ON FROBENIUS, K-THEORY, AND CHARACTERISTIC NUMBERS

MARKUS ROST

`preliminary version`

Contents

1. Introduction

This text is in a very preliminary status.

The starting point was my proof of the degree formula. For the prime 2 this used the Hilbert scheme $\operatorname{Hilb}(2,X)$ and its canonical line bundle $\overline{L}$. One has for X smooth and proper of dimension d:

$$\deg\bigl(c_1(\overline{L})^{2d}\bigr) = \frac{1}{2}\deg\bigl(c_d(-T_X)\bigr) \tag{1}$$

This gives a simple short proof that for any X the Segre number $\deg\bigl(c_d(-T_X)\bigr)$ is 2-divisible.

Let's have a look at the situation in characteristic 2. In this case there exist a canonical smooth divisor $j\colon\ \overline{\mathbf{P}} \to \operatorname{Hilb}(2,X)$ which represents $\overline{L}$. It fits into a commutative diagram

$$\begin{array}{ccc} \mathbf{P}(\Omega_X) & \xrightarrow{i} & \operatorname{Bl}_\Delta(X\times X) \\ \rho\big\downarrow & & \bar\rho\big\downarrow \\ \overline{\mathbf{P}} & \xrightarrow{j} & \operatorname{Hilb}(2,X) \end{array}$$

Here $\operatorname{Bl}_\Delta(X\times X)$ is the blow up of the diagonal and i is the inclusion of the exceptional fiber which is the projective tangent bundle of X.

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Moreover $\overline{\rho}$ is the standard double cover. The morphism ρ is radicial of degree 2. The pullback of $\overline{L}$ to $\mathrm{Bl}_\Delta(X \times X)$ is the canonical line bundle of the blow up, and hence the pullback of $\overline{L}$ to $\mathbf{P}(\Omega_X)$ is the canonical line bundle of the projective bundle.

One may take also the following point of view: The canonical line bundle has a connection with respect to itself with trivial 2-curvature. By general principles it descends canonically to something, and that something is $\overline{\mathbf{P}}$.

Let $L = j^*(\overline{L})$. Now, since $\overline{\mathbf{P}}$ represents $c_1(\overline{L})$, we get the following variant of (1):

$$\deg\bigl(c_1(L)^{2d-1}\bigr) = \frac{1}{2}\deg\bigl(c_d(-T_X)\bigr)$$

Hence $\frac{1}{2}$ of the Segre number is the degree of a zero cycle on $\overline{\mathbf{P}}$.

One can do somewhat better: There is a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\pi} & \mathbf{P}(\Omega_X) \\ {\scriptstyle F}\downarrow & & {\scriptstyle \rho}\downarrow \\ X^{(2)} & \xleftarrow{\pi^{(2)}\circ\eta} & \overline{\mathbf{P}} \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \eta}\downarrow \\ X^{(2)} & \xleftarrow{\pi^{(2)}} & \mathbf{P}(\Omega_X)^{(2)} \end{array}$$

Here F and $\eta \circ \rho$ are the relative Frobenius morphisms. One finds

$$(2) \qquad 2\bigl(\pi^{(2)} \circ \eta\bigr)_*\bigl(c_1(L)^{2d-1}\bigr) = c_d(-T_{X^{(2)}}) = F_*\bigl(c_d(-T_X)\bigr)$$

Hence one gets in characteristic 2 not only the 2-divisibility of the image of the Segre class in $\mathrm{CH}_0(\operatorname{Spec} k) = \mathbf{Z}$ (the Segre number) but of its image in $\mathrm{CH}_0\bigl(X^{(2)}\bigr)$.

In the case of curves this is long known: The line bundle $\Omega_{X^{(2)}}$ has a canonical square root, namely $F_*(\mathcal{O}_X)/\mathcal{O}_{X^{(2)}}$. (See [2], [1], [5].)

`Note: (2) is not yet contained in the text.`

There is another result in this text, Proposition 1, which together with Riemann-Roch and the Hattori-Stong theorem yields the following: Let $P \in \mathbf{Z}[c_1, \dots, c_d]$ be a polynomial of degree d (with $\deg c_i = i$), and suppose that there is a p-power q such that for any compact almost complex manifold X of dimension $2d$ the number $P(X)/q$ is integral. Then for any smooth variety X in characteristic p of dimension d, the number $P(X)/q$ is the degree of an integral zero cycle on $X^{(p^d)}$.

