ON THE ADJUNCT OF AN ENDOMORPHISM

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Contents

Introduction

Let R be a ring (unital, commutative), let M be a R-module and let $f \in \mathrm{End}(M)$ be an endomorphism.

For $k \geq 0$ we consider endomorphisms

$$A_k(f) \in \mathrm{End}(M \otimes \Lambda^{k+1} M)$$

defined linearly from $\Lambda^k f$ with (co)-product operations of the exterior algebra.

For an explicit description of $A_k(f)$ see (2). For $A_1(f)$ see 1.7.

If M is a locally free R-module of rank n, then $A_{n-1}(f)$ yields the adjunct $f^\#$ of f. In short, this text is based on a simple observation: When the adjunct is considered as element of

$$\mathrm{End}(M \otimes \Lambda^n M)$$

rather than of $\mathrm{End}(M)$, there is no duality needed and the definition and proofs of basic properties extend smoothly to arbitrary R-modules.

The resulting generalization of the standard relation $\det(f) = ff^\# = f^\# f$ to the $A_k(f)$ is formulated in Proposition 2. The proof is immediate from the definition and the functoriality of the product and the co-product of the exterior algebra.

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The Cayley-Hamilton theorem generalizes accordingly, see Corollary 5. Here we follow the standard method of expanding $A_k(f - t \cdot 1_M)$ as polynomial in t.

Corollary 8 generalizes the standard expression of $f^\#$ as a polynomial in f.

The proofs are formulated on a quite formal functorial level and worked out in detail, even when a inspection of explicit formulas might appear simpler.

1. The endomorphisms $A_k(f)$

Let M be a R-module.

1.1. **Notation for elements in the exterior algebra.** Let K be a finite ordered set. For an K-tuple

$$x \in M^K$$

and a subset I of K of length r we use the notation

$$x_I = x_{i_1} \wedge \cdots \wedge x_{i_r} \in \Lambda^r M$$

where $i_1 < \cdots < i_r$ are the elements of I.

For instance, if $K = \{0, 1, \ldots, n\}$, then

$$\begin{aligned} x_K &= x_0 \wedge \cdots \wedge x_n \\ &= (-1)^i x_i \wedge x_{K\setminus\{i\}} = (-1)^{n-i} x_{K\setminus\{i\}} \wedge x_i \end{aligned}$$

1.2. **Multiplication and co-multiplication of the exterior algebra.** For the exterior algebra of M

$$\Lambda M = \bigoplus_{k\geq 0} \Lambda^k M$$

we denote by

$$\begin{aligned} \mu &\colon \Lambda M \otimes \Lambda M \to \Lambda M \\ \delta &\colon \Lambda M \to \Lambda M \mathbin{\hat{\otimes}} \Lambda M \end{aligned}$$

its product and co-product and by

$$\begin{aligned} \mu_{m,n} &\colon \Lambda^m M \otimes \Lambda^n M \to \Lambda^{m+n} M \\ \delta_{m,n} &\colon \Lambda^{m+n} M \to \Lambda^m M \otimes \Lambda^n M \end{aligned}$$

the corresponding components.

The product μ is given by

$$\mu(\omega \otimes \eta) = \omega \wedge \eta$$

and the co-product δ is the R-algebra homomorphism to the graded tensor product with

$$\delta(x) = x \otimes 1 + 1 \otimes x \qquad (x \in M)$$

Explicitly one has

$$\delta_{m,n}(x_K) = \sum_{I \subset K,\, |I|=m} \varepsilon_I x_I \otimes x_{K\setminus I} \qquad (\varepsilon_I x_I \wedge x_{K\setminus I} = x_K)$$

with $K = \{1, \ldots, n+m\}$ and appropriate signs ε_I as indicated on the right.

