ON THE GALOIS COHOMOLOGY OF SPIN(14)

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Preliminary Notes

Note from May/June 2006

I am very grateful to Skip Garibaldi for comments. They led to several corrections and additions.

In the version from 1999 I had claimed without proof $\mathrm{ed}(\mathrm{Spin}_{13}) = 6$. I have now added a new section (Section 10) containing a proof.

Abstract

Let k be a field with $\operatorname{char} k \neq 2$. For $i = 6, 7$ we define invariants

$$h_i \colon H^1\big(k, \mathrm{Spin}(14)\big) \to H^i(k, \mathbf{Z}/2)/(-1)H^{i-1}(k, \mathbf{Z}/2).$$

Further we show that the natural map

$$H^1\big(k, (G_2 \times G_2) \rtimes \mu_8\big) \to H^1\big(k, \mathrm{Spin}(14)\big)$$

is surjective.

One concludes that the essential dimension of Spin(14) is equal to 7.

Similar considerations are done for Spin(12). We also present the list of essential dimensions of the split groups $\mathrm{Spin}(n)$ for $n \leq 14$.

Contents

Note from May/June 2006 1
Abstract 1
1. The Arason invariant 2
2. Reduced squares 3
3. Lambda operations 3
4. Multiplicative transfer 5
5. The invariants h_6 and h_7 5
6. A reduction lemma 7
7. 14-dimensional spinors 8
8. The essential dimension of Spin(14) 9
9. On the cohomology of Spin(12) 9
10. On the cohomology of Spin(13) 11
11. The essential dimension of split $\mathrm{Spin}(n)$ for $n \leq 14$ 16
References 16

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1. The Arason invariant

1.1. **The invariants e_i, $i \le 3$.** Let

$$e_i \colon I^i(k)/I^{i+1}(k) \to H^i(k, \mathbf{Z}/2), \quad i = 0, \dots, 3$$

be the first invariants on the graded Witt ring given by dimension, discriminant, the Hasse-Witt invariant, and Arason's invariant, cf. [1, 26].

1.2. **The split groups of type D_n.** We denote by $\mathrm{SO}(n,n)$ the automorphism group of the quadratic form

$$\sum_1^n (x_i^2 - y_i^2).$$

Furthermore, $\mathrm{Spin}(n,n)$ denotes the universal cover of $\mathrm{SO}(n,n)$ and

$$\mathrm{PSO}(n,n) = \mathrm{SO}(n,n)/\{\pm 1\} = \mathrm{Spin}(n,n)/\mu$$

denotes the corresponding adjoint group. Here μ is the center of $\mathrm{Spin}(n,n)$. One has $\mu = \mu_2 \times \mu_2$ if n is even and $\mu = \mu_4$ if n is odd.

If n is odd, every split group of type D_n is isomorphic to one of $\mathrm{Spin}(n,n)$, $\mathrm{SO}(n,n)$, $\mathrm{PSO}(n,n)$.

1.3. **Galois cohomology of $\mathrm{SO}(n,n)$.** The set $H^1\big(k, \mathrm{SO}(n,n)\big)$ consists of the isomorphism classes of $2n$-dimensional quadratic forms with trivial discriminant. We consider $H^1\big(k, \mathrm{SO}(n,n)\big)$ as a subset of $I^2(k) \subset W(k)$.

The image of

$$H^1\big(k, \mathrm{SO}(n,n)\big) \to H^1\big(k, \mathrm{PSO}(n,n)\big)$$

consists of the similarity classes of the quadratic forms in $H^1\big(k, \mathrm{SO}(n,n)\big)$. For $u \in H^1\big(k, \mathrm{Spin}(n,n)\big)$ let q_u be the corresponding quadratic form.

The image of

$$H^1\big(k, \mathrm{Spin}(n,n)\big) \to H^1\big(k, \mathrm{SO}(n,n)\big)$$

consists of those classes in $H^1\big(k, \mathrm{SO}(n,n)\big)$ with trivial Hasse-Witt invariant.

1.4. **The invariant $\tilde{e}_3$ in $K_3^M/2$.** Let $K_n^M k$ be Milnor's K-group [18].

By Merkurjev's theorem [2, 16, 31] the invariant e_2 is bijective. Furthermore, Milnor's homomorphism

$$s_3 \colon K_3^M k/2 \to I^3(k)/I^4(k)$$

is bijective (cf. [11, 17, 18, 25]).

Putting things together yields natural maps

$$\tilde{e}_3 \colon H^1\big(k, \mathrm{Spin}(n,n)\big) \to K_3^M k/2.$$

For $u \in H^1\big(\mathrm{Spin}(n,n)\big)$ the class $\tilde{e}_3(u)$ depends alone on q_u. For $u \in H^1\big(\mathrm{Spin}(8,8)\big)$ the corresponding quadratic form q_u is a 3-fold Pfister form (cf. [5, 15, 20, 26]); if $q_u = \langle\langle a, b, c \rangle\rangle$, then $\tilde{e}_3(u) = \{a, b, c\}$. Furthermore, the maps $\tilde{e}_3$ behave additively with respect to the natural inclusions

$$\mathrm{Spin}(n,n) \times \mathrm{Spin}(m,m) \to \mathrm{Spin}(n+m, n+m).$$

These properties determine the family of maps $\tilde{e}_3$ uniquely.

2. Reduced squares

It has been observed by Serre that for any $n \geq 2$ there is a natural map

$$P\colon K_n^M k/2 \to K_{2n}^M k/(2K_{2n}^M k + \{-1\}^{n-1} K_{n+1}^M k)$$

characterized by

$$P\Big(\sum_i x_i\Big) = \sum_{i<j} x_i x_j \bmod (2K_{2n}^M k + \{-1\}^{n-1} K_{n+1}^M k)$$

where x_i are symbols. (An element $x \in K_n^M k/2$ is called a symbol if it is of the form $x = \{a_1, \ldots, a_n\}$ for some $a_i \in k^*$.)

To define the operation P one checks that the right hand side of this formula does not depend on the presentation of an element as a sum of symbols. This follows easily from the definition of Milnor's K-theory and the identity $\{a, a\} = \{a, -1\}$, cf. [18].

Let

$$\alpha_n\colon K_n^M k/2 \to H^n(F, \mathbf{Z}/2)$$

be the norm residue homomorphism [18]. Milnor's conjecture (cf. [30]) asserts that α_n is bijective. With Milnor's conjecture, the operations P give rise to corresponding maps

$$H^n(k, \mathbf{Z}/2) \to H^{2n}(k, \mathbf{Z}/2)/(-1)^{n-1} H^{n+1}(k, \mathbf{Z}/2).$$

Combining this with the fact that $(-1)H^{2n-1}(k, \mathbf{Z}/2)$ is in the kernel of the natural maps $H^{2n}(k, \mathbf{Z}/2) \to H^{2n}(k, \mathbf{Z}/4)$, one obtains operations

$$H^n(k, \mathbf{Z}/2) \to H^{2n}(k, \mathbf{Z}/4).$$

In the case $n = 2$ this operation is nothing else than the Pontryagin square, cf. [3, 4, 32, 33]. For $n > 2$ I don't know any explanation of the operations P by an operation defined on the cohomology of topological spaces.

