On The Tensor Product of Two Composition Algebras

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1 Introduction

Let $C_1 \otimes_F C_2$ be the tensor product of two composition algebras over a field F with $\mathrm{char}(F) \neq 2$. R. Brauer [7] and A. A. Albert [1], [2], [3] seemed to be the first mathematicians who investigated the tensor product of two quaternion algebras. Later their results were generalized to this more general situation by B. N. Allison [4], [5], [6] and to biquaternion algebras over rings by Knus [12].

In the second section we give some new results on the Albert form of these algebras. We also investigate the F-quadric defined by this Albert form, generalizing a result of Knus ([13]).

Since Allison regarded the involution $\sigma = \gamma_1 \otimes \gamma_2$ as an essential part of the algebra $C = C_1 \otimes C_2$, he only studied automorphisms of C which are compatible with σ. In the last section we show that any automorphism of C that preserves a certain biquaternion subalgebra also is compatible with σ. As a consequence, if C is the tensor product of two octonion algebras, we show that C does not satisfy the Skolem-Noether Theorem.

Let F be a field and C a unital, nonassociative F-algebra. Then C is a *composition algebra* if there exists a nondegenerate quadratic form $n : C \to F$ such that $n(x \cdot y) = n(x)n(y)$ for all $x, y \in C$. The form n is uniquely determined by these conditions and is called the *norm* of C. We will write $n = n_C$. Composition algebras only exist in ranks 1, 2, 4 or 8 (see [10]). Those of rank 4 are called *quaternion algebras*, and those of rank 8 *octonion algebras*. A composition algebra C has a *canonical involution* γ given by $\gamma(x) = t(x)1_C - x$, where the *trace* map $t : C \to F$ is given by $t(x) = n(1, x)$.

An example of an 8-dimensional composition algebra is Zorn's algebra of vector matrices $\mathrm{Zor}(F)$ (see [14, p. 507] for the definition). The norm form of $\mathrm{Zor}(F)$ is given by the determinant and is a hyperbolic form.

Composition algebras are *quadratic*; i.e., they satisfy the identities

$$\begin{aligned} x^2 - t(x)x + n(x)1_C &= 0 \text{ for all } x \in C, \\ n(1_C) &= 1, \end{aligned}$$

and are *alternative* algebras; i.e., $xy^2 = (xy)y$ and $x^2y = x(xy)$ for all $x, y \in C$. In particular, $n(x) = \gamma(x)x = x\gamma(x)$ and $t(x)1_C = \gamma(x) + x$.

For any composition algebra D over F with $\dim_F(D) \leq 4$, and any $\mu \in F^\times$, the F-vector space $D \oplus D$ becomes a composition algebra via the multiplication

$$(u,v)(u',v') = (uu' + \mu\gamma(v')v, v'u + v\gamma(u'))$$

for all $u, v, u', v' \in D$, with norm

$$n((u,v)) = n_D(u) - \mu n_D(v).$$

This algebra is denoted by $\mathrm{Cay}(D,\mu)$. Note that the embedding of D into the first summand of $\mathrm{Cay}(D,\mu)$ is an algebra monomorphism. The norm form of $\mathrm{Cay}(D,\mu)$ is obviously isometric to $\langle 1, -\mu\rangle \otimes n_D$. Since two composition algebras are isomorphic if and only if their norm forms are isometric, we see that if C is a composition algebra whose norm form satisfies $n_C \cong \langle 1, -\mu\rangle \otimes n_D$ for some D, then $C \cong \mathrm{Cay}(D,\mu)$. In particular, $\mathrm{Zor}(F) \cong \mathrm{Cay}(D,1)$ for any quaternion algebra D since $\langle 1, -1\rangle \otimes n_D$ is hyperbolic. A composition algebra is *split* if it contains an isomorphic copy of $F \oplus F$ as a composition subalgebra, which is the case if and only if it contains zero divisors.

2 Albert Forms

From now on we consider only fields F with $\mathrm{char}(F) \neq 2$. It is well known that any norm of a composition algebra is a Pfister form, and conversely, any Pfister form is the norm of some composition algebra.

Let C be a composition algebra. Define $C' = \langle F1\rangle^\perp = \{x \in C : t_1(x) = n_1(x,1) = 0\}$. Then $n' = n|_{C'}$ is the *pure norm* of C. Note that

$$\begin{aligned} C' &= \{x \in C : x = 0 \text{ or } x \notin F1_C \text{ and } x^2 \in F1_C\} \\ &= \{x \in C : \gamma(x) = -x\}. \end{aligned}$$

Moreover, C is split if and only if its norm n is hyperbolic, two composition algebras are isomorphic if and only if their norms are isometric, and C is a division algebra if and only if n is anisotropic.

We now investigate tensor products of two composition algebras. Following Albert, we associate to the tensor product $C = C_1 \otimes C_2$ of two composition algebras with $\dim(C_i) = r_i$ and $n_{C_i} = n_i$ the $(r_1 + r_2 - 2)$-dimensional form $n_1' \perp \langle -1\rangle n_2'$ of determinant -1. This definition, for C_1 or C_2 an octonion algebra, was first given by Allison in [5]. In the Witt ring $W(F)$, obviously this form is equivalent to $n_1 - n_2$. Like the norm form of a composition algebra, this *Albert form* contains crucial information about the tensor product algebra C. For biquaternion algebras, this is well-known ([1, Thm. 3], [11, Thm. 3.12]). We introduce some notation and terminology. If q is a quadratic form and if $\mathbb{H} = \langle 1, -1\rangle$ is the hyperbolic plane, then $q = q_0 \perp i\mathbb{H}$ for some anisotropic form q_0 and integer i. The integer i is called the *Witt index* of q and is denoted $i_W(q)$. In the proof of the following proposition, we use the notion of linkage of Pfister forms (see [8, Sec. 4]). Recall that two n-fold Pfister forms q_1 and q_2 are *r-linked* if there is an r-fold Pfister form h with $q_i = h \otimes q_i'$ for some Pfister forms q_i'. Finally, we call a two-dimensional commutative F-algebra that is separable over F a *quadratic étale algebra.* Note that any quadratic étale algebra is either a quadratic field extension of F or is isomorphic to $F \oplus F$. Part of the following result has been proved in [9, Thm. 5.1].

Proposition 2.1. *Let C_1 and C_2 be octonion algebras over F with norms n_1 and n_2, and let $i = i_W(N)$ be the Witt index of the Albert-form $N = n_1' \perp \langle -1 \rangle n_2'$.*

(i) $i = 0 \iff$ *C_1 and C_2 do not contain isomorphic quadratic étale subalgebras.*

(ii) $i = 1 \iff$ *C_1 and C_2 contain isomorphic quadratic étale subalgebras, but no isomorphic quaternion subalgebras.*

(iii) $i = 3 \iff$ *C_1 and C_2 contain isomorphic quaternion subalgebras, but C_1 and C_2 are not isomorphic.*

(iv) $i = 7 \iff$ $C_1 \cong C_2$.

