NOTES ON SCHUR FUNCTORS

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1. Introduction and Overview

We consider the Schur functors, see [1].

Let V be a locally free R-module of finite rank. The exterior power algebra is denoted as

$$\Lambda V = \bigoplus_{k \geq 0} \Lambda^k V$$

In the following V is fixed and we drop it from the notations.

The simplest non-trivial example of a Schur functor is

$$T^{a,b} = \big(\Lambda^a \otimes \Lambda^b\big)/Q$$

where Q is generated by the "quadratic relations". In the notation of [1, Section 8.1] one has $T^{a,b} = E^\lambda$ where λ is the conjugate partition of (a,b) (the Young diagram has two columns of sizes a,b).

Let us describe Q. We assume $a \geq b$. For $0 \leq r \leq b$ let

$$A_r \colon \Lambda^a \otimes \Lambda^r \otimes \Lambda^{b-r} \to \Lambda^a \otimes \Lambda^b$$

where

$$A_r = [\Phi]^{a,r,b-r}_{a,b}$$

is the graded component (by means of inclusion and projection) of

$$\Phi = (\mu \otimes \mu)(1 \otimes \sigma \otimes 1)(\Delta \otimes 1 \otimes 1) \colon \Lambda \otimes \Lambda \otimes \Lambda \to \Lambda \otimes \Lambda$$

Here μ is the multiplication and Δ is the comultiplication in the exterior algebra. Moreover σ is the unsigned switch involution:

$$\sigma(x \otimes y) = y \otimes x$$

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Thus

$$A_r(x \otimes y \otimes z) = \sum_i x_i y \otimes x'_i z \qquad \big(\sum_i x_i \otimes x'_i = [\Delta(x)]_{a-r,r}\big)$$

Note that

$$A_0(x \otimes y \otimes z) = x \otimes yz$$

By definition Q is generated by the "exchange relations". This means

$$Q = \sum_{r=1}^{b} \operatorname{im}(Q_r)$$

where

$$Q_r \colon \Lambda^a \otimes \Lambda^r \otimes \Lambda^{b-r} \to \Lambda^a \otimes \Lambda^b$$
$$Q_r(x \otimes y \otimes z) = A_r(x \otimes y \otimes z) - x \otimes yz$$

with $0 \le r \le b$ (one has $Q_0 = 0$). In [1, Section 8.1] the module Q is described more explicitly in terms of boxes and tensors $\wedge_i v_i$.

This seems to be the standard definition of $T^{a,b}$.

For many considerations it has some advantages, but there is a disadvantage. Namely if $\operatorname{rank} V \le a$, then $Q = 0$. (If $\operatorname{rank} V < a$, then obviously $T^{a,b} = 0$.) However that is obvious only at the second glance. So why not having a description of Q which makes this obvious?

(This kind of question was also the starting point for my text "On the adjunct of an endomorphism" on the adjunct and the Cayley-Hamilton theorem.)

Indeed, one has

Proposition 1. *For $0 \le r \le b$ let*

$$R_r \colon \Lambda^{a+r} \otimes \Lambda^{b-r} \to \Lambda^a \otimes \Lambda^b$$
$$R_r = [(1 \otimes \mu)(\Delta \otimes 1)]_{a,b}^{a+r,b-r}$$

and put

$$R = \sum_{r=1}^{b} \operatorname{im}(R_r)$$

Then $Q = R$.

There is a variant:

Proposition 2. *For $0 \le r \le b$ let*

$$R'_r \colon \Lambda^{b-r} \otimes \Lambda^{a+r} \to \Lambda^a \otimes \Lambda^b$$
$$R'_r = [(\mu \otimes 1)(1 \otimes \Delta)]_{a,b}^{b-r,a+r}$$

and put

$$R' = \sum_{r=1}^{b} \operatorname{im}(R'_r)$$

Then $Q = R'$.

Proofs are given in the text.

My feeling is that the exchange relations are the most basic or elementary relations (I always use them to check some ideas), but the R_r are more convenient for some generalities.