3. Lambda operations

Let $\widehat{W}(k)$ be the Grothendieck (-Witt) ring of quadratic forms over k. One defines λ-operations

$$\lambda^i\colon \widehat{W}(k) \to \widehat{W}(k)$$

in the usual fashion (see for instance [13]):

For a quadratic form $\varphi\colon V \to k$ let $\lambda^i\varphi\colon \bigwedge^i V \to k$ be its i-th exterior power. One has $\lambda^0\varphi = \langle 1 \rangle$ and $\lambda^1\varphi = \varphi$. The form λ^2 is also given by the Killing form on the Lie algebra $\mathrm{so}(\varphi)$ (at least if $\bar{\mathbf{Q}} \subset k$).

One forms the formal power series

$$\lambda_t\varphi = \sum_{i \geq 0} t^i \lambda^i \varphi.$$

Then

$$\lambda_t(\varphi \perp \psi) = \lambda_t\varphi \otimes \lambda_t\psi.$$

The series λ_t extends to $\widehat{W}(k)$ by

$$\lambda_t(\varphi - \psi) = \lambda_t\varphi \otimes (\lambda_t\psi)^{-1}$$

and the operations λ^i on $\widehat{W}(k)$ are defined by

$$\lambda_t(x) = \sum_{i\geq 0} t^i \lambda^i(x)$$

for $x \in \widehat{W}(k)$.

We are mainly interested in λ^2. Note that

$$\begin{aligned}
\lambda^0(x) &= 1,\\
\lambda^1(x) &= x,\\
y^2 &= \dim y + 2\lambda^2(y),\\
\lambda^2(x+y) &= \lambda^2(x) + xy + \lambda^2(y),\\
\lambda^2(x-y) &= \lambda^2(x) - y(x-y) - \lambda^2(y),\\
\lambda^2(x-y) &= \lambda^2(x) - xy + \dim y + \lambda^2(y),\\
\lambda^2(\langle a\rangle x) &= \lambda^2(x)
\end{aligned}$$

for x, $y \in \widehat{W}(k)$ and $a \in k^*$.

Let $\widehat{I}(k) \subset \widehat{W}(k)$ be the fundamental ideal of zero dimensional virtual quadratic forms. The projection $\widehat{W}(k) \to W(k)$ induces identifications $\widehat{I}^n(k) = I^n(k)$ for $n > 0$. $\widehat{I}^n(k)$ is additively generated by elements of the form

$$\langle\langle a_1, \dots, a_n\rangle\rangle - \langle\langle 1\rangle\rangle^n = \langle\langle a_1, \dots, a_{n-1}\rangle\rangle - \langle a_n\rangle\langle\langle a_1, \dots, a_{n-1}\rangle\rangle.$$

Lemma 3.1. *Let φ be an n-fold Pfister form and $x = \varphi - \langle\langle 1\rangle\rangle^n$. Then*

$$\lambda^2(x) = \langle\langle -1\rangle\rangle^{n-1} x.$$

Proof. Write $\varphi = \psi\langle\langle a\rangle\rangle$ where ψ is an $(n-1)$-fold Pfister form and where $a \in k^*$. Then $x = \psi - \langle a\rangle\psi$ and one finds

$$\begin{aligned}
\lambda^2(x) &= \lambda^2(\psi - \langle a\rangle\psi)\\
&= \lambda^2(\psi) - \langle a\rangle\psi x - \lambda^2(\psi)\\
&= -\langle a\rangle\langle\langle -1\rangle\rangle^{n-1} x\\
&= \langle\langle -1\rangle\rangle^{n-1}\langle -a\rangle x = \langle\langle -1\rangle\rangle^{n-1} x
\end{aligned}$$

Here one uses $\psi^2 = \langle\langle -1\rangle\rangle^{n-1}\psi$, $\langle -a\rangle x = -\langle a\rangle x$ if $\dim x = 0$, and $\langle -a\rangle\langle\langle a\rangle\rangle = \langle\langle a\rangle\rangle$. □

Corollary 3.2. *Let φ be an n-fold Pfister form. Then*

$$\begin{aligned}
\lambda^2(\varphi) &\simeq \varphi'\langle\langle -1\rangle\rangle^{n-1},\\
\lambda^2(\varphi') &\simeq \varphi'(\langle\langle -1\rangle\rangle^{n-1})'. \qquad \square
\end{aligned}$$

We define operations

$$\begin{aligned}
P' &\colon I^n(k) \to I^{2n}(k),\\
P'(x) &= \lambda^2(x) - \langle\langle -1\rangle\rangle^{n-1} x.
\end{aligned}$$

It follows from Lemma 3.1 and $\lambda^2(x+y) = \lambda^2(x) + xy + \lambda^2(y)$ that indeed $P'(x) \in I^{2n}(k)$.

These operations lift the operations P to the Witt ring.

4. Multiplicative transfer

Let L/F be separable field extension. In addition to the restriction map

$$r_{L/F}\colon W(F) \to W(L), \quad [\varphi] \mapsto [\varphi_L]$$

and the corestriction map

$$c_{L/F}\colon W(L) \to W(F), \quad [\psi] \mapsto [\operatorname{trace}_{L/F} \varphi]$$

one may define a multiplicative transfer map

$$N_{L/F}\colon W(L) \to W(F).$$

This map is analogous to the multiplicative transfer in cohomology, cf. [6, 12, 29].

We are interested in the case $[L : F] = 2$. Let σ denote the generator of the Galois group. Then for a quadratic form $\psi\colon W \to L$ the form $N_{L/F}(\psi)$ is given by the restriction of $\psi \otimes {}^\sigma\psi\colon W \otimes {}^\sigma W \to L$ to the subspace of invariants $(W \otimes {}^\sigma W)^\sigma$.

Suppose $L = F(\sqrt{a})$. One has the following rules

$$\begin{aligned}
\dim_F\bigl(N_{L/F}(\psi)\bigr) &= (\dim_L \psi)^2,\\
N_{L/F}(\langle\alpha\rangle) &= \langle N_{L/F}(\alpha)\rangle,\\
N_{L/F}(x+y) &= N_{L/F}(x) + c_{L/F}(x\sigma(y)) + N_{L/F}(y),\\
N_{L/F}(x-y) &= N_{L/F}(x) - c_{L/F}(x\sigma(y)) + N_{L/F}(y),\\
\lambda^2\bigl(c_{L/F}(x)\bigr) &= c_{L/F}\bigl(\lambda^2(x)\bigr) + aN_{L/F}(x),\\
N_{L/F}(\langle\langle\alpha\rangle\rangle) &= \langle\langle a\rangle\rangle + \begin{cases} \langle\langle \operatorname{trace}\alpha, -aN_{L/F}(\alpha)\rangle\rangle & \text{if } \operatorname{trace}\alpha \neq 0,\\ 0 & \text{if } \operatorname{trace}\alpha = 0,\end{cases}\\
N_{L/F}(\langle\langle\alpha_1,\dots,\alpha_n\rangle\rangle) &= \langle\langle a\rangle\rangle^n + \begin{cases} \prod_i \langle\langle \operatorname{trace}\alpha_i, -aN_{L/F}(\alpha_i)\rangle\rangle & \text{if } \operatorname{trace}\alpha_i \neq 0,\\ 0 & \text{else.}\end{cases}
\end{aligned}$$

In particular, if -1 is a square in F, then

$$N_{L/F}\bigl(I^n(L)\bigr) \subset I^{2n}(F)$$

for $n \geq 2$.

