}$, relative tolerance $\mathrm{rtol} = 10^{-2}$ and maximal stepsize $\triangle t = 1.0$, homogeneous Neumann boundary conditions and fixed phase conditions. The initial data and the reference function come from a simulation: First we solved the nonfrozen system (2.1) until time $t = 150$, as explained in Example 2.1, then we solved the freezing system (10.37) from $t_0 = 150$ to $T = 500$, where the initial data and the reference function is chosen as the solution of the nonfrozen equation (2.1) at time $t = 150$, cf. Figure 2.2(a)–2.2(c). This general procedure is also displayed in Figure 10.3(a), that shows a space-time diagram on the line $x_1 \in [-20, 20]$ for $x_2 = 0$ and $0 \leqslant t \leqslant 500$.

(**3**): For the parameter values (2.6) we know from Example 2.1 that the QCGL exhibits twisted and untwisted scroll waves and scroll ring solutions $u(x,t)$ for space dimension $d = 3$, cf. Figure 2.3 for a scroll ring solution.

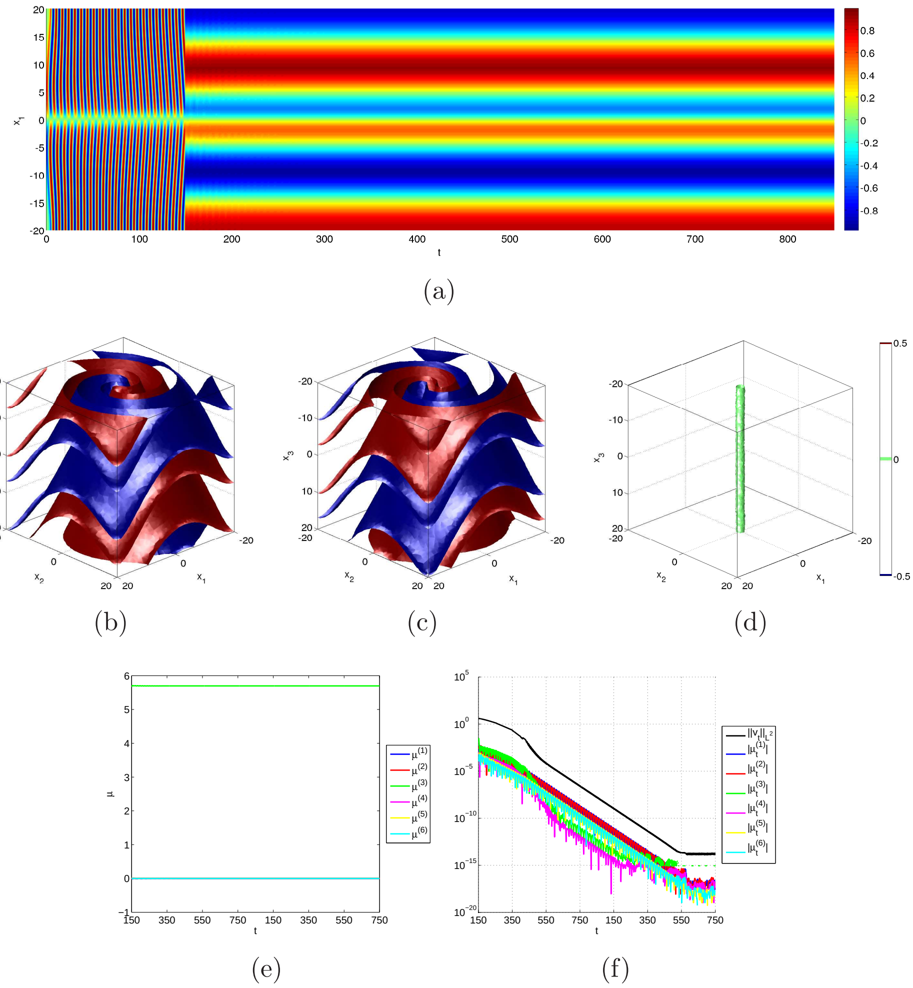


Figure 10.4: Frozen scroll ring of QCGL and λ-ω system for $d = 3$

Figure 10.4(b)–10.4(d) shows the real part (b), imaginary part (c) and the absolute value (d) for the approximation of the profile $v_\star$ of the untwisted scroll wave in $\mathbb{R}^3$ as the solution of (10.37) on a cube with edge length $L = 40$ centered in the origin at time $t = 850$. Figure 10.4(e) shows the translational and rotational velocities. $\mu^{(1)} = \lambda^{(1)}$, $\mu^{(2)} = \lambda^{(2)}$ and $\mu^{(3)} = \lambda^{(3)}$ denotes the translational velocity in x_1-, x_2- and x_3-direction, respectively, and $\mu^{(4)} = S_{12}$, $\mu^{(5)} = S_{13}$ and $\mu^{(6)} = S_{23}$ denote the rotational velocities in the (x_1, x_2)-, (x_1, x_3)- and (x_2, x_3)-plane. Their values at time $t = 850$ are

$$\begin{aligned} \mu^{(1)} &= -0.00123, \quad \mu^{(2)} = 0.004219, \quad \mu^{(3)} = 5.697, \\ \mu^{(4)} &= 0.0004585, \quad \mu^{(5)} = 0.0004447, \quad \mu^{(6)} = -0.0003909. \end{aligned} \tag{10.41}$$

Figure 10.4(f) shows once more that neither the approximation v of the profile $v_\star$ nor the velocities $\mu^{(1)}, \ldots, \mu^{(6)}$ vary in time any more.

We find out that the freezing method yields a traveling wave instead of a rotating wave, since the rotational velocities are approximately zero and only $\mu^{(3)}$, that describes the velocity in x_3-direction, is unequal 0. This phenomena is caused by the periodic boundary conditions on the (x_1, x_2)-faces. Due to the boundary condition the scroll ring can be considered either as a traveling wave, that drifts along the x_3-axis, or as a rotating wave, that rotates about the x_3-axis. This states that the isotropy group is nontrivial for $G = \mathrm{SE}(3)$, but it is indeed trivial in $G = \mathrm{SO}(3)$. This fact was already discussed in [46]. Thus, we perform a second computation using the freezing method with $G = \mathrm{SO}(3)$. Therefore, we modify (10.37) as follows: We set $\lambda_l = 0$ in (10.37a) and omit the phase conditions (10.37c). Consequently, the τ-equation in (10.37d) has the solution $\tau(t) = 0$. Using the same computational settings as before in the case $G = \mathrm{SE}(3)$, the freezing method in $G = \mathrm{SO}(3)$ yields at time $t = 850$ the velocities

$$\begin{aligned} &\mu^{(1)} = 0, \quad \mu^{(2)} = 0, \quad \mu^{(3)} = 0, \\ &\mu^{(4)} = -0.8934, \quad \mu^{(5)} = 0.002114, \quad \mu^{(6)} = -0.001088. \end{aligned} \tag{10.42}$$

The reconstruction equation, the rank-deficient least squares problem and the velocities yield the axis of rotation, cf. (10.35),

$$a_{\mathrm{rot}}^{3D}(r) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} -0.001088 \\ -0.002114 \\ -0.8934 \end{pmatrix}, \; r \in \mathbb{R}.$$

The temporal period, that the scroll ring in $\mathbb{R}^3$ needs for exact one rotation, is given by

$$T^{3D} = \frac{2\pi}{\left|\sqrt{(\mu^{(4)})^2 + (\mu^{(5)})^2 + (\mu^{(6)})^2}\right|} = 7.0329.$$

Since the numerical results for $G = \mathrm{SO}(3)$ are very similar to those from Figure 10.4 for $G = \mathrm{SE}(3)$, we omit to present separate figures for this situation.

For both computations of (10.37) with $d = 3$ we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 1.6$, the BDF method of order 2 with absolute tolerance atol $= 10^{-5}$, relative tolerance rtol $= 10^{-2}$ and maximal stepsize $\triangle t = 1.0$, homogeneous Neumann boundary conditions on all faces and fixed phase conditions. The initial data and the reference function come again from a simulation: First we solved the nonfrozen system (2.1) until time $t = 150$, as explained in Example 2.1, then we solved the freezing system (10.37) from $t_0 = 150$ to $T = 850$, where the initial data and the reference function is chosen as the solution of the nonfrozen equation (2.1) at time $t = 150$, cf. Figure 2.3(a)–2.3(c). This general procedure is also displayed in Figure 10.4(a), that shows a space-time diagram on the line $x_1 \in [-20, 20]$ for $x_2 = x_3 = 0$ and $0 \leqslant t \leqslant 850$.

Example 10.10 (λ-ω system)**.** Consider the freezing system for λ-ω system from Example 2.2

$$\text{(10.43a)}\quad v_t = \alpha\triangle v + v\left(\lambda\left(|v|^2\right) + i\omega\left(|v|^2\right)\right) + \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}(x_jD_i - x_iD_j)v + \sum_{l=1}^{d}\lambda_l D_l v, \quad v(\cdot,t_0) = v_0$$

$$\text{(10.43b)}\quad 0 = \mathrm{Re}\,\left(v-\hat{v},(x_jD_i - x_iD_j)\hat{v}\right)_{L^2}, \quad i=1,\ldots,d-1,\ j=i+1,\ldots,d$$

$$\text{(10.43c)}\quad 0 = \mathrm{Re}\,\left(v-\hat{v},D_l\hat{v}\right)_{L^2}, \quad l=1,\ldots,d$$

$$\text{(10.43d)}\quad \begin{pmatrix} R_t \\ \tau_t \end{pmatrix} = \begin{pmatrix} RS \\ R\lambda \end{pmatrix}, \quad \begin{pmatrix} R(t_0) \\ \tau(t_0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix},$$

with $v:\mathbb{R}^d\times[0,\infty[\to\mathbb{C}$, $d\in\{2,3\}$, $\alpha\in\mathbb{C}$, $\lambda:[0,\infty[\to\mathbb{R}$ and $\omega:[0,\infty[\to\mathbb{R}$.

(**1**): For the parameter values (2.8) we know from Example 2.2 that the λ-ω system exhibits rigidly rotating spiral wave solutions $u(x,t)$ for space dimension $d=2$, cf. Figure 2.4.

Figure 10.5(b)–10.5(d) shows the real part (b), imaginary part (c) and the absolute value (d) for the approximation of the profile $v_\star$ of the spiral wave in $\mathbb{R}^2$ as the solution of (10.43) on a circular disk of radius $R=50$ centered in the origin at time $t=550$. Figure 10.5(e) shows the translational and rotational velocities. $\mu^{(1)}=\lambda^{(1)}$ and $\mu^{(2)}=\lambda^{(2)}$ denotes the translational velocity in x_1- and x_2-direction, respectively, and $\mu^{(3)}=S_{12}$ denotes the rotational velocity. Their values at time $t=550$ are

$$\text{(10.44)}\qquad \mu^{(1)} = 0.0005907, \quad \mu^{(2)} = 0.001609, \quad \mu^{(3)} = -0.9091.$$

Since $S_{12}<0$, the spiral wave rotates counter clockwise. The reconstruction equation and the velocities yield the center of rotation

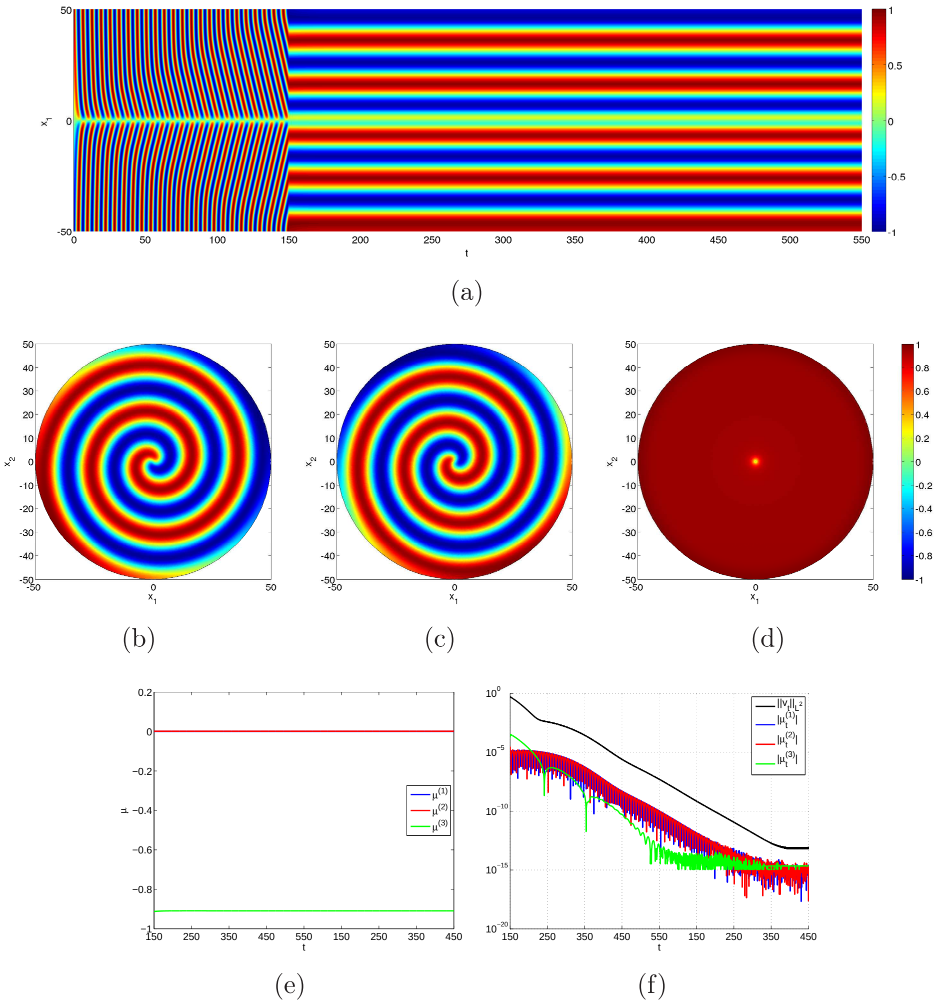
$$c_{\mathrm{rot}}^{2D} = \frac{1}{\mu^{(3)}}\begin{pmatrix}\mu^{(2)} \\ -\mu^{(1)}\end{pmatrix} = \begin{pmatrix}-\frac{0.001609}{0.9091} \\ \frac{0.0005907}{0.9091}\end{pmatrix} = \begin{pmatrix}-0.001770 \\ 0.000650\end{pmatrix}.$$

The temporal period, that the spiral wave needs for exact one rotation, is given by

$$T^{2D} = \frac{2\pi}{|\mu^{(3)}|} = 6.9114.$$

Figure 10.5(f) shows that neither the approximation v of the profile $v_\star$ nor the velocities $\mu^{(1)}$, $\mu^{(2)}$ and $\mu^{(3)}$ vary in time any more. For the computation of (10.43) with $d=2$ we used continuous piecewise linear finite elements with maximal stepsize $\triangle x=0.5$, the BDF method of order 2 with absolute tolerance atol $=10^{-5}$, relative tolerance rtol $=10^{-2}$ and maximal stepsize $\triangle t=0.1$, homogeneous Neumann boundary conditions and fixed phase conditions. The initial data and the reference function come again from a simulation: First we solved the nonfrozen system (2.7) until time $t=150$, as explained in Example 2.2, then we solved the freezing system (10.43) from $t_0=150$ to $T=550$, where the initial data and the reference function is chosen as the solution of the nonfrozen equation (2.7) at time $t=150$, cf. Figure 2.4(a)–2.4(c). This general procedure is also displayed in Figure

10.5(a), that shows a space-time diagram on the line $x_1 \in [-50, 50]$ for $x_2 = 0$ and $0 \leqslant t \leqslant 550$.



(a) (b) (c) (d) (e) (f)

Figure 10.5: Frozen spiral of λ-ω system for $d = 2$

(**2**): For a discussion about the scroll ring in Figure 10.4(a)–10.4(f) we refer to Example 10.9(3).

Example 10.11 (Barkley model)**.** Consider the freezing system for the Barkley model from Example 2.3

(10.45a)
$$v_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \triangle v + \begin{pmatrix} \frac{1}{\varepsilon} v_1 \left(1 - v_1\right) \left(v_1 - \frac{v_2 + b}{a}\right) \\ g(v_1) - v_2 \end{pmatrix}$$
$$+ \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij} (x_j D_i - x_i D_j) v + \sum_{l=1}^{d} \lambda_l D_l v, \quad v(\cdot, t_0) = v_0$$

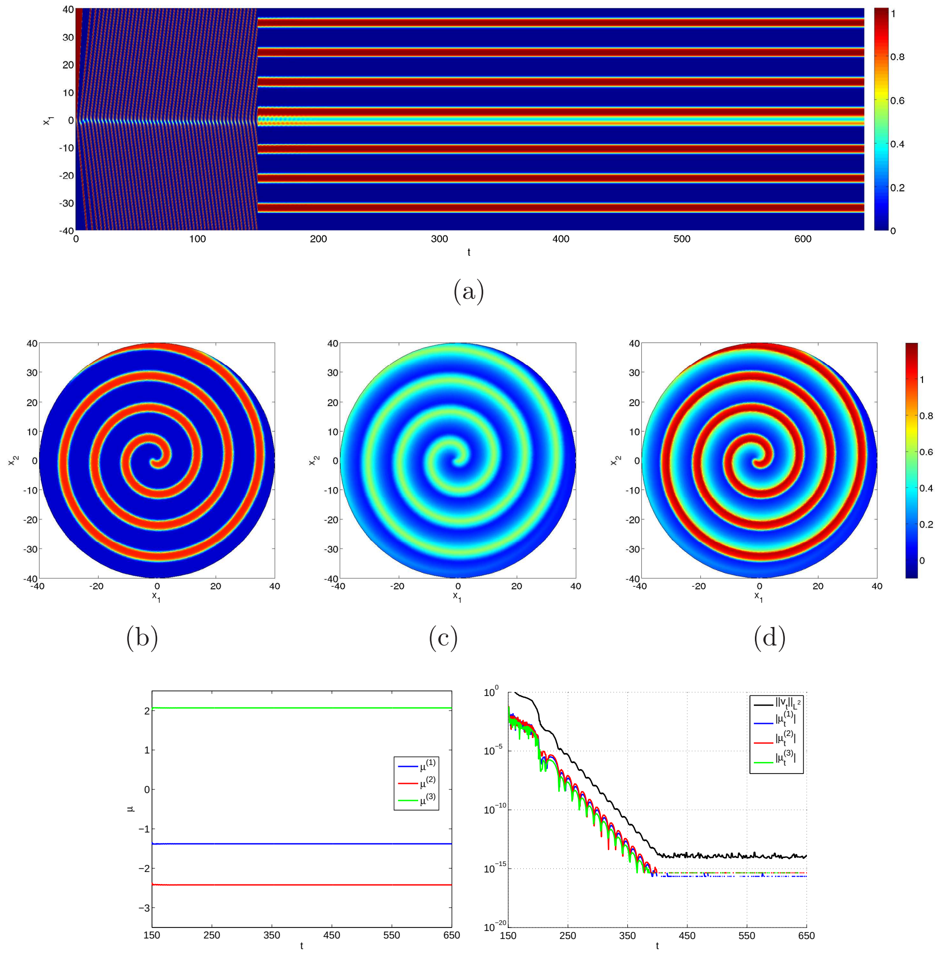
(10.45b)
$$0 = (v - \hat{v}, (x_j D_i - x_i D_j)\hat{v})_{L^2}, \quad i = 1, \ldots, d-1,\ j = i+1, \ldots, d$$

(10.45c)
$$0 = (v - \hat{v}, D_l \hat{v})_{L^2}, \quad l = 1, \ldots, d$$

(10.45d)
$$\begin{pmatrix} R_t \\ \tau_t \end{pmatrix} = \begin{pmatrix} RS \\ R\lambda \end{pmatrix}, \quad \begin{pmatrix} R(t_0) \\ \tau(t_0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix},$$

with $v = (v_1, v_2)^T$, $v : \mathbb{R}^d \times [0, \infty[\to \mathbb{R}^2$, $d \in \{2, 3\}$, $0 \leqslant D << 1$, $0 < \varepsilon << 1$, $0 < a, b \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$.

(**1**): For the parameter values (2.10) we know from Example 2.3 that the Barkley model exhibits rigidly rotating spiral wave solutions $u(x,t)$ for space dimension $d = 2$, cf. Figure 2.5.