Proof. By [8, Props. 4.4, 4.5], the Witt index of $n_1 \perp \langle -1 \rangle\, n_2$ is either 0 or 2^r, where r is the linkage number of $n_1 \perp \langle -1 \rangle n_2$. Note that the Witt index of N is one less than the Witt index of $n_1 \perp \langle -1 \rangle n_2$ since $n_1 \perp \langle -1 \rangle n_2 = \mathbb{H} \perp N$. If $C_1 \cong C_2$, then $n_1 \cong n_2$, so $i = 7$. Conversely, if $i = 7$, then $n_1 \perp \langle -1 \rangle n_2$ is hyperbolic, so $n_1 \cong n_2$, which forces $C_1 \cong C_2$. If C_1 and C_2 are not isomorphic but contain a common quaternion algebra Q, then $C_i = \mathrm{Cay}(Q, \mu_i)$ for some i. Therefore, $n_1 = n_Q \otimes \langle 1, -\mu_1 \rangle$ and $n_2 = n_Q \otimes \langle 1, -\mu_2 \rangle$. These descriptions show that n_1 and n_2 are 2-linked, so $i = 3$. Conversely, if $i = 3$, then n_1 and n_2 are 2-linked but not isometric. If $\langle\langle a, b \rangle\rangle$ is a factor of both n_1 and n_2, then $n_1 = \langle\langle a, b, c \rangle\rangle$ and $n_2 = \langle\langle a, b, d \rangle\rangle$ for some $c, d \in F^\times$. If $Q = (-a, -b)$, we get $C_1 = \mathrm{Cay}(Q, -c)$ and $\mathrm{Cay}(Q, -d)$, so C_1 and C_2 contain a common quaternion algebra. If C_1 and C_2 contain a common quadratic étale algebra $F[t]/(t^2 - a)$ but no common quaternion algebra, then $\langle 1, -a \rangle$ is a factor of n_1 and n_2, which means they are 1-linked. If n_1 and n_2 are 2-linked, then the previous step shows that C_1 and C_2 have a common quaternion subalgebra, which is false. Conversely, if n_1 and n_2 are 1-linked but not 2-linked, then C_1 and C_2 do not have a common quaternion subalgebra, and if $\langle\langle a \rangle\rangle$ is a common factor to n_1 and n_2, then C_1 and C_2 both contain the étale algebra $F[t]/(t^2 - a)$. □

Proposition 2.2. *Let C_1 be an octonion algebra over F and C_2 be a quaternion algebra over F, with norms n_1 and n_2. Again consider the Witt index i of the Albert form $N = n_1' \perp \langle -1 \rangle n_2'$.*

(i) $i = 0 \iff$ *C_1 and C_2 do not contain isomorphic quadratic étale subalgebras.*

(ii) $i = 1 \iff$ *C_1 and C_2 contain isomorphic quadratic étale subalgebras, but C_2 is not a quaternion subalgebra of C_1.*

(iii) $i = 3 \iff$ *$C_1 \cong \mathrm{Cay}(C_2, \mu)$ for a suitable $\mu \in F^\times$, and C_2 is a division algebra.*

(iv) $i = 5 \iff$ *$C_1 \cong \mathrm{Zor}(F)$ and $C_2 \cong M_2(F)$.*

Proof. In the case that both algebras C_1 and C_2 are division algebras, this is an immediate consequence of [9, Lemma 3.2]. If both C_1 and C_2 is split, then clearly N has Witt index 5. If C_2 is a division algebra and $C_1 = \mathrm{Cay}(C_2, \mu)$ for some μ, then n_2 is anisotropic and $N \perp \mathbb{H} = n_2 \otimes \langle 1, -\mu\rangle \perp \langle -1\rangle\, n_2 = 4\mathbb{H} \perp \langle -\mu\rangle\, n_2$, so N has Witt index 3. Note that the converse is easy, since if $i = 3$, then n_2 is isomorphic to a subform of n_1, which forces n_2 to be a factor of n_1. If $n_1 = \langle 1, a\rangle \otimes n_2$, then $C_1 \cong \mathrm{Cay}(C_2, -a)$, so C_2 is a subalgebra of C_1. If C_1 and C_2 contain a common quadratic étale algebra $F[t]/(t^2 - a)$ but C_2 is not a quaternion subalgebra of C_1, then n_1 and n_2 have $\langle\langle a\rangle\rangle$ as a common factor, so $i = 1$. Finally, if N is isotropic, there are $x_i \in C_i$, both skew, with $n_1(x_1) = n_2(x_2)$. Then, as $t_1(x_1) = 0 = t_2(x_2)$, the algebras $F[x_1]$ and $F[x_2]$ are isomorphic, so C_1 and C_2 share a common quadratic étale subalgebra. This finishes the proof. ☐

If C is a biquaternion algebra; i.e., $C \cong C_1 \otimes C_2$ for two quaternion algebras C_1 and C_2, then the Albert form $n_1' \perp \langle -1\rangle n_2'$ is determined up to similarity by the isomorphism class of the algebra C ([11, Thm. 3.12]). Allison generalizes this result ([5, Thm. 5.4]) to tensor products of arbitrary composition algebras. However, he always considers the involution $\sigma = \gamma_1 \otimes \gamma_2$ as a crucial part of the algebra $C = C_1 \otimes C_2$. Allison proves that $(C_1 \otimes C_2, \gamma_1 \otimes \gamma_2)$ and $(C_3 \otimes C_4, \gamma_3 \otimes \gamma_4)$ are isotopic algebras if and only if they have similar Albert forms, for the cases that C_1, C_3 are octonion and C_2, C_4 quaternion or octonion algebras.

The fact that any F-algebra isomorphism $\varphi : (C_1 \otimes C_2, \gamma_1 \otimes \gamma_2) \to (C_3 \otimes C_4, \gamma_3 \otimes \gamma_4)$ between arbitrary products of composition algebras yields an isometry $n_1' \perp \langle -1\rangle n_2' \cong \mu(n_3' \perp \langle -1\rangle n_4')$ for a suitable $\mu \in F^\times$ is easy to see. Also, since for $C = C_1 \otimes C_2$ the map $\langle\ ,\ \rangle : C \times C \to F$ given by $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = n_1(x_1, y_1) \otimes n_2(x_2, y_2)$ is a nondegenerate symmetric bilinear form on C such that $\langle \sigma(x), \sigma(y)\rangle = \langle x, y\rangle$, the equation $\langle zx, y\rangle = \langle x, \sigma(z)y\rangle$ holds (that is, an *invariant form* cf. [6, p. 144]), and $\tau : C \times C \to k$, $\tau(x, y) = \langle x, \sigma(y)\rangle$ an associative nondegenerate symmetric bilinear form which is proper, it follows easily that $n_1 \otimes n_2 \cong n_3 \otimes n_4$.

Suppose that we have two algebras that each are a tensor product of an octonion algebra and a quaternion algebra. We obtain a necessary and sufficient condition for when their Albert forms are similar. We use the notation $D(q)$ to denote the elements of $F^\times$ represented by a quadratic form q.

Proposition 2.3. *Let C_1, C_2 be octonion algebras and Q_1, Q_2 quaternion algebras over F. Let N_1 and N_2 be the Albert forms of $C_1 \otimes Q_1$ and $C_2 \otimes Q_2$, respectively. If $N_1 \cong \mu N_2$ for some $\mu \in F^\times$, then $Q_1 \cong Q_2$. Moreover, there is a quaternion algebra Q and elements $c, d \in F^\times$*

such that $C_1 \cong \mathrm{Cay}(Q,c)$*,* $C_2 \cong \mathrm{Cay}(Q,d)$*,* $\mathrm{Cay}(Q_1,\mu) \cong \mathrm{Cay}(Q,cd)$*, and* $-\mu c \in D(n_{C_2})$*. Conversely, if there is a quaternion algebra* Q *and elements* $c, d, \mu \in F^\times$ *such that* C_1*,* $Q_1 = Q_2$*, and* C_2 *satisfy the conditions of the previous sentence, then* $N_1 \cong \mu N_2$*.*

Proof. Suppose that $N_1 \cong \mu N_2$ for some $\mu \in F^\times$. If $c : W(F) \to \mathrm{Br}(F)$ is the Clifford invariant, then $c(N_1) = c(\mu N_2) = c(N_2)$. Since c is trivial on $I^3(F)$, we have $c(N_1) = c(-n_{Q_1})$ and $c(N_2) = c(-n_{Q_2})$ (see [15, Ch. 5.3]). Therefore, $c(n_{Q_1}) = c(n_{Q_2})$. However, the Clifford invariant of the norm form of a quaternion algebra is the class of the quaternion algebra, by [15, Cor. V.3.3]. This implies that $c(N_1) = [Q_1]$ and $c(\mu N_2) = [Q_2]$. Since $[Q_1] = [Q_2]$, we get $Q_1 \cong Q_2$. As a consequence of this, $n_{Q_1} \cong n_{Q_2}$. Thus,

$$\begin{aligned} n_{C_1} \perp -\langle 1, -\mu\rangle \otimes n_{Q_1} &\cong n_{C_1} \perp (-n_{Q_1} \perp \mu n_{Q_1}) \\ &\cong \mu\,(n_{C_2} \perp -n_{Q_2}) \perp \mu n_{Q_1} \cong \mu n_{C_1} \perp (-\mu n_{Q_1} \perp \mu n_{Q_2}) \\ &\cong \mu n_{C_2} \perp 4\mathbb{H}. \end{aligned}$$