5. The invariants h_6 and h_7

For this section it is assumed for simplicity that $\sqrt{-1} \in k$.

We define

$$\begin{aligned}
h_6\colon H^1\bigl(k, \operatorname{Spin}(7,7)\bigr) &\to H^6(k, \mathbf{Z}/2),\\
h_6(u) &= \alpha_6 \circ P \circ \tilde{e}_3(u).
\end{aligned}$$

The invariant $h_6(u)$ depends only on q_u.

By the remarks of Section 3 one can lift this invariant to $I^6(k)$.

In some cases the invariant h_6 can be described explicitly. For a Pfister form φ one denotes by φ' its pure subform (one has $\varphi = \langle 1\rangle \perp \varphi'$). Let a_i, b_i, $c \in k^*$, $i = 1$, 2, 3, and put

$$q = c(\langle\langle a_1, a_2, a_3\rangle\rangle' \perp -\langle\langle b_1, b_2, b_3\rangle\rangle') \tag{1}$$

Then $q = q_u$ for some $u \in H^1\bigl(k, \operatorname{Spin}(7,7)\bigr)$ and for any such u one finds

$$h_6(u) = (a_1, a_2, a_3, b_1, b_2, b_3). \tag{2}$$

Lemma 5.1. *Let $u \in H^1\bigl(k, \operatorname{Spin}(7,7)\bigr)$. If q_u is isotropic, then $h_6(u) = 0$.*

Proof. If q_u is isotropic, the q_u has a representation (1) with $a_1 = b_1$, see [19, Satz 14, Zusatz] or [24]. The claim follows from (2). □

Proposition 5.2. *Let $u \in H^1\big(k, \mathrm{Spin}(7,7)\big)$ and let c be a nonzero value of q_u. The element*

$$h_6(u) \cup (c) \in H^7(k, \mathbf{Z}/2)$$

does not depend on the choice of c.

Proof (Variant 1). Write $q = q_u$. If q is isotropic, then $h_6(u) = 0$ by Lemma 5.1. We may therefore assume that q is anisotropic. Let $c = q(v)$ and $c' = q(v')$ be two values of q with v, v' linearly independent. Then c/c' is a norm from the quadratic extension L splitting the 2-dimensional subform $q|(vk + v'k)$. Say $c/c' = N_{L/k}(\lambda)$. Then

$$\begin{aligned} h_6(u) \cup (c) - h_6(u) \cup (c') &= h_6(u) \cup (c/c') \\ &= h_6(u) \cup N_{L/k}\big((\lambda)\big) \\ &= N_{L/k}\big(h_6(u_L) \cup (\lambda)\big) \\ &= N_{L/k}\big(0 \cup (\lambda)\big) = 0 \end{aligned}$$

since q_L is isotropic and by Lemma 5.1. □

Proof (Variant 2). Write $q = q_u$ as $q\colon V \to k$. Then any $x = [v] \in \mathbf{P}V$ determines an element

$$q(x) \in \kappa(x)/\big(\kappa(x)^*\big)^2.$$

Let $\xi \in \mathbf{P}V$ be the generic point and consider

$$\omega = h_6(u) \cup \big(q(\xi)\big) \in H^7\big(k(\mathbf{P}V), \mathbf{Z}/2\big).$$

The element ω is unramified on $\mathbf{P}V$, except possibly at the divisor

$$Z = \{q = 0\} \subset \mathbf{P}V$$

Here the residue is a multiple of (in fact, equal to)

$$h_6(u)_{k(Z)} \in H^6\big(k(Z), \mathbf{Z}/2\big)$$

But the quadratic form $q_{k(Z)}$ is isotropic, whence $h_6(u)_{k(Z)} = 0$ by Lemma 5.1. Hence ω is unramified everywhere on $\mathbf{P}V$ and therefore $\omega = (\omega_0)_{k(\mathbf{P}V)}$ for some $\omega_0 \in H^7(k, \mathbf{Z}/2)$. The claim follows by specialization. □

Proposition 5.2 gives rise to an invariant

$$\begin{gathered} h_7\colon H^1\big(k, \mathrm{Spin}(7,7)\big) \to H^7(k, \mathbf{Z}/2), \\ h_7(u) = h_6(u) \cup \big(q_u(v)\big) \end{gathered}$$

where $q_u(v)$ is any nonzero value of q_u.

As for h_6, the invariant $h_7(u)$ depends only on q_u. If $q_u = q$ with q as in (1), then

$$h_7(u) = (a_1, a_2, a_3, b_1, b_2, b_3, c).$$

This computation shows that the invariant h_7 is non-trivial.

In the next two statements (Proposition 5.3, Lemma 5.4) we assume that k contains the algebraic closure of $\mathbf{Q}$. This assumption is made to be sure that we can neglect some universal constants arising in decompositions of Killing forms and of $\lambda^2(q)$. I have not tried to figure out the best possible conditions.

Proposition 5.3. *Assume $\bar{\mathbf{Q}} \subset k$. Any value of h_6 and of h_7 is a symbol.*

Proof. It suffices to consider h_6. Let $u \in H^1\big(k, \mathrm{Spin}(7,7)\big)$ and write $q = q_u$. Then $h_6(u)$ is represented by 92-dimensional form

$$\lambda^2 q \perp \langle 1 \rangle.$$

The form $\lambda^2 q$ is also given by the Killing form on $\mathrm{so}(q)$.

We may assume that u is induced from an element $x \in H^1\big(k, (G_2 \times G_2) \rtimes \mu_8\big)$, see Corollary 7.3. Let $\mathfrak{g} \subset \mathrm{so}(q)$ be the Lie algebra of type $G_2 + G_2$ corresponding to x. Its Killing form is the trace of the Killing form of a Lie algebra of type G_2 over some quadratic extension. In view of the next Lemma, this form is hyperbolic.

Therefore the 92-dimensional form $\lambda^2 q \perp \langle 1 \rangle$ contains a 28-dimensional hyperbolic subform. Thus $h_6(u)$ is represented by a $92 - 28 = 64$-dimensional quadratic form, which therefore must be a multiple of a 6-fold Pfister form. □

This proof indicates that one may represent $h_6(u)$ by a form on the spinor representation S, cf. below. In fact there is a natural way to represent $h_6(u)$ as $N_{L/k}(\psi)$ on S, where ψ/L is the 3-fold Pfister form corresponding to a reduction $x \in H^1\big(k, (G_2 \times G_2) \rtimes \mu_8\big)$ of u, cf. [24].