Note that

$$\mu_{m,n} \circ \delta_{m,n} = \binom{m+n}{m}$$

The (co-)product is (co)-associative. We use the following notations:

$$\mu_{m,n,k} = \mu_{m+n,k} \circ (\mu_{m,n} \otimes 1_{\Lambda^k M}) = \mu_{m,n+k} \circ (1_{\Lambda^m M} \otimes \mu_{n,k})$$
$$\delta_{m,n,k} = (\delta_{m,n} \otimes 1_{\Lambda^k M}) \circ \delta_{m+n,k} = (1_{\Lambda^m M} \otimes \delta_{n,k}) \circ \delta_{m,n+k}$$

Remark 1. If M is locally free of finite rank, the homomorphism $\delta_{m,n}$ is the "functorial dual" of $\mu_{m,n}$. This means that with respect to the canonical isomorphisms $(\Lambda^k M)^\vee = \Lambda^k(M^\vee)$ the dual of $\delta_{m,n}$ is the homomorphism $\mu_{m,n}$ for the dual of M:

$$((\delta_{m,n})_M)^\vee = (\mu_{m,n})_{(M^\vee)}$$

1.3. **The operator Φ.** Let

$$\Phi\colon \operatorname{End}(\Lambda^k M) \to \operatorname{End}(M \otimes \Lambda^{k+1} M)$$
$$\Phi(\varphi) = (1_M \otimes \mu_{1,k}) \circ (\tau \otimes \varphi) \circ (1_M \otimes \delta_{1,k})$$

where $\tau \in \operatorname{End}(M \otimes M)$ is the switch involution. Thus

$$\Phi(\varphi)(x \otimes s_L) = \sum_{i=0}^{k} (-1)^i s_i \otimes x \wedge \varphi(s_{L\setminus\{i\}})$$

for $x \in M$ and $s \in M^L$, $L = \{0, \ldots, k\}$.

Sometimes it is convenient to use the following variants. Let

$$\Psi\colon \operatorname{End}(\Lambda^k M) \to \operatorname{Hom}(M \otimes \Lambda^{k+1} M, \Lambda^{k+1} M \otimes M)$$
$$\Psi(\varphi) = (\mu_{1,k} \otimes 1_M) \circ (1_M \otimes \varphi \otimes 1_M) \circ (1_M \otimes \delta_{k,1})$$

and

$$\Psi^t\colon \operatorname{End}(\Lambda^k M) \to \operatorname{Hom}(\Lambda^{k+1} M \otimes M, M \otimes \Lambda^{k+1} M)$$
$$\Psi^t(\varphi) = (1_M \otimes \mu_{k,1}) \circ (1_M \otimes \varphi \otimes 1_M) \circ (\delta_{1,k} \otimes 1_M)$$

so that

$$\Psi(\varphi)(x \otimes s_L) = \sum_{i=0}^{k} (-1)^{k-i} x \wedge \varphi(s_{L\setminus\{i\}}) \otimes s_i$$
$$\Psi^t(\varphi)(s_L \otimes x) = \sum_{i=0}^{k} (-1)^i s_i \otimes \varphi(s_{L\setminus\{i\}}) \wedge x$$

Then

$$\sigma \circ \Psi(\varphi) = \Psi^t(\varphi) \circ \sigma = (-1)^k \Phi(\varphi)$$

where

$$\sigma\colon \Lambda^{k+1} M \otimes M \to M \otimes \Lambda^{k+1} M$$

is the switch.

1.4. **The endomorphisms P_n.** For $n \geq 1$ let

$$P_n = \Phi(1_{\Lambda^{n-1} M}) \in \operatorname{End}(M \otimes \Lambda^n M)$$

Thus

$$P_n(x \otimes s_N) = \sum_{i=1}^{n} (-1)^{i-1} s_i \otimes x \wedge s_{N\setminus\{i\}} \tag{1}$$

for $x \in M$ and $s \in M^N$, $N = \{1, \ldots, n\}$. We put $P_0 = 0$.