Another example of such divisibilities has been provided by Deligne: For a smooth surface X in characteristic 2, let $F\colon X \to X^{(2)}$ be the relative Frobenius. Then one has in $\mathrm{CH}_0\big(X^{(2)}\big)$:

$$4c_2\big(F_*(\mathcal{O}_X)\big) = (c_1^2 + c_2)(T_{X^{(2)}}) \tag{3}$$

Taking degrees, one gets

$$\deg\big(c_2\big(F_*(\mathcal{O}_X)\big)\big) = 3\,\mathrm{Todd}(X)$$

`Note:` (3) `is not yet contained in the text.`

I have learned Lemma 2 from Deligne. I don't have a reference for it. I am also wondering about a reference for Lemma 1, Lemma 3 and, in particular, for Lemma 4.

2. Preliminaries

References: [4, Section 7].

For the Frobenius maps we use the following notations. Let X be a scheme in characteristic p.

The absolute Frobenius is denoted by

$$f = f_X \colon X \to X$$

If $X = \operatorname{Spec} R$ is affine, then f is given by the p-th power map

$$\varphi\colon R \to R$$
$$\varphi(a) = a^p$$

For a sheaf M of $\mathcal{O}_X$-modules let

$$M^{[p]} = f^*(M)$$

be the pull back of M along f. Similarly, in the affine case $X = \operatorname{Spec} R$ we denote for R-modules V

$$V^{[p]} = V \otimes_{R,\varphi} R$$

Here the tensor product is understood so that $va \otimes b = v \otimes a^p b$ and the R-module structure on $V^{[p]}$ is given by $(v \otimes a)b = v \otimes ab$.

The symmetric algebras are denoted by

$$S^\bullet M = \bigoplus_{k \geq 0} S^k M, \quad S^\bullet V = \bigoplus_{k \geq 0} S^k V$$

There is a natural morphism

$$j_M \colon M^{[p]} \to S^p M$$

given locally by

$$j_V \colon V^{[p]} \to S^p V$$
$$j_V(v \otimes a) = v^p a$$

Let

$$B(M) = S^\bullet M / \langle j_M(M^{[p]}) \rangle$$

be the quotient of the symmetric algebra of M by the ideal sheaf generated by the image of j_M.

Similarly we understand $B(V)$....

The natural multiplication map

$$B(M) \otimes B(M') \to B(M \oplus M')$$

is an isomorphism.

For a line bundle L one has

$$B(L) = S^\bullet L / \langle L^{\otimes p} \rangle = \mathcal{O}_X \oplus L^{\otimes 1} \oplus L^{\otimes} \oplus \cdots \oplus L^{\otimes (p-1)}$$

Lemma 1. *Suppose that M is a vector bundle on X of rank n. Then*

(1) *j_M is a monomorphism.*
(2) *If M is invertible ($n = 1$), then j_M is an isomorphism, so that $M^{[p]} = M^{\otimes p}$.*
(3) *$B(M)$ is a vector bundle of rank p^n.*

Proof. The question being local, we may assume that $X = \operatorname{Spec} R$ and that M is given by a free R-module V with basis e_i, $i = 1, \ldots, n$. Then $S^\bullet V$ is the polynomial ring $R[e_1, \ldots, e_n]$. Moreover $V^{[p]}$ is free with basis $e_i \otimes 1$, $i = 1, \ldots, n$ and j_V is given by $j_V(e_i \otimes 1) = e_i^p$.

From this (1) and (2) are clear. As for (3), note that

$$B(V) = R[e_1, \ldots, e_n] / \langle e_1^p, \ldots, e_n^p \rangle = \bigotimes_{i=1}^{n} R[e_i] / \langle e_i^p \rangle$$

`Better:`

$$B(L_1 \oplus \cdots \oplus L_n) = B(L_1) \otimes \cdots \otimes B(L_n)$$

□

Remark 1. Let $K^0(X)$ denote the Grothendieck group of vector bundles on X. There is the ring homomorphism $f^*\colon K^0(X) \to K^0(X)$ induced by the absolute Frobenius. Since f^* is on line bundles the p-th power map (cf. Lemma 1 (2)), it follows that f^* is the p-th Adams operation in K-theory.