The forms n_{C_1} and $\langle 1, -\mu\rangle \otimes n_{Q_1}$ are Pfister forms. The line above shows that these Pfister forms are 2-linked, in the terminology of [8]. Therefore, there is a 2-fold Pfister form $\langle\langle -a, -b\rangle\rangle$ with $n_{C_1} \cong \langle\langle -a, -b, -c\rangle\rangle$ and $\langle 1, -\mu\rangle \otimes n_{Q_1} \cong \langle\langle -a, -b, -e\rangle\rangle$ for some $c, e \in F^\times$. An elementary calculation shows that

$$\langle\langle -a, -b, -c\rangle\rangle \perp -\langle\langle -a, -b, -e\rangle\rangle \cong 4\mathbb{H} \perp \langle -c, e\rangle \otimes \langle\langle -a, -b\rangle\rangle\,.$$

Therefore, $\mu n_{C_2} \cong \langle -c, e\rangle \otimes \langle\langle -a, -b\rangle\rangle$. Thus, $n_{C_2} \cong -\mu c\,\langle\langle -a, -b, -ce\rangle\rangle$. Since n_{C_2} and $\langle\langle -a, -b, -ce\rangle\rangle$ are Pfister forms, we get $n_{C_2} \cong \langle\langle -a, -b, -ce\rangle\rangle$. If we set $d = ce$ and let Q be the quaternion algebra $(a, b)_F$, then the isomorphisms $n_{C_1} \cong \langle\langle -a, -b, -c\rangle\rangle$ and $n_{C_2} \cong \langle\langle -a, -b, -d\rangle\rangle$ give $C_1 \cong \mathrm{Cay}(Q, c)$ and $C_2 \cong \mathrm{Cay}(Q, d)$. Moreover, $n_{C_2} \cong -\mu c n_{C_2}$, so$-\mu c \in D(n_{C_2})$. Finally, the isomorphism $\langle 1, -\mu\rangle \otimes n_{Q_2} \cong \langle\langle -a, -b, -e\rangle\rangle$ gives $\mathrm{Cay}(Q_1, \mu) \cong \mathrm{Cay}(Q, e) \cong \mathrm{Cay}(Q, cd)$.

It is a short calculation to show that if $C_1 = \mathrm{Cay}(Q, c)$, $C_2 = \mathrm{Cay}(Q, d)$, and $Q_1 = Q_2$ is a quaternion algebra with $\mathrm{Cay}(Q_1, \mu) \cong \mathrm{Cay}(Q, -dc)$, then $n'_{C_1} \perp \langle -1\rangle\, n'_{Q_1} \cong \mu(n'_{C_2} \perp \langle -1\rangle\, n'_{Q_2})$. □

The argument of the previous proposition does not work for a tensor product of two octonion algebras since the Albert form is an element of $I^3(F)$, whose Clifford invariant is trivial.

Corollary 2.4. *With the notation in the previous proposition, suppose that* $N_1 \cong \mu N_2$ *for some* $\mu \in F^\times$*. If one of* C_1 *and* C_2 *is split, then the other algebra is isomorphic to* $\mathrm{Cay}(Q_1, \mu)$*.*

Proof. We saw in the proof of the previous proposition that

$$n_{C_1} \perp -\langle 1, -\mu\rangle \otimes n_{Q_1} \cong \mu n_{C_2} \perp 4\mathbb{H}.$$

Suppose that C_2 is split. Then $n_{C_1} \perp -\langle 1, -\mu\rangle \otimes n_{Q_1}$ is hyperbolic, so $n_{C_1} \cong \langle 1, -\mu\rangle \otimes n_{Q_2}$. Therefore, $C_1 \cong \mathrm{Cay}(Q_1, \mu)$. On the other hand, if C_1 is split, then $n_{C_1} \cong 4\mathbb{H}$, so by Witt cancellation, $-\mu n_{C_2} \cong \langle 1, -\mu\rangle \otimes n_{Q_1}$. Since n_{C_2} and $\langle 1, -\mu\rangle \otimes n_{Q_1}$ are both Pfister forms, this implies that $n_{C_2} \cong \langle 1, -\mu\rangle \otimes n_{Q_1}$, and so $C_2 \cong \mathrm{Cay}(Q_1, \mu)$. □

In Proposition 2.3 above, it is possible for $N_1 \cong \mu N_2$ without $C_1 \cong C_2$. Moreover, the quaternion algebra Q of the proposition need not be isomorphic to Q_1. We verify both these claims in the following example.

Example 2.5. In this example we produce nonisomorphic octonion algebras C_1 and C_2 and a quaternion algebra Q_1 that is not isomorphic to a subalgebra of either C_1 or C_2, and such that the Albert forms of $C_1 \otimes_F Q_1$ and $C_2 \otimes_F Q_1$ are similar. To do this we produce nonisometric Pfister forms $\langle\langle x, y, z\rangle\rangle$ and $\langle\langle x, y, w\rangle\rangle$ and elements u, v, μ with $\langle\langle x, y, zw\rangle\rangle \cong \langle\langle u, v, \mu\rangle\rangle$ such that the Witt index of $\langle\langle x, y, z\rangle\rangle \perp -\langle\langle u, v, \mu\rangle\rangle$ and $\langle\langle x, y, w\rangle\rangle \perp -\langle\langle u, v, \mu\rangle\rangle$ are both 2, and $\mu z \in D(\langle\langle x, y, w\rangle\rangle)$. We then set $Q = (-x, -y)$, $C_1 = \mathrm{Cay}(Q, -z)$, $C_2 = \mathrm{Cay}(Q, -w)$, and $Q_1 = (-u, -v)$. From Proposition 2.3, we have $N_1 \cong \mu N_2$. However, Proposition 2.2 shows that Q_1 is not isomorphic to a subalgebra of either C_1 or C_2. Moreover, C_1 and C_2 are not isomorphic since their norm forms are not isometric. Note that Q and C_2 are not isomorphic since Q_1 is not a subalgebra of C_1.

Let k be a field of characteristic not 2, and let $F = k(x, y, z, w)$ be the rational function field in 4 variables over k. Set $\mu = xyzw$, $n_1 = \langle\langle x, y, z\rangle\rangle$, and $n_2 = \langle\langle x, y, w\rangle\rangle$. By embedding F in the Laurent series field $k(x, y, z)((w))$, we see that n_1 and $n_2 = \langle\langle x, y\rangle\rangle \perp w\langle\langle x, y\rangle\rangle$ are not isomorphic over this field by Springer's theorem [15, Prop. VI.1.9], so n_1 and n_2 are not isomorphic over F. Also, $\mu z = z^2(xyw)$, which is clearly represented by n_3. Set $Q_1 = (-zw, -xzw)$. A short calculation shows that $\langle\langle x, y, zw\rangle\rangle = \langle\langle zw, xzw, \mu\rangle\rangle$. Finally, for the Witt indices, we have

$$\begin{aligned} n_1 \perp -n_{Q_1} &= \langle 1, x, y, xy, z, xz, yz, xyz\rangle \perp -\langle 1, zw, xzw, x\rangle \\ &= 2\mathbb{H} \perp \langle y, xy, z, xz, yz, xyz, -zw, -xzw\rangle \\ &= 2\mathbb{H} \perp \langle y, xy, z, xz, yz, xyz\rangle \perp w\langle -z, -xz\rangle . \end{aligned}$$

The Springer theorem shows that this form has Witt index 2. Similarly,

$$\begin{aligned} n_3 \perp -n_{Q_1} &= \langle 1, x, y, xy, w, xw, yw, xyw\rangle \perp -\langle 1, zw, xzw, x\rangle \\ &= 2\mathbb{H} \perp \langle y, xy, w, xw, yw, xyw, -zw, -xzw\rangle \\ &= 2\mathbb{H} \perp \langle y, xy\rangle \perp w\langle 1, x, y, xy, -z, -xz\rangle \end{aligned}$$

has Witt index 2.