An interesting example is branching: write $V = V_1 \oplus L$ for a line bundle L and see what you get. One doesn't get here always the Schur modules of V_1, but extensions of them.

Another interesting topic is the description of Schur modules as submodules rather than quotient modules (see [1, Section 8.1, p. 109]). One may describe this as follows: Since $T^{a,b}$ is a strict polynomial functor, there is a pairing

$$S_{a+b}\big(\mathrm{Hom}(V,U)\big) \otimes T^{a,b}(V) \to T^{a,b}(U)$$

which yields a morphism

$$T^{a,b}(V) \to S^{a+b}\big(V \otimes U^{\#}\big) \otimes T^{a,b}(U)$$

where $^{\#}$ denotes the dual.

Suppose $\operatorname{rank} U = a$. Then

$$T^{a,b}(U) = \Lambda^a U \otimes \Lambda^b U$$

and we get a morphism

$$T^{a,b}(V) \to S^{a+b}\big(V \otimes U^{\#}\big) \otimes \Lambda^a U \otimes \Lambda^b U$$

Now choose a basis f_i of $U^{\#}$ and apply

$$(f_1 \wedge \cdots \wedge f_a) \otimes (f_1 \wedge \cdots \wedge f_b)$$

to the terms on the right. This results in a morphism

$$T^{a,b}(V) \to S^{a+b}\big(V \otimes U^{\#}\big)$$

That's essentially the embedding in [1, Section 8.1, Corollary of proof, p. 111].

Everything generalizes without much problem to any Schur functor

$$T^{a_1,\ldots,a_h} = \big(\Lambda^{a_1} \otimes \cdots \otimes \Lambda^{a_h}\big)/Q \qquad (a_1 \geq \cdots \geq a_h)$$

This is clear since the module Q is generated by the quadratic relations for the T^{a_i,a_j}. If I am not mistaken, one may restrict here to the quadratic relations for successive terms, that is for the $T^{a_i,a_{i+1}}$. And that follows perhaps from a defining relation for $\mathrm{SL}(3,\mathbf{Z})$, see the end of my text "Notes on strict bicommutative Hopf algebras".

A further interesting related topic is to determine all strict polynomial functors of the form

$$\Lambda^{a_1} \otimes \cdots \otimes \Lambda^{a_r} \to \Lambda^{b_1} \otimes \cdots \otimes \Lambda^{b_s} \qquad \Big(\sum_i a_i = \sum_j b_j\Big)$$

(I think that is not too difficult, at least for $r, s \leq 2$). Moreover one may ask for instance whether all morphisms of strict polynomial functors of the form

$$\Lambda^c \otimes X \to \Lambda^a \otimes \Lambda^b$$

are given by a morphism

$$X \to \bigoplus_{c+e+f=a+b} \Lambda^e \otimes \Lambda^f$$

followed by the corresponding component of

$$(\mu \otimes \mu)(1 \otimes \sigma \otimes 1)(\Delta \otimes 1 \otimes 1)\colon \Lambda \otimes \Lambda \otimes \Lambda \to \Lambda \otimes \Lambda$$

By the way, I came to learn Schur functors and Young diagrams in detail because I wanted to understand the Pluecker relations for the embedding

$$\mathrm{Gr}(r, V) \to \mathbf{P}(\Lambda^r V)$$

for $r = 3$. I had never realized how complicated this is already in the case $r = 2$. See [1, Section 8.4], in particular [1, Proposition 2, p. 126].

2. On the Hopf algebra structure

2.1. **The action of integral matrices.** Let V be a locally free R-module of finite rank. The exterior power algebra is denoted as usual by

$$\Lambda^\bullet V = \bigoplus_{k\geq 0} \Lambda^k V$$

There is a natural isomorphism

$$J_n\colon (\Lambda^\bullet V)^{\otimes n} \to \Lambda^\bullet(V^n)$$

given by the product. To describe J_n precisely, let $e_i = (0,\dots,0,1,0,\dots,0)$ be the standard basis of R^n and let

$$j_i\colon \Lambda^\bullet V \to \Lambda^\bullet(V\otimes R^n)$$
$$j_i = \Lambda^\bullet(v\mapsto v\otimes e_i)$$

be the morphism induced from the inclusion of the i-th summand. Then J_n is given by

$$J_n\colon (\Lambda^\bullet V)^{\otimes n} \to \Lambda^\bullet(V\otimes R^n)$$
$$J_n(x_1\otimes\cdots\otimes x_n) = j_1(x_1)\cdots j_n(x_n)$$