Lemma 5.4. *Assume $\bar{\mathbf{Q}} \subset k$. Let $\mathfrak{g}$ be a Lie algebra of type G_2 and let φ be the associated 3-fold Pfister form. Then the Killing form on $\mathfrak{g}$ is hyperbolic.*

Proof. Let V be the 7-dimensional representation of $\mathfrak{g}$. Then

$$\mathfrak{g} \perp V = \textstyle\bigwedge^2 V.$$

Let further ψ denote the Killing form on $\mathfrak{g}$ and let φ be the associated 3-fold Pfister form. Then

$$\psi \perp \varphi' = \lambda^2(\varphi') = \varphi' \langle\langle -1, -1 \rangle\rangle'$$

by Corollary 3.2. The claim follows. □

Our considerations in the construction of the invariants h_6, h_7 may be also applied to the group $\mathrm{SO}(6)$. This leads to invariants

$$H^1\big(k, \mathrm{SO}(6)\big) \to H^i(k, \mathbf{Z}/2)$$

for $i = 4, 5$, given by

$$c(\langle\langle a_1, a_2 \rangle\rangle' \perp \langle\langle b_1, b_2 \rangle\rangle') \mapsto (a_1, a_2, b_1, b_2),$$
$$c(\langle\langle a_1, a_2 \rangle\rangle' \perp \langle\langle b_1, b_2 \rangle\rangle') \mapsto (a_1, a_2, b_1, b_2, c).$$

The latter coincides with the invariant

$$\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle \mapsto (a_1, a_2, a_3, a_4, a_5)$$

defined by Serre.

6. A reduction lemma

Let G be an algebraic group over k and let $i\colon H \subset G$ be a subgroup. For $x \in H^1(k, G)$ we denote by P_x a corresponding G-torsor.

Lemma 6.1. *Let $x \in H^1(k, G)$. Then x is in the image of*

$$i_*\colon H^1(k, H) \to H^1(k, G)$$

if and only if the variety P_x/H has a k-rational point.

Proof. Indeed, if $x = i_*(y)$, then $P_x \simeq P_y \times_H G$ and P_x/H has the k-rational point given by $[P_y, 1] \bmod H$.

Conversely, if $z \in P_x/H$ is k-rational, then the fiber of z under $P \to P_x/H$ is an H-torsor Q with $Q \times_H G \simeq P_x$. □

This simple lemma is the basis of many structure theorems on quadratic forms and algebras. It applies usually when there is a "small" representation of G, i.e., a representation $G \to \mathrm{GL}(V)$ with $\dim V < \dim G$.

A fairly simple example is given by $G = \mathrm{O}(n)$ and $H = \mathrm{O}(n-1) \times \mu_2$: Let $x \in H^1\big(k, \mathrm{O}(n)\big)$; if $q_x \colon V \to k$ is the corresponding quadratic form, then P_x/H is naturally isomorphic to $U = \mathbf{P}V \setminus \{q_x = 0\}$. Since U has a rational point, it follows that x has a reduction to H.

Her majesty E_8 does not have a small representation.

7. 14-dimensional spinors

Let $\mathrm{Spin}(7,7) \to \mathrm{GL}(S)$ be one of the spinor representations ($\dim S = 64$) and let $\mathrm{PSO}(7,7) \to \mathrm{PGL}(S)$ be the induced homomorphism. We denote $G = \mathrm{PSO}(7,7)$ and define $H \subset G$ as the image of

$$(G_2 \times G_2) \rtimes \mathbf{Z}/2 \to \mathrm{PSO}(7,7)$$

given by

$$(g,h)\epsilon^n \mapsto \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho(h) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n$$

where $\rho \colon G_2 \to \mathrm{Spin}(7)$ is the standard representation.

We need the following fact, see [7, 9, 21, 24].

Proposition 7.1. *The action of G on $\mathbf{P}S$ has an open and dense orbit U. If k is algebraically closed, then the isotropy group H_u of $u \in U$ is conjugate to H. In particular, $U = G/H$.*

Now let $x \in H^1(k, G)$. Then $X_x = P_x \times_G \mathbf{P}S$ is a Brauer-Severi variety whose Brauer class coincides with the Tits class $t(x) \in H^2(k, \mu_4)$ of x. Further, the variety $U_x = P_x \times_G U = P_x/H$ is a dense open subscheme of X_x. It follows that P_x/H has k-rational points if and only if $t(x) = 0$ (to be sure, let us assume that k is infinite). Lemma 6.1 shows

Corollary 7.2. *An element $x \in H^1(k, G)$ has an H-reduction if and only if $t(x) = 0$.* □

Let $\widetilde{H}$ be the preimage of H under $\mathrm{Spin}(7,7) \to \mathrm{PSO}(7,7)$. One finds (see [24])

$$\widetilde{H} = (G_2 \times G_2) \rtimes \mu_8$$

where $\mu_8 \subset \mathrm{Spin}(7,7)$ is the normalizer of $G_2 \times G_2$.

Corollary 7.3. *The homomorphism*

$$H^1\big(k, (G_2 \times G_2) \rtimes \mu_8\big) \to H^1\big(k, \mathrm{Spin}(7,7)\big)$$

is surjective.

Proof. This follows from a diagram chase in

$$\begin{array}{ccccccc} H^1(k,\mu_4) & \longrightarrow & H^1\big(k,(G_2\times G_2)\rtimes\mu_8\big) & \longrightarrow & H^1\big(k,(G_2\times G_2)\rtimes \mathbf{Z}/2\big) & \longrightarrow & H^2(k,\mu_4) \\ \| & & \downarrow & & \downarrow & & \| \\ H^1(k,\mu_4) & \longrightarrow & H^1\big(k,\mathrm{Spin}(7,7)\big) & \longrightarrow & H^1\big(k,\mathrm{PSO}(7,7)\big) & \xrightarrow{t} & H^2(k,\mu_4) \end{array}$$

□

It can be shown that there exist a field k and $x \in H^1\big(k,\mathrm{Spin}(7,7)\big)$ such that x has no reduction to the subgroup $(G_2\times G_2)\times\mu_4$. This means that the appearing forms of $G_2\times G_2$ are necessarily of type $R_{\ell/k}(G_2)$ with ℓ/k a quadratic *field* extension. Examples have been provided in [8] using residue arguments and in [10] using computations of the K-theory of certain homogeneous varieties.

8. The essential dimension of Spin(14)

We denote by $\mathrm{ed}(G)$ the essential dimension of G, see [22].

Proposition 8.1. $\mathrm{ed}\big(\mathrm{Spin}(14)\big) = 7$.

Proof. $\mathrm{ed}\big(\mathrm{Spin}(14)\big) \geq 7$ follows from the non-triviality of the invariant h_7.

It remains to show $\mathrm{ed}\big(\mathrm{Spin}(14)\big) \leq 7$. By Corollary 7.3 it suffices show $\mathrm{ed}(\widetilde{H}) \leq 7$. To describe any $\widetilde{H}$-torsor one needs one parameter to describe a class $(a) \in H^1(k,\mu_8) = k^*/(k^*)^8$ and $3\cdot 2$ parameters to describe an octonion algebra

$$O(a_1+\sqrt{a}b_1, a_2+\sqrt{a}b_2, a_3+\sqrt{a}b_3)$$

over $k(\sqrt{a})$. □

9. On the cohomology of Spin(12)

We briefly sketch a proof of $\mathrm{ed}\big(\mathrm{Spin}(6,6)\big) = 6$.