For the remainder of this section we will also consider the case that $\mathrm{char}(F) = 2$. Let C_1 and C_2 be composition algebras over F of $\dim_F(C_i) = r_i \geq 2$, and let n_i be the norm form of C_i. Using the notation of [13], the subspace $Q(C_1, C_2) = \{u = x_1 \otimes 1 - 1 \otimes x_2 : t_1(x_1) = t_2(x_2)\}$

has dimension $r_1 + r_2 - 2$, and $Q(C_1, C_2) = \{z - (\gamma_1 \otimes \gamma_2)(z) : z \in C_1 \otimes C_2\}$ is the set of alternating elements of $C_1 \otimes C_2$ with respect to $\gamma_1 \otimes \gamma_2$. The nondegenerate quadratic form $N : Q(C_1, C_2) \to F$ given by $N(x_1 \otimes 1 - 1 \otimes x_2) = n_1(x_1) - n_2(x_2)$ is isometric to the Albert form $n_1' \perp \langle -1 \rangle n_2'$ of $C_1 \otimes C_2$.

Let $V_N \subset \mathbb{P}^{r_1+r_2-3}$ be the F-quadric defined via N. In the case that $\mathrm{char}(F) \neq 2$, V_N coincides with the open subvariety U_N of closed points $x_1 \otimes 1 - 1 \otimes x_2$ with $x_1 \notin F1$ and $x_2 \notin F1$. We now generalize [13, Prop.] in the following two propositions. We will make use of the following fact that comes from Galois theory: Let $F[z_i]$ be the commutative F-subalgebra of dimension two of C_i generated by $z_i \in C_i$ for $i = 1, 2$. Then there exists an isomorphism $\alpha : F[z_1] \xrightarrow{\sim} F[z_2]$ such that $\alpha(z_1) = z_2$ if and only if $n_1(z_1) = n_2(z_2)$ and $t_1(z_1) = t_2(z_2)$.

Proposition 2.6. *There exists a bijection Φ between the set of F-rational points of U_N and the set of triples (K_1, K_2, α), where K_i is a two-dimensional commutative subalgebra of C_i, and where $\alpha : K_1 \xrightarrow{\sim} K_2$ is an F-algebra isomorphism:*

$$\Phi : \{P \in U_N \mid P \text{ an } F\text{-rational point}\} \xrightarrow{\sim} \{(K_1, K_2, \alpha) \mid K_1, K_2, \alpha \text{ as above}\}$$
$$P = z_1 \otimes 1 - 1 \otimes z_2 \longmapsto \begin{pmatrix} F[z_1], F[z_2], \alpha : F[z_1] \xrightarrow{\sim} F[z_2] \\ z_1 \mapsto z_2 \end{pmatrix}$$

Proof. Any F-rational point $P \in U_N$ corresponds with an element $x_1 \otimes 1 - 1 \otimes x_2 \in Q(C_1, C_2)$ with $t_1(x_1) = t_2(x_2)$ and $n_1(x_1) = n_2(x_2)$. Then there exists an F-algebra isomorphism $\alpha : F[x_1] \xrightarrow{\sim} F[x_2]$ with $x_1 \mapsto x_2$. For $x_1 \otimes 1 - 1 \otimes x_2 = z_1 \otimes 1 - 1 \otimes z_2$ it can be easily verified that

$$\begin{pmatrix} F[x_1], F[x_2], \alpha : F[x_1] \xrightarrow{\sim} F[x_2] \\ x_1 \mapsto x_2 \end{pmatrix} = \begin{pmatrix} F[z_1], F[z_2], \beta : F[z_1] \xrightarrow{\sim} F[z_2] \\ z_1 \mapsto z_2 \end{pmatrix}$$

Therefore, the mapping Φ is well defined.

Given a triple (K_1, K_2, α), there are elements $z_i \in C_i'$ such that $K_i = F[z_i]$ and $\alpha : F[z_1] \xrightarrow{\sim} F[z_2]$ with $z_1 \mapsto z_2$. By the remark before the proposition, we have $n_1(z_1) = n_2(z_2)$ and $t_1(z_1) = t_2(z_2)$; thus $N(z_1 \otimes 1 - 1 \otimes z_2) = 0$, and the triple defines the F-rational point $P \in U_N$ corresponding to $z_1 \otimes 1 - 1 \otimes z_2$. So Φ is surjective.

To prove injectivity, suppose that $\Phi(x_1 \otimes 1 - 1 \otimes x_2) = \Phi(z_1 \otimes 1 - 1 \otimes z_2)$. Then $F[x_1] = F[z_1]$, $F[x_2] = F[z_2]$ and the maps $\alpha : F[x_1] \xrightarrow{\sim} F[x_2]$, $x_1 \mapsto x_2$ and $\beta : F[z_1] \xrightarrow{\sim} F[z_2]$, $z_1 \mapsto z_2$ are equal. Since $F[x_i] = F[z_i]$, write $x_1 = a + bz_1$ and $x_2 = c + dz_2$ with $a, b, c, d \in F$. We have $a = c$ since $t_1(x_1) = t_2(x_2)$. Therefore, we may replace x_1 with bz_1 and x_2 by dz_2 without changing $x_1 \otimes 1 - 1 \otimes x_2$. Thus, $x_1 \otimes 1 - 1 \otimes x_2 = bz_1 \otimes 1 - 1 \otimes dz_2$, and $n_1(x_1) = n_2(x_2)$, $n_1(z_1) = n_2(z_2)$ imply that $n_1(x_1) = b^2 n_1(z_1)$ and $n_2(x_2) = d^2 n_2(z_2)$. Therefore, $b^2 = d^2$, so $b = \pm d$. Now $x_2 = \alpha(x_1) = \alpha(bz_1) = bz_2$ yields $b = d$ and we get $x_1 \otimes 1 - 1 \otimes x_2 = b(z_1 \otimes 1 - 1 \otimes z_2)$ which shows that Φ is injective. ∎

In the case that $\mathrm{char}(F) = 2$, the set $U_N = \{x_1 \otimes 1 - 1 \otimes x_2 : x_1 \notin F1,\ x_2 \notin F1\}$ is a proper open subvariety of V_N. The proof of the previous proposition shows that Φ again is a bijection between the F-rational points of U_N and the triples (K_1, K_2, α), where K_i is a two-dimensional commutative F-subalgebra of C_i and $\alpha : K \xrightarrow{\sim} L$ an F-algebra isomorphism. We can say more in this situation.

Proposition 2.7. *Let* $\mathrm{char}(F) = 2$. *There exists an F-rational point in V_N if and only if there exists a triple (K_1, K_2, α) such that K_i is a quadratic étale subalgebra of C_i and $\alpha : K_1 \xrightarrow{\sim} K_2$ an F-algebra isomorphism. In addition, there exists an F-rational point in $V_N \cap \{t_1(x_1) = 0\}$ if and only if there exists a triple (K_1, K_2, α) such that K_1 and K_2 are purely inseparable quadratic extensions and $\alpha : K_1 \xrightarrow{\sim} K_2$ an F-algebra isomorphism.*

Proof. As pointed out before the proposition, there is a bijection between F-rational points in U_N and triples (K_1, K_2, α) with $K_i \subseteq C_i$ commutative subalgebras of dimension 2 over F. To prove the first statement, only one half needs further argument. Suppose V_N has an F-rational point. Since V_N is a quadric hypersurface, V_N is then birationally equivalent to $\mathbb{P}^{r_1+r_2-3}$. The F-rational points of projective space are dense, so there is an F-rational point in U_N. Therefore, we get a triple (K_1, K_2, α) with K_i a quadratic étale subalgebra of C_i.