(a) (b) (c) (d) (e) (f)

Figure 10.6: Frozen spiral of Barkley model for $d = 2$

Figure 10.6(b)–10.6(d) shows the first component (b), the second component (c) and the absolute value (d) for the approximation of the profile $v_\star$ of the spiral wave in $\mathbb{R}^2$ as the solution of (10.45) on a circular disk of radius $R = 40$ centered in the origin at time $t = 650$. Figure 10.6(e) shows the translational and rotational velocities. $\mu^{(1)} = \lambda^{(1)}$ and $\mu^{(2)} = \lambda^{(2)}$ denotes the translational velocity in x_1- and x_2-direction, respectively, and $\mu^{(3)} = S_{12}$ denotes the rotational velocity. Their values at time $t = 650$ are

$$\mu^{(1)} = -1.370, \quad \mu^{(2)} = -2.422, \quad \mu^{(3)} = 2.067. \tag{10.46}$$

Since $S_{12} > 0$, the spiral wave rotates clockwise. The reconstruction equation and the velocities yield the center of rotation

$$c_{\mathrm{rot}}^{2D} = \frac{1}{\mu^{(3)}}\begin{pmatrix}\mu^{(2)}\\ -\mu^{(1)}\end{pmatrix} = \begin{pmatrix}-\frac{2.422}{2.067}\\ \frac{1.370}{2.067}\end{pmatrix} = \begin{pmatrix}-1.1717\\ 0.6628\end{pmatrix}.$$

The temporal period, that the spiral wave needs for exact one rotation, is given by

$$T^{2D} = \frac{2\pi}{|\mu^{(3)}|} = 3.0398.$$

Figure 10.6(f) shows that neither the approximation v of the profile $v_\star$ nor the velocities $\mu^{(1)}$, $\mu^{(2)}$ and $\mu^{(3)}$ vary in time any more. For the computation of (10.45) with $d = 2$ we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.5$, the BDF method of order 2 with absolute tolerance atol $= 10^{-4}$, relative tolerance rtol $= 10^{-2}$ and maximal stepsize $\triangle t = 1.0$, homogeneous Neumann boundary conditions and fixed phase conditions. The initial data and the reference function come again from a simulation: First we solved the nonfrozen system (2.9) until time $t = 150$, as explained in Example 2.3, then we solved the freezing system (10.45) from $t_0 = 150$ to $T = 650$, where the initial data and the reference function are chosen as the solution of the nonfrozen equation (2.9) at time $t = 150$, cf. Figure 2.5(a)–2.5(c). This general procedure is also displayed in Figure 10.6(a), that shows a space-time diagram on the line $x_1 \in [-40, 40]$ for $x_2 = 0$ and $0 \leqslant t \leqslant 650$.

10.4 Numerical computations of the essential and the point spectrum

In this section we investigate numerically the eigenvalue problem

$$[\lambda I - \mathcal{L}]\, v(x) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2, \tag{10.47}$$

where $\mathcal{L}$ denotes the linearization about a rotating wave $v_\star$

$$[\mathcal{L}v]\,(x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x \in \mathbb{R}^d.$$

To solve the eigenvalue problem (10.47) numerically, we restrict equation (10.47) to a bounded domain and require additional boundary conditions. More precisely, in the examples below we compute the finite element approximation of the eigenvalue problem (10.47) on circular disk for $d = 2$ and on cubes for $d = 3$. In both cases we use homogeneous Neumann boundary conditions. The rotating wave $v_\star$, at which we linearize, is chosen as the solution v at the end time from Example 10.9, 10.10 and 10.11, that is an approximation of $v_\star$, i.e. we first solve the freezing system and use the solution as linearization point. The velocities of the rotating waves $v_\star$, that are needed for the matrix S, can be found in the Examples 10.9, 10.10 and 10.11. To solve the eigenvalue problem by the finite element method we use Comsol Multiphysics$^{\mathrm{TM}}$, [1]. The code involves the ARnoldi PACKage, short ARPACK, that was developed to solve large-scale eigenvalue problems. The ARPACK requires a real shift $\sigma \in \mathbb{R}$ for computing a prescribed number *neig* of eigenvalues that are closest to σ and satisfy a certain eigenvalue tolerance *etol*.

Example 10.12 (Ginzburg-Landau equation)**.** Consider the eigenvalue problem for the real-valued version of the cubic-quintic complex Ginzburg-Landau equation (QCGL) from Example 2.1 and Example 10.9

$$\begin{pmatrix}\alpha_1 & -\alpha_2\\ \alpha_2 & \alpha_1\end{pmatrix}\triangle v(x)+\langle S_\star x+\lambda_\star,\nabla v(x)\rangle+Df\left(v_\star(x)\right)v(x)=\lambda v(x), \tag{10.48}$$

with $v:\mathbb{R}^d\to\mathbb{C}^2$, $d\in\{2,3\}$ and $f:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$f\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}(u_1\mu_1-u_2\mu_2)+(u_1\beta_1-u_2\beta_2)\left(u_1^2+u_2^2\right)+(u_1\gamma_1-u_2\gamma_2)\left(u_1^2+u_2^2\right)^2\\ (u_1\mu_2+u_2\mu_1)+(u_1\beta_2+u_2\beta_1)\left(u_1^2+u_2^2\right)+(u_1\gamma_2+u_2\gamma_1)\left(u_1^2+u_2^2\right)^2\end{pmatrix},$$

where $u=u_1+iu_2$, $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$, $\gamma=\gamma_1+i\gamma_2$ and $u_i,\alpha_i,\beta_i,\gamma_i\in\mathbb{R}$ for $i=1,2$. The pattern $v_\star$ must be considered as a function $v_\star:\mathbb{R}^d\to\mathbb{R}^2$ instead of $v_\star:\mathbb{R}^d\to\mathbb{C}$.

(**1**): For the parameter values from (2.4) we have already seen in Example 2.1 and in Example 10.9 that the Ginzburg-Landau equation exhibits spinning soliton solutions for space dimensions $d=2$ and $d=3$, that this parameter values satisfy our assumptions (A1)–(A9) for every $p\in]4-2\sqrt{2},4+2\sqrt{2}[$, i.e. $p=2,3,4,5,6$, and that the solitons are exponentially localized in the sense of Theorem 1.8 for the real valued system and in the sense of Corollary 8.1 for the complex-valued equation with maximal decay rate (2.3). In the examples below we approximate solutions (λ,v) of the eigenvalue problem (10.48), where (λ,v) consists of an eigenvalue $\lambda\in\mathbb{C}$ and its corresponding eigenfunction $v:\mathbb{R}^d\to\mathbb{C}^2$. In particular, we point out that

$$\lambda_\star=\begin{pmatrix}\mu^{(1)}\\ \mu^{(2)}\end{pmatrix},\; S_\star=\begin{pmatrix}0 & \mu^{(3)}\\ -\mu^{(3)} & 0\end{pmatrix},\text{ if } d=2 \tag{10.49}$$

and

$$\lambda_\star=\begin{pmatrix}\mu^{(1)}\\ \mu^{(2)}\\ \mu^{(3)}\end{pmatrix},\; S_\star=\begin{pmatrix}0 & \mu^{(4)} & \mu^{(5)}\\ -\mu^{(4)} & 0 & \mu^{(6)}\\ -\mu^{(5)} & -\mu^{(6)} & 0\end{pmatrix},\text{ if } d=3. \tag{10.50}$$

Recall the following values that are relevant to discuss the results concerning the linearization and its associated eigenvalue problem:

$$\begin{aligned}&v_\infty=\begin{pmatrix}0\\0\end{pmatrix},\quad \mu=-\frac{1}{2},\quad \sigma\left(Df(v_\infty)\right)=\{\mu,\bar{\mu}\}=\left\{-\frac{1}{2}\right\},\\ &b_0=-s\left(Df(v_\infty)\right)=-\operatorname{Re}\mu=\frac{1}{2}.\end{aligned} \tag{10.51}$$

Moreover, we know from Example 10.9 for $d=2$, cf. (10.38) and (10.49),

$$\sigma(S_\star)=\{\pm\sigma_1 i\},\quad \sigma_1=\mu^{(3)}=1.0270, \tag{10.52}$$

and for $d=3$, cf. (10.39) and (10.50),

$$\sigma(S_\star)=\{0,\pm\sigma_1 i\},\quad \sigma_1=\sqrt{\left(\mu^{(4)}\right)^2+\left(\mu^{(5)}\right)^2+\left(\mu^{(6)}\right)^2}=0.68576. \tag{10.53}$$

For the computation of the eigenvalue problem (10.48) we use in both cases $d=2$ and $d=3$ continuous piecewise linear finite elements with maximal stepsize $\triangle x=0.25$ (if $d=2$) and $\triangle x=0.8$ (if $d=3$), homogeneous Neumann boundary conditions and the following parameters for the eigenvalue solver

$$\text{neig}=800,\quad \sigma=-1,\quad \text{etol}=10^{-7}. \tag{10.54}$$

The profile $v_\star$ and the velocities $(S_\star,\lambda_\star)$ come actually from a simulation: First we solve the freezing system (10.37) until time 400 for $d=2$ and until time 500 for $d=3$, as explained in Example 10.9, then we solve the eigenvalue problem (10.48), where the profile $v_\star$ and the velocities $(S_\star,\lambda_\star)$ are chosen as the solution of (10.37) at the last time instance, cf. Figure 10.1(b)-(e) and Figure 10.2(b)-(e).

Figure 10.7 and 10.9 show an approximation $\sigma^{\text{approx}}(\mathcal{L})$ of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ linearized about the spinning soliton $v_\star$ for $d=2$, see Figure 10.7(b), and $d=3$, see Figure 10.9(b), as well as the exact informations about $\sigma(\mathcal{L})$ that we know from our theoretical results in Figure 10.7(a) for $d=2$ and Figure 10.9(a) for $d=3$. In Figure 10.8 and 10.10 there are visualized the real parts of the first component of certain eigenfunctions v associated to different eigenvalues belonging to the point spectrum $\sigma_{\text{point}}(\mathcal{L})$. Let us start to discuss the numerical results in detail:

By Theorem 9.10 we can expect that

$$\left\{\lambda=-\omega^2\alpha_1+\mu_1+i\left(-\omega^2\alpha_2\pm\mu_2-n\sigma_1\right)\mid \omega\in\mathbb{R},\ n\in\mathbb{Z}\right\}\subseteq\sigma_{\text{ess}}(\mathcal{L}) \tag{10.55}$$

with σ_1 from (10.52) for $d=2$ and from (10.53) for $d=3$. The case $d=2$ was also discussed in [15, Section 8]. In both cases the essential spectrum $\sigma_{\text{ess}}(\mathcal{L})$ forms a zig-zag-structure, that is illustrated by the red lines in Figure 10.7(a) for $d=2$ and in Figure 10.9(a) for $d=3$. The distance between two neighboring tips of the cones equals σ_1, that can easily by seen in (10.55). The minimal distance between the essential spectrum and the imaginary axis equals $b_0=\frac{1}{2}$, since, cf. (10.51),

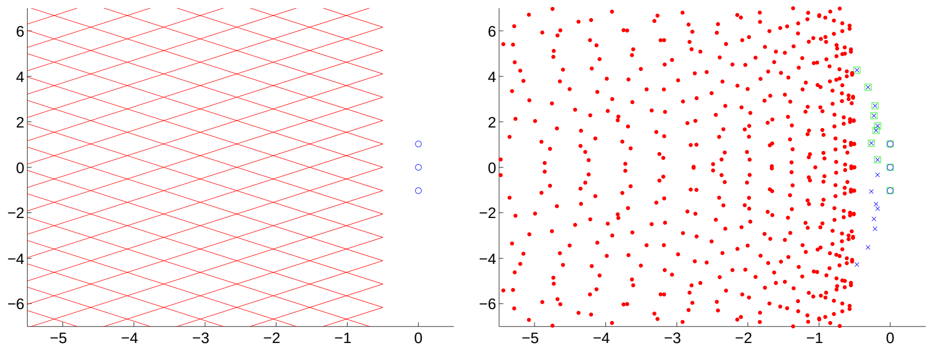
$$\operatorname{Re}\sigma(\mathcal{L})\leqslant -b_0=\operatorname{Re}\mu=-\frac{1}{2}.$$

The dispersion relation from (9.15) states that $\lambda\in\mathbb{C}$ belongs to $\sigma_{\text{ess}}(\mathcal{L})$ if there exist $\omega\in\mathbb{R}$ and $n\in\mathbb{Z}$ such that

$$\lambda+\frac{1}{2}\left(\omega^2+1\right)+i\left(\frac{\omega^2}{2}+n\sigma_1\right)=0.$$

Figure 10.7(b) and Figure 10.9(b) show an approximation $\sigma_{\text{ess}}^{\text{approx}}(\mathcal{L})$ of the essential spectrum, that is represented by the red dots. In both cases, $\sigma_{\text{ess}}^{\text{approx}}(\mathcal{L})$ gives a good approximation for $\sigma_{\text{ess}}(\mathcal{L})$, given by the red lines in Figure 10.7(a) and Figure 10.9(a). An application of Theorem 9.4 yields the eigenvalues of $\mathcal{L}$, that are due to the SE(d)-action, and the shape of their eigenfunctions. In Example 9.6 and 9.7, they are explicitly computed for the case $d=2$ and $d=3$, respectively. In both cases there are three eigenvalues $\lambda_1,\lambda_2,\lambda_3\in\sigma_{\text{point}}(\mathcal{L})$, that are located on the imaginary axis, compare Corollary 9.5 and see Figure 9.1. They are given by

$$\lambda_1=i\sigma_1,\quad \lambda_2=-i\sigma_1,\quad \lambda_3=0.$$



(a) exact spectrum (b) numerical spectrum

Figure 10.7: Essential and point spectrum of QCGL for a spinning soliton with $d = 2$

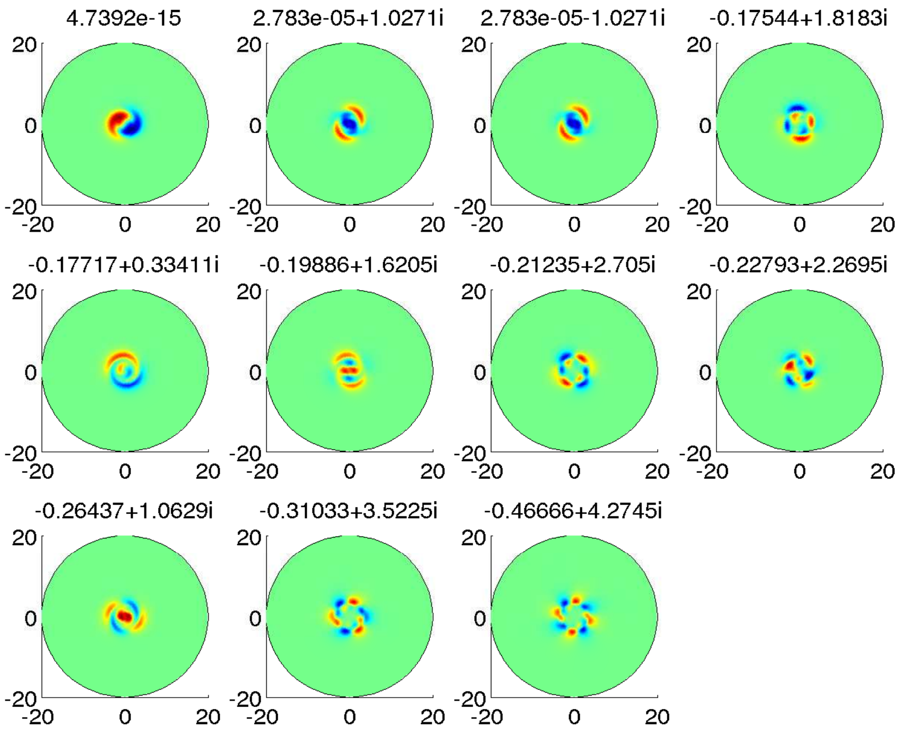
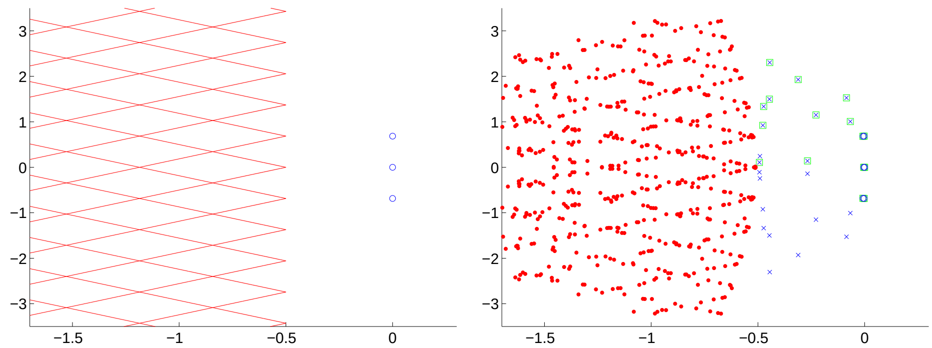


Figure 10.8: Eigenfunctions of QCGL for a spinning soliton with $d = 2$

Every eigenvalue has algebraic multiplicity 1 for $d = 2$ and algebraic multiplicity 2 for $d = 3$. They are visualized by the blue circles in Figure 10.7(a) for $d = 2$ and in Figure 10.9(a) for $d = 3$. Their corresponding approximations are illustrated also by the blue circles in Figure 10.7(b) and in Figure 10.9(b). Indeed, as indicated in Corollary 9.5, Theorem 9.4 does not give a complete characterization of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$, i.e. in Figure 10.7(a) and 10.9(a) there can in general exist further isolated eigenvalues between the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ and the imaginary axis. This is also motivated by our numerical observations. In case $d = 2$, the approximation of $\sigma(\mathcal{L})$ contains in addition 8 complex-conjugated pairs of isolated eigenvalues, represented by the blue crosses in Figure 10.7(b). Similarly, in case $d = 3$, the approximation of $\sigma(\mathcal{L})$ admits additionally 11 complex-conjugated



(a) exact spectrum (b) numerical spectrum

Figure 10.9: Essential and point spectrum of QCGL for a spinning soliton with $d = 3$

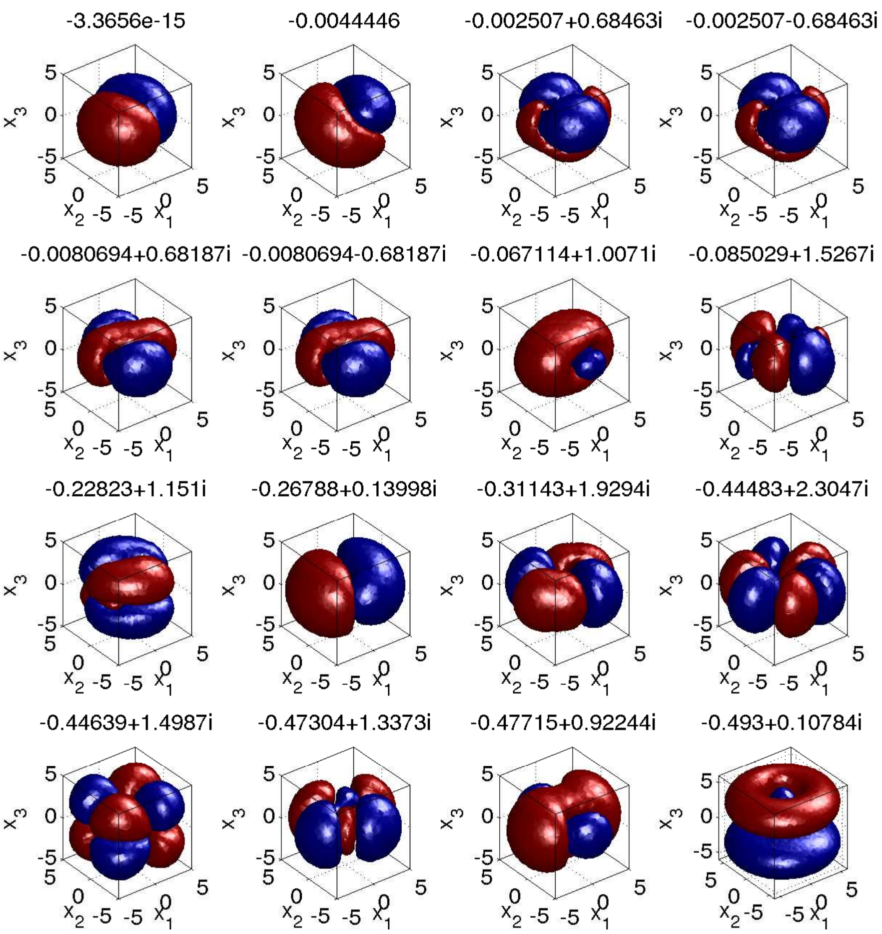


Figure 10.10: Eigenfunctions of QCGL for a spinning soliton with $d = 3$

pairs of isolated eigenvalues, see Figure 10.9(b). Both, the eigenvalues visualized by the blue circles and by the blue crosses form together an approximation $\sigma_{\mathrm{point}}^{\mathrm{approx}}(\mathcal{L})$ of the point spectrum of $\mathcal{L}$. In Figure 10.7(b) and Figure 10.9(b) there are some isolated eigenvalues labeled by a green square. Their associated eigenfunctions are visualized in Figure 10.8 and 10.10. To be more precisely, in the pictures there are illustrated the real parts of the first component of the corresponding eigenfunction $v : \mathbb{R}^d \to \mathbb{C}^2$. The first three eigenfunctions in Figure 10.8 and the first six eigenfunctions in Figure 10.10 are approximations of the eigenfunctions from Theorem 9.4, see also Example 9.6 for $d = 2$ and Example 9.7 for $d = 3$. The remaining eigenfunctions are associated to an eigenvalue from the point spectrum. Note that the first eigenfunction in Figure 10.8 and the third eigenfunction in Figure 10.10 are approximations of the rotational term $\langle S_\star x, \nabla v_\star(x)\rangle$. An application of Theorem 9.8 shows that every eigenfunction with associated eigenvalue $\lambda \in \mathbb{C}$ satisfying

$$\operatorname{Re}\lambda \geqslant -\frac{b_0}{3} = -\frac{1}{6} \approx -0.1667$$

decays exponentially in space with the same rate as in (2.3). In case $d = 2$, this property is satisfied only for the first three eigenfunctions associated to the approximations of λ_1, λ_2 and λ_3, see Figure 10.8. In case $d = 3$, this property is satisfied for the first six eigenfunctions associated to the approximations λ_1, λ_2, λ_3 and additionally for the *7th* and *8th* eigenfunction. Nevertheless, every eigenfunction with associated eigenvalue $\lambda \in \sigma_{\text{point}}^{\text{approx}}(\mathcal{L})$ seems to be exponentially decaying in space. In particular, Corollary 9.9 states that also the rotational term $\langle S_\star x, \nabla v_\star(x)\rangle$ decays exponentially in space with the same rate as in (2.3).