We define $H \subset \mathrm{SO}(6,6)$ as the image of

$$\mathrm{SL}(6)\rtimes\mathbf{Z}/2 \to \mathrm{SO}(6,6)$$

given by

$$g\epsilon^n \mapsto \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}^n.$$

Here we understand coordinates (x,y) with respect to the quadratic form $\sum_i x_iy_i$. The preimage $\widetilde{H}$ of H in $\mathrm{Spin}(6,6)$ is

$$\widetilde{H} = \mathrm{SL}(6)\rtimes\mu_4.$$

By the mentioned theorem of Pfister ([19, Satz 14, Zusatz] or [24]), any $\mathrm{Spin}(6,6)$-torsor admits an $\widetilde{H}$-reduction. Since any hermitian form can be diagonalized, the map

$$H^1\big(k,\mathrm{SO}(6)\times\mu_4\big) \to H^1(k,\widetilde{H})$$

is surjective. Hence

Corollary 9.1. $\mathrm{ed}\big(\mathrm{Spin}(6,6)\big) \leq 6$.

We define invariants in $H^5(\mathbf{Z}/2)$, $H^6(\mathbf{Z}/2)$ by a variant of the previous method. It is based on the following facts:

Lemma 9.2. *Let $a \in k^*$. Then the kernel of*

$$W(k) \to W(k), \quad x \mapsto \langle\langle a \rangle\rangle x$$

is generated by 2-dimensional forms of the form $\langle\langle N_{\ell/k}(\alpha)\rangle\rangle$ with $\alpha \in \ell^*$, $\ell = k(\sqrt{a})$.

Proof. Well known... □

Lemma 9.3. *Let a, $b \in k^*$ and let x, $y \in W(k)$. If*

$$\langle\langle a \rangle\rangle x = \langle\langle b \rangle\rangle y,$$

then there exist $z \in W(k)$ with

$$\langle\langle a \rangle\rangle x = \langle\langle a \rangle\rangle z = \langle\langle b \rangle\rangle z = \langle\langle b \rangle\rangle y.$$

Moreover, any such z may be written as a sum of 2-dimensional forms of the form $\langle\langle N_{\ell/k}(\alpha)\rangle\rangle$ with $\alpha \in \ell^*$, $\ell = k(\sqrt{ab})$.

Proof. Let φ be a quadratic form representing x, let $K = k(\sqrt{b})$, and suppose that $\langle\langle a \rangle\rangle \varphi_K$ is split.

Since $\langle\langle a \rangle\rangle \varphi_K$ is isotropic, one has $\langle\langle a \rangle\rangle \varphi = \langle\langle a \rangle\rangle (c\langle\langle d \rangle\rangle + \varphi')$ such that $\langle\langle a, d \rangle\rangle_K$ is isotropic. To see this, let $\varphi = \langle a_1, \dots, a_n \rangle$ and let

$$q \colon V = L^n \to k$$

$$q(\lambda_1, \dots, \lambda_n) = \sum_i a_i N_{L/k}(\lambda_i)$$

with $L = k(\sqrt{a})$. Note that $q = \langle\langle a \rangle\rangle \varphi$ and that $q(\lambda v) = N_{L/k}(\lambda) q(v)$ for $\lambda \in L$. If q_K is isotropic, there exists a 2-dimensional L-submodule W of V such that $q|W$ is isotropic over K. Next note that $q|W = c\langle\langle a, d \rangle\rangle$ for some c, d.

We may assume $\varphi = c\langle\langle d \rangle\rangle \perp \varphi'$. There exists e such that

$$\langle\langle a, d \rangle\rangle = \langle\langle a, e \rangle\rangle = \langle\langle b, e \rangle\rangle$$

Then $\langle\langle a \rangle\rangle \varphi = c\langle\langle a, e \rangle\rangle + \langle\langle a \rangle\rangle \varphi'$. The claim follows by induction on $\dim \varphi$ and Lemma 9.2. □

Let $I_2(k) \subset I(k)$ be the subset of elements which are split over *some* quadratic extension. One defines an operation

$$Q \colon I_2(k) \to I_2(k),$$

$$Q(\langle\langle a \rangle\rangle x) = \langle\langle a \rangle\rangle \lambda^2(x).$$

This map is well defined by Lemma 9.2 and Lemma 9.3.

We assume that -1 is a square. Let $u \in H^1\big(k, \mathrm{Spin}(6,6)\big)$. Then

$$q_u = a\langle\langle b \rangle\rangle (\langle\langle c, d \rangle\rangle' - \langle\langle e, f \rangle\rangle')$$

and

$$Q(q_u) = \langle\langle b, c, d, e, f \rangle\rangle$$

Hence we an invariant

$$k_5 \colon H^1\big(k, \mathrm{Spin}(6,6)\big) \to H^5(k, \mathbf{Z}/2).$$

If q_u is isotropic, then $q_u = a\langle\langle b, c', d' \rangle\rangle \perp \langle 1, -1 \rangle$. This shows $Q(q_u) = 0$. By the same argument as in the proof of Proposition 5.2 we get an invariant

$$k_6 \colon H^1\big(k, \mathrm{Spin}(6,6)\big) \to H^6(k, \mathbf{Z}/2),$$

$$k_6(u) = k_5(u) \cup \big(q_u(v)\big)$$

where $q_u(v)$ is any nonzero value of q_u.

If $q_u = a\langle\langle b\rangle\rangle(\langle\langle c,d\rangle\rangle' - \langle\langle e,f\rangle\rangle')$, then

$$k_6(u) = (a,b,c,d,e,f).$$

This shows that k_6 is nontrivial.

Corollary 9.4. $\operatorname{ed}\bigl(\operatorname{Spin}(6,6)\bigr) \geq 6$.

10. On the cohomology of Spin(13)

Let

$$\begin{gathered} q\colon V = k^{13} \to k \\ q(x_1,\dots,x_{13}) = (x_1^2+x_2^2+x_3^2) - (x_4^2+x_5^2+x_6^2) \\ - (x_7^2+x_8^2+x_9^2) + (x_{10}^2+x_{11}^2+x_{12}^2) - x_{13}^2 \end{gathered}$$

An element of $H^1(k,\operatorname{SO}(q))$ is given by a 13-dimensional quadratic form q' with

$$q' \perp \langle 1\rangle \in H^1(k,\operatorname{SO}(7,7)) \subset I^2 \subset W(k)$$

Let G be the subgroup of $\operatorname{SO}(q)$ generated by (matrix notation with respect to $k^{13} = k^3\times k^3\times k^3\times k^3\times k$)

$$U(g,h) = \begin{pmatrix} g & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$V(\alpha,\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & \alpha\beta \end{pmatrix}, \quad W(\eta) = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with g, $h \in \operatorname{SO}(3)$, α, $\beta \in \mu_2$ and $\eta \in \mu_4$. One has

$$G = \bigl(\operatorname{SO}(3)\times\mu_2\bigr)^2 \rtimes \mu_4 \subset \operatorname{SO}(q)$$

with μ_4 acting via the projection $\mu_4 \to \mu_2 = \mathbf{Z}/2$ by permutation of the factors.

We consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{G} & \xrightarrow{\pi_G} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \tilde{j}} & & \downarrow{\scriptstyle j} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \operatorname{Spin}(q) & \xrightarrow{\pi} & \operatorname{SO}(q) & \longrightarrow & 1 \end{array} \tag{3}$$

where $\widetilde{G} \subset \operatorname{Spin}(q)$ is the preimage of G under the projection $\pi\colon \operatorname{Spin}(q) \to \operatorname{SO}(q)$ and $\tilde{j}$, j are the inclusions.