Let k be a field of characteristic p and let X be a scheme over k. The structure morphism of X is denoted by $\pi_X\colon X \to \operatorname{Spec} k$. The relative

Frobenius is described by the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{F} & X^{(p)} & \xrightarrow{W} & X \\ {\scriptstyle\pi_X}\downarrow & & {\scriptstyle\pi_{X^{(p)}}}\downarrow & & {\scriptstyle\pi_X}\downarrow \\ \operatorname{Spec} k & \xrightarrow{\mathrm{id}} & \operatorname{Spec} k & \xrightarrow{f_k} & \operatorname{Spec} k \end{array}$$

Here the right square is Cartesian and defines $X^{(p)}$ as the fiber product of X and k over k with respect to the absolute Frobenius, with W and $\pi_{X^{(p)}}$ the corresponding maps. One has $f_k \circ \pi_X = \pi_X \circ f_X$. The relative Frobenius F is the unique map with $W \circ F = f_X$ and $\pi_{X^{(p)}} \circ F = \pi_X$.

The morphism $W\colon X^{(p)} \to X$ is flat, since it is the pull back of the flat morphism f_k. If X is a localization of a scheme of finite type over k, then F is finite.

Lemma 2. *Suppose that X is smooth over k of dimension d. Then F is flat and finite of rank p^d.*

For the induced maps $F_*\colon K_0(X) \to K_0\big(X^{(p)}\big)$, $F^*\colon K_0\big(X^{(p)}\big) \to K_0(X)$ one has

$$F^*\big(F_*(x)\big) = [B(\Omega_{X/k})] \cdot x$$

for $x \in K_0(X)$.

`Better:`

$F^* \circ F_*$ is multiplication by $[B(\Omega_{X/k})]$.

Proof. For the first claim see [4]....

We consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{O}_X \otimes_k \mathcal{O}_X & \xrightarrow{\mu} & \mathcal{O}_X & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & & \downarrow & & {\scriptstyle\mathrm{id}}\downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X & \xrightarrow{\mu} & \mathcal{O}_X & \longrightarrow & 0 \end{array}$$

Here μ denotes the multiplication maps with kernels I, J and the vertical arrows are the natural maps. By definition one has $I/I^2 = \Omega_{X/k}$ and $J/J^2 = \Omega_{X/X^{(p)}}$. Moreover h induces an isomorphism $I/I^2 \to J/J^2$. Let

$$\operatorname{gr}_I(\mathcal{O}_X \otimes_k \mathcal{O}_X) = \bigoplus_{n\ge 0} I^n/I^{n+1}, \qquad \operatorname{gr}_J(\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X) = \bigoplus_{n\ge 0} J^n/J^{n+1},$$

be the sheaves of graded rings associated to the filtrations induced by I, J, respectively, and let

$$\operatorname{gr}(h)\colon \operatorname{gr}_I(\mathcal{O}_X \otimes_k \mathcal{O}_X) \to \operatorname{gr}_J(\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X)$$

be the homomorphism induced from h. Since X is smooth, the natural ring homomorphism

$$\alpha\colon S^\bullet(I/I^2) \to \mathrm{gr}_I(\mathcal{O}_X \otimes_k \mathcal{O}_X)$$

with α the identity on I/I^2 is an isomorphism. For $x \in J$ one has $x^p = 0$; namely, for $a \in \mathcal{O}_X$ one has $(a\otimes 1 - 1\otimes a)^p = a^p \otimes 1 - 1 \otimes a^p = 0$ since a^p is in $\mathcal{O}_{X^{(p)}}$. Hence $\mathrm{gr}(h)\circ\alpha$ factors trough a ring homomorphism

$$\beta\colon B(I/I^2) \to \mathrm{gr}_J(\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X)$$

It is not difficult to see by local considerations that β is an isomorphism. Composing β with the inverse of $B(h)\colon B(I/I^2) \to B(J/J^2)$ we obtain an isomorphism

$$\beta\colon B(J/J^2) \to \mathrm{gr}_J(\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X)$$

of $\mathcal{O}_X$-modules.