Let further Q_n be the composite of

$$M \otimes \Lambda^n M \xrightarrow{\mu} \Lambda^{n+1} M \xrightarrow{\delta} M \otimes \Lambda^n M$$

that is,

$$Q_n = \delta_{1,n} \circ \mu_{1,n} \in \mathrm{End}(M \otimes \Lambda^n M)$$

Obviously, if $\Lambda^{n+1} M = 0$, then $Q_n = 0$.

Lemma 1. *One has*

$$P_n + Q_n = 1_{M \otimes \Lambda^n M}$$

In particular, if $\Lambda^{n+1} M = 0$, *then* P_n *is the identity morphism.*

Proof. This is a consequence of the basic axiom for graded bialgebras. Explicitly:

$$Q_n(x \otimes s_N) = \delta_{1,n}(x \wedge s_N) = x \otimes s_N + \sum_{i=1}^{n} (-1)^i s_i \otimes x \wedge s_{N \setminus \{i\}}$$

□

Remark 2. Since $\mu_{1,n} \circ \delta_{1,n} = n + 1$ one has

$$Q_n^2 = (n + 1) Q_n$$

and

$$(P_n - 1)(P_n + n) = 0$$

Moreover

$$\mu_{1,n} \circ P_n = -n \mu_{1,n}$$

1.5. **The endomorphisms $A_k(f)$.** Let

$$f \in \mathrm{End}(M)$$

be an endomorphism of M.

We define

$$A_k(f) = \Phi(\Lambda^k f) \in \mathrm{End}(M \otimes \Lambda^{k+1} M)$$

Hence

$$A_k(f)(x \otimes s_L) = \sum_{i=0}^{k} (-1)^i s_i \otimes x \wedge (\Lambda^k f)(s_{N \setminus \{i\}}) \tag{2}$$

for $x \in M$ and $s \in M^L$, $L = \{0, \ldots, k\}$.

Proposition 2. *For* $n \geq 1$ *one has*

$$P_n \circ (1_M \otimes \Lambda^n f) = (f \otimes 1_{\Lambda^n M}) \circ A_{n-1}(f)$$
$$(1_M \otimes \Lambda^n f) \circ P_n = A_{n-1}(f) \circ (f \otimes 1_{\Lambda^n M})$$

Proof. This follows quickly by inspection of the explicit expressions (1) and (2). For a formal proof it is convenient consider instead of P_n and $A_{n-1}(f)$ the endomorphisms

$$\begin{aligned} P'_n &= \Psi(1_{\Lambda^{n-1} M}) = (-1)^{n-1} \sigma^{-1} \circ P_n \\ &= (\mu_{1,n-1} \otimes 1_M) \circ (1_M \otimes \delta_{n-1,1}) \\ A'_{n-1}(f) &= \Psi(\Lambda^{n-1} f) = (-1)^{n-1} \sigma^{-1} \circ A_{n-1}(f) \\ &= (\mu_{1,n-1} \otimes 1_M) \circ (1_M \otimes \Lambda^{n-1} f \otimes 1_M) \circ (1_M \otimes \delta_{n-1,1}) \end{aligned}$$

respectively. Then the first claim follows from the functoriality of the co-product:

$$\begin{aligned} P'_n \circ (1_M \otimes \Lambda^n f) &= (\mu_{1,n-1} \otimes 1_M) \circ (1_M \otimes \delta_{n-1,1}) \circ (1_M \otimes \Lambda^n f) \\ &= (\mu_{1,n-1} \otimes 1_M) \circ (1_M \otimes \Lambda^{n-1} f \otimes f) \circ (1_M \otimes \delta_{n-1,1}) \\ &= (1_{\Lambda^n M} \otimes f) \circ A'_{n-1}(f) \end{aligned}$$

Similarly for the second claim:

$$\begin{aligned} (\Lambda^n f \otimes 1_M) \circ P'_n &= (\Lambda^n f \otimes 1_M) \circ (\mu_{1,n-1} \otimes 1_M) \circ (1_M \otimes \delta_{n-1,1}) \\ &= (\mu_{1,n-1} \otimes 1_M) \circ (f \otimes \Lambda^{n-1} f \otimes 1_M) \circ (1_M \otimes \delta_{n-1,1}) \\ &= A'_{n-1}(f) \circ (f \otimes 1_{\Lambda^n M}) \end{aligned}$$

□

1.6. **The adjunct.** To simplify notation, we consider f and $\Lambda^n f$ as endomorphisms of $M \otimes \Lambda^n M$ by the action on the first resp. second factor.