(**2**): For the parameter values from (2.5) we have already seen in Example 2.1 and in Example 10.9 that this equation exhibits rigidly rotating spiral wave solutions for space dimensions $d = 2$ and that this parameter values satisfy our assumptions (A1)–(A9) for every $p \in]4 - 2\sqrt{2}, 4 + 2\sqrt{2}[$, i.e. $p = 2, 3, 4, 5, 6$. But the spiral wave is not localized in the sense of Theorem 1.8, since condition (1.20) seems not to be satisfied. But note that the spiral wave seems to be Archimedean far away from the center of rotation, i.e. $v_\star$ satisfies (9.16). For the discussion about the spectrum of $\mathcal{L}$ linearized at the spiral wave, we recall (10.51). Moreover, we know from Example 10.9, cf. (10.40) and (10.49),

$$\sigma(S_\star) = \{\pm\sigma_1 i\}, \quad \sigma_1 = \mu^{(3)} = 1.323. \tag{10.56}$$

For the computation of the eigenvalue problem (10.48) we use again continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.5$, homogeneous Neumann boundary conditions and the same parameter values as in (10.54) for the eigenvalue solver. The profile $v_\star$ and the velocities $(S_\star, \lambda_\star)$ come again from a simulation: First we solve the freezing system (10.37) until time 500, as explained in Example 10.9, then we solve the eigenvalue problem (10.48), where the profile $v_\star$ and the velocities $(S_\star, \lambda_\star)$ are chosen as the solution of (10.37) at time $t = 500$, cf. Figure 10.3(b)-(e).

Figure 10.11 shows an approximation of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ linearized about the spiral wave $v_\star$. In Figure 10.12 there are visualized the real parts of the first component of certain eigenfunctions associated to different eigenvalues belonging to the point spectrum $\sigma_{\text{point}}(\mathcal{L})$. We next discuss the numerical results in detail:

Since the spiral wave does not satisfied the condition (1.20), we cannot apply Theorem 1.8, Corollary 8.1, Theorem 9.8, Corollary 9.9 and Theorem 9.10. Nevertheless, the spiral wave is Archimedean far away from the center of rotation and therefore we can apply the theory from Section 9.5, that yields, cf. (9.21),

$$\left\{\lambda^l(ik) + i\sigma_1\mathbb{Z} \mid k \in \mathbb{R},\, l = 1,2\right\} \subseteq \sigma_{\text{ess}}(\mathcal{L}).$$

In contrast to the essential spectrum of localized rotating patterns, where the essential spectrum contains infinite many cones that form a zig-zag structure, the essential spectrum for a spiral wave contains infinitely many parabolas $\lambda^l(i\cdot)$ that are all opened to the left. The distance between two neighboring turning points of the parabolas equals σ_1, that can easily be seen in (9.21).

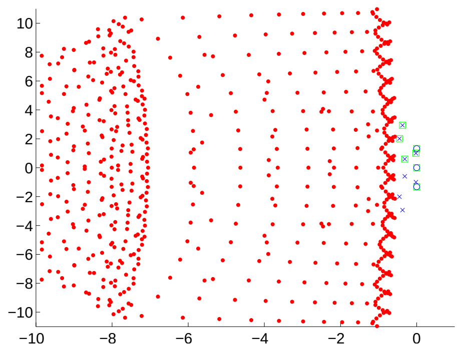


Figure 10.11: Essential and point spectrum of QCGL for a spiral wave with $d = 2$

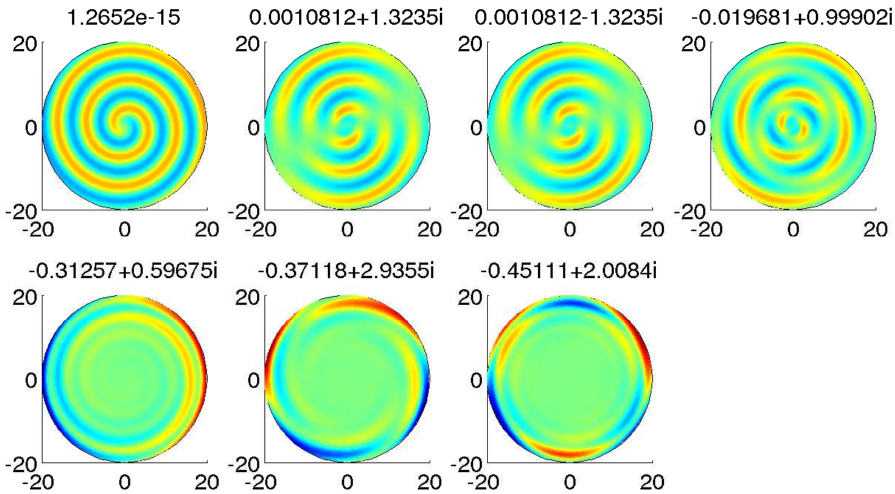


Figure 10.12: Eigenfunctions of QCGL for a spiral wave with $d = 2$

Figure 10.11 shows an approximation $\sigma_{\text{ess}}^{\text{approx}}(\mathcal{L})$ of the essential spectrum at a spiral wave, represented by the red dots. On the left we have an approximation of these infinitely many parabolas. In particular, the distance of their turning points is approximatively σ_1. In the middle we have against our theoretical knowledge infinitely many horizontal lines and on the right we observe the Floquet exponents in the comoving frame, see also [38, Figure 19(b)]. Note that neither a smaller spatial stepsize nor a larger domain has any effect on the shape of the approximation for the spectrum. This means that we can expect that also $\sigma_{\text{ess}}(\mathcal{L})$ contains some horizontal lines and the Floquet exponents. An application of Theorem 9.4 yields informations about the part of the point spectrum $\sigma_{\text{point}}(\mathcal{L})$ that is located on the imaginary axis and is due to the SE(2)-action. These eigenvalues are given by

$$\lambda_1 = i\sigma_1, \quad \lambda_2 = -i\sigma_1, \quad \lambda_3 = 0,$$

and have algebraic multiplicity 1. Their corresponding approximations are illustrated by the blue circles in Figure 10.11. Since Theorem 9.4 does not yield a complete characterization of the point spectrum $\sigma_{\text{point}}(\mathcal{L})$, there can in general exist further eigenvalues belonging to $\sigma_{\text{point}}(\mathcal{L})$. The approximation of $\sigma(\mathcal{L})$ contains 4 complex-conjugated pairs of isolated eigenvalues, represented by the blue crosses in Figure 10.11. Note that one pair of these eigenvalues, $\lambda \approx -10^{-2} \pm i$, is very close to the imaginary axis. The approximation of the point spectrum, denoted by $\sigma_{\text{point}}^{\text{approx}}(\mathcal{L})$, is compound by the eigenvalues visualized by the blue circles and by the blue crosses. In Figure 10.11 there are some isolated eigenvalues labeled by a green square. Their associated eigenfunctions are visualized in Figure 10.12. As before, there are illustrated the real parts of the first components of the corresponding eigenfunction. The first three eigenfunctions in Figure 10.12 are approximations of the eigenfunctions from Theorem 9.4, see also Example 9.6. The fourth eigenfunction belongs to the green boxed eigenvalue that is closest to the imaginary axis. Note that the first eigenfunction is an approximation of the rotational term $\langle S_\star x, \nabla v_\star(x)\rangle$. In particular, all these eigenfunctions doesn't decay in space.

(**3**): For the parameter values from (2.6) we have already seen in Example 2.1(3) and in Example 10.9(3) that the Ginzburg-Landau equation exhibits twisted and untwisted scroll wave as well as scroll ring solutions for space dimensions $d = 3$ and that this parameter values satisfy our assumptions (A1)–(A8) for every $1 < p < \infty$ but not the spectral assumption (A9). But neither the scroll wave nor the scroll ring is not localized in the sense of Theorem 1.8, since condition (1.20) seems neither to be satisfied. In the x_1-x_2-plane both, the scroll wave and the scroll ring, seems in a certain sense to be Archimedean far away from the center of rotation, compare (9.16). This motivates that the approach for the computation of the essential spectrum for the spiral wave extends similarly to scroll waves and scroll rings. For the discussion about the spectrum of $\mathcal{L}$ linearized at a scroll ring we know from Example 10.9(3), cf. (10.41) and (10.50)

$$\sigma(S_\star) = \{0, \pm\sigma_1 i\}, \quad \sigma_1 = \sqrt{\left(\mu^{(4)}\right)^2 + \left(\mu^{(5)}\right)^2 + \left(\mu^{(6)}\right)^2} = 0.0006387 \tag{10.57}$$

for $G = \text{SE}(3)$ and, cf. (10.42) and (10.50)

$$\sigma(S_\star) = \{0, \pm\sigma_1 i\}, \quad \sigma_1 = \sqrt{\left(\mu^{(4)}\right)^2 + \left(\mu^{(5)}\right)^2 + \left(\mu^{(6)}\right)^2} = 0.8934032 \tag{10.58}$$

for $G = \text{SO}(3)$. For the computation of the eigenvalue problem (10.48) we use again continuous piecewise linear finite elements with maximal stepsize $\triangle x = 1.6$, homogeneous Neumann boundary conditions on the side surfaces, periodic boundary conditions on the faces $x_3 = \mp 20$ and the same parameter values as in (10.54) for the eigenvalue solver. The profile $v_\star$ and the velocities $(S_\star, \lambda_\star)$ come again from a simulation: First we solve the freezing system (10.37) until time 850, as explained in Example 10.9(3), then we solve the eigenvalue problem (10.48), where the profile

$v_\star$ and the velocities $(S_\star, \lambda_\star)$ are chosen as the solution of (10.37) at time $t = 850$, cf. Figure 10.4(b)-(e). This procedure we perform for both cases, $G = \mathrm{SE}(3)$ and $G = \mathrm{SO}(3)$. The values for $(S_\star, \lambda_\star)$ one also obtains from (10.41) and (10.42) in combination with (10.50).

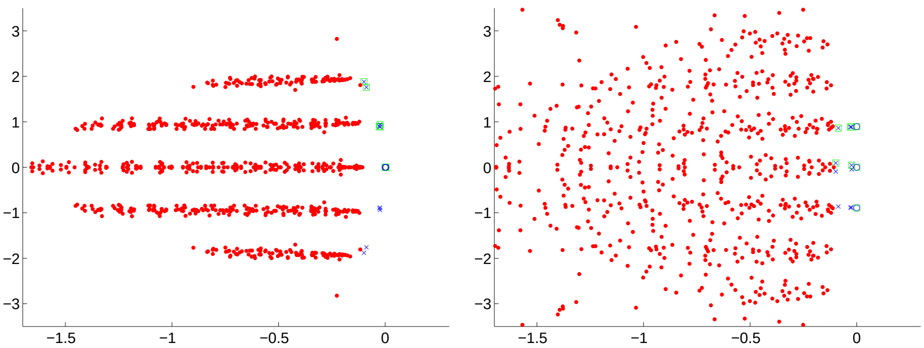
Figure 10.13 shows an approximation of $\sigma^{\mathrm{approx}}(\mathcal{L})$ of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ linearized at a scroll wave $v_\star$ for $G = \mathrm{SE}(3)$, see Figure 10.13(a), and $G = \mathrm{SO}(3)$, see Figure 10.13(b). In Figure 10.12 and 10.12 there are visualized the real parts of the first components of some eigenfunction v associated to different eigenvalues belonging to the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ for $G = \mathrm{SE}(3)$, see Figure 10.12, and for $G = \mathrm{SO}(3)$, see Figure 10.12. We now discuss the numerical results in more detail:

Since the parameters does not satisfy the assumption (A9) and the scroll wave does not satisfy the condition (1.20), we cannot apply Theorem 1.8, Corollary 8.1, Theorem 9.8, Corollary 9.9 and Theorem 9.10. Therefore, we can neither expect that the pattern $v_\star$ and the eigenfunctions v (belonging to some eigenvalue $\lambda \in \sigma_{\mathrm{point}}(\mathcal{L})$) decay exponentially nor that the essential spectrum has a zig-zag-structure. Nevertheless, since the scroll ring is at least in a certain sense Archimedean far away from the center of rotation, we can expect from the theory from Section 9.5, that also the essential spectrum for a scroll ring contains infinitely many parabolas that are opened to the left, compare (9.21).

Figure 10.13 shows an approximation $\sigma_{\mathrm{ess}}^{\mathrm{approx}}(\mathcal{L})$ of the essential spectrum at a scroll ring, represented by the red dots, for $G = \mathrm{SE}(3)$, see Figure 10.13(a), and $G = \mathrm{SO}(3)$, see Figure 10.13(b). As already motivated in Example 10.9(3), the scroll ring can either be considered as a rotating wave that rotates about the x_3-axis or as a traveling wave that travels along the x_3-axis. In case of $G = \mathrm{SE}(3)$ the freezing method yields a traveling wave, since the rotational velocities are almost zero, compare (10.41), and therefore Figure 10.13 can be considered as the spectrum at a traveling wave in three space dimensions. In case of $G = \mathrm{SO}(3)$ the freezing method yields a rotating wave, since the translational velocities are set to 0, compare (10.42). In case of a traveling wave with $G = \mathrm{SE}(3)$ the essential spectrum seems to contain only horizontal lines, see Figure 10.13(a). Contrary, in case of a rotating wave with $G = \mathrm{SO}(3)$ the essential spectrum contains infinitely many (filled) parabolas that are located in $\mathrm{Re}\,\lambda \leqslant -0.1$. In particular, the distance of their turning points is approximatively σ_1. An application of Theorem 9.4 yields informations about the part of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ on the imaginary axis, that is due to the SE(3)-action. These eigenvalues are given by

$$\lambda_1 = i\sigma_1, \quad \lambda_2 = -i\sigma_1, \quad \lambda_3 = 0,$$

and have algebraic multiplicity 2, compare Corollary 9.5 and Figure 9.1. Their corresponding approximations are illustrated by the blue circles in Figure 10.13(a) for $G = \mathrm{SE}(3)$ and in Figure 10.13(b) for $G = \mathrm{SO}(3)$. In case of $G = \mathrm{SE}(3)$ there is only one zero eigenvalue on the imaginary axis, see Figure 10.13(a). In case of $G = \mathrm{SO}(3)$, Figure 10.13(b) can be considered as the spectrum at a rotating wave in three space dimensions and the blue circles corresponds the approximation of $\pm\sigma_1 i$ and 0. Since Theorem 9.4 does not yield a complete characterization of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$, there can in general exist further eigenvalues belonging to $\sigma_{\mathrm{point}}(\mathcal{L})$. In both cases, $G = \mathrm{SE}(3)$ and $G = \mathrm{SO}(3)$, the approximation $\sigma^{\mathrm{approx}}(\mathcal{L})$



(a) (b)

Figure 10.13: Essential and point spectrum of QCGL and λ-ω for an (untwisted) scroll ring with $d = 3$ and with respect to the Lie group $G = \mathrm{SE}(3)$ (a) and $G = \mathrm{SO}(3)$ (b)

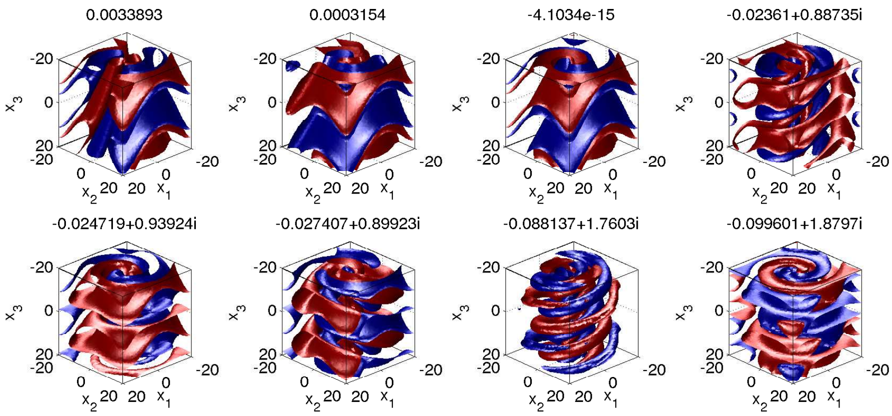


Figure 10.14: Eigenfunctions of QCGL and λ-ω for an (untwisted) scroll ring with $d = 3$ and with respect to the Lie group $G = \mathrm{SE}(3)$

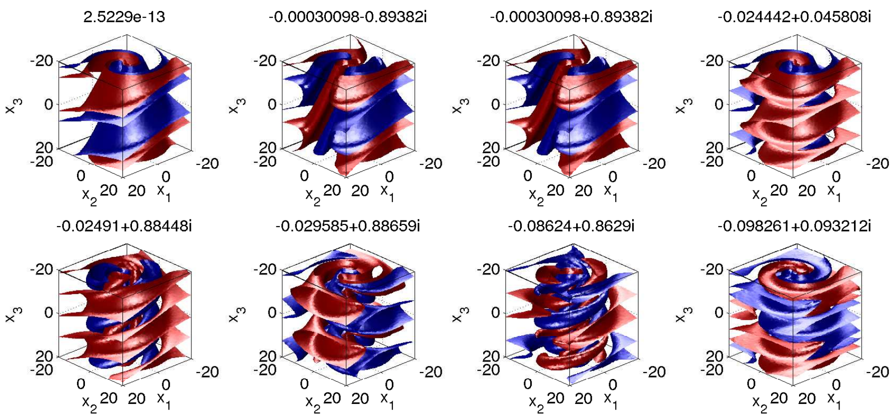


Figure 10.15: Eigenfunctions of QCGL and λ-ω for an (untwisted) scroll ring with $d = 3$ and with respect to the Lie group $G = \mathrm{SO}(3)$

of $\sigma(\mathcal{L})$ contains in addition 5 complex-conjugated pairs of isolated eigenvalues, represented by the blue crosses in Figure 10.13. The approximation of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$, denoted by $\sigma_{\mathrm{point}}^{\mathrm{approx}}(\mathcal{L})$, is compound by the eigenvalues visualized by the blue circles and by the blue crosses. In Figure 10.13 there are some isolated eigenvalues labled by a green square. Their associated eigenfunctions are visualized in Figure 10.12 for $G = \mathrm{SE}(3)$ and in Figure 10.12 for $G = \mathrm{SO}(3)$. As before, there are illustrated the real parts of the first components of the corresponding eigenfunction. In Figure 10.12, the first three eigenfunctions are approximations of the eigenfunctions belonging to the eigenvalue 0. All these eigenfunctions doesn't decay in space. In Figure 10.12, the first three eigenfunctions are approximations of the eigenfunctions from Theorem 9.4, see also Example 9.7. Since the algebraic multiplicity of the corresponding eigenvalues is equal 2, we expect three more eigenfunctions. It seems that they do not appear in our numerical results, since the spatial stepsize is indeed too large. Note that the first eigenfunction in Figure 10.12 is an approximation of the rotational term $\langle S_\star x, \nabla v_\star(x)\rangle$. In particular, also these eigenfunctions doesn't decay in space.

Example 10.13 (λ-ω system)**.** Consider the eigenvalue problem for the real-valued version of the λ-ω system from Example 2.2 and Example 10.10

$$\begin{pmatrix}\alpha_1 & -\alpha_2\\ \alpha_2 & \alpha_1\end{pmatrix}\triangle v(x) + \langle S_\star x + \lambda_\star, \nabla v(x)\rangle + Df\left(v_\star(x)\right)v(x) = \lambda v(x), \tag{10.59}$$

with $v:\mathbb{R}^d\to\mathbb{C}^2$, $d\in\{2,3\}$ and $f:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$f\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}u_1\lambda\left(u_1^2+u_2^2\right) - u_2\omega\left(u_1^2+u_2^2\right)\\ u_1\omega\left(u_1^2+u_2^2\right) + u_2\lambda\left(u_1^2+u_2^2\right)\end{pmatrix},$$

where $\lambda,\omega:[0,\infty[\to\mathbb{R}$, $u=u_1+iu_2$, $\alpha=\alpha_1+i\alpha_2$ and $u_i,\alpha_i\in\mathbb{R}$ for $i=1,2$. The pattern $v_\star$ must be considered as a function $v_\star:\mathbb{R}^d\to\mathbb{R}^2$ instead of $v_\star:\mathbb{R}^d\to\mathbb{C}$.

(**1**): For the parameter values from (2.8) we have already seen in Example 2.2 and in Example 10.10 that the λ-ω system exhibits rigidly rotating spiral wave solutions for space dimension $d=2$ and that this parameter values satisfy the assumptions (A1)–(A8) for every $1<p<\infty$ with $v_\infty=(0,0)^T$, but neither condition (1.20) nor assumption (A9), since $Df(0,0)$ contains the eigenvalue 1 with algebraic multiplicity 2. Therefore, the spiral wave is not localized in the sense of Theorem 1.8. But the spiral wave seems to be Archimedean far away from the center of rotation, i.e. $v_\star$ satisfies (9.16). Recall the following values from Example 10.10 that are relevant to discuss the results concerning the linearization and its associated eigenvalue problem, cf. (10.44) and (10.49),

$$\sigma(S_\star) = \{\pm\sigma_1 i\}, \quad \sigma_1 = \mu^{(3)} = -0.9091. \tag{10.60}$$

In order to approximate solutions of the eigenvalue problem (10.59) we use continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.5$, homogeneous Neumann boundary conditions and the same parameters as in (10.54) for the eigenvalue solver. The profile $v_\star$ and the velocities $(S_\star,\lambda_\star)$ come again from a simulation: First we solve the freezing system (10.43) until time 550, as explained in Example

10.10, then we solve the eigenvalue problem (10.59), where the profile $v_\star$ and the velocities $(S_\star, \lambda_\star)$ are chosen as the solution of (10.43) at time $t = 550$, cf. Figure 10.5(b)-(e). The values for $(S_\star, \lambda_\star)$ can also be received from (10.44) and (10.49).