There is the natural functor associating to a R-module A the exterior power algebra $\Lambda^\bullet(V\otimes A)$ of $V\otimes A$. We will consider the restriction of this functor to the free R-modules $R^n = R\otimes_{\mathbf{Z}} \mathbf{Z}^n$ and to the morphisms $R^n\to R^m$ given by integral matrices.

We use the abbreviations

$$H = \Lambda^\bullet V, \quad H_k = \Lambda^k V$$

Let $\mathcal{Z}$ be the category with objects $\mathbf{Z}^n$ $(n\geq 0)$ and with morphisms

$$\mathrm{Hom}_{\mathcal{Z}}(\mathbf{Z}^m,\mathbf{Z}^n) = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^m,\mathbf{Z}^n) = M(n,m)$$

the $\mathbf{Z}$-linear homomorphisms (or integral $n\times m$-matrices).

We consider the functor

$$F\colon \mathcal{Z} \to R\text{-algebras}$$
$$F(\mathbf{Z}^n) = \Lambda^\bullet(V\otimes_{\mathbf{Z}}\mathbf{Z}^n) = (\Lambda^\bullet V)^{\otimes n} = H^{\otimes n}$$

Here we used the isomorphisms J_n for the identifications.

This way we get an action of integral matrices on the tensor powers of H (by acting on the $\mathbf{Z}^n$). We denote this action by

$$M(n,m)\times H^{\otimes m} \to H^{\otimes n}$$
$$(A,x)\mapsto [A](x)$$

The basic morphisms S,μ,Δ (the antipode, the product and the coproduct of H as a graded bicommutative Hopf algebra) have the descriptions

$$S = [-1], \quad \mu = [1,1], \quad \Delta = \begin{bmatrix}1\\1\end{bmatrix}$$

Using the matrix notation the sometimes tiring computations in terms of S,μ,Δ can be written in a more compact form.

For example, the operation

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}: H^{\otimes 2} \to H^{\otimes 2}$$

is in concrete terms the morphism

$$(\mu \otimes 1) \circ (1 \otimes \Delta)\colon (\Lambda^\bullet V)^{\otimes 2} \to (\Lambda^\bullet V)^{\otimes 2}$$
$$x \otimes y \mapsto \sum_i x y_i \otimes y_i' \qquad \Big(\sum_i y_i \otimes y_i' = \Delta(y)\Big)$$

See my text "Notes on strict bicommutative Hopf algebras" for more examples.

2.2. **Rewriting the exchange relations.** The antipode S acts on H_r by multiplication with $(-1)^r$. The involution τ acts like this:

$$\tau\colon H_r \otimes H_s \to H_s \otimes H_r$$
$$\tau(x \otimes y) = (-1)^{rs} y \otimes x$$

For $r = s$ this gives

$$\sigma = \tau(S \otimes 1) = \tau(1 \otimes S)\colon H_r \otimes H_r \to H_r \otimes H_r$$

where σ is the unsigned switch.

It follows that in the definition of A_r we may use

$$\Phi_1 = (\mu \otimes \mu)\big(1 \otimes \tau(1 \otimes S) \otimes 1\big)(\Delta \otimes 1 \otimes 1)\colon H^{\otimes 3} \to H^{\otimes 2}$$

instead of Φ. The advantage is that Φ_1 is expressible in terms of the Hopf algebra structure (which doesn't contain the unsigned switch). In terms of a matrix action:

$$\Phi_1 = \begin{bmatrix}1 & -1 & 0\\1 & 0 & 1\end{bmatrix}$$

By the way, we could use

$$\Phi_2 = \begin{bmatrix}1 & 1 & 0\\-1 & 0 & 1\end{bmatrix} = (\mu \otimes \mu)\big(1 \otimes \tau(S \otimes 1) \otimes 1\big)(\Delta \otimes 1 \otimes 1)$$

as well: The graded components $[\,]_{a,b}^{a,r,b-r}$ of Φ, Φ_1, Φ_2 are all equal to A_r.