Let M be a $\mathcal{O}_X$-module. Then

$$F^*\big(F_*(M)\big) = M \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X = M \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X$$

The J-filtration induces a filtration on $F^*\big(F_*(M)\big)$ with associated graded module

$$M \otimes_{\mathcal{O}_X} \mathrm{gr}_J(\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X)$$

... □

Let E be a finitely generated field over k. We denote by $E^p \subset E$ the image of the p-th power homomorphism $E \to E$ and by

$$\overline{E} = kE^p \subset E$$

the subfield generated by E^p and k. If E is generated as a field over k by $x_1, \ldots, x_N$, then the same is true over $\overline{E}$. Since $x_i^p \in \overline{E}$, it follows that E is finite over $\overline{E}$.

Let

$$E^{(p)} = E^p \otimes_{k^p} k$$

We consider the maps

$$E^p \xrightarrow{f} E^{(p)} \xrightarrow{F} E$$

with $f(a) = a \otimes 1$ and $F(a \otimes b) = ab$. Thus $F \circ f$ is the natural inclusion. Let $\mathfrak{m}$ be the kernel of F.

One has $\mathfrak{m}^p = 0$. Indeed, let $x \in E^{(p)}$. Then x^p is in the field

$$E^p \otimes_{k^p} k^p = E^p$$

If $F(x) = 0$, then $F(x^p) = 0$ and therefore $x^p = 0$.

The ideal $\mathfrak{m}$ is the unique maximal ideal of $E^{(p)}$. Its residue class field is

$$\overline{E} = F\big(E^{(p)}\big) = kE^p \subset E$$

We denote by $\ell(E/k)$ the length of the ring $E^{(p)}$.

Lemma 3. *The extension $E/\overline{E}$ is finite.*

The ring $E^{(p)}$ has finite length.

Let $[E:\overline{E}] = \dim_{\overline{E}} E$, let ℓ be the length of $E^{(p)}$, and let d be the transcendence degree of E/k. Then

$$[E:\overline{E}] = \ell p^d$$

Proof. Let $k \subset F \subset E$ be an intermediate field with F/k separable and E/F finite. For instance, if $x_1, \dots, x_d$ is a transcendence basis of E/k, one may take $F = k(x_1, \dots, x_d)$. Consider the diagram

$$\begin{array}{ccccc} E^{(p)} & \xrightarrow{r_E} & \overline{E} & \longrightarrow & E \\ \uparrow & & \uparrow & & \uparrow \\ F^{(p)} & \xrightarrow{r_F} & \overline{F} & \longrightarrow & F \end{array}$$

Since F/k is separable, $F^{(p)}$ is a field and r_F is an isomorphism. Hence

$$\dim_{F^{(p)}} E^{(p)} = \operatorname{length}(E^{(p)})[\overline{E}:\overline{F}]$$

On the other hand

$$\dim_{F^{(p)}} E^{(p)} = [E^p : F^p] = [E:F]$$

Since F/k is separable, one has

$$[F:\overline{F}] = p^d$$

(For instance, if $F = k(x_1, \dots, x_d)$, then $\overline{F} = k(x_1^p, \dots, x_d^p)$.) Finally

$$[E:F][F:\overline{F}] = [E:\overline{E}][\overline{E}:\overline{F}]$$

The claim is now immediate. □

Lemma 4. *Let X/k be of finite type. Then for W^*, $F_*\colon \mathrm{CH}_r(X) \to \mathrm{CH}_r\big(X^{(p)}\big)$ one has*

$$F_* = p^r W^*$$

on the cycle groups

$$C_r(X) = \bigoplus_{x \in X_{(r)}} \mathbf{Z}$$

See also [9, Proposition 2] for smooth schemes of finite type over a finite field.

Proof. Let $x \in X$ be a point with $\dim \overline{\{x\}} = r$ and let $y = F(x)$. Let further $O_{X,x}$, $\mathfrak{m}_x$, $\kappa_x = O_{X,x}/\mathfrak{m}_x$ be the local ring at x, its maximal ideal, and its residue class field, respectively. Then

$$F_*([x]) = [\kappa_x : \kappa_y][y]$$

and
$$W^*([x]) = \operatorname{length}\big(O_{X^{(p)},y} \otimes_{O_{X,x}} \kappa_x\big)[y]$$
Since
$$O_{X^{(p)},y} \otimes_{O_{X,x}} \kappa_x = \big(O_{X,x} \otimes_{k,\varphi} k\big) \otimes_{O_{X,x}} \kappa_x = \kappa_x \otimes_{k,\varphi} k = \kappa_x^{(p)}$$
the claim follows from Lemma 3.

