A PFISTER FORM INVARIANT FOR ETALE ALGEBRAS

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Let k be a field of characteristic different from 2. We denote by $GW(k)$ the Witt-Grothendieck ring of k.

For $a \in k^*$ we denote $k_a = k[t]/(t^2 - a)$. Further we denote

$$[[a]] = \langle 1 \rangle - \langle a \rangle$$

This is a 0-dimensional element of $GW(k)$, with image $\langle\langle a \rangle\rangle$ in the Witt ring.

We fix a quadratic extension $L = k_a$.

The multiplicative norm in GW for quadratic extensions. The conjugation in L is denoted by $\alpha \mapsto \bar{\alpha}$. We consider the additive trace map

$$T_{L/k} \colon GW(L) \to GW(k)$$

and the multiplicative norm map

$$N_{L/k} \colon GW(L) \to GW(k)$$

One has

$$N_{L/k}(x + y) = N_{L/k}(x) + T_{L/k}(x\bar{y}) + N_{L/k}(y)$$

and

$$N_{L/k}(\langle \alpha \rangle) = \langle N_{L/k}(\alpha) \rangle$$

for $\alpha \in L^*$.

Lemma 1. *For $\alpha \in L^*$ one has*

$$N_{L/k}(\langle\langle \alpha \rangle\rangle) = \langle\langle T_{L/k}(\alpha), -aN_{L/k}(\alpha) \rangle\rangle + [[a]]$$

Proof.

$$\begin{aligned} N_{L/k}(\langle\langle \alpha \rangle\rangle) &= N_{L/k}(\langle 1, -\alpha \rangle) \\ &= N_{L/k}(\langle 1 \rangle) + T_{L/k}(\langle -\bar{\alpha} \rangle) + N_{L/k}(\langle -\alpha \rangle) \\ &= \langle 1 \rangle + \langle -T_{L/k}(\alpha) \rangle \langle 1, aN_{L/k}(\alpha) \rangle + \langle N_{L/k}(\alpha) \rangle \\ &= \langle 1, -T_{L/k}(\alpha) \rangle \langle 1, aN_{L/k}(\alpha) \rangle + \langle N_{L/k}(\alpha) \rangle [[a]] \end{aligned}$$

□

Corollary 1. *Let $n > 0$ and let f be a n-fold Pfister form over L. Then*

$$N_{L/k}(f) - [[a]]^n$$

is a $2n$-fold Pfister form over k.

Proof. For $\alpha \in L^*$ one has $\langle\langle -aN_{L/k}(\alpha) \rangle\rangle [[a]] = 0$. Hence Lemma 1 gives

$$N_{L/k}(\langle\langle \alpha_1, \dots, \alpha_n \rangle\rangle) = \prod_i \langle\langle T_{L/k}(\alpha_i), -aN_{L/k}(\alpha_i) \rangle\rangle + [[a]]^n$$

and the claim is clear. □

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Invariants for etale algebras over a quadratic extension. Let Θ be a cohomological (or Witt group) invariant for etale algebras E over L of some fixed degree n. That is, for each field F over k and each etale extension E of $F \otimes L$ of degree n we are given an element $\Theta(E) \in H(F)$, compatible with the restriction maps for morphisms $F \to F'$. We denote by Θ_L the induced invariant given by $\Theta_L(E) = \Theta(E)_L \in H(F \otimes L)$.

Proposition 1. *Suppose* $\Theta_L = 0$*. Then* Θ *is constant.*

Proof. A versal parameter space for our objects E is the Weil restriction $Z = R_{L/k}(\mathbf{A}^n \setminus \Delta)$ of the versal parameter space $\mathbf{A}^n \setminus \Delta$ (with Δ the discriminant locus of normed polynomials of degree n). One has $R_{L/k}(\mathbf{A}^n) = \mathbf{A}^{2n}$. Thus Z is an open subvariety of $\mathbf{A}^{2n}$. Its complement Δ' is a divisor birational isomorphic to $\mathbf{A}^n \times \Delta \times \operatorname{Spec} L$.

The invariant Θ defines a versal class $\Theta_{\text{gen}} \in H\big(k(Z)\big)$ unramified on Z. If $\Theta_L = 0$, then Θ_{gen} is unramified in Δ' as well, since $k(\Delta')$ contains L.