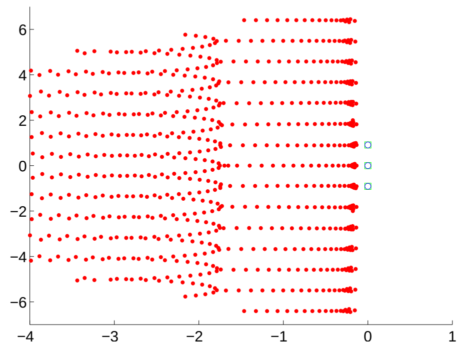


Figure 10.16: Essential and point spectrum of the λ-ω system for a spiral wave with $d = 2$

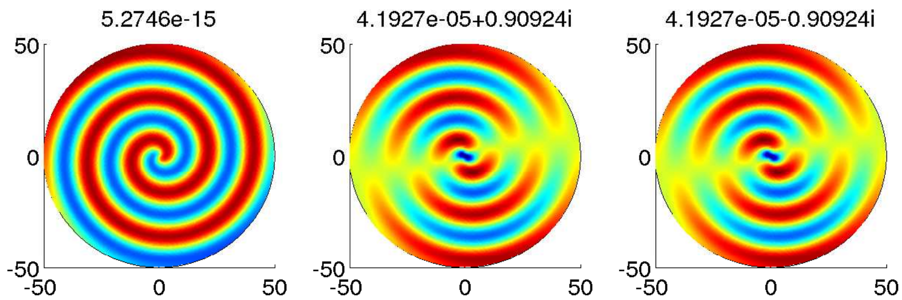


Figure 10.17: Eigenfunctions of the λ-ω system for a spiral wave with $d = 2$

Figure 10.13 show an approximation of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ linearized about the spiral wave $v_\star$. In Figure 10.13 there are visualized the real parts of the first component of some eigenfunctions associated to different eigenvalues belonging to the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$. Let us discuss the numerical results:

Since the spiral wave does not satisfy the condition (1.20), we cannot apply Theorem 1.8, Corollary 8.1, Theorem 9.8, Corollary 9.9 and Theorem 9.10. However, we can act on the assumption that the spiral wave is Archimedean far away from the center of rotation and therefore we can apply the theory from Section 9.5. We infer from (9.21) that

$$\left\{\lambda^l(ik) + i\sigma_1\mathbb{Z} \mid k \in \mathbb{R},\, l = 1,2\right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

This means that the essential spectrum for a spiral wave contains infinitely many parabolas $\lambda^l(i\cdot)$ that are all opened to the left. The distance between two neighboring turning points of the parabolas equals σ_1, that can easily be seen in (9.21).

Figure 10.13 shows an approximation $\sigma_{\mathrm{ess}}^{\mathrm{approx}}(\mathcal{L})$ of the essential spectrum at a spiral wave, represented by the red dots. On the left of the picture, there is illustrated the approximation of these infinitely many parabolas. In particular, the distance of two neighboring turning points is as expected approximatively σ_1. In the middle we have against our theoretical knowledge infinitely many horizontal lines.

Using Theorem 9.4 we deduce informations about the part of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ that is located on the imaginary axis and is due to the SE(2)-action. All these eigenvalues are given by

$$\lambda_1 = i\sigma_1, \quad \lambda_2 = -i\sigma_1, \quad \lambda_3 = 0,$$

and have algebraic multiplicity 1. Their corresponding approximations are illustrated by the blue circles in Figure 10.13. As everyone knows, Theorem 9.4 does not yield a complete characterization of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$. This means that there maybe exist further eigenvalues belonging to $\sigma_{\mathrm{point}}(\mathcal{L})$. The approximation of $\sigma(\mathcal{L})$ contains no further isolated eigenvalues. The approximation of the point spectrum, denoted by $\sigma_{\mathrm{point}}^{\mathrm{approx}}(\mathcal{L})$, contains only the eigenvalues visualized by the blue circles. The three eigenfunctions belonging to the eigenvalues labeled by a green square in Figure 10.13, are visualized in Figure 10.13. These pictures show the real parts of the first component of the eigenfunctions. These three eigenfunctions are approximations of the eigenfunctions from Theorem 9.4, see also Example 9.6. In particular, the first eigenfunction is an approximation of the rotational term $\langle S_\star x, \nabla v_\star(x)\rangle$, that obviously does not decay in space.

(**2**): For a discussion about the spectrum at a scroll ring in Figure 10.13, 10.12 and 10.12 we refer to Example 10.12(3).

Example 10.14 (Barkley model)**.** Consider the eigenvalue problem for the Barkley model from Example 2.3 and Example 10.11

$$\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \triangle v(x) + \langle S_\star x + \lambda_\star, \nabla v(x)\rangle + Df\left(v_\star(x)\right) v(x) = \lambda v(x), \tag{10.61}$$

with $v : \mathbb{R}^d \to \mathbb{C}^2$, $d \in \{2,3\}$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} u_1 \left(1-u_1\right)\left(u_1 - \frac{u_2+b}{a}\right) \\ g(u_1) - u_2 \end{pmatrix},$$

where $u = (u_1, u_2)^T \in \mathbb{R}^2$, $0 \leqslant D << 1$, $0 < \varepsilon << 1$, $0 < a, b \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $v_\star : \mathbb{R}^d \to \mathbb{R}^2$.

(**1**): For the parameter values from (2.10) we have already seen in Example 2.3 and Example 10.11 that the Barkley model exhibit rigidly rotating wave solutions for space dimension $d = 2$. In the following we consider also the case $D = 0.1$. Note that the Barkley model (2.9) is a mixed hyperbolic-parabolic system for $D = 0$ and a parabolic system for $D = 0.1$. The parameter values (2.10) with $D = 0$ satisfy only our assumptions (A1) and (A5)–(A7). If we choose $D = 0.1$ in (2.10), then the parameters even satisfy the assumptions (A1)–(A8) for every

$$1.2699 \approx \frac{22}{2\sqrt{10}+11} < p < \frac{22}{-2\sqrt{10}+11} \approx 4.7054,$$

with $v_\infty = (0,0)^T$, but neither condition condition (1.20) nor our assumption (A9), since (A9) requires that $b < 0$. Therefore, in both cases, $D = 0$ and $D = 0.1$, the spiral wave is not localized in the sense of Theorem 1.8. But in case of $D = 0.1$ the spiral wave seems to be Archimedean far away from the center of rotation, i.e. $v_\star$ satisfies (9.16). In order to discuss the results concerning the linearization and its associated eigenvalue problem, we recall the translational and rotational velocities of the spiral wave: For $D=0$ we have already observed in Example 10.11 that the velocities are given by (10.46). Repeating the computations from Example 2.3 and Example 10.11 with exactly the same setting but with $D=0.1$ instead of $D=0$ yields the following velocities

$$\mu^{(1)} = -3.195, \quad \mu^{(2)} = -1.570, \quad \mu^{(3)} = 1.957.$$

Thus, compare (10.49),

$$\sigma(S_\star) = \{\pm\sigma_1 i\}, \quad \sigma_1 = \mu^{(3)} = 2.067, \text{ for } D=0, \tag{10.62}$$

$$\sigma(S_\star) = \{\pm\sigma_1 i\}, \quad \sigma_1 = \mu^{(3)} = 1.957, \text{ for } D=0.1. \tag{10.63}$$

For the computation of the eigenvalue problem (10.61) we use in both cases, i.e. for $D=0$ and $D=0.1$, continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.5$, homogeneous Neumann boundary conditions and the same parameter values as in (10.54) for the eigenvalue solver. The profile $v_\star$ and the velocities $(S_\star,\lambda_\star)$ come again from a simulation: First we solve the freezing system (10.45) until time 500, as explained in Example 10.11, then we solve the eigenvalue problem (10.61), where the profile $v_\star$ and the velocities $(S_\star,\lambda_\star)$ are chosen as the solution of (10.45) at time $t=500$, cf. Figure 10.6(b)-(e) for the case $D=0$.

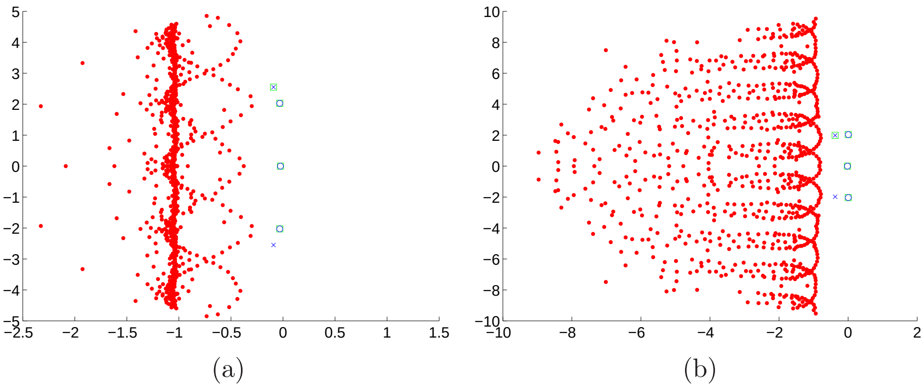


Figure 10.18: Essential and point spectrum of the Barkley model for a spiral wave with $d=2$ for the hyperbolic-parabolic case with $D=0$ (a) and for the parabolic case with $D=0.1$ (b)

Figure 10.18 shows an approximation of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ linearized about the spiral wave $v_\star$ for $D=0$ (a) and $D=0.1$ (b). In Figure 10.14 there are visualized the real parts of the first component of certain eigenfunctions associated to different eigenvalues belonging to the point spectrum $\sigma_{\text{point}}(\mathcal{L})$ for $D=0$. Analogously, Figure 10.14 contains the corresponding eigenfunctions for $D=0.1$. We next discuss the numerical results in detail:

For both, $D=0$ and $D=0.1$, the spiral wave does not satisfied the condition (1.20). Therefore, Theorem 1.8, Corollary 8.1, Theorem 9.8, Corollary 9.9 and Theorem 9.10 are not applicable. Nevertheless, in the parabolic case with $D=0.1$

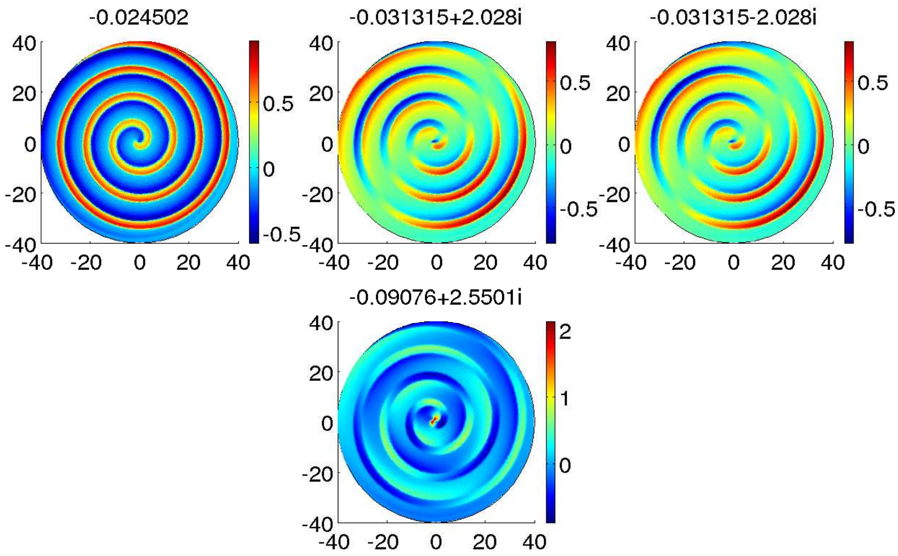


Figure 10.19: Eigenfunctions of the Barkley model for a spiral wave with $d = 2$ (for $D = 0$)

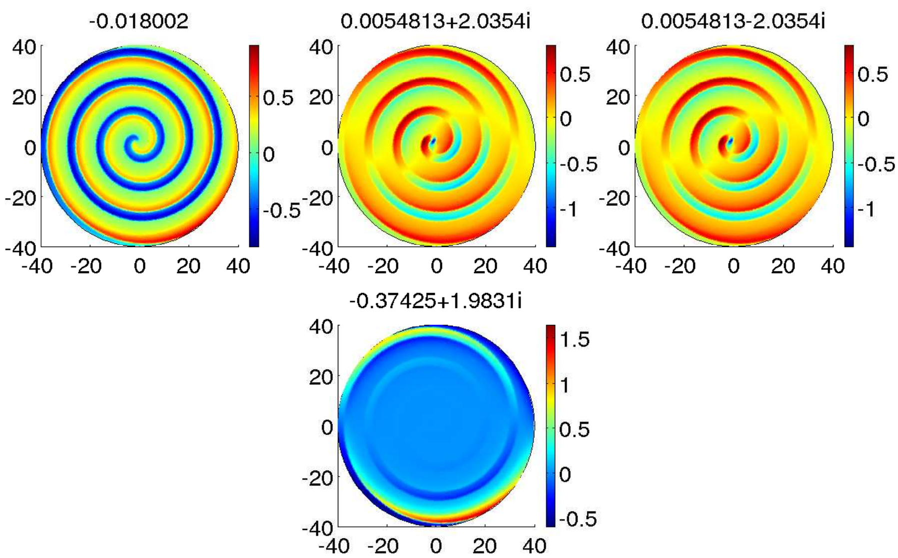


Figure 10.20: Eigenfunctions of the Barkley model for a spiral wave with $d = 2$ (for $D = 0.1$)

the spiral wave is Archimedean far away from the center of rotation and therefore we can apply the theory from Section 9.5. This yields, cf. (9.21),

$$\left\{\lambda^l(ik) + i\sigma_1\mathbb{Z} \mid k \in \mathbb{R},\, l = 1, 2\right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

meaning that the essential spectrum contains infinitely many parabolas $\lambda^l(i\cdot)$ that are all opened to the left. The distance between two neighboring turning points of the parabolas equals σ_1, that can easily be seen in (9.21). In the hyperbolic-parabolic case with $D = 0$, where the diffusion matrix is degenerated and doesn't have full rank, the situation much more involved. The parabolic part generates identically to the case above such a set of parabolas. But in case of $D = 0$ it is well known, that the hyperbolic part generates additionally a vertical line which is located in the left half-plane and belongs to the essential spectrum. Note that the theory from Section 9.5 does not cover the case for degenerated diffusion matrices.

Figure 10.18(a)-(b) shows the approximations $\sigma_{\mathrm{ess}}^{\mathrm{approx}}(\mathcal{L})$ of the essential spectrum at a spiral wave for $D = 0$ (a) and $D = 0.1$ (b), represented by the red dots. In both pictures we observe on the left an approximation of these infinitely many parabolas. In the left picture we additionally observe an approximation of the vertical line belonging to the essential spectrum that is due to the hyperbolic part. Both, for $D = 0$ and $D = 0.1$, the distance of their turning points is approximatively σ_1. Furthermore, an application of Theorem 9.4 provides a certain part of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ that is located on the imaginary axis and is due to the SE(2)-action. These eigenvalues are given by

$$\lambda_1 = i\sigma_1, \quad \lambda_2 = -i\sigma_1, \quad \lambda_3 = 0,$$

and have algebraic multiplicity 1. Their corresponding approximations are illustrated by the blue circles in Figure 10.18(a) for $D = 0$ and in Figure 10.18(b). Note that Theorem 9.4 does not yield a complete characterization of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$, meaning that there can exist further eigenvalues belonging to $\sigma_{\mathrm{point}}(\mathcal{L})$. The approximations of $\sigma(\mathcal{L})$ contain in both cases, $D = 0$ and $D = 0.1$, one complex-conjugated pair of isolated eigenvalues, represented by the blue crosses in Figure 10.18(a)-(b). The approximation of the point spectrum, denoted by $\sigma_{\mathrm{point}}^{\mathrm{approx}}(\mathcal{L})$, is compound by the eigenvalues visualized by the blue circles and by the blue crosses. The eigenfunctions of the isolated eigenvalues, that are labled by a green square in Figure 10.18(a) and Figure 10.18(b), are visualized in Figure 10.14 and Figure 10.14, respectively. Similar as before, there are illustrated the real parts of the first components of the corresponding eigenfunction. The first three eigenfunctions in Figure 10.14 and Figure 10.14 are approximations of the eigenfunctions from Theorem 9.4, see also Example 9.6. The fourth eigenfunction belongs in both cases to the remaining eigenvalue, which is labeled by a green box. We notice that in both cases the first eigenfunction is an approximation of the rotational term $\langle S_\star x, \nabla v_\star(x)\rangle$. Finally, we observe that none of these eigenfunctions decay in space.

10.5 Decompose and freeze method for multi-structures

In this section we introduce the **decompose and freeze method**. This method can be considered as an extension of the freezing approach to multi-structures, e.g. multi-fronts and multi-pulses for $d = 1$ and multi-solitons for $d = 2$. For some literature about the decompose and freeze method we refer to [17] and [99] for one-dimensional multi-structures and to [16] for one- and two-dimensional multi-structures.

Let E denote a module acting on the state space $(X, \|\cdot\|)$ via left multiplication,

$$\bullet : E \times X \to X, \quad (\varphi, u) \mapsto \varphi \bullet u.$$

Moreover, let

$$b : G \to GL(E), \quad \gamma \mapsto b(\gamma)$$

denote the action of the Lie group G on E. Considering the mapping

$$b(\cdot)\varphi : G \to E, \quad \gamma \mapsto b(\gamma)\varphi, \quad \varphi \in E,$$

we require that the actions a and b satisfy the identities

$$a(\gamma)\,(\varphi \bullet u) = (b(\gamma)\varphi) \bullet (a(\gamma)u)\,, \qquad \forall\,\gamma \in G\;\forall\,\varphi \in E\;\forall\,u \in X, \tag{10.64}$$

$$b(\gamma)\,(\varphi\psi) = (b(\gamma)\varphi)\,(b(\gamma)\psi)\,, \qquad \forall\,\gamma \in G\;\forall\,\varphi, \psi \in E. \tag{10.65}$$

In practice, given an explicit representation for the action a, one derives a representation for the action b from (10.64), which then must satisfy the property from (10.65). In particular, if we apply (10.65) with $\psi = \varphi^{-1}$ we obtain the equality $1_E = b(\gamma)\,(\varphi\varphi^{-1}) = (b(\gamma)\varphi)\,(b(\gamma)\,(\varphi^{-1}))$, where 1_E denotes the unit element of E. This yields the inverse

$$(b(\gamma)\varphi)^{-1} = b(\gamma)\left(\varphi^{-1}\right).$$

In the following we briefly explain the main concept of the decompose and freeze approach for multi-structures, [17], [99], [16]:

Consider a general equivariant evolution equation (10.9). We introduce new functions $\gamma_j(t) \in G$ and $v_j(t) \in Y$ for $j = 1, \ldots, m$ ($m \in \mathbb{N}$ and $m \geqslant 2$) such that the solution u of (10.9) can be written as

$$u(t) = \sum_{j=1}^{m} a(\gamma_j(t))v_j(t),\; 0 \leqslant t < T. \tag{10.66}$$

Here, $\gamma_j(t) \in G$ denotes the time-dependent position of the pattern $v_j(t) \in Y$.