`Better:`

For a morphism $h\colon Z \to X$ there are the Cartesian diagrams
$$\begin{array}{ccccc} Z^{(p)} & \xrightarrow{h^{(p)}} & X^{(p)} & \xrightarrow{\pi_{X^{(p)}}} & \operatorname{Spec} k \\ {\scriptstyle W}\big\downarrow & & {\scriptstyle W}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Z & \xrightarrow{h} & X & \xrightarrow{\pi_X} & \operatorname{Spec} k \end{array}$$
If h is a closed immersion, then $W^* \circ h_* = h_* \circ W^*$. If h is a open immersion, then $h^* \circ W^* = W^* \circ h^*$, see [3], [8]. Thus we may replace X by $\operatorname{Spec} \kappa_x$. This case follows easily from Lemma 3. □

`This can be generalized as follows:`

Let S be a scheme over a field of characteristic p and let X be a scheme over S. The structure morphism of X is denoted by $\pi_X\colon X \to S$. The relative Frobenius is described by the following commutative diagram
$$\begin{array}{ccccc} X & \xrightarrow{F} & (X/S)^{(p)} & \xrightarrow{W} & X \\ {\scriptstyle \pi_X}\big\downarrow & & {\scriptstyle \pi_{X^{(p)}}}\big\downarrow & & {\scriptstyle \pi_X}\big\downarrow \\ S & \xrightarrow{\mathrm{id}} & S & \xrightarrow{f_S} & S \end{array}$$

Lemma 5. *Let S be smooth over k of dimension e and let X/S be of finite type. Then for W^*, $F_*\colon \mathrm{CH}_r(X) \to \mathrm{CH}_r\big((X/S)^{(p)}\big)$ one has*
$$F_* = p^{r-e} W^* \qquad \text{if } r \geq e$$
$$p^{e-r} F_* = W^* \qquad \text{if } r \leq e$$
on the cycle groups
$$C_r(X) = \bigoplus_{x \in X_{(r)}} \mathbf{Z}$$

Proof. `No proof yet.`

□

Let $\pi_X\colon X \to \operatorname{Spec} k$ be a scheme of finite type over k. Let $K_0(X)$ denote the Grothendieck group of coherent $\mathcal{O}_X$-module sheaves on X. Let $K_0(X)_{(d)} \subset K_0(X)$ be the subgroup generated by sheaves M with $\dim \operatorname{supp} M \leq d$.

Proposition 1.

$$(\pi_X)_*\big(K_0(X)_{(d)}\big) \subset (\pi_{X^{(p^d)}})_*\big(\mathrm{CH}_0(X^{(p^d)})\big) \otimes \mathbf{Z}_{(p)}$$

Proof. For $d \geq 0$ and a $\mathcal{O}_X$-module sheaf M on X let

$$\Theta_d(M) = F_*(M) - p^d f^*(M)$$

This is a sheaf on $X^{(p)}$. Note that

$$(\pi_{X^{(p)}})_*\big(\Theta_d(M)\big) = (1-p^d)(\pi_X)_*(M)$$

`This needs proof!!!`

Clearly $\dim\operatorname{supp}\Theta_d(M) \leq \dim\operatorname{supp} M$. The two diagrams

$$\begin{array}{ccccc} \mathrm{CH}_d(X) & \longrightarrow & K_0'(X)_{(d)}/K_0'(X)_{(d-1)} & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle F_*, f^*} & & \Big\downarrow{\scriptstyle F_*, f^*} & & \\ \mathrm{CH}_d(X^{(p)}) & \longrightarrow & K_0'(X^{(p)})_{(d)}/K_0'(X^{(p)})_{(d-1)} & \longrightarrow & 0 \end{array}$$

for F_*, f^*, respectively commute and have exact rows. By Lemma 4 one has $\dim\operatorname{supp}\Theta_d(M) \leq d-1$ if $\dim\operatorname{supp} M \leq d$. Argue directly for sheaves!!!