We describe the image

$$J = j_*\bigl(H^1(k,G)\bigr) \subset H^1(k,\operatorname{SO}(q))$$

Lemma 10.1. *The set J consists exactly of the (isomorphism classes of) quadratic forms q' of the type*

$$q' = \tilde{q} \perp \langle -\det(\tilde{q})\rangle \tag{4}$$

with

$$\tilde{q} = (T_{K/k})_*\big(\langle s\rangle\langle 1, -\lambda\rangle\langle -\mu_1, -\mu_2, \mu_1\mu_2\rangle\big) \tag{5}$$

with $K = k[s]/(s^2 - b)$ for some $b \in k^\times$ and λ, μ_1, $\mu_2 \in K^\times$.

Proof. Note that $G \subset \mathrm{SO}(q)$ leaves the subspace $V' = k^{12} \times \{0\} \subset V$ invariant. Let

$$\ell\colon G \to \mathrm{O}(q|V')$$
$$\ell(g) = j(g)|V'$$

Then

$$j(g) = \big(\ell(g), \det(\ell(g))\big) \in \mathrm{O}(q|V') \times \mathrm{O}(1) \subset \mathrm{O}(q)$$

This yields the decomposition (4).

It remains to show that $\ell_*\big(H^1(k, G)\big) \subset H^1(\mathrm{O}(q|V'))$ consists of the forms $\tilde{q}$ as in (5).

Elements of $H^1(k, G)$ are the isomorphism classes of triples (K', φ, φ_1), where $K' = k[t]/(t^4 - b)$ is a Galois μ_4-algebra and where φ, φ_1 are quadratic forms over the quadratic subextension $K = k[s] \subset K'$, $s = t^2$ with φ of rank 3 and determinant 1 and with φ_1 of rank 1. Let

$$H = \big(\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)\big) \cap \mathrm{SO}(3) \simeq \mu_2 \times \mu_2$$

and

$$G' = (H \times \mu_2)^2 \rtimes \mu_4 \subset G$$

Since quadratic forms (over K) can be diagonalized, it follows that $H^1(k, G') \to H^1(k, G)$ is surjective.

The claim follows from Corollary 10.3 below. □

Lemma 10.2. *Let*

$$G'' = (\mu_2)^2 \rtimes \mu_4$$

generated by μ_4 and elements α, β with the relations

$$\alpha^2 = \beta^2 = (\alpha\beta)^2 = 1, \quad \zeta\alpha\zeta^{-1} = \beta, \quad \zeta\beta\zeta^{-1} = \alpha$$

for a generator ζ of μ_4.

Let

$$q_0\colon k^2 \to k$$
$$q_0(x, y) = x^2 - y^2$$

and let

$$\varphi\colon G'' \to \mathrm{O}(q_0)$$

be the homomorphism with

$$\varphi(\alpha) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varphi(\eta) = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}$$

with α, $\beta \in \mu_2$ and $\eta \in \mu_4$.

Let $\xi \in H^1(k, G'')$ and write the corresponding Galois G''-algebra as

$$E_\xi = k[t, s, x, y]/(t^4 - b, s - t^2, x^2 - u - sv, y^2 - u + sv)$$

with b, u, $v \in k$, $b \neq 0$, $u^2 - bv^2 \neq 0$. Here the action of G'' is given by

$$\begin{aligned} \zeta(t) &= \zeta t, \quad \zeta(s) = -s, \quad \zeta(x) = y, \quad \zeta(y) = x \\ \alpha(t) &= t, \quad \alpha(s) = s, \quad \alpha(x) = -x, \quad \alpha(y) = y \\ \beta(t) &= t, \quad \beta(s) = s, \quad \beta(x) = x, \quad \beta(y) = -y \end{aligned}$$

Then the associated quadratic form $q_\xi = \varphi_*(\xi) \in H^1(k, \mathrm{O}(q_0))$ is given by

$$q_\xi = (T_{K/k})_*\big(\langle s\rangle\langle u + sv\rangle\big)$$

with $K = k[s] \subset E_\xi$.

Proof. One has (more or less by definition)

$$q_u = (q_0 \otimes_k E)|(k^2 \otimes_k E)^{G''}$$

with G'' acting on k^2 via $\mathrm{O}(q_0)$ and on E as Galois algebra, respectively.

The claim follows from the following explicit computation (for a related consideration see Garibaldi's Lens notes from May 2006, Example 16.5):

One finds that $(k^2 \otimes_k E)^{G''}$ is the free k-module with basis

$$X = (xt, -yt), \qquad Y = (xt^3, yt^3) = (xts, yts)$$

For c, $d \in k$ one has with $\lambda = x^2 = u + sv$ and $\bar\lambda = y^2 = u - sv$

$$\begin{aligned} q_0(cX + dY) &= \big(xt(c + ds)\big)^2 - \big(yt(-c + ds)\big)^2 \\ &= \lambda s(c + ds)^2 + \bar\lambda(-s)(c - ds)^2 \\ &= T_{K/k}\big(\lambda s(c + ds)^2\big) \end{aligned}$$

□

Corollary 10.3. *Let n, $m \geq 0$, let $U = (\mu_2)^n$ and let*

$$\Phi\colon U \to \mathrm{O}(1)^m \subset \mathrm{O}(m)$$

be some homomorphism. Let

$$G'' = U^2 \rtimes \mu_4$$

with μ_4 acting via the projection $\mu_4 \to \mu_2 = \mathbf{Z}/2$ by permutation of the factors and let

$$\varphi\colon G'' \to \mathrm{O}(m, m)$$

$$\varphi(u_1, u_2) = \begin{pmatrix} \Phi(u_1) & 0 \\ 0 & \Phi(u_2) \end{pmatrix}$$

$$\varphi(\zeta) = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}$$

for $(u_1, u_2) \in U^2$ and a generator ζ of μ_4.

Let $\xi \in H^1(k, G'')$ and write the corresponding Galois G''-algebra as

$$E_\xi = k[t, s, x_i, y_i; i = 1, \ldots, n]/(t^4 - b, s - t^2, x_i^2 - u_i - sv_i, y_i^2 - u_i + sv_i)$$

with b, u_i, $v_i \in k$, $b \neq 0$, $u_i^2 - bv_i^2 \neq 0$ (with obvious G'' action, see Lemma 10.2).

Then the associated quadratic form $q_\xi = \varphi_*(\xi) \in H^1(k, \mathrm{O}(m, m))$ is given by

$$q_\xi = (T_{K/k})_*\big(\langle s\rangle\langle \mu_1, \ldots, \mu_m\rangle\big)$$

with $K = k[s] \subset E_\xi$ *and with*

$$\mu_j = \prod_{i=1}^{n} \lambda_i^{\Phi_{ij}} \in K, \quad j = 1, \dots, m$$

where

$$\lambda_i = u_i + s v_i$$

and where $\Phi_{ij} = 0,\ 1$ *is defined by*

$$\Phi(\alpha_1, \dots, \alpha_n) = \big(\prod_{i=1}^{n} \alpha_i^{\Phi_{ij}}\big)_{j=1,\dots,m}$$

Proof. One easily reduces to the case $m = 1$, $n = 1$ and $\Phi = \mathrm{id}$, which is treated in Lemma 10.2. □

Proposition 10.4. *The natural map* $\tilde{j}_* \colon H^1(\widetilde{G}) \to H^1(\mathrm{Spin}(q))$ *is surjective.*