We now perform a time-dependent partition of unity. For this purpose, we assume $\varphi \in E$ such that the inverse of $\sum_{j=1}^{m} b(\gamma_j)\varphi \in E$ with respect to the multiplication in E exists for every $\gamma_1, \ldots, \gamma_m \in G$ and we denote this inverse $\left(\sum_{j=1}^{m} b(\gamma_j)\varphi\right)^{-1}$ by $\frac{1}{\sum_{j=1}^{m} b(\gamma_j)\varphi}$. Inserting the ansatz (10.66) into (10.9) and using the abbreviation $\gamma_j^k(t) = \gamma_j^{-1}(t) \circ \gamma_k(t)$ we obtain

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(a(\gamma_j(t))v_{j,t}(t) + d\left[a(\gamma_j(t))v_j(t)\right]\gamma_{j,t}(t)\Big) = \sum_{j=1}^{m}\frac{d}{dt}\left[a(\gamma_j(t))v_j(t)\right] \\
=&\frac{d}{dt}\left[\sum_{j=1}^{m} a(\gamma_j(t))v_j(t)\right] = u_t(t) = F(u(t)) = F\left(\sum_{j=1}^{m} a(\gamma_j(t))v_j(t)\right) \\
=&\sum_{j=1}^{m} F\left(a(\gamma_j(t))v_j(t)\right) + \left[F\left(\sum_{k=1}^{m} a(\gamma_k(t))v_k(t)\right) - \sum_{k=1}^{m} F\left(a(\gamma_k(t))v_k(t)\right)\right] \\
=&\sum_{j=1}^{m} a(\gamma_j(t))F(v_j(t)) + a(\gamma_j(t))\Bigg(\frac{\varphi}{\sum_{k=1}^{m} b(\gamma_j^k(t))\varphi}
\end{aligned}$$

$$\bullet \left[F\left(\sum_{k=1}^{N} a(\gamma_j^k(t))v_k(t)\right) - \sum_{k=1}^{N} F\left(a(\gamma_j^k(t))v_k(t)\right)\right]\Bigg),$$

where in the last equation we used the following relation

$$\begin{aligned}
&F\left(\sum_{k=1}^{m} a(\gamma_k(t))v_k(t)\right) - \sum_{k=1}^{m} F\left(a(\gamma_k(t))v_k(t)\right) \\
&= \frac{\sum_{j=1}^{m} b(\gamma_j(t))\varphi}{\sum_{k=1}^{m} b(\gamma_k(t))\varphi} \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_k(t))v_k(t)\right) - \sum_{k=1}^{m} F\left(a(\gamma_k(t))v_k(t)\right)\right] \\
&= \sum_{j=1}^{m} \frac{b(\gamma_j(t))\varphi}{\sum_{k=1}^{m} b(\gamma_k(t))\varphi} \bullet \left[F\left(\sum_{k=1}^{m} a(\mathbb{1})a(\gamma_k(t))v_k(t)\right) - \sum_{k=1}^{m} F\left(a(\mathbb{1})a(\gamma_k(t))v_k(t)\right)\right] \\
&= \sum_{j=1}^{m} \frac{b(\gamma_j(t))\varphi}{\sum_{k=1}^{m} b(\gamma_k(t))\varphi} \bullet \left[F\left(a(\gamma_j(t))\sum_{k=1}^{m} a(\gamma_j^{-1}(t)\circ\gamma_k(t))v_k(t)\right)\right. \\
&\qquad \left. - \sum_{k=1}^{m} F\left(a(\gamma_j(t))a(\gamma_j^{-1}(t)\circ\gamma_k(t))v_k(t)\right)\right] \\
&= \sum_{j=1}^{m} \frac{b(\gamma_j(t))\varphi}{\sum_{k=1}^{m} b(\gamma_k(t))\varphi} \bullet a(\gamma_j(t))\left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t))v_k(t)\right)\right. \\
&\qquad \left. - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t))v_k(t)\right)\right] \\
&= \sum_{j=1}^{m} \frac{1}{\sum_{k=1}^{m} b(\gamma_k(t))\varphi} \bullet a(\gamma_j(t))\Bigg(\varphi \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t))v_k(t)\right)\right. \\
&\qquad \left. - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t))v_k(t)\right)\right]\Bigg) \\
&= \sum_{j=1}^{m} \left(\sum_{k=1}^{m} b(\mathbb{1})b(\gamma_k(t))\varphi\right)^{-1} \bullet a(\gamma_j(t))\Bigg(\varphi \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t))v_k(t)\right)\right. \\
&\qquad \left. - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t))v_k(t)\right)\right]\Bigg) \\
&= \sum_{j=1}^{m} \left(b(\gamma_j(t))\sum_{k=1}^{m} b(\gamma_j^{-1}(t)\circ\gamma_k(t))\varphi\right)^{-1} \bullet a(\gamma_j(t))\Bigg(\varphi \\
&\qquad \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t))v_k(t)\right) - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t))v_k(t)\right)\right]\Bigg) \\
&= \sum_{j=1}^{m} \left(b(\gamma_j(t))\left(\sum_{k=1}^{m} b(\gamma_j^k(t))\varphi\right)^{-1}\right) \bullet a(\gamma_j(t))\Bigg(\varphi \\
&\qquad \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t))v_k(t)\right) - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t))v_k(t)\right)\right]\Bigg)
\end{aligned}$$

$$= \sum_{j=1}^{m} a(\gamma_j(t)) \Bigg(\frac{\varphi}{\sum_{k=1}^{m} b(\gamma_j^k(t))\varphi} \bullet \Bigg[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t)) v_k(t) \right) - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t)) v_k(t) \right) \Bigg] \Bigg)$$

Requiring equality of the summands in $\sum_{j=1}^{m}$ and applying $a(\gamma_j^{-1}(t))$ on both sides in the j-th equation we obtain

(10.67)
$$\begin{aligned} v_{j,t}(t) =& F(v_j(t)) - a(\gamma_j^{-1}(t)) d\left[a(\gamma_j(t)) v_j(t)\right] \gamma_{j,t}(t) + \frac{\varphi}{\sum_{k=1}^{m} b(\gamma_j^k(t))\varphi} \\ & \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t)) v_k(t) \right) - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t)) v_k(t) \right) \right] \end{aligned}$$

for $j = 1, \ldots, m$. Next, we introduce $\mu_j(t) \in \mathfrak{g} = T_{\mathbb{1}} G$ via

(10.68)
$$\gamma_{j,t}(t) = dL_{\gamma_j(t)}(\mathbb{1}) \mu_j(t),\ 0 < t < T$$

for every $j = 1, \ldots, m$. Using (10.15) once more, equation (10.67) can be written as

(10.69)
$$\begin{aligned} v_{j,t}(t) =& F(v_j(t)) - d\left[a(\mathbb{1}) v_j(t)\right] \mu_j(t) + \frac{\varphi}{\sum_{k=1}^{m} b(\gamma_j^k(t))\varphi} \\ & \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t)) v_k(t) \right) - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t)) v_k(t) \right) \right]. \end{aligned}$$

To compensate the extra variables $\mu_j(t)$ for $j = 1, \ldots, m$, we require as in the derivation of the freeze method for single structures $q = \dim \mathfrak{g}$ phase conditions $\Psi(v_j(t), \mu_j(t)) = 0$ for every $j = 1, \ldots, m$, where Ψ is defined by (10.17). Finally, we impose the initial conditions $\gamma_j(0) = \gamma_j^0 \in G$ for equation (10.68) and $v_j(0) = v_j^0$ such that

$$u_0(x) = \sum_{j=1}^{m} a(\gamma_j^0) v_j^0(x),\ x \in \mathbb{R}^d.$$

This leads to the abstract formulation of the decompose and freeze method as a **coupled nonlinear system of differential algebraic evolution equations (SDAE)**

(10.70a)
$$\begin{aligned} v_{j,t}(t) =& F(v_j(t)) - d\left[a(\mathbb{1}) v_j(t)\right] \mu_j(t) + \frac{\varphi}{\sum_{k=1}^{m} b(\gamma_j^k(t))\varphi}, \qquad v_j(0) = v_j^0, \\ & \bullet \left[F\left(\sum_{k=1}^{m} a(\gamma_j^k(t)) v_k(t) \right) - \sum_{k=1}^{m} F\left(a(\gamma_j^k(t)) v_k(t) \right) \right] \end{aligned}$$

(10.70b)
$$0 = \Psi(v_j(t), \mu_j(t)),$$

(10.70c)
$$\gamma_{j,t}(t) = dL_{\gamma_j(t)}(\mathbb{1}) \mu_j(t), \qquad \gamma_j(0) = \gamma_j^0,$$

with abbreviation $\gamma_j^k(t) := \gamma_j^{-1}(t)\circ\gamma_k(t)$. The equations (10.70a) and (10.70c) must be satisfied for $t > 0$ and (10.70b) for $t \geqslant 0$. In applications (10.70a) is now a coupled PDE, (10.70b) are algebraic constraints and (10.70c) is an ODE. In contrast to the freezing method for single-structures, the system (10.70a) and the reconstruction equation (10.70c) are coupled, i.e. they must be solved simultaneously. The algebraic constraint can be substituted once more by one of the phase conditions, that we have discussed in Section 10.2.

Example 10.15 (Reaction diffusion systems, Part 4)**.** We continue with Example 10.8. Let the Banach space $(X, \|\cdot\|)$ be still given by $\left(L^p(\mathbb{R}^d,\mathbb{K}^N), \|\cdot\|_{L^p}\right)$ for $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and $1<p<\infty$. Let $E = C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$, then the module $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$ acts on X via multiplication

$$\bullet : C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})\times X \to X, \quad (\varphi,u)\mapsto \varphi\bullet u := \varphi u.$$

The representation for the SE(d)-action a on X and property (10.64) yields that the SE(d)-action b on $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$ must also be given by

$$b(\cdot)\varphi : \mathrm{SE}(d)\to C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R}), \quad \gamma=(R,\tau)\mapsto [b(\gamma)\varphi](\cdot) := \varphi(R^{-1}(\cdot-\tau)),$$

i.e. a and b formally coincide. Now, it is straightforward to check that (10.65) is satisfied. Further, let $\varphi\in C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$ be a positive radial bump function such that the main mass is located near zero and $0<\varphi(x)\leqslant 1$ for $x\in\mathbb{R}^d$, e.g.

$$\varphi(x) = \mathrm{sech}\,(\beta\,|x|)\,, \text{ for some } \beta>0,$$

then the expression $\sum_{j=1}^m b(\gamma_j)\varphi\in C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$ is invertible in $C_{\mathrm{ub}}(\mathbb{R}^d,\mathbb{R})$ for every $\gamma_1,\ldots,\gamma_m\in\mathrm{SE}(d)$.

We introduce new functions $\gamma_j(t) = (R_j(t),\tau_j(t))\in\mathrm{SE}(d)$ and $v_j(\cdot,t)\in Y$ for every $j=1,\ldots,m$ such that the solution u of (10.10) can be written as

$$u(x,t) = \sum_{j=1}^m a(\gamma_j(t))v_j(x,t) = \sum_{j=1}^m v\left(R_j^{-1}(t)\,(x-\tau_j(t))\,,t\right), \tag{10.71}$$

for $t\geqslant 0$ and $x\in\mathbb{R}^d$ with $d\geqslant 2$. Inserting the freezing ansatz (10.71) into (10.10), using the partition of unity explained above and requiring equality for every summand, yields analogously to (10.67)

$$\begin{aligned} v_{j,t}(x,t) =& A\triangle v_j(x,t) + f(v_j(x,t)) - a(\gamma_j^{-1}(t))d\left[a(\gamma_j(t))v_j(x,t)\right]\gamma_{j,t}(t)\\ &+\frac{\varphi(x)}{\sum_{k=1}^m b(\gamma_j^k(t))\varphi(x)}\left[f\left(\sum_{k=1}^m a(\gamma_j^k(t))v_k(x,t)\right)\right.\\ &\left.-\sum_{k=1}^m f\left(a(\gamma_j^k(t))v_k(x,t)\right)\right], \end{aligned} \tag{10.72}$$

for $t>0$, $x\in\mathbb{R}^d$, $j=1,\ldots,m$ and abbreviation $\gamma_j^k(t) := \gamma_j^{-1}(t)\circ\gamma_k(t)$. Introducing $\mu_j(t) = (S_j(t),\lambda_j(t))\in\mathfrak{se}(d) = T_{\mathbb{1}}\mathrm{SE}(d)$ via (10.68), i.e.

$$\begin{pmatrix}R_{j,t}(t)\\ \tau_{j,t}(t)\end{pmatrix} = \gamma_{j,t}(t) = dL_{\gamma_j(t)}(\mathbb{1})\mu_j(t) = \begin{pmatrix}R_j(t)S_j(t)\\ R_j(t)\lambda_j(t)\end{pmatrix}, \quad \begin{pmatrix}R_j(0)\\ \tau_j(0)\end{pmatrix} = \begin{pmatrix}R_j^0\\ \tau_j^0\end{pmatrix},$$

where $\gamma_j(t) = (R_j(t), \tau_j(t)) \in \mathrm{SE}(d)$, the v_j-equations (10.72) can be transformed into (10.69) with $d\,[a(\mathbb{1})v_j(x,t)]\,\mu_j(t)$ given by (10.11). To compensate the extra variables $\mu_j(t)$ we require once more $\frac{d(d+1)}{2} = \dim \mathfrak{se}(d)$ phase conditions $\Psi(v_j(\cdot,t), \mu_j(t)) = 0$ for every $j = 1, \ldots, m$, where Ψ is a function as in (10.25). Possibilities for the choice of phase condition were discussed in Example 10.7. Thus, the decompose and freeze method yields the **coupled nonlinear system of partial differential algebraic evolution equations (SPDAE)**

$$\begin{aligned} v_{j,t}(x,t) =& A\triangle v_j(x,t) + f(v_j(x,t)) + \langle S_j(t)x + I_d\lambda_j(t), \nabla v_j(x,t)\rangle \\ &+ \frac{\varphi(x)}{\sum_{k=1}^m b(\gamma_j^k(t))\varphi(x)} \left[f\left(\sum_{k=1}^m a(\gamma_j^k(t))v_k(x,t)\right) \right. \\ &\left. - \sum_{k=1}^m f\left(a(\gamma_j^k(t))v_k(x,t)\right)\right], \quad v_j(x,0) = v_j^0(x) \end{aligned} \tag{10.73a}$$

$$0 = \Psi(v_j(\cdot,t), \mu_j(t)), \tag{10.73b}$$

$$\begin{pmatrix} R_{j,t}(t) \\ \tau_{j,t}(t) \end{pmatrix} = \begin{pmatrix} R_j(t)S_j(t) \\ R_j(t)\lambda_j(t) \end{pmatrix}, \quad \begin{pmatrix} R_j(0) \\ \tau_j(0) \end{pmatrix} = \begin{pmatrix} R_j^0 \\ \tau_j^0 \end{pmatrix}, \tag{10.73c}$$

where $\gamma_j(t) = (R_j(t), \tau_j(t)) \in \mathrm{SE}(d)$, $\mu_j(t) = (S_j(t), \lambda_j(t)) \in \mathfrak{se}(d)$ and the argument $\gamma_j^k(t)$ is equal to

$$\begin{aligned} \gamma_j^k(t) :=& \gamma_j^{-1}(t) \circ \gamma_k(t) = (R_j(t), \tau_j(t))^{-1} \circ (R_k(t), \tau_k(t)) \\ =& \left(R_j^{-1}(t), -R_j^{-1}(t)\tau_j(t)\right) \circ (R_k(t), \tau_k(t)) \\ =& \left(R_j^{-1}(t)R_k(t), R_j^{-1}(t)\left(\tau_k(t) - \tau_j(t)\right)\right). \end{aligned}$$

10.6 Numerical examples of multi-solitons

In this section we apply the decompose and freeze method to investigate numerically the interaction of multi-solitons. To be more precise, we consider the cubic-quintic complex Ginzburg-Landau equation (QCGL) in two space dimensions and study interaction processes of several spinning solitons. For this purpose we analyze both the nonfrozen equation and the decompose and freeze system. Interaction processes of the QCGL in the nonfrozen case for $d = 2$ and $d = 3$ can be found in [78]. Let us briefly discuss the numerical settings.

Generation of initial data. Consider the QCGL from Examples 2.1, 10.9 and 10.12

$$\begin{aligned} u_t =& \alpha\triangle u + u\left(\mu + \beta|u|^2 + \gamma|u|^4\right) \\ u(0) =& u_0 \end{aligned} \tag{10.74}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $u_0 : \mathbb{R}^d \to \mathbb{C}$, $d \in \{2,3\}$, $\alpha, \beta, \gamma, \mu \in \mathbb{C}$ and $\operatorname{Re}\alpha > 0$. For the parameter values (2.4) we already know from Example 2.1(1) and 10.9(1) that single spinning solitons exists, cf. Figure 2.1 for an illustration.

To investigate the interaction of several spinning solitons we need appropriate initial data, that come originally from a simulation. To generate the initial data we solve (10.74) on a circular disk of radius $R = 20$ until time $t = 150$. For the numerical computations we use the parameters from (2.4), continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.5$, the BDF method of order 2 with absolute tolerance $\mathrm{atol} = 10^{-4}$, relative tolerance $\mathrm{rtol} = 10^{-2}$ and maximal stepsize $\triangle t = 0.1$, homogeneous boundary conditions and initial data

$$u_0^{2D}(x_1, x_2) = \frac{1}{5}(x_1 + ix_2)\exp\left(-\frac{x_1^2 + x_2^2}{49}\right).$$

To avoid confusions in the sequal we denote the corresponding solution by $u^{\mathrm{id}}(x,t)$.

Compuational settings for the nonfrozen system. Now we investigate the interaction of several spinning solitons in the nonfrozen system (10.74). For this purpose we start a second computation and solve (10.74) on a circular disk of radius $R = 20$ until time $t = 150$. For the numerical computations we use again the parameters from (2.4), continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.5$, the BDF method of order 2 with absolute tolerance $\mathrm{atol} = 10^{-6}$, relative tolerance $\mathrm{rtol} = 10^{-2}$ and maximal stepsize $\triangle t = 0.1$ as well as homogeneous boundary conditions. As initial data u_0 for (10.74) we take the sum of $m \geqslant 2$ such solitons, shifted a certain distance apart with a possibly shifted phase, i.e. we consider initial data of the form

$$u_0(x) = \sum_{j=1}^{m} u_j^0\left(\tilde{R}_j^{-1}(x - \tilde{\tau}_j)\right),\ x \in \mathbb{R}^d, \tag{10.75}$$

with $u_j^0(x) := u^{\mathrm{id}}(x, 150)$ for $|x| \leqslant R$ and $u_j^0(x) = 0$ for $|x| > R$, $j = 1, \ldots, m$. This means that $u_j^0(x)$ is the single spinning soliton that we have computed before and which is illustrated in Figure 2.1. The constants $\tilde{\theta}_j$ and $\tilde{\tau}_j$, that are needed to construct the initial data from (10.75) via

$$\tilde{R}_j := \tilde{R}(\tilde{\theta}_j) := \begin{pmatrix} \cos\tilde{\theta}_j & -\sin\tilde{\theta}_j \\ \sin\tilde{\theta}_j & \cos\tilde{\theta}_j \end{pmatrix} \in \mathbb{R}^{2,2}, \quad \tilde{\theta}_j \in \mathbb{R}, \quad \tilde{\tau}_j \in \mathbb{R}^2, \tag{10.76}$$

are chosen explicitly in the examples below for every $j = 1, \ldots m$. Now, the solution $u(x,t)$ of (10.74) describes an interaction process of multi-solitons in the nonfrozen case.

In the numerical computations occur three different situations: If the distance of the centers of rotation $\tilde{\tau}_j$ is large enough and the phases are shifted identically, the solitons repel each other and a multi-structure consisting of m spinning solitons stabilizes. This behavior we call a **weak interaction**. If the distance is small and the phases are shifted identically, the solitons collide into a single spinning soliton. This behavior we call a **strong interaction**. If the phases are shifted differently and the distance of the centers of rotation is small enough, a permanent collision process will occur. This behavior we call a **phase shift interaction**.

Computational settings for the decompose and freeze system. We are now interested into that what happens with the shape of the profiles and their corresponding velocities during the interaction process of the spinning solitons. To investigate this numerically, we consider the coupled nonlinear system of partial differential equations for the QCGL

$$
\begin{aligned}
v_{j,t}(x,t) &= A\triangle v_j(x,t) + f(v_j(x,t)) + \langle S_j(t)x + I_d\lambda_j(t), \nabla v_j(x,t)\rangle \\
&\quad + \frac{\varphi(x)}{\sum_{k=1}^m b(\gamma_j^k(t))\varphi(x)}\left[f\left(\sum_{k=1}^m a(\gamma_j^k(t))v_k(x,t)\right) \right. \\
&\quad \left. - \sum_{k=1}^m f\left(a(\gamma_j^k(t))v_k(x,t)\right)\right], \quad v_j(x,0) = v_j^0(x)
\end{aligned} \tag{10.77a}
$$

$$
\begin{aligned}
0 &= \mathrm{Re}\,\left(v_j - \hat{v}_j, (x_k D_i - x_i D_k)\,\hat{v}_j\right)_{L^2}, \; i = 1,\dots,d-1, \; k = i-1,\dots,d \\
0 &= \mathrm{Re}\,\left(v_j - \hat{v}_j, D_l \hat{v}_j\right)_{L^2}, \; l = 1,\dots,d,
\end{aligned} \tag{10.77b}
$$

$$
\begin{pmatrix} R_{j,t}(t) \\ \tau_{j,t}(t) \end{pmatrix} = \begin{pmatrix} R_j(t)S_j(t) \\ R_j(t)\lambda_j(t) \end{pmatrix}, \quad \begin{pmatrix} R_j(0) \\ \tau_j(0) \end{pmatrix} = \begin{pmatrix} R_j^0 \\ \tau_j^0 \end{pmatrix}, \tag{10.77c}
$$

for $j = 1,\dots,m$, with parameters $\alpha,\beta,\gamma,\mu \in \mathbb{C}$, $\mathrm{Re}\,\alpha > 0$ and nonlinearity

$$
f : \mathbb{C} \to \mathbb{C}, \quad f(v) = v\left(\mu + \beta|v|^2 + \gamma|v|^4\right).
$$

In the examples below we compute the solution of (10.77) on a circular disk of radius $R = 20$ that is centered at the origin. For the numerical computations we used the same settings as for the nonfrozen case above, i.e. we use the parameters from (2.4), continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.5$, the BDF method of order 2 with absolute tolerance $\mathrm{atol} = 10^{-6}$, relative tolerance $\mathrm{rtol} = 10^{-2}$ and maximal stepsize $\triangle t = 0.1$ as well as homogeneous boundary conditions. Moreover, we equip (10.77) with initial data

$$
v_j^0(x) = u_j^0(x), \quad R_j^0 = \tilde{R}(\theta_j^0), \quad \theta_j^0 = \tilde{\theta}_j \in \mathbb{R}, \quad \tau_j^0 = \tilde{\tau}_j \in \mathbb{R}^2 \tag{10.78}
$$

for every $j = 1,\dots,m$. Finally, the bump function is given by

$$
\varphi(x) = \frac{2}{e^{b|x|} + e^{-b|x|}} = \frac{1}{\cosh b|x|} = \mathrm{sech} b|x|, \quad b \in \mathbb{R}, \tag{10.79}
$$

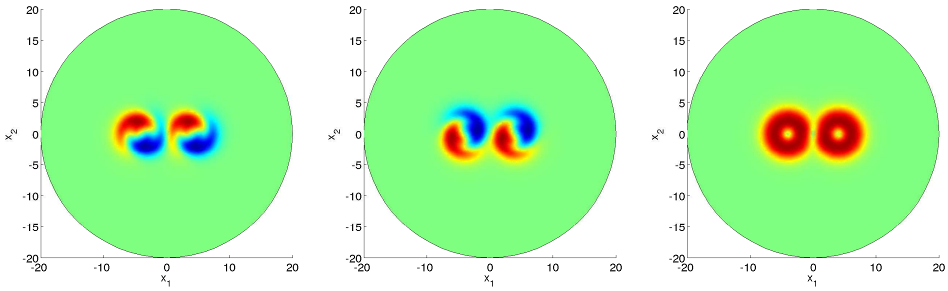
with $b = 0.5$ and the reference functions $\hat{v}_j(x) = v_j^0(x)$ for $j = 1, 2$.