Corollary 3. *Suppose* $\Lambda^{n+1} M = 0$. *Then*

$$\Lambda^n f = f \circ A_{n-1}(f) = A_{n-1}(f) \circ f$$

in $\mathrm{End}(M \otimes \Lambda^n M)$.

Proof. Follows from Proposition 2 and Lemma 1. □

Suppose M is a locally free R-module of rank n. Then $\Lambda^n M$ is an invertible R-module. A standard definition of the adjunct of f

$$f^\# \in \mathrm{End}(M)$$

is to take the adjoint of

$$\Lambda^{n-1} f \in \mathrm{End}(\Lambda^{n-1} M)$$

with respect to the non-degenerate pairing

$$M \otimes \Lambda^{n-1} M \xrightarrow{\mu} \Lambda^n M$$

Hence $f^\#$ is characterized by

$$(\#) \qquad f^\#(x) \wedge \eta = x \wedge (\Lambda^{n-1} f)(\eta) \qquad (x \in M,\ \eta \in \Lambda^{n-1} M)$$

The basic property

$$(\#\#) \qquad \det(f) \cdot 1_M = f f^\# = f^\# f$$

follows then from

$$(f^\# f)(x) \wedge \eta = f(x) \wedge (\Lambda^{n-1} f)(\eta) = (\Lambda^n f)(x \wedge \eta)$$

and $(f^\#)^\vee = (f^\vee)^\#$.

Lemma 4. *If* M *is a locally free* R*-module of rank* n*, then*

$$A_{n-1}(f) = f^\# \in \mathrm{End}(M \otimes \Lambda^n M) = \mathrm{End}(M)$$

Proof. It suffices to verify (#) with $f^\#$ replaced by $A_{n-1}(f)$. Instead of $A_{n-1}(f)$ we use again

$$A'_{n-1}(f) = \big((\mu_{1,n-1} \circ (1_M \otimes \Lambda^{n-1} f)) \otimes 1_M\big) \circ (1_M \otimes \delta_{n-1,1})$$

(cf. proof of Proposition 2). Note the general rule

$$\delta_{n-1,1}(\omega) \wedge \eta = (-1)^{n-1} \eta \otimes \omega \qquad (\omega \in \Lambda^n M,\ \eta \in \Lambda^{n-1} M)$$

for locally free R-modules of rank n. Therefore

$$\begin{aligned} A'_{n-1}(f)(x\otimes\omega)\wedge\eta &= (-1)^{n-1}\big(\big(\mu_{1,n-1}\circ(1_M\otimes\Lambda^{n-1}f)\big)\otimes 1_{\Lambda^n M}\big)(x\otimes\eta\otimes\omega) \\ &= (-1)^{n-1}x\wedge(\Lambda^{n-1}f)(\eta)\otimes\omega \end{aligned}$$

which was to be shown. □

Remark 3. Lemma 4 follows also from Corollary 3, since $f^{\#}$ is uniquely determined by (##) as a functor on triples (R, M, f).

1.7. **Example: The case $n = 2$.** The general expression for $A_1(f)$ is

$$A_1(f)(x\otimes s\wedge t) = s\otimes x\wedge f(t) - t\otimes x\wedge f(s)$$

It is easy to see that $A_1(f)$ and $f\otimes 1_{\Lambda^2 M}$ do not commute in general. On other hand suppose that M is locally free of rank 2. Then one gets indeed

$$\begin{aligned} \big(fA_1(f)\big)(x\otimes s\wedge t) &= f(s)\otimes x\wedge f(t) - f(t)\otimes x\wedge f(s) \\ &= x\otimes f(s)\wedge f(t) \\ &= (x\otimes s\wedge t)\det(f) \end{aligned}$$

using $x\wedge f(s)\wedge f(t) = 0$ and

$$\begin{aligned} \big(A_1(f)f\big)(x\otimes s\wedge t) &= s\otimes f(x)\wedge f(t) - t\otimes f(x)\wedge f(s) \\ &= (s\otimes x\wedge t - t\otimes x\wedge s)\det(f) \\ &= (x\otimes s\wedge t)\det(f) \end{aligned}$$

using $x\wedge s\wedge t = 0$.

