THE DISCRIMINANT ALGEBRA OF A CUBIC ALGEBRA

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Introduction

In this note we describe a construction of the discriminant algebra $D(A)$ of a flat cubic algebra A over any ring R (all rings and algebras are unital and commutative). The algebra $D(A)$ is a flat quadratic algebra. Its discriminant is the discriminant of A.

The discriminant algebra of a cubic binary form has been constructed by Manjul Bhargava. His construction should be the same as the one considered here via the correspondence (binary cubic forms) $\leftrightarrow$ (cubic algebras).

There are many further related questions and constructions not considered here. For example: If A is etale, then $A \otimes_R A = A \oplus A \otimes_R D(A)$. What is the relation between $A \otimes_R A$ and $D(A)$ for arbitrary A?

1. The discriminant of a quadratic algebra

Let L be a flat quadratic algebra over R.

Consider the bilinear form

$$b\colon L \times L \to R, \qquad b(x,y) = \operatorname{trace}_{L/R}(xy)$$

We define the *discriminant* of L as

$$\operatorname{disc}(L) = \det b \in (\textstyle\bigwedge^2 L)^{-\otimes 2}$$

2. Verschiebung

Let $H = L/R$. Since $\bigwedge^2 L = H$, the discriminant $\operatorname{disc}(L)$ can also be considered as an element of $H^{\otimes -2}$.

Let

$$p\colon L \to H$$

be the projection. For any element $e \in H^{\otimes -2}$ we define the quadratic algebra L_e by $L_e = L$ as R-module and with new product

$$x * y = xy + e\big(p(x) \otimes p(y)\big)$$

We call L_e the Verschiebung (something like "translational displacement", if you really insist on a translation) of L with e.

For the discriminant one finds:

$$\operatorname{disc}(L_e) = \operatorname{disc}(L) + 4e$$

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3. Cubic algebras

Now let A be a flat cubic algebra over R.

Let $N_A\colon A \to R$ be the norm map and write

$$N_A(x+t) = t^3 + T_A(x)t^2 + S_A(x)t + N_A(x)$$

with $x \in A$ and $t \in R$ where $T_A \in A^\vee$ is the trace and $S_A \in S^2 A^\vee$ is a quadratic form on A.

Let $M = A/R$ and let

$$\pi\colon A \to M$$

be the projection.

3.1. **The discriminant.** Consider the bilinear form

$$b\colon A \times A \to R, \qquad b(x,y) = T_A(xy)$$

We define the *discriminant* of A as

$$\operatorname{disc}(A) = \det b \in (\textstyle\bigwedge^3 A)^{-\otimes 2}$$

Since $\bigwedge^3 A = \bigwedge^2 M$, the discriminant $\operatorname{disc}(A)$ can also be considered as an element of $(\bigwedge^2 M)^{\otimes -2}$.

3.2. **The reduced trace form.** Consider the expression

$$T_A(x^2) - S_A(x)$$

for $x \in A$. One finds that it is invariant under $x \mapsto x + a$ with $a \in R$. Hence there exists a quadratic form

$$q_A\colon M \to R$$

with

$$q_A\bigl(\pi(x)\bigr) = T_A(x^2) - S_A(x)$$

for $x \in A$.

Let

$$K(A) = C_0(q_A)$$

be the even Clifford algebra of q_A. Note that

$$K(A)/R = C_0(q_A)/R = \textstyle\bigwedge^2 M$$

For the discriminants one finds

$$\operatorname{disc}(q_A) = \operatorname{disc}\bigl(K(A)\bigr) = -3\operatorname{disc}(A) \in (\textstyle\bigwedge^2 M)^{\otimes -2}$$

We call $K(A)$ the *Kummer discriminant algebra* of A. This is suggested by

Lemma. *The following are equivalent (at least for a local ring R):*

(1) *$K(A)$ is etale and split, i.e., $K(A) \simeq R \oplus R$.*
(2) *$1/3 \in R$ and $A \simeq R[t]/(t^3 - a)$ for some $a \in R^*$.*

The proof is left to the reader.

3.3. **The discriminant algebra.** We are now ready to define the *discriminant algebra* $D(A)$ as the Verschiebung

$$D(A) = K(A)_{\mathrm{disc}(A)}$$

Then indeed

$$\mathrm{disc}\big(D(A)\big) = \mathrm{disc}\big(K(A)\big) + 4\,\mathrm{disc}(A) = \mathrm{disc}(A)$$

We conclude with

Lemma. *The following are equivalent (at least for a local ring R):*

(1) *$D(A)$ is etale and split, i.e., $K(A) \simeq R \oplus R$.*
(2) *A is etale and cyclic.*

The proof is left to the reader.

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