Let $u(x,t)$ denote the solution of the nonfrozen system (10.74) and let $v(x,t)$ denote the solution of the decompose and freeze system (10.77), then we can expect that the time evolution of the interaction process is approximatively

$$
u(x,t) \approx \sum_{j=1}^m v_j\left(R_j^{-1}(t)\left(x - \tau_j(t)\right), t\right), \tag{10.80}
$$

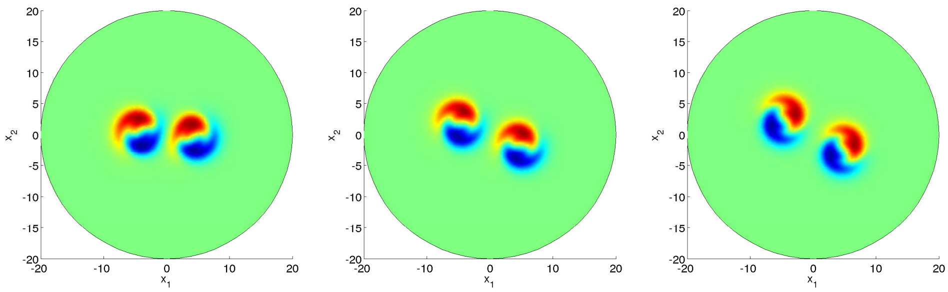
i.e. the right hand side, called the **superposition**, approximates the solution of (10.74), where $(v_j, (R_j, \tau_j))$ denotes the solution of (10.77). This will be checked in the examples below. For the numerical computations we use Comsol Multiphysics$^{\mathrm{TM}}$, [1].

Example 10.16 (Weak interaction of two spinning solitons in 2D)**.** In this example we investigate the weak interaction of two spinning solitions in two space dimensions. To make ourself familiar with the behavior of weak interaction involving two spinning solitons we first discuss the results of the nonfrozen system (10.74). Afterwards we discuss the results of the decompose and freeze system (10.77).



(a) (b) (c)

Figure 10.21: Initial data for weak interaction of 2 spinning solitons in $2D$



(a) (b) (c)

Figure 10.22: Time evolution for weak interaction of 2 spinning solitons in $2D$

Figure 10.22 shows the real part of the time evolution for the weak interaction of two spinning solitons in $\mathbb{R}^2$ as the solution of (10.74) at time $t = 12.6$, 75.0 and 150.0 in (a)-(c). For the numerical computation we used (10.76) with $m = 2$ and

$$\tilde{\theta}_1 = \tilde{\theta}_2 = 0, \quad \tilde{\tau}_1 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad \tilde{\tau}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{10.81}$$

i.e. the initial data for (10.74) are the sum of two spinning solitons centered at $\pm(4, 0)$ and without phase shift. Figure 10.21 shows the real part (a), imaginary part (b) and the absolute value (c) of the initial function u_0 from (10.75). The colorbars in Figure 10.21 and 10.22 are scaled from -1.65 (blue) to 1.65 (red). In Figure 10.22 we observe that the solitons repel from each other and they change their position clockwise. We next discuss the results obtained by the decompose and freeze method, cf. (10.77).

Figure 10.23 illustrates the corresponding results for the decompose and freeze method (10.77) in $\mathbb{R}^2$ for $m = 2$. For the numerical computation we used the initial data (10.78) with parameters from (10.81). Figure 10.23(d)-(e) shows the real parts of the single profiles v_1 in (d) and v_2 in (e) at time $t = 150$. We observe that each of these profiles possess the shape of a spinning soliton. Figure 10.23(f) illustrates the time evolution for the positions of these two profiles from $t = 0$ to $t = 150$. The

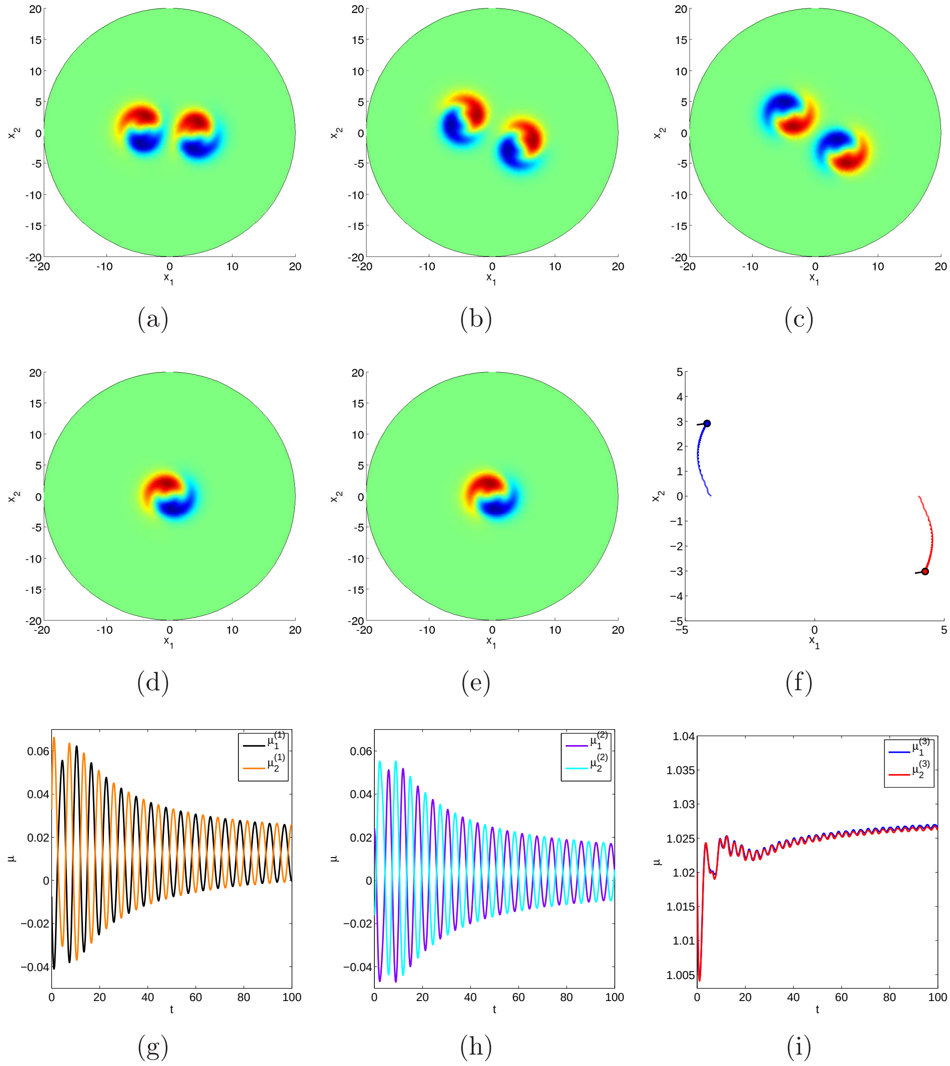


Figure 10.23: Weak interaction of 2 spinning solitons in $2D$ with decompose and freeze method

blue and the red line describes the curve for the position of v_1 and v_2, respectively. The positions at the end time $t = 150$ are represented by the blue and the red circle. The pointers that are fixated at each circle represent the current phase position. We observe that the positions travel clockwise on a circle. Since the positions at time $t = 150$ are

$$p_1(150) = \begin{pmatrix} -4.155 \\ 2.916 \end{pmatrix}, \quad p_2(150) = \begin{pmatrix} 4.265 \\ -3.021 \end{pmatrix},$$

we deduce that $|p_1(150) - p_2(150)| = 10.3026$ and expect that the circle has approximatively the radius $R_{\mathrm{circ}} = 5.1513$. Therefore, the distance of the solitons has grown up from 8 to 10.3026, meaning that the solitons repel. In particular, the phases seem to coincide. Figure 10.23(g)-(i) shows the velocities: the translational velocities in x_1-direction (g) and in x_2-direction (h) as well as the angular velocities in the x_1-x_2-plane (i). Note that $\mu_j = \left(\mu_j^{(1)}, \mu_j^{(2)}, \mu_j^{(3)}\right)^T$ are the velocities for the jth profile v_j, $j=1,2$. We observe e.g. in Figure 10.23(g) that $\mu_1^{(1)}(t)$ and $\mu_2^{(1)}(t)$ are periodic in time with period $T_{x_1}^{2D}\approx 6.2$ and that $\mu_1^{(1)}(t)$ and $\mu_2^{(1)}(t)$ are shifted from each other by the value 3.1, which equals the half period length. A similar behavior we observe in Figure 10.23(h) for $\mu_1^{(2)}$ and $\mu_2^{(2)}$. In contrary to the translational velocities, the angular velocities $\mu_1^{(3)}$ and $\mu_2^{(3)}$ are just also periodic and their periods coincide but there seems to be no shift between their curves. Figure 10.23(a)-(c) shows the time evolution for the real part of the superposition, cf. (10.80), at time $t=12.6$, 75.0 and 150.0. Since the superposition can be considered as an approximation for the solution of (10.74), we compare the results illustrated in Figure 10.22(a)-(c) with those from Figure 10.23(a)-(c). Here, we observe that the decompose and freeze method after long time yields a certain phase shift, but the centers of rotation are good approximated. Altogether, the decompose and freeze method can reproduce the weak interaction of two spinning solitons.

Example 10.17 (Strong interaction of two spinning solitons in 2D)**.** In this example we investigate the strong interaction of two spinning solitons in two space dimensions. To establish a better understanding for the strong interaction of two spinning solitons we first discuss the results for the nonfrozen system (10.74) and then we discuss the results of the decompose and freeze system (10.77).

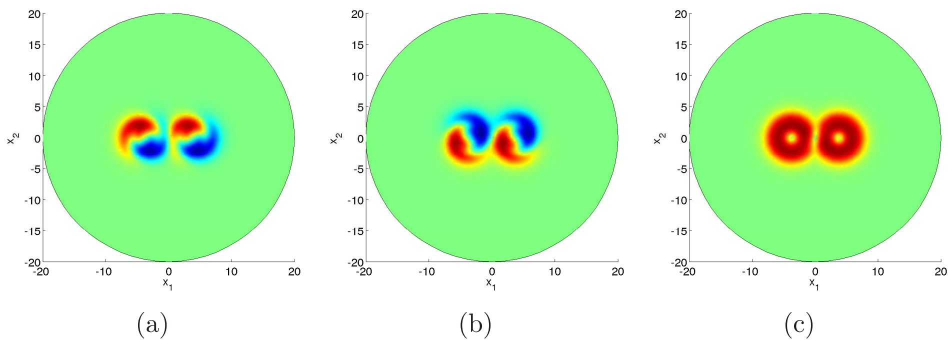


Figure 10.24: Initial data for strong interaction of 2 spinning solitons in $2D$

Figure 10.25 shows the real parts of the time evolution for the strong interaction of two spinning solitons in $\mathbb{R}^2$ as the solution of (10.74) at time $t=4.2$, 10.8, 11.7 in (a)-(c) and at time $t=15.3$, 18.0, 35.1 in (d)-(f). For the numerical computation we used (10.76) with $m=2$ and

$$\tilde{\theta}_1=\tilde{\theta}_2=0,\quad \tilde{\tau}_1=\begin{pmatrix}-3.75\\0\end{pmatrix},\quad \tilde{\tau}_2=\begin{pmatrix}3.75\\0\end{pmatrix}, \tag{10.82}$$

i.e. the two spinning solitons are now centered at $\pm(3.75,0)$ and without phase shift. Figure 10.24 shows the real part (a), imaginary part (b) and the absolute value (c) of the initial function u_0 from (10.75). The colorbar is again scaled from -1.65 (blue) to 1.65 (red). In Figure 10.25 we observe that the clockwise rotating solitons collide and produce a single spinning soliton that rotates about the origin with the same velocity as each of these two solitons before the collision. We next discuss the results obtained by the decompose and freeze method, cf. (10.77).

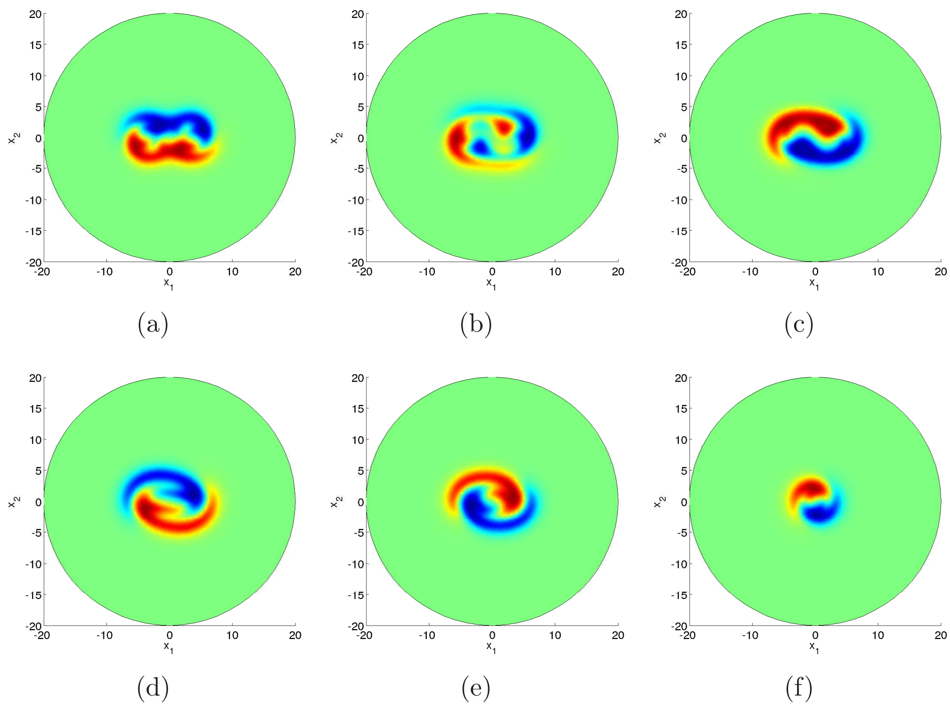


Figure 10.25: Time evolution for strong interaction of 2 spinning solitons in $2D$

Figure 10.26 illustrates the correspondig results of the decompose and freeze method (10.77) in $\mathbb{R}^2$ for $m = 2$. For the numerical computation we used the initial data (10.78) with parameters from (10.82). Figure 10.26(d)-(e) shows the real parts of the single profiles v_1 in (d) and v_2 in (e) at time $t = 150$. At first glance the profiles v_1 and v_2 look very strange but on closer inspection we observe that their sum equals the real part of a single spinning solition. This tells us that every profile v_1 and v_2 contains a certain portion of the single spinning soliton. Figure 10.26(f) illustrates the time evolution for the positions of these two profiles from $t = 0$ to $t = 150$. The blue and the red line describes the curve for the position of v_1 and v_2, respectively. The positions at the end time $t = 150$ are represented by the blue and the red circle. The pointers that are fixated at each circle represent the current phase position. We observe, in contrast to the weak interaction, that the positions in case of strong interaction travel counter-clockwise on a circle. Since the positions at time $t = 150$ are

$$p_1(150) = \begin{pmatrix} -0.1013 \\ -1.0114 \end{pmatrix}, \quad p_2(150) = \begin{pmatrix} 0.1250 \\ 0.9204 \end{pmatrix},$$

we deduce that their distance is given by $|p_1(150) - p_2(150)| = 1.9450$. Therefore, we expect that the circle has approximatively the radius $R_{\text{circ}} = 0.9725$. Note that the knowledge about the distance of the positions doesn't permit us to make any conclusions about the distance of the solitons and vice versa. In particular, the phases seem to coincide. Note that the lines forming the boundary of the circle are a little bit wavy. This seems to be caused by the homogeneous Neumann boundary conditions. In case of homogeneous Dirichlet boundary conditions these lines form a smooth circular curve. Figure 10.26(g)-(i) show the velocities: the translational

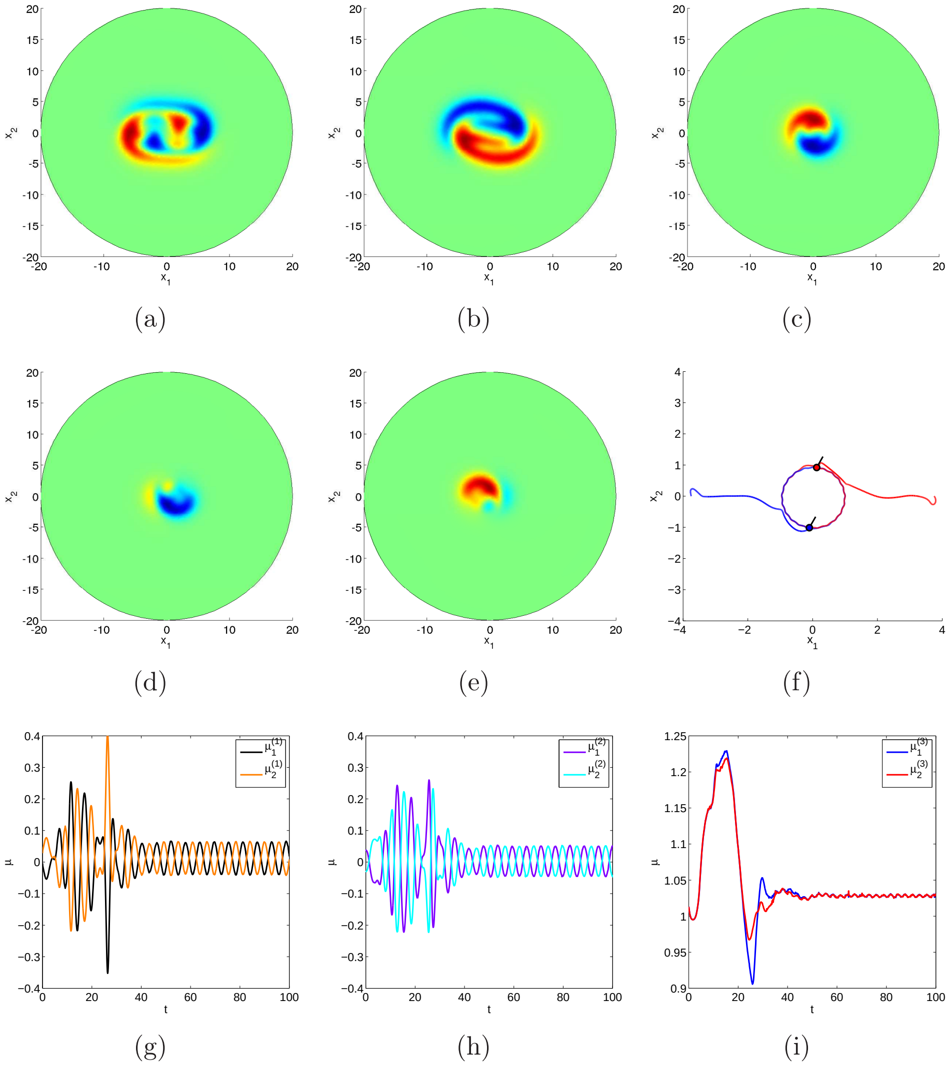


Figure 10.26: Strong interaction of 2 spinning solitons in $2D$ with decompose and freeze method

velocities in x_1-direction (g) and in x_2-direction (h) as well as the angular velocities in the x_1-x_2-plane (i). Recall that $\mu_j = \left(\mu_j^{(1)}, \mu_j^{(2)}, \mu_j^{(3)}\right)^T$ are the velocities for the jth profile v_j, $j = 1, 2$. Similar to the case of weak interaction, we observe e.g. in Figure 10.26(g) that $\mu_1^{(1)}(t)$ and $\mu_2^{(1)}(t)$ are periodic in time with period $T_{x_1}^{2D} \approx 6.0$ and that $\mu_1^{(1)}(t)$ and $\mu_2^{(1)}(t)$ approximatively satisfy the property $\mu_1^{(1)}(t) = -\mu_2^{(1)}(t)$. A similar behavior can be discovered in Figure 10.26(h) for $\mu_1^{(2)}$ and $\mu_2^{(2)}$. Note that the collision process that takes time from $t = 0$ to $t \approx 35$ can also be observed in the curves of the velocities. Figure 10.26(a)-(c) shows the time evolution for the real part of the superposition, cf. (10.80), at time $t = 10.8$, 15.3 and 35.1. A comparison of the results illustrated in Figure 10.26(a)-(c) with those from Figure 10.25(b),(d),(f) shows that the decompose and freeze method yields a very good reproduction of the collision process.