Thus Θ_{gen} is unramified on all of $\mathbf{A}^{2n}$. □

Serre's splitting principle. A cohomological (or Witt group) invariant for etale algebras E over k (of some fixed degree) which vanishes on multiquadratic extensions is trivial.

(A multiquadratic extension is a product of extensions of degree 1 or 2.)

The reduced trace form S_E. For an etale extension E of degree n let q_E be its trace form, let

$$\omega_n = \begin{cases} \langle 2, 2, ..., 2\rangle \ (\operatorname{rank} m) & \text{if } n = 2m \\ \langle 1, 2, ..., 2\rangle \ (\operatorname{rank} m+1) & \text{if } n = 2m+1 \end{cases}$$

and let

$$S_E = q_E - \omega_n$$

to be considered as an element of $GW(k)$. One has $\dim S_E = m$.

For extensions E, F of even degree and $a \in k^*$ one has

$$S_{E \times k} = S_E$$
$$S_{E \times F} = S_E + S_F$$
$$S_{k_a} = \langle 2a \rangle$$

If E is multiquadratic, then S_E is given by an honest quadratic form of dimension m. Although this is not true in general, S_E has some properties of a m-dimensional quadratic form. Here are two of these properties (due to Serre):

$$\lambda^i(S_E) = 0 \quad \text{if } i > m$$

$$w_i(S_E) = 0 \quad \text{if } i > m$$

They are immediate from Serre's splitting principle. Theorem 1 below yields another one.

The Pfister form $P(E)$. Let

$$P(E) = P(E/k) = \sum_{i=0}^{m} \lambda^i(S_E) = \sum_{i=0}^{\infty} \lambda^i(S_E)$$

Note that $\dim P(E) = 2^m$. If E is multiquadratic, then it is easy to see that $P(E)$ is a m-fold Pfister form.

We first note some elementary properties: Let E, F be of even degree and $a \in k^*$. Then

$$\begin{aligned} P(E \times k) &= P(E) \\ P(E \times F) &= P(E)P(F) \\ P(k_a) &= \langle 1, 2a\rangle = \langle\langle -2a\rangle\rangle \end{aligned}$$

These statements are clear from the definitions and the standard properties of the λ-operations. For the multiplicativity property one may also resort to Serre's splitting principle and reduce to the trivial case of multiquadratic extensions.

Theorem 1. *$P(E)$ is a m-fold Pfister form.*

Proof. One may assume that k has no proper field extensions of odd degree. Then, by the multiplicativity property of P, we may further assume that E is a field extension of a quadratic extension $L = k_a$.

Let r be the degree of E/L. The case $r = 1$ is clear. Otherwise $r = 2s$.

Let

$$\Theta(E) = P(E/k) - N_{L/k}\big(P(E/L)\big) + [[a]]^s$$

By induction on the degree and by Corollary 1 it suffices to show $\Theta(E) = 0$.

The class $\Theta(E)$ defines an invariant as considered in Proposition 1. One has

$$\Theta(E)_L = P(E \times \bar{E}) - P(E)P(\bar{E}) = 0$$

(with $^-$ denoting the conjugation of L/k). Thus Θ is constant.

It remains to check that $\Theta(E) = 0$ for some specific E. Thus we may assume that $E = (F \times \cdots \times F) \otimes L$ for any given quadratic extension F of k. Using the multiplicativity properties of P and $N_{L/k}$ and Lemma 1 one reduces to the case $E = F \otimes L$.

Take $F = k \times k$. Then $S_{E/k} = \langle 2a, 2a\rangle$ and therefore $P(E/k) = \langle\langle -2a, -2a\rangle\rangle = \langle\langle -1, -a\rangle\rangle$. Furthermore, $S_{E/L} = \langle 2\rangle$, $P_{E/L} = \langle\langle -2\rangle\rangle$. Therefore $N_{L/k}\big(P(E/L)\big) = \langle\langle -4, -4a\rangle\rangle + [[a]]$ by Lemma 1. The claim follows.