2. The Cayley-Hamilton theorem

In this section we exploit Proposition 2 using a standard method: Replace f by $f - t\cdot 1_M$ and take the coefficients of the resulting polynomials in t.

2.1. **The endomorphisms $L_{n,k}(f)$.** Let

$$\begin{aligned} \Theta_r\colon\ \mathrm{End}(\Lambda^k M) &\to \mathrm{End}(\Lambda^{k+r}M) \\ \Theta_r(\varphi) &= \mu_{k,r}\circ(\varphi\otimes 1_{\Lambda^r M})\circ\delta_{k,r} \\ &= \mu_{r,k}\circ(1_{\Lambda^r M}\otimes\varphi)\circ\delta_{r,k} \end{aligned}$$

and for $f\in\mathrm{End}(M)$ and $0\le k\le n$ let

$$L_{n,k}(f) = \Theta_{n-k}(\Lambda^k f)\in\mathrm{End}(\Lambda^n M)$$

Particular cases are

$$\begin{aligned} L_{n,0}(f) &= 1_{\Lambda^n M} \\ L_{n,n}(f) &= \Lambda^n f \end{aligned}$$

Explicitly one has

$$L_{n,k}(f)(x_N) = \sum_{I\subset N,\,|I|=k} f^{I(1)}(x_1)\wedge\cdots\wedge f^{I(n)}(x_n) \tag{3}$$

with $N = \{1,\ldots,n\}$ and

$$I(i) = \begin{cases} 1 & i\in I \\ 0 & i\notin I \end{cases}$$

It follows easily that

$$\Lambda^n(f + t\cdot 1_M) = \sum_{k=0}^{n} L_{n,k}(f)t^{n-k} \tag{4}$$

in $\mathrm{End}(\Lambda^n M)[t]$.

In particular, if M is locally free of rank n, the elements

$$L_{n,k}(f) \in \mathrm{End}\big(\Lambda^n M\big) = R$$

are the (unsigned) coefficients of the characteristic polynomial of f.

2.2. **The Cayley-Hamilton theorem.** Here is a general form of the Cayley-Hamilton theorem.

Corollary 5. *For any R-module M, any $f \in \mathrm{End}(M)$ and $n \geq 0$ one has*

$$\sum_{k=0}^{n}(-1)^k f^{n-k} P_n L_{n,k}(f) = 0$$
$$\sum_{k=0}^{n}(-1)^k L_{n,k}(f) P_n f^{n-k} = 0$$

in $\mathrm{End}(M \otimes \Lambda^n M)$

Proof. Follows from Proposition 2 and the expansion (4) with a standard argument used in proofs of the Cayley-Hamilton theorem. For instance, write the first relation of Proposition 2 as

$$\beta(f) = f\alpha(f)$$

Then

$$\beta(f - t) = (f - t)\alpha(f - t)$$

gives

$$\left(\sum_{k=0}^{n} f^{n-k}t^k\right)\beta(f - t) = (f^{n+1} - t^{n+1})\alpha(f - t)$$