Proof. Let $u \in H^1(k, \mathrm{Spin}(q))$ and let $q_u \in H^1(k, \mathrm{SO}(q))$ be the associated quadratic form. Then

$$q_u \perp \langle 1 \rangle \in I^3$$

By the results on 14-dimensional forms in I^3 one has

$$q_u \perp \langle 1 \rangle = (T_{K/k})_*(\langle s \rangle \varphi')$$

with $K = k[s]/(s^2 - b)$ for some $b \in k^\times$ and with φ a 3-fold Pfister form over K (and with $\varphi = \langle 1 \rangle \perp \varphi'$). Since the left hand side represents 1, there exists a value $-\lambda$ of φ' with $T_{K/k}(-s\lambda) = 1$. As for any (invertible) value $-\lambda$ of φ', one has $\varphi = \langle\langle \lambda, \mu_1, \mu_2 \rangle\rangle$ for some $\mu_1,\ \mu_2 \in K^\times$. Note that

$$(T_{K/k})_*(\langle -s\lambda \rangle) = \langle 1, -N_{K/k}(\lambda) \rangle$$

Thus

$$q_u = (T_{K/k})_*\big(\langle s \rangle \langle\langle \lambda \rangle\rangle \langle\langle \mu_1, \mu_2 \rangle\rangle'\big) \perp \langle -N_{K/k}(\lambda) \rangle$$

By Lemma 10.1 it follows that $q_u \in J$. A diagram chase (see diagram (3)) involving the coboundary maps $H^1(k, G)$, $H^1(k, \mathrm{SO}(q)) \to H^2(k, \mu_2)$ shows that there exists $\tilde{u} \in H^1(k, \widetilde{G})$ such that $\tilde{j}(\tilde{u})$, $u \in H^1(k, \mathrm{Spin}(q))$ have the same image in $H^1(k, \mathrm{SO}(q))$. Another diagram chase shows that we can arrange $\tilde{j}(\tilde{u}) = u$. □

We next compute $\widetilde{G} \subset \mathrm{Spin}(q) \subset C(q)$ inside the Clifford algebra. Let $e_1, \dots, e_{13}$ be the standard base of V.

Let ζ be a primitive 4-th root of unity.

For $v,\ w \in V$ with $q(v) = 1$, $q(w) = -1$ and $v \perp w$ let

$$\omega(v, w) = \frac{1 + \zeta w v}{\sqrt{2}}$$

Then $\omega(v, w)\omega(w, v) = 1$ and therefore $\omega(v, w) \in \mathrm{Spin}(q)$. Moreover $\omega(v, w)^2 = \zeta w v$ and $\omega(v, w)^4 = -1$. Furthermore $\omega(v, w)v = v\omega(v, w)^{-1}$ and $\omega(v, w)w = w\omega(v, w)^{-1}$. Also $\omega(v, w)v\omega(v, w)^{-1} = \zeta w$ and $\omega(v, w)w\omega(v, w)^{-1} = \zeta v$.

Consider the element

$$\omega = \omega(e_1, e_7)\omega(e_2, e_8)\omega(e_3, e_9)\omega(e_{10}, e_4)\omega(e_{11}, e_5)\omega(e_{12}, e_6) \in \mathrm{Spin}(q)$$

Its image in $\mathrm{SO}(q)$ is $\pi(\omega) = W(\zeta)$. Moreover

$$\omega^4 = 1$$

Next let

$$\tilde{\alpha} = e_4e_5e_6e_{13}, \quad \tilde{\beta} = -\zeta e_{10}e_{11}e_{12}e_{13}$$

Both elements are in $\mathrm{Spin}(q)$ and $\pi(\tilde{\alpha}) = V(-1,1)$ and $\pi(\tilde{\beta}) = V(1,-1)$. Moreover

$$\begin{aligned}
\tilde{\alpha}^2 &= 1\\
\tilde{\beta}^2 &= 1\\
\tilde{\alpha}\tilde{\beta} &= -\tilde{\beta}\tilde{\alpha}\\
(\tilde{\alpha}\tilde{\beta})^2 &= -1\\
\omega\tilde{\alpha}\omega^{-1} &= \tilde{\beta}\\
\omega\tilde{\beta}\omega^{-1} &= -\tilde{\alpha}\\
\omega\tilde{\alpha}\tilde{\beta}\omega^{-1} &= \tilde{\alpha}\tilde{\beta}\\
\tilde{\alpha}\omega\tilde{\alpha}^{-1} &= \tilde{\alpha}\tilde{\beta}\omega\\
\tilde{\alpha}\omega^2\tilde{\alpha}^{-1} &= -\omega^2
\end{aligned}$$

Let H be the subgroup generated by ω and $\tilde{\alpha}$. Then $\tilde{\beta} \in H$ and

$$H = (\mu_4 \times \mu_4) \rtimes \mu_2$$

with the μ_2 generated by $\tilde{\alpha}$ and $\mu_4 \times \mu_4$ generated by ω and $\tilde{\alpha}\omega\tilde{\alpha}^{-1}$.

Note further that the diagonal embedding $\mathrm{SO}(3) \to \mathrm{SO}(3,3)$ lifts to $\mathrm{Spin}(3,3)$. Thus the connected component of G lifts (uniquely) to $\mathrm{Spin}(q)$. This yields:

Lemma 10.5. *One has*

$$\widetilde{G} \simeq \big(\mathrm{SO}(3)\big)^2 \rtimes_\varphi H$$

where H acts by permutation of the factors via $\varphi\colon H \to \mathbf{Z}/2$, $\varphi(\tilde{\alpha}) = 0$, $\varphi(\omega) = 1$.

(I was surprised about the simple structure of H. There ought to be a better approach to the subgroup $\widetilde{G}$ of $\mathrm{Spin}(13)$ than just by a computation starting from G.)

Proposition 10.6. $\mathrm{ed}(\widetilde{G}) \le 6$

Proof. Elements of $H^1(k,H)$ are given by Galois H-algebras which can be written as

$$L = k[z,x,y]/(z^2 - a, x^4 - u - sv, y^4 - u + sv)$$

with a, u, $v \in k$, $a \neq 0$, $u^2 - av^2 \neq 0$. For the generic case we may assume $v \neq 0$ and replace s by sv and a by av^2. Then $v = 1$. Therefore H-torsors are parameterized by a and u and we have $\mathrm{ed}(H) \le 2$.

Thus an element of $H^1(k,\widetilde{G})$ is given by a Galois H-algebra

$$L = k[z,x,y]/(z^2 - a, x^4 - u - s, y^4 - u + s)$$

and a quadratic form of rank 3 and determinant 1 over $K = k[t] \subset L$ with $t = (xy)^2$ and $t^2 = u^2 - a$. Thus $\mathrm{ed}(\widetilde{G}) \le \mathrm{ed}(H) + 2 \cdot 2$. □

Corollary 10.7. $\mathrm{ed}(\mathrm{Spin}(q)) \le 6$

Proof. This is clear from Proposition 10.4 and Proposition 10.6. □

11. The essential dimension of split $\mathrm{Spin}(n)$ for $n \leq 14$

Let Spin_n denote a split form of $\mathrm{Spin}(n)$.