For the second statement, an F-rational point in $V_N \cap \{t_1(x_1) = 0\}$ corresponds with an element $x_1 \otimes 1 - 1 \otimes x_2$ such that $n_1(x_1) = n_2(x_2)$ and $t_1(x_1) = t_2(x_2) = 0$, so $F[x_i]$ is a purely inseparable extension. There exists an isomorphism $\alpha : F[x_1] \xrightarrow{\sim} F[x_2]$ with $\alpha(x_1) = x_2$ and thus a triple $(F[x_1], F[x_2], \alpha)$. Conversely, if there is a triple (K_1, K_2, α) with K_i purely inseparable, there are $x_i \in C_i$ with $t_i(x_i) = 0$ such that $K = F[x_1]$, $L = F[x_2]$ and $\alpha : F[x_1] \xrightarrow{\sim} F[x_2]$, $x_1 \mapsto x_2$, so $n_1(x_1) = n_2(x_2)$, and $x_1 \otimes 1 - 1 \otimes x_2$ defines an F-rational point in $V_N \cap \{t_1(x_1) = 0\}$. □

3 Automorphisms

Let (Q_i, γ_i) be a quaternion algebra with its standard involution, and let $C_i = \mathrm{Cay}(Q_i, \mu_i)$ be an octonion algebra. We also write γ_i for the standard involution on C_i. Let $C = C_1 \otimes_F C_2$ and $\sigma = \gamma_1 \otimes \gamma_2$. Let $A = Q_1 \otimes_F Q_2$, an F-central associative subalgebra of C. In this section we will prove that any automorphism of C that preserves A is compatible with the involution σ. As a consequence, we show that the Skolem-Noether theorem does not hold for A.

Let

$$\begin{aligned} e &= (0,1) \otimes (1,0), \\ f &= (1,0) \otimes (0,1), \\ g &= (0,1) \otimes (0,1). \end{aligned}$$

We can decompose C as an F-vector space as

$$C = A \oplus Ae \oplus Af \oplus Ag.$$

Note that an automorphism of C is determined by its action on A and on $\{e, f, g\}$. Elementary calculations show that $Ae = eA = A(Ae) = (Ae)A$, and similar relations hold for f and g.

We point out some properties of this decomposition. We have, for $a \in A$,

$$\begin{aligned} ea &= (\gamma_1 \otimes \mathrm{id})(a)e \\ fa &= (\mathrm{id} \otimes \gamma_2)(a)f \\ ga &= \sigma(a)g. \end{aligned}$$

For example, if $a = (x, 0) \otimes (y, 0)$, then

$$\begin{aligned} ga &= (0,1) \otimes (0,1) \cdot (x,0) \otimes (y,0) = (0, \gamma_1(x)) \otimes (0, \gamma_2(y)) \\ &= (\gamma_1(x), 0) \otimes (\gamma_2(y), 0)g = \sigma(a)g. \end{aligned}$$

The other properties follow similarly.

Lemma 3.1. *For any $d, a \in A$, we have $(dg)a = \sigma(a)(dg)$. Also, $a(dg) = (da)g$.*

Proof. Let $d = \sum_i (u_i, 0) \otimes (v_i, 0)$. For $a = (x, 0) \otimes (y, 0)$, we have

$$(dg)a = \sum_i (0, u_i) \otimes (0, v_i) \cdot (x, 0) \otimes (y, 0) = \sum_i (0, u_i\gamma_1(x)) \otimes (0, v_i\gamma_2(y))$$

and

$$\sigma(a)(dg) = (\gamma_1(x), 0) \otimes (\gamma_2(x), 0) \sum_i (0, u_i) \otimes (0, v_i) = \sum_i (0, u_i\gamma_1(x)) \otimes (0, v_i\gamma_2(y)).$$

This proves the first assertion. For the second, we see that

$$a(dg) = (x, 0) \otimes (y, 0) \cdot \sum_i (0, u_i) \otimes (0, v_i) = \sum_i (0, u_i x) \otimes (0, v_i y)$$

and

$$\begin{aligned} (da)g &= \left(\sum_i (u_i, 0) \otimes (v_i, 0) \cdot (x, 0) \otimes (y, 0) \right) g \\ &= \left(\sum_i (u_i x, 0) \otimes (v_i y, 0) \right) g = \sum_i (0, u_i x) \otimes (0, v_i y). \end{aligned}$$

□

Lemma 3.2. *The right multiplication maps R_e, R_f, and R_g are injective. If $\varphi \in \mathrm{Aut}_F(C)$, then $R_{\varphi(e)}$, $R_{\varphi(f)}$, and $R_{\varphi(g)}$ are injective.*

Proof. The second statement is obvious from the first, so we prove the first. The arguments are similar, so we only prove that R_e is injective. Let $c = \sum_i (u_i, v_i) \otimes (w_i, x_i) \in C$. Then

$$\begin{aligned} ce &= \sum_i (u_i, v_i) \otimes (w_i, x_i) \cdot (0,1) \otimes (1,0) \\ &= \sum_i (\mu_1 v_i, u_i) \otimes (w_i, x_i), \end{aligned}$$

and

$$\begin{aligned} (ce)e &= \sum_i (\mu_1 v_i, u_i) \otimes (w_i, x_i) \cdot (0,1) \otimes (1,0) \\ &= \sum_i (\mu_1 u_i, \mu_1 v_i) \otimes (w_i, x_i) = \mu_1 c. \end{aligned}$$

Since $R_e \circ R_e$ is multiplication by the scalar μ_1, we see that R_e is injective. By similar calculations, we see that R_f^2 is multiplication by μ_2 and R_g^2 is multiplication by $\mu_1\mu_2$, so R_f and R_g are also injective. □

One fact we will use often is that if Q is a quaternion algebra with standard involution γ, and if $a \in Q$ satisfies $ab = \gamma(b)a$ for all $b \in Q$, then $a = 0$. This can be verified by an elementary calculation. The following lemma is a slight generalization of this fact.

Lemma 3.3. *If $x \in A$ satisfies $xa = (\gamma_1 \otimes \mathrm{id})(a)x$ for all $a \in A$, then $x = 0$. Similarly, if $xa = (\mathrm{id} \otimes \gamma_2)(a)x$ for all a, then $x = 0$.*

Proof. Suppose x satisfies the first condition above. If $a \in Q_2$, then $xa = ax$, so $x \in C_A(Q_2) = Q_1$. We then let $a \in Q_1$, so $xa = \gamma_1(a)x$. Since this is true for all $a \in Q_1$, the remark before the lemma shows that $x = 0$. A similar argument works for the second statement. □

In the proof of the following theorem, we make use of the following well known result: if $A = Q_1 \otimes_F Q_2$ is a biquaternion algebra with involution $\sigma = \gamma_1 \otimes \gamma_2$, where γ_i is the standard involution on Q_i, then any automorphism of A that is compatible with σ either preserves the Q_i or interchanges them. To aid the reader we recall one proof of this fact. If φ is such an automorphism, then φ restricts to a Lie algebra automorphism of $\mathrm{Skew}(A, \sigma)$, the space of skew-symmetric elements of A with respect to σ. This Lie algebra is the direct sum of the two simple Lie subalgebras $\mathrm{Skew}(Q_1, \gamma_1)$ and $\mathrm{Skew}(Q_2, \gamma_2)$. Since these are the unique simple Lie ideals of $\mathrm{Skew}(A, \sigma)$, the map φ must preserve them or interchange them. As $Q_i = F \oplus \mathrm{Skew}(Q_i, \gamma_i)$, this clearly shows that φ preserves the Q_i or interchanges them.

We now give the main result of this section.