Example 10.18 (Phase shift interaction of two spinning solitons in 2D)**.** In the following example we analyze the phase shift interaction of two spinning solitons in two space dimensions. Similar as in the examples above, we first discuss the results for the nonfrozen system (10.74), then the results for the decompose and freeze system (10.77).

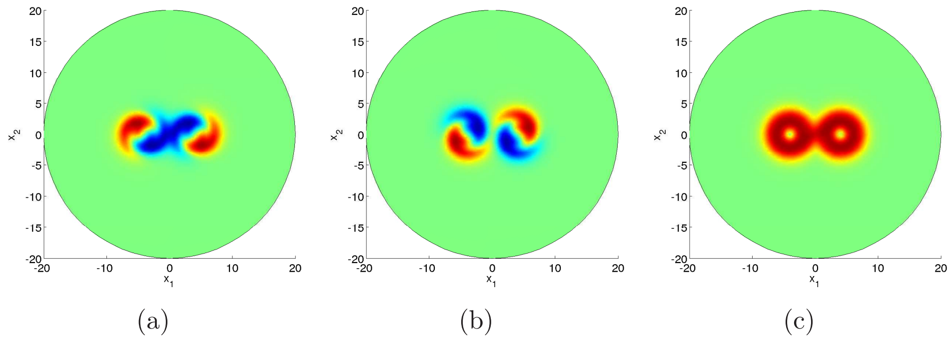


Figure 10.27: Initial data for phase shift interaction of 2 spinning solitons in $2D$

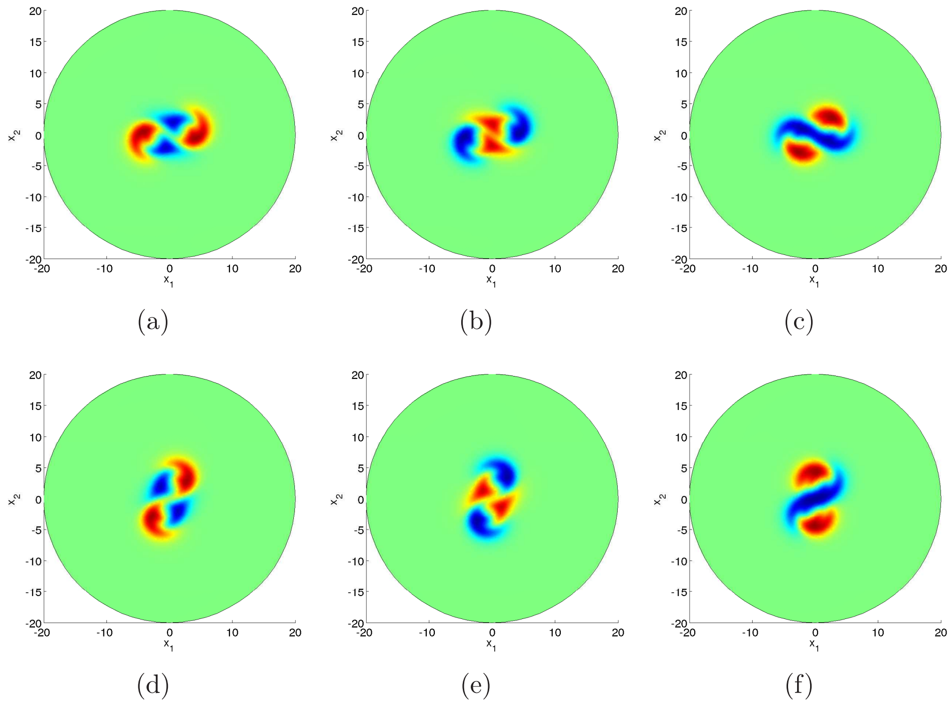


Figure 10.28: Time evolution for phase shift interaction of 2 spinning solitons in $2D$

Figure 10.28 illustrates the real parts of the time evolution for the phase shift interaction of two spinning solitons in $\mathbb{R}^2$ as the solution of (10.74) at time $t = 23.7$, 26.4, 27.9 in (a)-(c) and at time $t = 72.6$, 75.3, 77.1 in (d)-(f). For the numerical computation we used (10.76) with $m = 2$ and

$$\tilde{\theta}_1 = 0, \quad \tilde{\theta}_2 = \pi, \quad \tilde{\tau}_1 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad \tilde{\tau}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{10.83}$$

i.e. the first soliton is centered at $\pm(-4,0)$ without phase shift and the second soliton is centered at $(4,0)$ and rotated by 180 degrees. Figure 10.27 shows the real part (a), imaginary part (b) and the absolute value (c) of the initial function u_0 from (10.75). The colorbar reaches from -1.65 (blue) to 1.65 (red). In Figure 10.25 we observe that each soliton rotates clockwise about its respective center of rotation. Moreover, the real part, imaginary part and absolute value of the complete structure rotates additionally about the origin. This is in strong contrast to the examples of weak and strong interaction above. In particular, the phase shift seems to prevent the collision of the solitons. We next discuss the results obtained by the decompose and freeze method, cf. (10.77).

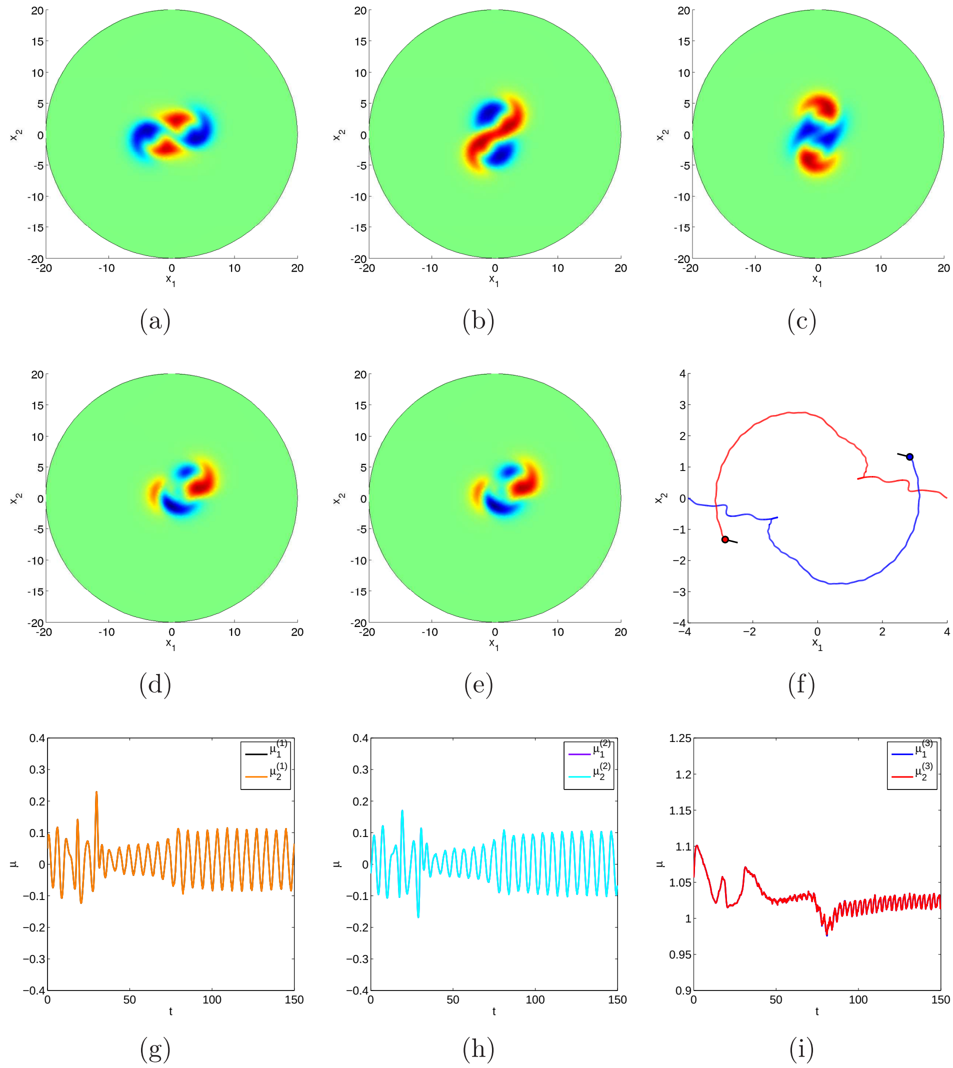


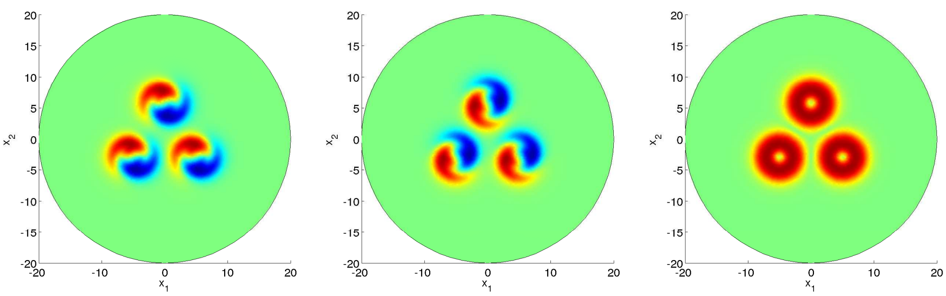
Figure 10.29: Phase shift interaction of 2 spinning solitons in $2D$ with decompose and freeze method

Figure 10.29 visualizes the numerical results of the decompose and freeze method (10.77) in $\mathbb{R}^2$ for $m=2$. For the numerical computation we used the initial data (10.78) with parameters from (10.83). Figure 10.29(d)-(e) show the real part of the single profiles v_1 in (d) and v_2 in (e) at time $t=150$. Even though we have a phase-shift interaction without interruption, we observe, similarly to the case of weakly interacting solitons, that both profiles coincide with each other. Figure 10.29(f) illustrates the time evolution for the positions of these two profiles from $t = 0$ to $t = 150$. The blue and the red line describes once more the curve for the position of v_1 and v_2, respectively. The positions at the end time $t = 150$ are represented by the blue and the red circle. The pointers that are fixated at each circle represent the current phase position. Similar to the case of strong interaction, the positions travel counter-clockwise on a circle. The positions at time $t = 150$ are

$$p_1(150) = \begin{pmatrix} 2.845 \\ 1.324 \end{pmatrix}, \quad p_2(150) = \begin{pmatrix} -2.854 \\ -1.327 \end{pmatrix}$$

and hence their distance is given by $|p_1(150) - p_2(150)| = 6.2854$. Consequently, we expect that the circle has approximatively the radius $R_{\text{circ}} = 3.1427$. In particular, as indicated by the pointers the time evolution preserves the initial shift. Figure 10.29(g)-(i) show the velocities: the translational velocities in x_1-direction (g), in x_2-direction (h) and the angular velocities in the x_1-x_2-plane (i). Recall that $\mu_j = \left(\mu_j^{(1)}, \mu_j^{(2)}, \mu_j^{(3)}\right)^T$ are the velocities for the jth profile v_j, $j = 1, 2$. The velocities $\mu_1^{(1)}(t)$ and $\mu_2^{(1)}(t)$ are again periodic in time with period $T_{x_1}^{2D} \approx 5.8$, but in contrast to the strong interaction, the translational velocities in x_1-direction now satisfy approximatively $\mu_1^{(1)}(t) = \mu_2^{(1)}(t)$, i.e. with positive sign. Figure 10.29(a)-(c) show the time evolution for the real part of the superposition, cf. (10.80), at time $t = 26.4$, 72.6 and 77.1. Note that the colorbar is now scaled from -1.7 (blue) to 1.7 (red). A comparison of the results from Figure 10.29(a)-(c) with those from Figure 10.28(b),(d),(f) shows that the decompose and freeze method provides us a good reproduction of the phase-shift collision process but admits a certain phase difference, that develops over long time.

Example 10.19 (Weak interaction of three spinning solitons in 2D)**.** We now expand the investigations from Example 10.16 and analyze the weak interaction of three spinning solitons in two space dimensions. As usual, we first discuss the results for the nonfrozen system (10.74) and afterward we discuss the results of the decompose and freeze system (10.77)



(a) (b) (c)

Figure 10.30: Initial data for weak interaction of 3 spinning solitons in $2D$

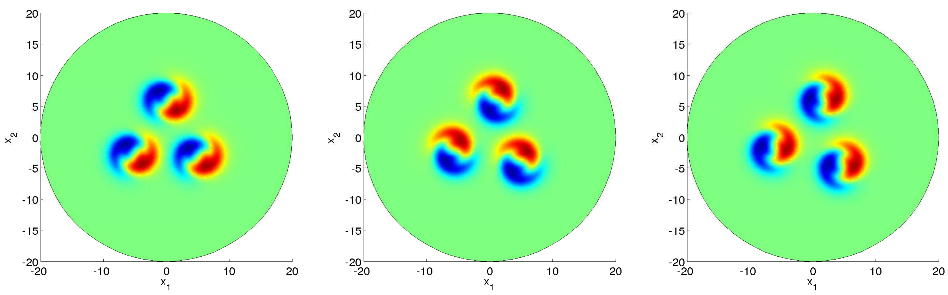


Figure 10.31: Time evolution for weak interaction of 3 spinning solitons in $2D$

Figure 10.31 shows the real part of the time evolution for the weak interaction of three spinning solitons in $\mathbb{R}^2$ as the solution of (10.74) at time $t = 15.0$, 75.0 and 150.0 in (a)-(c). For the numerical computation we used (10.76) with $m = 3$ and

$$\begin{aligned}\tilde{\tau}_1 &= \begin{pmatrix} r\cos\frac{\pi}{2} \\ r\sin\frac{\pi}{2}\end{pmatrix}, \quad \tilde{\tau}_2 = \begin{pmatrix} r\cos\frac{7\pi}{6} \\ r\sin\frac{7\pi}{6}\end{pmatrix}, \quad \tilde{\tau}_3 = \begin{pmatrix} r\cos\frac{11\pi}{6} \\ r\sin\frac{11\pi}{6}\end{pmatrix}, \\ \tilde{\theta}_1 &= \tilde{\theta}_2 = \tilde{\theta}_3 = 0, \quad r = \frac{2\cdot 5}{\sqrt{3}},\end{aligned} \tag{10.84}$$

i.e. as initial data for (10.74) we use the sum of three not phase shifted spinning solitons that are put on the vertices of an equilateral triangle. The numerator of r, that equals 10, describes the distance of two different $\tilde{\tau}_j$. The constant $r = \frac{2\cdot 5}{\sqrt{3}} \approx 5.7735$ itself denotes the radius of the circumcircle, which is known to be the circumradius. In complex notation the centers $\tilde{\tau}_j$ are located on the circle $re^{i\varphi}$ with $\varphi = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$. Figure 10.30 shows the real part (a), imaginary part (b) and the absolute value (c) of the initial function u_0 from (10.75). The colorbars in Figure 10.30 and 10.31 are scaled from -1.65 (blue) to 1.65 (red). Similar to the case of two weakly interacting solitons from Example 10.16, we observe that the solitons repel of each other and that they change their positions clockwise. We next discuss the results obtained by the decompose and freeze method, cf. (10.77).

Figure 10.32 illustrates the corresponding results for the decompose and freeze method (10.77) in $\mathbb{R}^2$ for $m = 3$. For the numerical computation we used the initial data (10.78) with parameters from (10.84). Figure 10.32(d)-(f) shows the real parts of the single profiles v_1 in (d), v_2 in (e) and v_3 in (f) at time $t = 150$. We observe that each of these profiles possess the shape of a spinning soliton. Figure 10.32(j) illustrates the time evolution for the positions of these three profiles from $t = 0$ to $t = 150$. The blue, the red and the green line describes the curve for the position of v_1, v_2 and v_3, respectively. The positions at the end time $t = 150$ are represented by the blue, red and green circle. The pointers that are fixated at each circle represent the current phase position. We observe that the positions travel clockwise on a circle and the phases are equal at the end time. Since the positions at time $t = 150$ are

$$p_1(150) = \begin{pmatrix} 2.331 \\ 5.625 \end{pmatrix}, \quad p_2(150) = \begin{pmatrix} -6.050 \\ -0.832 \end{pmatrix}, \quad p_3(150) = \begin{pmatrix} 3.734 \\ -4.862 \end{pmatrix},$$

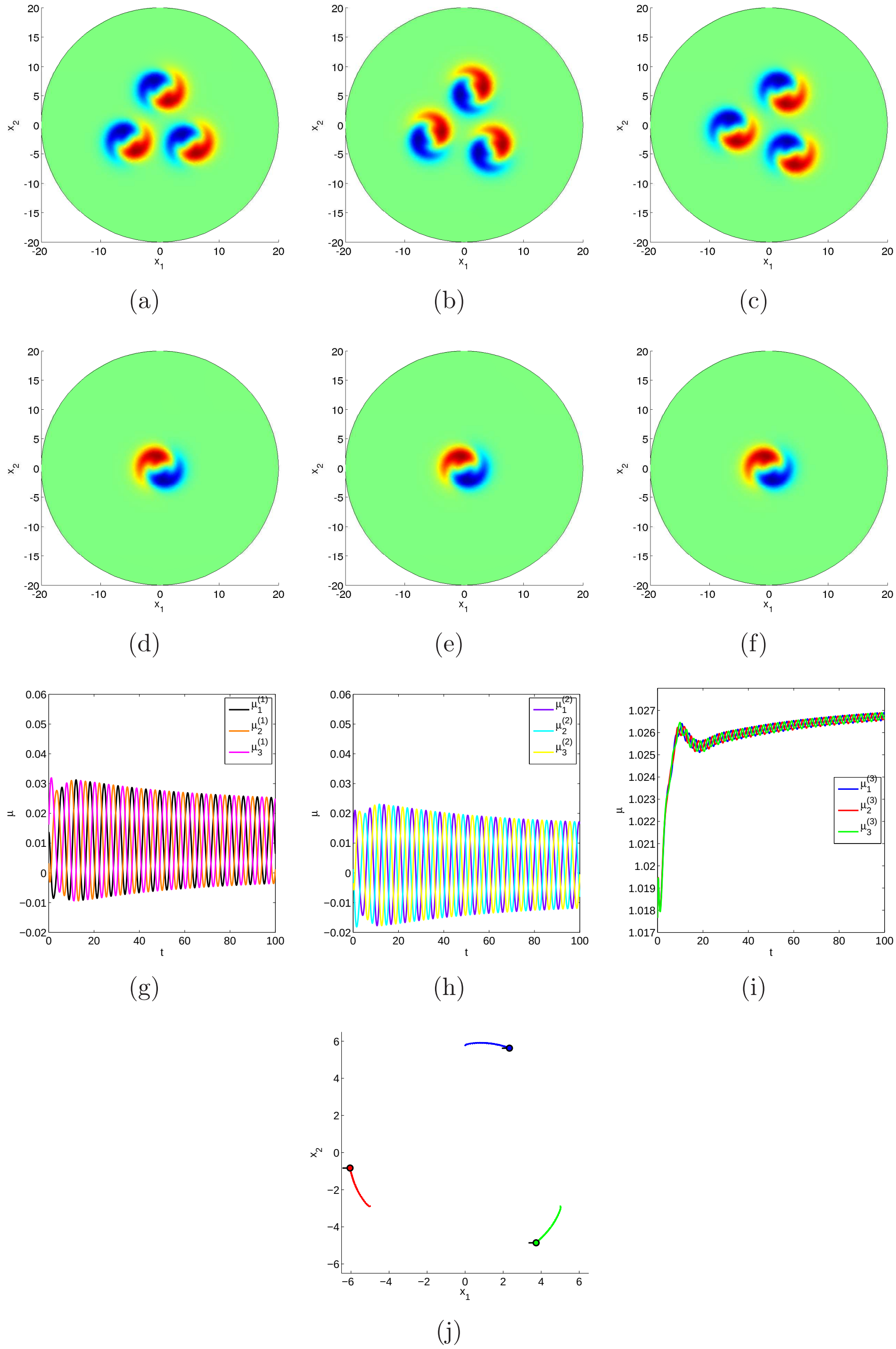
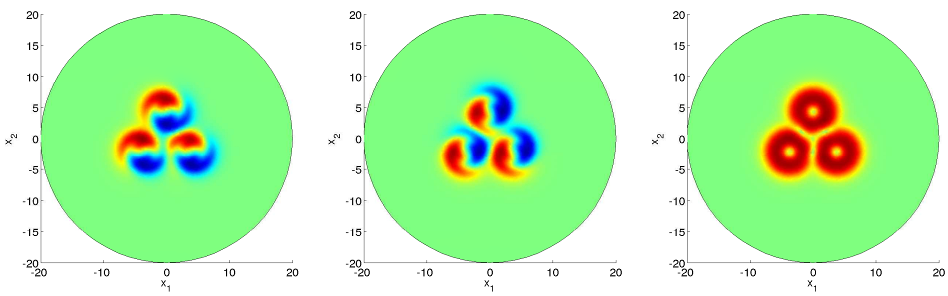


Figure 10.32: Weak interaction of 3 spinning solitons in $2D$ with decompose and freeze method.

we deduce that $|p_i(150) - p_j(150)| = 10.58$ for every $i, j \in \{1, 2, 3\}$ with $i \neq j$. Therefore, the distance of two different centers has grown up from 10 to 10.58, meaning that the solitons repel. In particular, we observe that the circumradius increases from 5.7735 to $R_{\text{circ}} \approx \frac{\sqrt{3}}{3} \cdot 10.58 = 6.10$. Figure 10.32(g)-(i) shows the velocities: the translational velocities in x_1-direction (g) and in x_2-direction (h) as well as the angular velocities in the x_1-x_2-plane (i). Note that $\mu_j = \left(\mu_j^{(1)}, \mu_j^{(2)}, \mu_j^{(3)}\right)^T$ are the velocities for the jth profile v_j, $j = 1, 2, 3$. We observe e.g. in Figure 10.32(g) that $\mu_1^{(1)}(t)$, $\mu_2^{(1)}(t)$ and $\mu_3^{(1)}(t)$ are periodic in time with period $T_{x_1}^{2D} \approx 6.1$, which is approximatively 2π, and that $\mu_1^{(1)}(t)$, $\mu_2^{(1)}(t)$ and $\mu_3^{(1)}(t)$ are shifted from each other by the value 2.03, which is approximatively $\frac{2\pi}{3}$, i.e. one third of the period length. The same behavior we observe in Figure 10.32(h) for $\mu_1^{(2)}$, $\mu_2^{(2)}$ and $\mu_3^{(2)}$. The angular velocities $\mu_1^{(3)}$, $\mu_2^{(3)}$ and $\mu_3^{(3)}$ are periodic with period $T_{(x_1,x_2)}^{2D} \approx 3.0$, that could be π, approximatively. But in contrast to Example 10.16, their curves are also shifted from each other by the value 1.0, which is approximatively $\frac{\pi}{3}$, i.e. one third of the period length. Figure 10.32(a)-(c) shows the time evolution for the real part of the superposition, cf. (10.80), at time $t = 15.0$, 75.0 and 150.0. Since the superposition can be considered as an approximation for the solution of (10.74), we compare the results illustrated in Figure 10.22(a)-(c) with those from Figure 10.32(a)-(c). Here, we observe that the decompose and freeze method after long time yields a certain phase shift, but the centers of rotation are good approximated.