Theorem.

$$\begin{aligned}
\mathrm{ed}(\mathrm{Spin}_n) &= 0 \quad \textit{for } n \leq 6,\\
\mathrm{ed}(\mathrm{Spin}_7) &= 4,\\
\mathrm{ed}(\mathrm{Spin}_8) &= 5,\\
\mathrm{ed}(\mathrm{Spin}_9) &= 5,\\
\mathrm{ed}(\mathrm{Spin}_{10}) &= 4,\\
\mathrm{ed}(\mathrm{Spin}_{11}) &= 5,\\
\mathrm{ed}(\mathrm{Spin}_{12}) &= 6,\\
\mathrm{ed}(\mathrm{Spin}_{13}) &= 6,\\
\mathrm{ed}(\mathrm{Spin}_{14}) &= 7.
\end{aligned}$$

Proof. (Sketch) The cases $n = 12$, 14 have been just considered. It is not difficult to extend our considerations to the case $n = 11$.

As for $n = 13$: By corollary 10.7 one has $\mathrm{ed}(\mathrm{Spin}_{13}) \leq 6$. The invariant h_6 restricted to Spin_{13} is nontrivial, for example for

$$q \perp \langle 1 \rangle = b_1\big(\langle\langle a_1, a_2, a_3 \rangle\rangle' - \langle\langle b_1, b_2, b_3 \rangle\rangle'\big)$$

Hence $\mathrm{ed}(\mathrm{Spin}_{13}) \geq 6$.

For $n = 7$, 10 one uses that any Spin_n-torsor admits a reduction to $G_2 \times \mu_2$ resp. to $G_2 \times \mu_4$. For $n = 8$, 9 one may use the fact that

$$\mathrm{Spin}_8 \to \mathrm{Spin}_9 \to F_4$$

induce surjections on H^1 at the prime 2 and Serre's $H^5(\mathbf{Z}/2)$-invariant for F_4, cf. [27, III. Annexe, § 3.4] or [28, III. Appendix 2, 3.4] and [14, § 40], [23]. For $n \leq 6$ note that any n-dimensional quadratic form with trivial e_1-, e_2-invariants is split. □

References

[1] J. K. Arason, *Cohomologische Invarianten quadratischer Formen*, J. Algebra **36** (1975), no. 3, 448–491.

[2] ———, *A proof of Merkurjev's theorem*, Quadratic and Hermitian forms (Hamilton, Ont., 1983) (C. R. Riehm and I. Hambleton, eds.), CMS Conf. Proc., vol. 4, Amer. Math. Soc., Providence, R.I., 1984, pp. 121–130.

[3] S. Eilenberg and S. Mac Lane, *On the groups $H(\pi, n)$. II. Methods of computation*, Ann. of Math. (2) **60** (1954), 49–139.

[4] ———, *On the groups $H(\pi, n)$. III*, Ann. of Math. (2) **60** (1954), 513–557.

[5] R. Elman and T. Y. Lam, *Pfister forms and K-theory of fields*, J. Algebra **23** (1972), 181–213.

[6] L. Evens, *A generalization of the transfer map in the cohomology of groups*, Trans. Amer. Math. Soc. **108** (1963), 54–65.

[7] V. Gatti and E. Viniberghi, *Spinors of 13-dimensional space*, Adv. in Math. **30** (1978), no. 2, 137–155.

[8] D. W. Hoffmann and J.-P. Tignol, *On 14-dimensional quadratic forms in I^3, 8-dimensional forms in I^2, and the common value property*, Doc. Math. **3** (1998), 189–214 (electronic).

[9] J. Igusa, *On a certain class of prehomogeneous vector spaces*, J. Pure Appl. Algebra **47** (1987), no. 3, 265–282.

[10] O. Izhboldin and N. Karpenko, *Some new examples in the theory of quadratic forms*, Preprint, 1997, http://wwwmath.uni-muenster.de/math/u/karpenko/Publications/.

[11] B. Jacob and M. Rost, *Degree four cohomological invariants for quadratic forms*, Invent. Math. **96** (1989), no. 3, 551–570.

[12] B. Kahn, *Classes de Stiefel-Whitney de formes quadratiques et de représentations galoisiennes réelles*, Invent. Math. **78** (1984), no. 2, 223–256.

[13] M. Karoubi, *K-theory*, Springer-Verlag, Berlin, 1978, An introduction, Grundlehren der Mathematischen Wissenschaften, Band 226.

[14] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.

[15] T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1980, Revised second printing, Mathematics Lecture Note Series.

[16] A. S. Merkurjev, *On the norm residue symbol of degree* 2, Dokl. Akad. Nauk SSSR **261** (1981), no. 3, 542–547 (Russian), [Soviet Math. Dokl. **24** (1981), no. 3, (1982), 546–551].

[17] A. S. Merkurjev and A. A. Suslin, *Norm residue homomorphism of degree three*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 2, 339–356 (Russian), [Math. USSR Izv. **36** (1991), no. 2, 349–367], also: LOMI-preprint (1986).

[18] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/1970), 318–344.

[19] A. Pfister, *Quadratische Formen in beliebigen Körpern*, Invent. Math. **1** (1966), 116–132.

[20] ———, *Quadratic forms with applications to algebraic geometry and topology*, Cambridge University Press, Cambridge, 1995.

[21] V. L. Popov, *A classification of spinors of dimension fourteen*, Trudy Moskov. Mat. Obshch. **37** (1978), 173–217, 270, [Trans. Mosc. Math. Soc. (1980), Issue 1, 181–232].

[22] Z. Reichstein, *On the notion of essential dimension for algebraic groups*, preprint, 1998, http://ucs.orst.edu/~reichstz/pub.html.

[23] M. Rost, *A descent property for Pfister forms*, preprint, 1998, http://www.physik.uni-regensburg.de/~rom03516/descent.html, submitted to Journal of the Ramanujan Mathematical Society.

[24] ———, *On* 14*-dimensional quadratic forms, their spinors, and the difference of two octonion algebras*, notes, 1994, http://www.physik.uni-regensburg.de/~rom03516/14-dim-abstract.html.

[25] ———, *On Hilbert Satz 90 for K_3 for quadratic extensions*, preprint, 1988, http://www.physik.uni-regensburg.de/~rom03516/K3-88.html.

[26] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der mathematischen Wissenschaften, vol. 270, Springer-Verlag, Berlin, 1985.

[27] J.-P. Serre, *Cohomologie galoisienne*, fifth ed., Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin, 1994.

[28] ———, *Galois cohomology*, Springer-Verlag, Berlin, 1997, Translated from the French by Patrick Ion and revised by the author.

[29] J. M. Shapiro, *A Riemann-Roch type theorem for the Witt and Milnor rings of a field*, J. Pure Appl. Algebra **15** (1979), no. 3, 293–304.

[30] V. Voevodsky, *The Milnor conjecture*, preprint, 1996, Max-Planck-Institute for Mathematics, Bonn, http://www.math.uiuc.edu/K-theory/0170.

[31] A. R. Wadsworth, *Merkurjev's elementary proof of Merkurjev's theorem*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983) (S. J. Bloch, R. K. Dennis, E. M. Friedlander, and M. R. Stein, eds.), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, R.I., 1986, pp. 741–776.

[32] J. H. C. Whitehead, *A certain exact sequence*, Ann. of Math. (2) **52** (1950), 51–110.

[33] ———, *On the theory of obstructions*, Ann. of Math. (2) **54** (1951), 68–84.

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