Theorem 3.4. *If $\varphi \in \mathrm{Aut}_F(C)$ satisfies $\varphi(A) = A$, then $\varphi \circ \sigma = \sigma \circ \varphi$.*

Proof. We first prove that either

$$\begin{aligned}
\varphi(e) &\in Q_1 e,\\
\varphi(f) &\in Q_2 f,\\
\varphi(g) &\in Ag,
\end{aligned}$$

or

$$\begin{aligned}
\varphi(e) &\in Q_2 f,\\
\varphi(f) &\in Q_1 e,\\
\varphi(g) &\in Ag.
\end{aligned}$$

For simplicity, we write $\tau_1 = (\gamma_1 \otimes \mathrm{id})\,|_A$ and $\tau_2 = (\mathrm{id} \otimes \gamma_2)\,|_A$. We begin by writing

$$\begin{aligned}
\varphi(e) &= a_1 + b_1 e + c_1 f + d_1 g,\\
\varphi(f) &= a_2 + b_2 e + c_2 f + d_2 g,\\
\varphi(g) &= a_3 + b_3 e + c_3 f + d_3 g
\end{aligned}$$

with the coefficients in A. Let $b \in A$. Applying φ to the equations $eb = \tau_1(b)e$, $fb = \tau_2(b)f$, and $gb = \sigma(b)g$ and setting $a = \varphi(b)$ gives

$$\begin{aligned}
\varphi(e)a &= \tau_1'(a)\varphi(e),\\
\varphi(f)a &= \tau_2'(a)\varphi(f),\\
\varphi(g)a &= \sigma'(a)\varphi(g),
\end{aligned}$$

where $\sigma' = (\varphi\sigma\varphi^{-1})\,|_A$ and $\tau_i' = (\varphi\tau_i\varphi^{-1})\,|_A$. In particular, looking at the g-coefficients, we get

$$\begin{aligned}
(d_1 g)a &= \tau_1'(a)(d_1 g),\\
(d_2 g)a &= \tau_2'(a)(d_2 g),\\
(d_3 g)a &= \sigma'(a)(d_3 g).
\end{aligned}$$

Now, we have $(dg)a = \sigma(a)(dg) = (d\sigma(a))g$ for all $d \in A$ by Lemma 3.1. So, we get

$$\begin{aligned}
d_1\sigma(a) &= d_1\tau_1'(a),\\
d_2\sigma(a) &= d_2\tau_2'(a),\\
d_3\sigma(a) &= d_3\sigma'(a).
\end{aligned} \tag{1}$$

It is clear that $\sigma \neq \tau_i'$ as τ_i' is not an involution. If A is a division algebra, then this forces $d_1 = d_2 = 0$. We prove that $d_1 = d_2 = 0$ in general in Corollary 3.9 below. From this we see that im$\varphi \subseteq A \oplus Ae \oplus Af \oplus (d_3 A)\, g$. Since φ is surjective, we get $d_3 A = A$ (using Lemma 9), so d_3 is a unit in A. From Equation 1, this forces $\sigma' = \sigma$. Thus, $(\varphi\sigma\varphi^{-1})\,|_A = \sigma|_A$, so $\varphi|_A$ is compatible with $\sigma|_A$. Note that this forces φ either to fix the Q_i or to interchange them, as pointed out before the theorem. So, $(\varphi\tau_1\varphi^{-1})\,|_A$ is either τ_1 or τ_2, and similarly for $(\varphi\tau_2\varphi^{-1})\,|_A$.

We next consider the coefficients a_3, b_3 and c_3. Since $\varphi \circ \sigma|_A = \sigma \circ \varphi|_A$, we have

$$\begin{aligned} a_3 a &= \sigma(a) a_3, \\ (b_3 e) a &= \sigma(a)(b_3 e), \\ (c_3 f) a &= \sigma(a)(c_3 f). \end{aligned}$$

for all $a \in A$. Clearly $a_3 = 0$. Writing $b_3 = \sum_i (u_i, 0) \otimes (v_i, 0)$ with $\{u_i\}$ an F-independent set, for $a = (x, 0) \otimes (y, 0)$ we have

$$\begin{aligned} (b_3 e) a &= \sum_i (0, u_i) \otimes (v_i, 0) \cdot (x, 0) \otimes (y, 0) \\ &= \sum_i (0, u_i \gamma_1(x)) \otimes (v_i y, 0), \\ \sigma(a)(b_3 e) &= (\gamma_1(x), 0) \otimes (\gamma_2(y), 0) \cdot \sum_i (0, u_i) \otimes (v_i, 0) \\ &= \sum_i (0, u_i \gamma_1(x)) \otimes (\gamma_2(y) v_i, 0). \end{aligned}$$

Setting these two equal with $x = 1$, and using the independence of $\{u_i\}$ yields $v_i y = \gamma_2(y) v_i$, so $v_i = 0$ for all i. Thus, $b_3 = 0$. Similarly, $c_3 = 0$. We have thus proven that $\varphi(g) = d_3 g$.

We next show that $\varphi(e) \in Q_1 e$ or $\varphi(e) \in Q_2 f$. From the condition $eb = \tau_1(b) e$, applying φ, we see that

$$\begin{aligned} a_1 a &= \tau_1'(a) a_1, \\ (b_1 e) a &= \tau_1'(a)(b_1 e), \\ (c_1 f) a &= \tau_1'(a)(c_1 f), \end{aligned}$$

and τ_1' is either τ_1 or τ_2, as we saw above. We see that $a_1 = 0$ by Lemma 3.3. Suppose that $\tau_1' = \tau_1$. For c_1, write $c_1 = \sum_i (u_i, 0) \otimes (v_i, 0)$. We have $(c_1 f) a = \tau_1(a)(c_1 f)$. For $a = (x, 0) \otimes (y, 0)$ we have

$$\begin{aligned} \sum_i (u_i, 0) \otimes (0, v_i) \cdot (x, 0) \otimes (y, 0) &= \sum_i (u_i x, 0) \otimes (0, v_i \gamma_2(y)), \\ (\gamma_1(x), 0) \otimes (y, 0) \sum_i (u_i, 0) \otimes (0, v_i) &= \sum_i (\gamma_1(x) u_i, 0) \otimes (0, v_i y). \end{aligned}$$

By assuming that the $\{v_i\}$ are F-independent, and setting $y = 1$, we get $u_i x = \gamma_1(x) u_i$, which forces each $u_i = 0$, so $c_1 = 0$. Since $a_1 = c_1 = d_1 = 0$, we have $\varphi(e) \in Ae$. On the other hand, if $\tau_1' = \tau_2$, a similar calculation shows that $\varphi(e) = c_1 f \in Af$. We further need to show that in the first case that $b_1 \in Q_1$ and that $c_1 \in Q_2$ in the second case. For the first case write $b_1 = \sum_i (u_i, 0) \otimes (v_i, 0)$ with $\{u_i\}$ an F-independent set. Since we have $(b_1 e)a = \tau_1(a)(b_1 e)$, for $a = (x, 0) \otimes (y, 0)$,

$$(b_1 e)a = \sum_i (0, u_i) \otimes (v_i, 0) \cdot (x, 0) \otimes (y, 0) = \sum_i (0, u_i \gamma_1(x)) \otimes (v_i y, 0)$$

and

$$\tau_1(a)(b_1 e) = (\gamma_1(x), 0) \otimes (y, 0) \sum_i (0, u_i) \otimes (v_i, 0) = \sum_i (0, u_i \gamma_1(x)) \otimes (y v_i, 0).$$

By setting $x = 1$ and using the independence of the $\{u_i\}$, we see that $v_i y = y v_i$ for all $y \in Q_2$. Thus, each $v_i \in F$. Therefore, $b_1 = \sum_i (u_i, 0) \otimes (v_i, 0) = (\sum_i u_i v_i, 0) \otimes (1, 0) \in Q_1$. The argument for c_1 is similar. Finally, similar calculations show that $\varphi(f) \in Q_1 e$ or $\varphi(f) \in Q_2 f$.