One may also take $F = k_{-1}$. In this case the trace forms of E/L and E/k are hyperbolic. Thus $S_{E/k} = \langle -2, -2\rangle$ and $P(E/k) = \langle\langle 2, 2\rangle\rangle$ is hyperbolic. Moreover $S_{E/L} = \langle -2\rangle$, $P_{E/L} = \langle\langle 2\rangle\rangle$ and $N_{L/k}\big(P(E/L)\big) = \langle\langle 4, -4a\rangle\rangle + [[a]]$ by Lemma 1. Again the claim follows. □

Proposition 2. *For the invariant e_m of $P(E)$ in $H^m(k, \mathbf{Z}/2)$ one has*

$$e_m\big(P(E)\big) = \sum_{i=0}^{m} (-1)^{m-i} w_i(S_E)$$

Proof. By Serre's splitting principle one may assume that E is multiquadratic. Both sides are multiplicative, and so one may assume $E = k_a$. In this case $q_E = \langle 2, 2a\rangle$, $P(E) = \langle\langle -2a\rangle\rangle$ and $w_1(S_E) = (2a)$. □

Remark. I don't know any "explanation" of Theorem 1. What is the nature of $P(E)$? One can play around and base everything on $S'_E = \langle -1\rangle S_E$ instead of S_E. Then

$$e_m\big(P(E)\big) = w_m(S'_E)$$

Let us normalize $P(E)$ to have dimension 0 and write

$$P'(E) = P(E) - \langle\!\langle 1\rangle\!\rangle^m$$

Then

$$P'(E) = \sum_{i=0}^{m} (-1)^i \lambda^i(S'_E)$$

This way the Pfister form invariant appears as a sort of Euler characteristic.

Remark. One has $w(q_E) = w(S_E)\big(1+(2)\big)^m = w(S_E)\big(1+m(2)\big)$. It follows that

$$e_m\big(P(E)\big) = \sum_{i=0}^{m} (-1)^{m-i} w_i(q_E) + m(2) w_{m-1}(q_E)$$

Furthermore:

$$e_m\big(P(E)\big) = \sum_{i=0}^{m} (-2)^{m-i} w_i^{\mathrm{gal}}(q_E)$$

I have not made up my mind about the preferred way to look at $P(E)$.

Remark. If one develops the material in terms of S'_E and $P'(E)$, then it seems natural to consider the normalized Pfister forms

$$[[a_1,\dots,a_n]] = [[a_1]]\cdots[[a_n]] = \langle\!\langle a_1,\dots,a_n\rangle\!\rangle - \langle\!\langle 1\rangle\!\rangle^n$$

of dimension 0. One gets the following variant of Lemma 1:

Lemma 2. *For $\alpha \in L^*$ one has*

$$N_{L/k}([[\alpha]]) = [[T_{L/k}(\alpha), -aN_{L/k}(\alpha)]] + [[2,a]]$$

□

Note that $[[2,a]]^2 = 0$ and $[[T_{L/k}(\alpha), -aN_{L/k}(\alpha)]][[2,a]] = 0$.

Corollary 2. *Let $n \geq 2$ and let f be a normalized n-fold Pfister form over L. Then $N_{L/k}(f)$ is a normalized $2n$-fold Pfister form over k.*

Example. Let E be of degree 4. Then

$$q_E = \langle 1\rangle + \langle c\rangle\langle -a, -b, ab\rangle$$

for some a, b, $c \in k^*$. One has $P(E) = \langle\!\langle a,b\rangle\!\rangle$.

Remark. Suppose k is ordered. Then the Pfister form $P(E)$ is positive definite if and only if E is totally real.

Remark. One has

$$\wedge q_E = P(E)\langle\!\langle -2\rangle\!\rangle^m$$

for even $n = 2m$ and

$$\wedge q_E = P(E)\langle\!\langle -1\rangle\!\rangle^{m+1}$$

for odd $n = 2m+1$. This follows easily from Serre's splitting principle.

Remark (Serre)**.** The Pfister form $P(E)$ represents $2^m d$. Here $m = [n/2]$ and d is the discriminant of E (and $n = \dim E$).

Beweis: this amounts to the equation $\langle 2^m d\rangle P(E) = P(E)$ in $W(k)$, which is true when E is multiquadratic, hence is true in general.

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