Comparing the coefficients of t^n yields

$$\sum_{k=0}^{n} f^{n-k}\beta_k(f) = 0$$

with

$$\beta(f - t) = \sum_{k=0}^{n} t^{n-k}\beta_k(f), \qquad \alpha(f - t) = \sum_{k=0}^{n-1} t^{n-1-k}\alpha_k(f)$$

and

$$\beta_k(f) = -\alpha_k(f) + f\alpha_{k-1}(f)$$

□

Note that f and $L_{n,k}(f)$ commute as they act separately on the factors of $M \otimes \Lambda^n M$. If $\Lambda^{n+1}M = 0$, then $P_n = 1$ and the two statements of Corollary 5 coincide. In particular, one gets the classical Cayley-Hamilton theorem:

Corollary 6. *If M is a locally free R-module of rank n, then*

$$\sum_{k=0}^{n} (-1)^k f^{n-k} L_{n,k}(f) = 0$$

in $\mathrm{End}(M \otimes \Lambda^n M) = \mathrm{End}(M)$. □

Remark 4. Let us make the first relation of Corollary 5 in the case $n = 2$ explicit. With

$$U = x \otimes s \wedge t$$

one has

$$\begin{aligned} P_2(U) &= s \otimes x \wedge t - t \otimes x \wedge s \\ L_{2,1}(f)(U) &= x \otimes (f(s) \wedge t + s \wedge f(t)) \end{aligned}$$

and

$$\begin{aligned} f^2 P_2 L_{2,0}(f)(U) &= f^2(s) \otimes x \wedge t - f^2(t) \otimes x \wedge s \\ f P_2 L_{2,1}(f)(U) &= f^2(s) \otimes x \wedge t - f(t) \otimes x \wedge f(s) \\ &\quad + f(s) \otimes x \wedge f(t) - f^2(t) \otimes x \wedge s \\ P_2 L_{2,2}(f)(U) &= f(s) \otimes x \wedge f(t) - f(t) \otimes x \wedge f(s) \end{aligned}$$

The terms just cancel each other out. The same happens in general when expanding the relations of Corollary 5 with the explicit expressions (1) and (3).

The significance of Corollary 5 comes from fact that in the formulation

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k f^{n-k} L_{n,k}(f) &= \sum_{k=0}^{n} (-1)^k f^{n-k} Q_n L_{n,k}(f) \\ \sum_{k=0}^{n} (-1)^k L_{n,k}(f) f^{n-k} &= \sum_{k=0}^{n} (-1)^k L_{n,k}(f) Q_n f^{n-k} \end{aligned}$$

all terms on the right hand sides factor through $\Lambda^{n+1}M$.

3. The endomorphisms $A_{k,h}(f)$

Finally we consider the endomorphisms $A_{k,h}(f)$ determined by the "t-expansion" of $A_k(f)$. They appear when computing $P_n L_{n,k}(f)$, $L_{n,k}(f) P_n$ and showed already up in the proof of Corollary 5. We also compute $Q_n A_{n-1,k}(f)$, $A_{n-1,k}(f) Q_n$.

3.1. **The endomorphisms $A_{k,h}(f)$.** For $0 \le h \le k$ let

$$A_{k,h}(f) = \Phi(L_{k,h}(f)) \in \mathrm{End}(M \otimes \Lambda^{k+1} M)$$

Note that

$$A_{k,h}(f) = (1_M \otimes \mu_{1,h,k-h}) \circ (\tau \otimes \Lambda^h f \otimes 1_{\Lambda^{k-h}M}) \circ (1_M \otimes \delta_{1,h,k-h})$$

and

$$A_{k,h}(f) = (1_M \otimes \mu_{h+1,k-h}) \circ (A_h(f) \otimes 1_{\Lambda^{k-h}M}) \circ (1_M \otimes \delta_{h+1,k-h}) \tag{5}$$

We understand $A_{k,h}(f) = 0$ for $h < 0$ or $h > k$.

Particular cases are

$$A_{k,0}(f) = P_{k+1}$$
$$A_{k,k}(f) = A_k(f)$$

From (4) it is clear that

$$A_k(f + t \cdot 1_M) = \sum_{h=0}^{k} A_{k,h}(f) t^{k-h} \tag{6}$$

in $\mathrm{End}(M \otimes \Lambda^{k+1} M)[t]$.