We now have proved that either

$$\begin{aligned} \varphi(e) &= be \text{ for some } b \in Q_1, \\ \varphi(f) &= cf \text{ for some } c \in Q_2, \\ \varphi(g) &= dg \text{ for some } d \in A. \end{aligned}$$

or

$$\begin{aligned} \varphi(e) &= cf \text{ for some } c \in Q_2, \\ \varphi(f) &= be \text{ for some } b \in Q_1, \\ \varphi(g) &= dg \text{ for some } d \in A. \end{aligned}$$

We claim that either of these cases implies that $\varphi \circ \sigma = \sigma \circ \varphi$. Note that $\sigma|_A \circ \tau_1 = \tau_1 \circ \sigma|_A = \tau_2$ and $\sigma|_A \circ \tau_2 = \tau_2 \circ \sigma|_A = \tau_1$. We consider the first case. Note that $\sigma(e) = -e$, $\sigma(f) = -f$, and $\sigma(g) = g$. We have

$$\begin{aligned} \varphi(\sigma(x + ye + zf + wg)) &= \varphi(\sigma(x) - e\sigma(y) - f\sigma(z) + g\sigma(w)) \\ &= \varphi(\sigma(x) - \tau_2(y)e - \tau_1(z)f + wg) \\ &= \varphi\sigma(x) - \varphi\tau_2(y)(be) - \varphi\tau_1(z)(cf) + \varphi(w)(dg) \end{aligned}$$

and

$$\begin{aligned}\sigma(\varphi(x+ye+zf+wg)) &= \sigma(\varphi(x)+\varphi(y)(be)+\varphi(z)(cf)+\varphi(w)(dg)) \\ &= \sigma\varphi(x)-(e\sigma(b))\sigma\varphi(y)-(f\sigma(c))\sigma\varphi(z)+(g\sigma(d)\sigma\varphi(w)) \\ &= \sigma\varphi(x)-(\tau_2(b)e)\sigma\varphi(y)-(\tau_1(c)f)\sigma\varphi(z)+(dg)\sigma\varphi(w) \\ &= \sigma\varphi(x)-(be)\sigma\varphi(y)-(cf)\sigma\varphi(z)+(dg)\sigma\varphi(w) \\ &= \sigma\varphi(x)-\tau_2\varphi(y)(be)-\tau_1\varphi(z)(cf)+\varphi(w)(dg).\end{aligned}$$

We used $b \in Q_1$ and $c \in Q_2$ to simplify the fourth line of the display above. Also, because $b \in Q_1$, we can see by a short calculation that $(be)a = \tau_1(a)(be)$ for all $a \in A$. Similarly, since $c \in Q_2$, we have $(cf)a = \tau_2(a)(cf)$. This shows how we obtained the fifth line of the display.

We will have $\varphi\circ\sigma = \sigma\circ\varphi$ once we know that $(\varphi\circ\sigma)|_A = (\sigma\circ\varphi)|_A$ and $\varphi|_A\circ\tau_i = \tau_i\circ\varphi|_A$. We saw earlier that $(\varphi\sigma\varphi^{-1})|_A = \sigma|_A$, so $(\varphi\circ\sigma)|_A = (\sigma\circ\varphi)|_A$. Moreover, we obtained the first case above by having $(\varphi\tau_i\varphi^{-1})|_A = \tau_i$. So, we have $\varphi\circ\sigma = \sigma\circ\varphi$. The second case is similar. □

We now need to prove that Equation 1 above forces $d_1 = d_2 = 0$. To do this it suffices to go to a splitting field of A. Thus, let $A = M_4(F)$. Recall that the first line of Equation 1 says that $d_1\sigma(a) = d_1\tau_1'(a)$ for all $a \in A$, and where $\tau_1' = \varphi\tau_1\varphi^{-1}$. We see that the left multiplication map L_{d_1} annihilates $X = \{\sigma(a) - \tau_1'(a) : a \in A\}$, and so it annihilates the right ideal I generated by X. Let

$$\begin{aligned}\mathcal{S} &= \mathrm{Sym}(A,\sigma), \\ \mathcal{K} &= \mathrm{Skew}(A,\sigma), \\ \mathcal{P} &= \{a \in A : \tau_1'(a) = -a\}, \\ \mathcal{T} &= \{a \in A : \tau_1'(a) = a\}.\end{aligned}$$

For simplicity, we write $\tau = \tau_1'$. It is a short argument to see that $\mathcal{T} = \varphi(Q_2)$, so $\mathcal{T}$ is a quaternion algebra. Our approach will be to show that the assumption $I \neq A$ forces $\mathcal{T}$ to be a commutative algebra, a contradiction to $\mathcal{T} = \varphi(Q_2)$. Thus, $A = I \subseteq \ker(L_{d_1})$, which forces $d_1 = 0$. Similarly we get $d_2 = 0$.

We now get information about I. We have $\mathcal{P} = \mathrm{Skew}(\varphi(Q_1), \varphi\gamma_1\varphi^{-1})\otimes Q_2$, so $\dim_F(\mathcal{P}) = 12$. By dimension count, $\dim_F(\mathcal{S}\cap\mathcal{P}) \geq 6$. If $a \in \mathcal{S}\cap\mathcal{P}$, then $\sigma(a) - \tau(a) = 2a \in X$. So $\mathcal{S}\cap\mathcal{P} \subseteq X \subseteq I$. The ideal I thus contains a 6-dimensional subspace of $\mathrm{Sym}(A,\sigma)$. We claim, and prove in the lemma below, that this forces $\dim_F(I) \geq 12$ and, if $\dim_F(I) = 12$, that $I \cap \mathcal{S} = \mathcal{S}\cap\mathcal{P}$ has dimension 6. Furthermore, the kernel of the map $\mathcal{P} \to \mathcal{S}$ given by $x \mapsto \sigma(x) + x$ is $\mathcal{P}\cap\mathcal{K}$. Since the image is contained in $I\cap\mathcal{S} = \mathcal{S}\cap\mathcal{P}$, we see that the image has dimension 6, so the kernel has dimension $12 - 6 = 6$. Thus, $\mathcal{K} \subseteq \mathcal{P}$. Moreover, by the descriptions of $\mathcal{P}$ and $\mathcal{T}$ above, we see that $\mathcal{T}\mathcal{P} \subseteq \mathcal{P}$. Therefore, $\mathcal{T}\mathcal{K} \subseteq \mathcal{P}$. Since

$\dim_F(\mathcal{S} \cap \mathcal{P}) = 6 = \dim(\mathcal{K} \cap \mathcal{P}) = \frac{1}{2}\dim_F(\mathcal{P})$, we have $\mathcal{P} = (\mathcal{S} \cap \mathcal{P}) \oplus \mathcal{K}$. From this, we see that if $t \in \mathcal{T}$ and $k \in \mathcal{K}$, then $tk \in \mathcal{P}$ and $tk + k\sigma(t) \in \mathcal{K} \subseteq \mathcal{P}$, so $tk - k\sigma(t) \in \mathcal{S} \cap \mathcal{P}$. We consider this situation after the following lemma.

Lemma 3.5. *Let I be a right ideal of $M_4(F)$. If $\dim_F(I) = 12$, then $\dim_F(I \cap \mathcal{S}) = 6$. If $\dim_F(I) < 12$, then $\dim_F(I \cap \mathcal{S}) < 6$.*

Proof. We first show that we may assume σ is the transpose T. Since σ is orthogonal, $\sigma = \mathrm{Int}(s) \circ T$ for some s with $s^T = s$. Then $\mathcal{S} = s\mathrm{Sym}(A, T)$. So,

$$\dim_F(I \cap \mathcal{S}) = \dim_F(I \cap s\mathrm{Sym}(A, T)) = \dim_F(s^{-1}I \cap \mathrm{Sym}(A, T)).$$

Thus, by replacing I by $s^{-1}I$, we may assume $\sigma = T$.

First suppose that $\dim_F(I) = 12$. Since I is generated by a single element, we see that there are $\alpha, \beta, \gamma, \delta \in F$, not all 0, with

$$I = \left\{ a \in A : \begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix} a = 0 \right\}.$$

Suppose that $s \in I \cap \mathcal{S}$. Write

$$s = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix}$$

for appropriate entries in F. The condition $\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix} s = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$ corresponds to a homogeneous system of 4 equations in the 10 variables $a, \ldots, j$. This system has coefficient matrix

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \beta & \gamma & \delta & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & \beta & 0 & \gamma & \delta & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & \beta & 0 & \gamma & \delta \end{pmatrix}.$$

Since at least one of $\alpha, \beta, \gamma, \delta$ is nonzero, this matrix clearly has rank 4. Therefore, its kernel is dimension 6. However, its kernel is $I \cap \mathcal{S}$, so $\dim_F(I \cap \mathcal{S}) = 6$, as desired.