Let us define for $d > 0$

$$\theta_d\colon K_0(X)_{(d)} \to K_0(X^{(p)})_{(d-1)} \otimes \mathbf{Z}_{(p)}$$
$$\theta_d([M]) = (1-p^d)^{-1}[\Theta_d(M)]$$

Then

$$(\pi_X)_* = (\pi_{X^{(p)}})_* \circ \theta_d\colon K_0(X)_{(d)} \to K_0(k) \otimes \mathbf{Z}_{(p)} = \mathbf{Z}_{(p)}$$

Consider the map

$$\bar{\theta}_d = \theta_1 \circ \theta_2 \circ \cdots \circ \theta_d\colon K_0(X)_{(d)} \to K_0(X^{(p^d)})_{(0)} \otimes \mathbf{Z}_{(p)}$$

Hence

$$(\pi_X)_*\big(K_0(X)_{(d)}\big) \subset (\pi_{X^{(p^d)}})_*\big(K_0(X^{(p^d)})_{(0)}\big) = (\pi_{X^{(p^d)}})_*\big(\mathrm{CH}_0(X^{(p^d)})\big)$$

□

`Simplify this!  For perfect` k `look at` $F' = (W^{-1}) \circ F\colon X \to X$. `Then` F' `acts with eigenvalue` p^r `on` $\mathrm{CH}_r(X)$`....`

Corollary 1. *If k is perfect of characteristic p, then*

$$(\pi_X)_*\big(K_0(X)\big) \subset (\pi_X)_*\big(\mathrm{CH}_0(X)\big) \otimes \mathbf{Z}_{(p)}$$

In other words, the Euler characteristic of a $\mathcal{O}_X$-module sheaf on X is the degree of a p-integral zero cycle on X.

3. Exploiting Riemann-Roch

[6][7]

4. Examples

Lemma 6. *Suppose* $\dim X = 1$. *Then*

$$F_*(\mathcal{O}_X - \Omega_{X/k}) = \mathcal{O}_{X^{(p)}} - \Omega_{X^{(p)}/k}$$
$$F_*(\mathcal{O}_X + \Omega_{X/k}) = p\big(\mathcal{O}_{X^{(p)}} + \Omega_{X^{(p)}/k}\big)$$

Moreover, if $p = 2$, *then*

$$2\big(F_*(\mathcal{O}_X) - 2\mathcal{O}_{X^{(2)}}\big) = \Omega_{X^{(2)}/k} - \mathcal{O}_{X^{(2)}}$$

and

$$F_*\big(F_*(\mathcal{O}_X) - 2\mathcal{O}_{X^{(2)}}\big) = F_*(\mathcal{O}_{X^{(2)}}) - 2\mathcal{O}_{X^{(4)}}$$

and if p *is odd, then*

$$F_*(\mathcal{O}_X) - p\mathcal{O}_{X^{(p)}} = \frac{p-1}{2}\big(\Omega_{X^{(p)}/k} - \mathcal{O}_{X^{(p)}}\big)$$

`Notation: bundles versus classes. Better notations!`

Lemma 7. *Suppose* $\dim X = 2$ *and* $p = 2$. *Then*

$$F_*(\mathcal{O}_X - \Omega^2_{X/k}) = 2\big(\mathcal{O}_{X^{(2)}} - \Omega^2_{X^{(2)}/k}\big)$$
$$4F_*(\mathcal{O}_X) = ...$$

Lemma 8. *Suppose* $\dim X = 2$ *and* $p = 3$. *Then*

$$....$$
$$9F_*(\mathcal{O}_X) = ...$$

Lemma 9. *Suppose* $\dim X = 2$ *and* $p > 3$. *Then*

$$....$$
$$p^2F_*(\mathcal{O}_X) = ...$$

5. Further Remarks

$[X \to X^{(p)}] \in \Omega(X^{(p)})$ new natural elements in bordism (= in any oriented cohomology theory).

Compress $[X]$: Over $\mathbf{Z}_{(p)}$, $[X]$ is represented by sums $[Y \to X^{(p^d)}].L$, $\dim Y = 0$, L= Lazard ring.

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

E-mail address: rost@math.uni-bielefeld.de

URL: http://www.math.uni-bielefeld.de/~rost