Lemma 7. *For $0 \leq k \leq n$ one has*

$$P_n L_{n,k}(f) = A_{n-1,k}(f) + f A_{n-1,k-1}(f)$$
$$L_{n,k}(f) P_n = A_{n-1,k}(f) + A_{n-1,k-1}(f) f$$

Proof. Follows from Proposition 2 by replacing f with $f + t \cdot 1_M$ and comparing the coefficients in t. See also the relation at the end of the proof of Proposition 2. □

Corollary 8. *For $0 \leq k < n$ one has*

$$A_{n-1,k}(f) = \sum_{h=0}^{k} (-f)^{k-h} P_n L_{n,h}(f) = \sum_{h=0}^{k} L_{n,h}(f) P_n (-f)^{k-h}$$

□

3.2. **More relations.** First we need a supplement for the endomorphisms $L_{n,k}(f)$.

Lemma 9. *One has*

$$L_{n,k}(f + t \cdot 1_M) = \sum_{h=0}^{k} \binom{n-h}{k-h} L_{n,h}(f) t^{k-h}$$

Proof. Follows from the definitions and

$$\mu_{h,k-h,n-k} \circ (\Lambda^h f \otimes 1_{\Lambda^{k-h} M} \otimes 1_{\Lambda^{n-k} M}) \circ \delta_{h,k-h,n-k} = \binom{n-h}{k-h} \mu_{h,n-h} \circ (\Lambda^h f \otimes 1_{\Lambda^{n-h} M}) \circ \delta_{h,n-h}$$

□

Lemma 9 yields the following generalization of (6).

Corollary 10. *For $0 \leq k \leq m$ one has*

$$A_{m,k}(f + t \cdot 1_M) = \sum_{h=0}^{k} \binom{m-h}{k-h} A_{m,h}(f) t^{k-h}$$

□

Lemma 11. *For $n \geq 1$ one has*

$$P_n A_{n-1}(f) = L_{n,n-1}(f) - f A_{n-1,n-2}(f)$$
$$A_{n-1}(f) P_n = L_{n,n-1}(f) - A_{n-1,n-2}(f) f$$

Proof. We prove only the first claim. One has

$$\begin{aligned} P_n A_{n-1}(f) &= \Psi^t(1_{\Lambda^{n-1}}) \circ \Psi(\Lambda^{n-1} f) \\ &= (1 \otimes \mu_{n-1,1}) \circ (Q_{n-1} \Lambda^{n-1} f \otimes 1) \circ (1 \otimes \delta_{n-1,1}) \end{aligned}$$

Since

$$Q_{n-1} \Lambda^{n-1} f = \Lambda^{n-1} f - f A_{n-2}(f)$$

by Proposition 2, one gets

$$P_n A_{n-1}(f) = L_{n,n-1}(f) - (f \otimes 1) \circ (1 \otimes \mu_{n-1,1}) \circ (A_{n-2}(f) \otimes 1) \circ (1 \otimes \delta_{n-1,1})$$

The claim follows from (5). □

Lemma 11 generalizes as follows.

Corollary 12. *For $0 \leq k \leq n-1$ one has*

$$Q_n A_{n-1,k}(f) = (n-k)[A_{n-1,k}(f) + f A_{n-1,k-1}(f) - L_{n,k}(f)]$$
$$A_{n-1,k}(f) Q_n = (n-k)[A_{n-1,k}(f) + A_{n-1,k-1}(f) f - L_{n,k}(f)]$$

Proof. Follows from Lemma 11, Lemma 9 and Corollary 10. □

Remark 5. Proposition 2 shows that

$$P_n (\Lambda^n f) P_n = f A_{n-1}(f) P_n = P_n A_{n-1}(f) f$$

is divisible by f from the left and from the right. With Lemma 11 one can make this more precise:

$$P_n (\Lambda^n f) P_n = f L_{n,n-1}(f) - f A_{n-1,n-2}(f) f$$

(f and $L_{n,n-1}(f)$ commute).

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