If $\dim_F(I) < 12$, then $\dim_F(I) \in \{0, 4, 8\}$. If $\dim_F(I) = 8$, then we can write $I = \{a \in A : xa = 0\}$ for some 2×4 matrix x of rank 2. By writing $xs = 0$ for a symmetric matrix s, we then obtain a coefficient matrix of size 8×10, and by writing it out we see that it has rank 8. Thus, its kernel has dimension 2, so $\dim_F(I \cap \mathcal{S}) = 2 < 6$. Clearly if $\dim_F(I) \leq 4$, then $\dim_F(I \cap \mathcal{S}) < 6$. This completes the proof of the lemma. $\square$

We are now consider the following situation. Let I be a right ideal of dimension 12 in A, let σ be an orthogonal involution on A, and define

$$R_{I,\sigma} = \{a \in A : ak - k\sigma(a) \in I \text{ for all } k \in \mathcal{K}\}.$$

Lemma 3.6. *If σ and τ are orthogonal involutions on A, then $R_{I,\sigma} = R_{I,\tau}$ for any I.*

Proof. Since τ and σ are both orthogonal, there is an $s \in A^*$ with $\tau = \mathrm{Int}(s) \circ \sigma$. Then $\mathrm{Skew}(A,\tau)s = \mathrm{Skew}(A,\sigma)$. Let $a \in R_{I,\tau}$. Then $ay - y\tau(a) \in I$ for all $y \in \mathrm{Skew}(A,\tau)$. So,$ay - ys\sigma(a)s^{-1} \in I$, so $a(ys) - (ys)\sigma(a) \in Is = I$. Thus, $a \in R_{I,\sigma}$. The reverse inclusion is similar. $\square$

Lemma 3.7. *If $J = \alpha I$ for some $\alpha \in A^*$, then $R_{J,\sigma} = \alpha R_{I,\sigma}\alpha^{-1}$.*

Proof. We point out any right ideal of A of dimension 12 can be written in the form αI for some $\alpha \in A^*$. Let $a \in R_{I,\sigma}$ and $x \in \mathrm{Skew}(A,\sigma)$. Then $\alpha^{-1}x\sigma(\alpha)^{-1} \in \mathrm{Skew}(A,\sigma)$. Therefore,

$$\begin{aligned}(\alpha a\alpha^{-1})x - x(\alpha a\alpha^{-1}) &= \alpha a\alpha^{-1}x - x\sigma(\alpha)^{-1}\sigma(a)\sigma(\alpha) \\ &= \alpha\left(a\alpha^{-1}x\sigma(\alpha)^{-1} - \alpha^{-1}x\sigma(\alpha)^{-1}\sigma(a)\right)\sigma(\alpha) \\ &\in \alpha I\sigma(\alpha) = \alpha I = J.\end{aligned}$$

Thus, $\alpha R_{I,\sigma}\alpha^{-1} \subseteq R_{J,\sigma}$. The reverse inclusion is similar. $\square$

Proposition 3.8. *With the notation above, $R_{I,\sigma}$ is a 4-dimensional commutative F-subalgebra of $M_4(F)$ and R has a two-sided ideal J of F-dimension 3 with $J^2 = 0$.*

Proof. By Lemma 3.6, we may assume that $\sigma = T$ is the transpose involution. By Lemma 3.7, the set $R_{I,\sigma}$ is, up to conjugation, independent of the ideal I, so we may choose I to be the right ideal consisting of all matrices whose first row is zero. We determine $R_{T,I}$ for this right ideal. Let $a = \sum_{i,j}\alpha_{ij}e_{ij} \in R_{I,\sigma}$. We note that $\{e_{ij} - e_{ji} : i > j\}$ is a basis for $\mathrm{Skew}(A,T)$. Let $k = e_{uv} - e_{vu}$ with $u > v$. Then

$$\begin{aligned}ak - ka^T &= \sum_{i,j}\alpha_{ij}e_{ij}(e_{uv} - e_{vu}) - \sum_{i,j}\alpha_{ij}(e_{uv} - e_{vu})e_{ji} \\ &= \sum_i \alpha_{iu}e_{iv} - \sum_i \alpha_{iv}e_{iu} - \sum_i \alpha_{iv}e_{ui} + \sum_i \alpha_{iu}e_{vi}.\end{aligned}$$

The first row of this matrix is $\alpha_{1u}e_{1v} - \alpha_{1v}e_{1u}$ if $v > 1$, and $\alpha_{1u}e_{1v} - \alpha_{1v}e_{1u} + \sum_i \alpha_{iu}e_{1i}$ if $v = 1$. In any case, the first row is zero. Thus, $\alpha_{1u} = \alpha_{1v} = 0$ for all $u > v > 1$, and if $v = 1$, we get $(\alpha_{1u} + \alpha_{1u})e_{11} + (\alpha_{uu} - \alpha_{11})e_{1u} + \sum_{i \neq u,1}\alpha_{iu}e_{1i} = 0$, so $\alpha_{iu} = 0$ if $i \neq u, 1$, and $\alpha_{uu} = \alpha_{11}$. Thus, the diagonal entries of a are all the same. Also, the only nonzero non-diagonal entries are in the first column. There is no restriction on α_{i1} for $i > 1$, so we have shown that

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & 0 & 0 & a \end{pmatrix} : a, b, c, d \in F \right\}.$$

From this description, it is clear that $\dim_F(R) = 4$ and that R is a commutative F-subalgebra of A. Moreover, if $J = I \cap R = \{a \in R : \mathrm{trace}(a) = 0\}$, then we see that J is an ideal of R of dimension 3, and that $J^2 = 0$. This completes the proof. □

We can now finish the proof of Theorem 3.4.

Corollary 3.9. *In Equation 1 above, we have* $d_1 = d_2 = 0$.

Proof. The first line of Equation 1 is that $d_1(\sigma(a) - \tau_1'(a)) = 0$ for all $a \in A$. We have shown above that the right ideal I generated by $\{\sigma(a) - \tau_1'(a) : a \in A\}$ has dimension at least 12, and that if it is dimension 12, then we have $\varphi(Q_2) \subseteq R_{I,\sigma}$. However, Proposition 3.8 shows that this set is a 4-dimensional commutative F-algebra. This is a contradiction since $\varphi(Q_2)$ is a quaternion algebra. Thus, $I = A$. Since $0 = d_1 I = d_1 A$, we get $d_1 = 0$. Similarly, $d_2 = 0$. □

From this result we can show that the Skolem-Noether theorem does not hold for the tensor product of two octonion algebras.

Corollary 3.10. *There exists simple* F*-subalgebras* B *and* B' *of* C *and an* F*-algebra isomorphism* $f : B \to B'$ *such that there is no* F*-algebra automorphism* φ *of* C *with* $\varphi|_B = f$.

Proof. Let f be an F-algebra automorphism of A that is not compatible with $\sigma|_A$; such maps exist since we can take f to be the inner automorphism of an element $t \in A$ with $\sigma(t)t \notin F$. The condition $\sigma(t)t \in F$ is precisely the condition needed to ensure that f is compatible with $\sigma|_A$. For example, we can take $t = 1 + i_1 i_2 \in A = Q_1 \otimes_F Q_2$ (where the standard generators of Q_r are i_r and j_r). If f extends to an automorphism φ of C, then $\varphi(A) = A$, so φ is compatible with σ. This forces $\varphi|_A = f$ to be compatible with $\sigma|_A$, and f is chosen so that this does not happen. □

Remark 3.11. Consider the slightly more general situation that $\varphi \in \mathrm{Aut}(C)$ satisfies $\varphi(A) = \tilde{A}$ with $\tilde{A} = \tilde{Q}_1 \otimes \tilde{Q}_2$ a biquaternion algebra, where $\tilde{Q}_i$ is a quaternion subalgebra of C_i, $i = 1, 2$, or of C_j, $j = 1, 2, i \neq j$. Then a similar argument shows that also $\varphi \circ \sigma = \sigma \circ \varphi$, i.e., $\varphi \in \mathrm{Aut}(C, \sigma) = (G_2 \times G_2) \rtimes \mathbb{Z}_2$. In particular, this means that for any $\varphi \in \mathrm{Aut}(C)$ which is not compatible with σ, the image of any biquaternion subalgebra A of the type investigated above (i.e., each factor a subalgebra of one of the octonion algebras) cannot be of this type again.

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