Example 10.20 (Strong interaction of three spinning solitons in 2D)**.** In the last example we expand the investigations from Example 10.17 and investigate the strong interaction of three spinning solitons in two space dimensions. We again first discuss the results for the nonfrozen system (10.74) and then we discuss the results of the decompose and freeze system (10.77).



(a) (b) (c)

Figure 10.33: Initial data for strong interaction of 3 spinning solitons in $2D$

Figure 10.34 shows the real part of the time evolution for the strong interaction of three spinning solitons in $\mathbb{R}^2$ as the solution of (10.74) at time $t = 2.1$, 5.1, 8.4 in (a)-(c) and at time $t = 12.0$, 19.8, 50.1 in (d)-(f). For the numerical computation we used (10.76) with $m = 3$ and

$$
\begin{aligned}
&\tilde{\tau}_1 = \begin{pmatrix} r\cos\frac{\pi}{2} \\ r\sin\frac{\pi}{2} \end{pmatrix}, \quad \tilde{\tau}_2 = \begin{pmatrix} r\cos\frac{7\pi}{6} \\ r\sin\frac{7\pi}{6} \end{pmatrix}, \quad \tilde{\tau}_3 = \begin{pmatrix} r\cos\frac{11\pi}{6} \\ r\sin\frac{11\pi}{6} \end{pmatrix}, \\
&\tilde{\theta}_1 = \tilde{\theta}_2 = \tilde{\theta}_3 = 0, \quad r = \frac{2\cdot 3.75}{\sqrt{3}},
\end{aligned} \tag{10.85}
$$

i.e. as initial data for (10.74) we use the sum of three not phase shifted spinning solitons that are put on the vertices of an equilateral triangle. The distance of two different $\tilde{\tau}_j$ is chosen to be 7.5 and thus the circumradius equals $r = \frac{2\cdot 3.75}{\sqrt{3}} \approx 4.3301$.

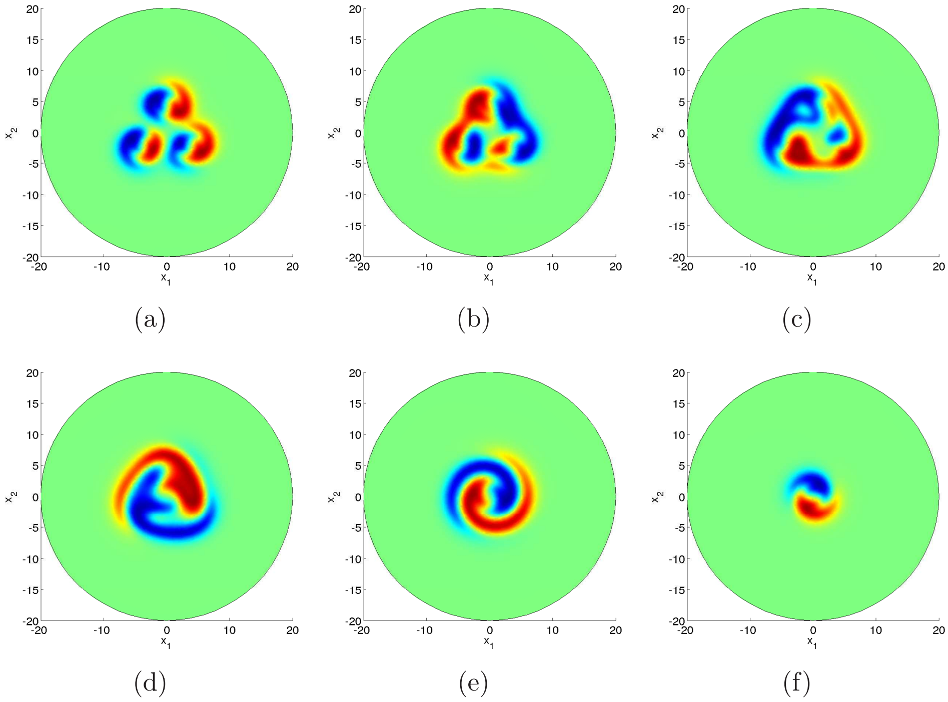


Figure 10.34: Time evolution for strong interaction of 3 spinning solitons in $2D$

Figure 10.33 shows the real part (a), imaginary part (b) and the absolute value (c) of the initial function u_0 from (10.75). The colorbars in Figure 10.33 and 10.34 are scaled from -1.65 (blue) to 1.65 (red). Similar to the case of two strongly interacting solitons from Example 10.17, we observe that the solitons collide into a single spinning soliton that rotates about the origin with the same velocity as each of these three solitions before the collision. We next discuss the results obtained by the decompose and freeze method, cf. (10.77).

Figure 10.35 illustrates the corresponding results for the decompose and freeze method (10.77) in $\mathbb{R}^2$ for $m = 3$. For the numerical computation we used the initial data (10.78) with parameters from (10.85). Figure 10.35(d)-(f) shows the real parts of the single profiles v_1 in (d), v_2 in (e) and v_3 in (f) at time $t = 150$. Similar as in Example 10.17 the profiles look very strange. But their superposition (10.80) that is depicted in Figure 10.35(c) shows the real part of a single spinning soliton. This tells us one more that every profile v_1, v_2 and v_3 contains a certain portion of the single spinning soliton. Figure 10.35(j) illustrates the time evolution for the positions of these three profiles from $t = 0$ to $t = 150$. The blue, the red and the green line describes the curve for the position of v_1, v_2 and v_3, respectively. The positions at the end time $t = 150$ are represented by the blue, red and green circle. The pointers that are fixated at each circle represent the current phase position. We observe that the positions travel clockwise on a circle, which is in contrast to the strong interaction of two spinning solitons from Example 10.17. Their phases are equal at the end time. Since the positions at time $t = 150$ are

$$p_1(150) = \begin{pmatrix} -1.1520 \\ -0.3108 \end{pmatrix}, \quad p_2(150) = \begin{pmatrix} 0.8207 \\ -0.8348 \end{pmatrix}, \quad p_3(150) = \begin{pmatrix} 0.2992 \\ 1.1330 \end{pmatrix},$$

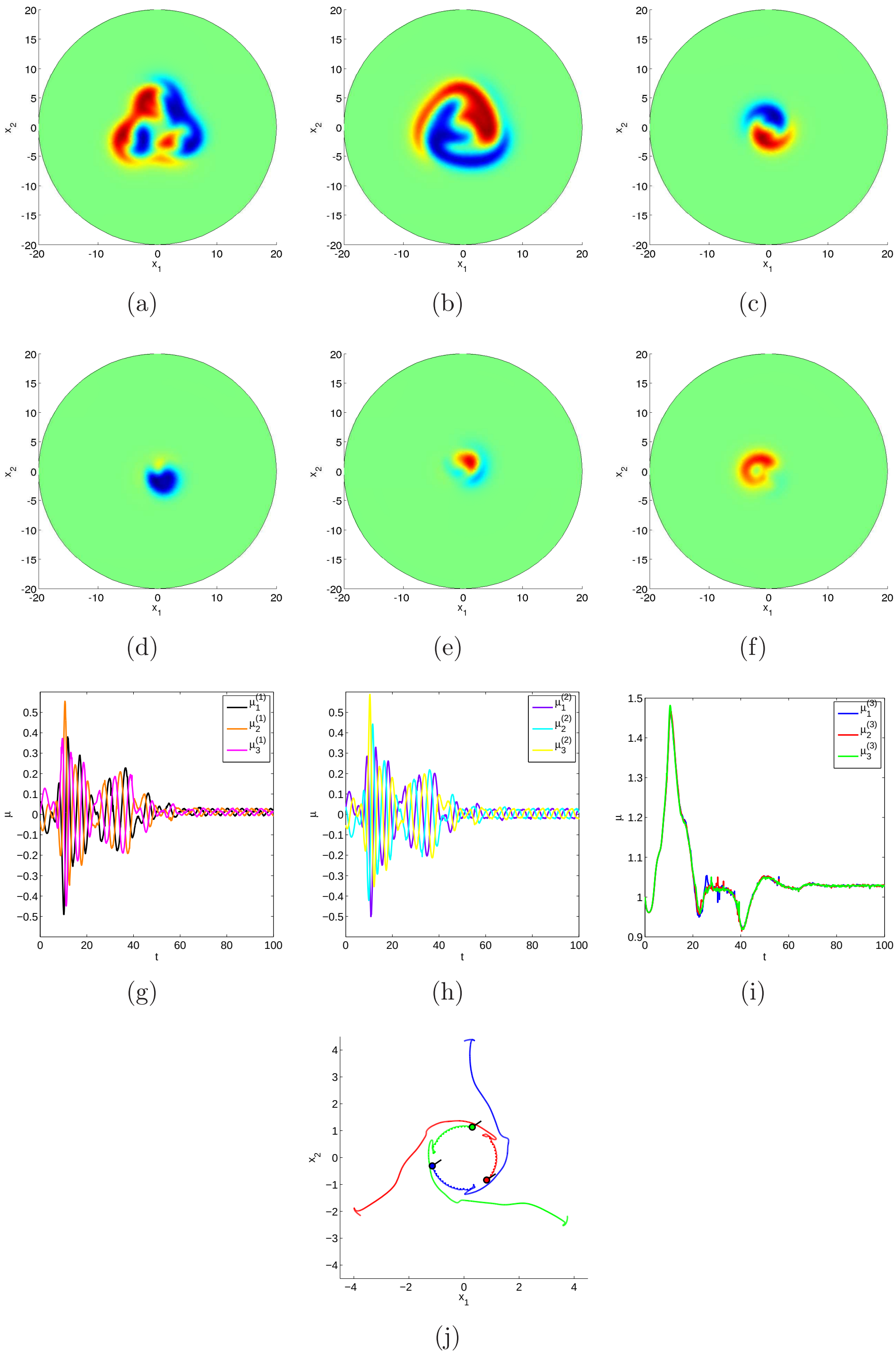


Figure 10.35: Strong interaction of 3 spinning solitons in $2D$ with decompose and freeze method

we deduce that $|p_i(150) - p_j(150)| = 2.04$ for every $i, j \in \{1, 2, 3\}$ with $i \neq j$. This shows that the distance of two different centers has plummeted from 7.5 to 2.04. In particular, we observe that the circumradius decreases from 4.3301 to $R_{\text{circ}} \approx \frac{\sqrt{3}}{3} \cdot 2.04 = 1.1778$. The boundary of the circle is wavy again, which is similar to the case of two strongly interaction solitons from Example 10.17. Figure

10.35(g)-(i) shows the velocities: the translational velocities in x_1-direction (g) and in x_2-direction (h) as well as the angular velocities in the x_1-x_2-plane (i). Note that $\mu_j = \left(\mu_j^{(1)}, \mu_j^{(2)}, \mu_j^{(3)}\right)^T$ are the velocities for the jth profile v_j, $j = 1, 2, 3$. Similar to the previous examples we observe in Figure 10.35(g) that $\mu_1^{(1)}(t)$, $\mu_2^{(1)}(t)$ and $\mu_3^{(1)}(t)$ are periodic in time with period $T_{x_1}^{2D} \approx 6.0$, which is approximatively 2π, and that $\mu_1^{(1)}(t)$, $\mu_2^{(1)}(t)$ and $\mu_3^{(1)}(t)$ are shifted from each other by the value 2.0, which is approximatively $\frac{2\pi}{3}$, i.e. one third of the period length. The same behavior we observe in Figure 10.35(h) for $\mu_1^{(2)}$, $\mu_2^{(2)}$ and $\mu_3^{(2)}$. The angular velocities $\mu_1^{(3)}$, $\mu_2^{(3)}$ and $\mu_3^{(3)}$ converge to 1.027. Moreover, we observe that their graphs are congruent with each other and that they are not periodic in time. Note that the collision process that takes time from $t = 0$ to $t = 70$ which can also be observed in the velocity diagrams from Figure 10.35(g)-(i). Figure 10.35(a)-(c) shows the time evolution for the real part of the superposition, cf. (10.80), at time $t = 5.1$, 12.0 and 50.1. Finally, comparing the results illustrated in Figure 10.25(b),(d),(f) with those from Figure 10.35(a)-(c), we realize that the decompose and freeze method gives a good reproduction of the strong interaction process of three spinning solitons.

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List of Symbols

$\arg z$	argument of $z \in \mathbb{C}$	39
$\operatorname{Re} z$	real part of $z \in \mathbb{C}^N$	39
$\operatorname{Im} z$	imaginary part of $z \in \mathbb{C}^N$	39
$\det A$	determinant of matrix A	39
$\exp(A)$	exponential mapping, matrix exponential	40
$\operatorname{Tr}(A)$	trace of a matrix $A \in \mathbb{C}^{N,N}$	48

differential operators

$A\triangle v(x)$	diffusion term for $A \in \mathbb{K}^{N,N}$	13
$\langle Sx, \nabla v(x)\rangle$	drift term, rotational term for $S \in \mathfrak{so}(d)$	6
$\langle Sx + I_d\lambda, \nabla v(x)\rangle$	drift and translation term	161
$\mathcal{L}$	linearized differential operator	159
$\mathcal{L}_0^{\mathrm{diff}}$	diffusion operator	13
$\mathcal{L}_0^{\mathrm{drift}}$	drift operator	13
$\mathcal{L}_0$, $\mathcal{L}_{\mathrm{OU}}$	Ornstein-Uhlenbeck operator, $\mathcal{L}_0 = \mathcal{L}_0^{\mathrm{diff}} + \mathcal{L}_0^{\mathrm{drift}}$	67
$\mathcal{L}_\infty$	constant coefficient perturbation of $\mathcal{L}_0$	123
$\mathcal{L}_{Q_\varepsilon}$	small variable coefficient perturbation of $\mathcal{L}_\infty$	136
$\mathcal{L}_Q$	variable coefficient perturbation of $\mathcal{L}_\infty$	131
$\frac{d}{dt}$	time derivative	7
$\triangle$	Laplacian, Laplace operator, $\triangle = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$	6
$x_jD_i - x_iD_j$	angular derivative in x_i-x_j-plane	7
D_i	partial derivatives w.r.t. x, $D_i = \frac{\partial}{\partial x_i}$	7
Df	total derivative of $f : \mathbb{R}^N \to \mathbb{R}^N$	9
$\|\cdot\|_{\mathcal{L}_0}$	norm on $\mathcal{D}^p_{\max}(\mathcal{L}_0)$	121

heat kernels and Green's functions

$G(x,\xi)$	complex Green's matrix function of $\mathcal{L}_\infty$	130
$H(x,\xi,t)$	complex heat kernel matrix of $\mathcal{L}_\infty$	45
$H_0(x,\xi,t)$	complex Ornstein-Uhlenbeck kernel, heat kernel matrix of $\mathcal{L}_0$	70
$H_\infty(x,\xi,t)$	complex heat kernel matrix of $\mathcal{L}_\infty$	125
$K(x-\xi,t)$	complex heat kernel matrix of $\mathcal{L}_0^{\mathrm{diff}}$	114
$K(\psi,t)$	complex integral kernels, K, K^i, K^{ji}	56
$\tilde{K}(\psi,t)$	complex integral kernels, $\tilde{K}$, $\tilde{K}^i$, $\tilde{K}^{ji}$	56

function spaces and norms

$\theta \in C(\mathbb{R}^d,\mathbb{R})$	weight function (of exponential growth rate)	10
p, q	index for the $W^{k,p}$-spaces, $1 \leqslant p \leqslant \infty$	42
Ω_T	space–time domain for $T > 0$, $\Omega_T = \mathbb{R}^d \times]0,T[$	42
$L^p(\mathbb{R}^d,\mathbb{K}^N)$	Lebesgue spaces with exponent $1 \leqslant p \leqslant \infty$	42
$W^{k,p}(\mathbb{R}^d,\mathbb{K}^N)$	Sobolev spaces of order $k \in \mathbb{N}_0$ with exponent $1 \leqslant p \leqslant \infty$	42
$L^p_\theta(\mathbb{R}^d,\mathbb{K}^N)$	exponentially weighted L^p–spaces	42
$W^{k,p}_\theta(\mathbb{R}^d,\mathbb{K}^N)$	exponentially weighted $W^{k,p}$–spaces	42
$W^{(2l,l),p}(\Omega_T,\mathbb{K}^N)$	space-time Sobolev spaces of order $(2l,l)$ with exponent p	42

special semigroups

notations for semigroup theory

$\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})$	part of $\sigma_{\mathrm{ess}}(\mathcal{L})$	144
$\\| \cdot \\|_{A_p}$	graph norm of A_p	77

notations for evolution equations and freezing

$a(\gamma)v$	group action of G on X, $a(\cdot)u : G \to X$ for $u \in X$ with derivative $d[a(\gamma)u]$	188
$b(\gamma)\varphi$	group action of $\varphi \in E$ in G	232
$\mathfrak{g}$	Lie algebra associated with G, $\mathfrak{g} = T_{\mathbb{1}}G$	188
q	dimension of Lie group $(G, \circ)$	188
$u_\star$	rotating wave, $u_\star : \mathbb{R}^d \times [0, \infty[\to \mathbb{K}^N$	7
$v_\star$	profile of rotating wave, $v_\star : \mathbb{R}^d \to \mathbb{K}^N$	7
$\hat{v}$	reference function, template function, $\hat{v} \in Y$	197
$x_\star$	fixed point under rotation, $x_\star \in \mathbb{R}^d$ from rotating wave $v_\star(e^{-tS_\star}(x - x_\star))$	194
E	E module that acts on X	231
F	operator for the evolution equation	188
$(G, \circ)$	Lie group with group operation $\circ$	188
H_v	isotropy group of an element $v \in Y$	198
L_γ	left multiplication with derivative $dL_\gamma(g)$	188
$\mathcal{N}$	kernel, null space	202
$\mathcal{O}_G(v)$	group orbit of $v \in Y$ in G	190
$\mathcal{R}$	range	202
$S_\star$	skew-symmetric matrix $S_\star \in \mathbb{R}^{d,d}$ from rotating wave $v_\star(e^{-tS_\star}(x - x_\star))$	194
X, Y	state spaces, e.g. $X = L^p(\mathbb{R}^d, \mathbb{K}^N)$ and $Y = W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$	188
$T_\gamma G$	tangential space of G at $\gamma \in G$	188
$\gamma, \gamma_1, \gamma_2$	elements of Lie group $(G, \circ)$	188
γ_∞	asymptotic phase	191
μ	$\mu \in \mathfrak{g}$, translational and angular velocities	189
$\sigma^{\mathrm{approx}}(\mathcal{L})$	numerical spectrum of $\mathcal{L}$	218
$\sigma^{\mathrm{approx}}_{\mathrm{point}}(\mathcal{L})$	numerical point spectrum of $\mathcal{L}$	220
$\sigma^{\mathrm{approx}}_{\mathrm{ess}}(\mathcal{L})$	numerical essential spectrum of $\mathcal{L}$	218
$\Psi(v(t), \mu(t))$	functional for phase conditions	196
$\Psi_{\mathrm{orth}}(v(t), \mu(t))$	orthogonal phase condition	198
$\Psi_{\mathrm{fix}}(v(t), \mu(t))$	fixed phase condition	197
$\mathbb{1}$	unit element of Lie group $(G, \circ)$	188

special constants

a_{rot}^{3D}	axis of rotation for $d = 3$	205
$a_{\min}$	$a_{\min} = (\rho(A^{-1}))^{-1}$ from (1.18)	9
$a_{\max}$	$a_{\max} = \rho(A)$ from (1.18)	9
a_0	$a_0 = -s(-A)$ from (1.18)	9
b_0	$b_0 = -s(Df(v_\infty))$ or $b_0 = -s(-B)$ from (1.18)	9
c_{rot}	center of rotation	202
v_∞	constant asymptotic state, zero of f, $v_\infty \in \mathbb{R}^N$	9

other expressions, functions and symbols

Index

M

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W