3. Proof of Proposition 1 ($R \subset Q$)

We show that the image of R_r $(1 \le r \le b)$ is contained in Q.

We may assume $r = b > 0$ (the factor Λ^{b-r} in Q_r and R_r gets just multiplied from the right to Λ^b). One has

$$R_b\colon \Lambda^{a+b} \to \Lambda^a \otimes \Lambda^b$$
$$R_b = [\Delta]_{a,b}^{a+b}$$

Since the multiplication

$$\mu\colon \Lambda^a \otimes \Lambda^b \to \Lambda^{a+b}$$

is surjective, it suffices to show that the image of

$$[\Delta\mu]_{a,b}^{a,b}\colon \Lambda^a \otimes \Lambda^b \to \Lambda^a \otimes \Lambda^b$$

is contained in Q.

One has (by the bialgebra axiom)

$$\begin{aligned}\Delta\mu &= (\mu\otimes\mu)(1\otimes\tau\otimes 1)(\Delta\otimes\Delta)\\ &= \Phi_1(1\otimes S\otimes 1)(1\otimes\Delta)\end{aligned}$$

Taking the graded components $[\,]_{a,b}^{a,b}$ this is modulo Q the same as

$$(1\otimes\mu)(1\otimes S\otimes 1)(1\otimes\Delta) = 0$$

(The last expression is trivial by the Hopf algebra axiom for the antipode.)

□

4. Proof of Proposition 1 ($Q \subset R$)

We show that the image of Q_r $(1 \le r \le b)$ is contained in

$$R = \sum_{r=1}^{b} \operatorname{im}(R_r)$$

Again we may assume $r = b > 0$. One has

$$A_b \colon \Lambda^a \otimes \Lambda^b \to \Lambda^a \otimes \Lambda^b$$

$$A_b = [\Phi_1']_{a,b}^{a,b}$$

with

$$\begin{aligned}\Phi_1' &= (\mu\otimes 1)(1\otimes\tau)(\Delta\otimes S)\\ &= \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\end{aligned}$$

The matrix $\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$ restricted to $\Lambda^a \otimes \Lambda^b$ is a morphism

$$\Lambda^a \otimes \Lambda^b \to \bigoplus_{r\ge 0} \Lambda^{a+r} \otimes \Lambda^{b-r}$$

The matrix $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ is on $\Lambda^{a+r} \otimes \Lambda^{b-r}$ just R_r (after projection to $\Lambda^a \otimes \Lambda^b$). So modulo R there just remains the term for $r = 0$. But that's the identity. Hence $Q_b = A_b - \mathrm{id}$ is trivial mod R. □

5. Proof of Proposition 2

We have to show

$$R = R' := \sum_{r=1}^{b} \operatorname{im}(R_r')$$

We will use the matrix identity

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}$$

Informally speaking, if $a \ge b$ the matrix on the left yields the R_r' and the second matrix yields R_r. If $a \le b$ it is the other way round.

To gain symmetry, we drop the condition $a \le b$ and put

$$k = \max(a, b)$$

Let $d = k + r$ with $r > 0$ and $c = a + b - d$. Taking the component $[\,]_{a,b}^{c,d}$ the matrix identity yields a decomposition

$$\Lambda^c \otimes \Lambda^d \to \bigoplus_{e+f=c,\, e\geq 0} \Lambda^{e+d} \otimes \Lambda^f \to \Lambda^a \otimes \Lambda^b$$

If $k = a$, the composition is R'_r and the map on the right is the sum of the R_{e+r}. If $k = b$, the composition is R_r and the map on the right is the sum of the R'_{e+r}. □

References

[1] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry.

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