Finitely Generated Subnormal Subgroups of $GL_n(D)$ Are Central [*]

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Abstract

Let D be an infinite division algebra of finite dimension over its centre. Assume that N is a subnormal subgroup of $GL_n(D)$ with $n \geq 1$. It is shown that if N is finitely generated, then N is central.

Let D be an infinite division algebra of degree m over its centre $Z(D) = F$. Denote by D' the commutator subgroup of the multiplicative group $D^* = D - \{0\}$. The aim of this note is to investigate the structure of finitely generated subnormal subgroups of $GL_n(D)$ with $n \geq 1$. Assume that $n \geq 2$ and N is a normal subgroup of $GL_n(D)$. It is shown in [1] that if N is finitely generated, then N is central. A similar result for finitely generated normal subgroups is obtained for the case $n = 1$ in [2]. Here we shall generalize some of the main results appeared in [1] and [2] to subnormal subgroups of $GL_n(D)$ with $n \geq 1$. To be more precise, assume that N is a subnormal subgroup of $GL_n(D)$ with $n \geq 1$. It is proved that if N is finitely generated, then N is central. Using this, it is also shown that $GL_n(D)/Z(GL_n(D))$ contains no non-trivial finitely generated subnormal subgroups. Furthermore, given an infinite subnormal subgroup N of $GL_n(D)$, it is proved that N contains no finitely generated maximal subgroups. Therefore, $GL_n(D)$ contains no finitely generated maximal subgroups. The reader may consult [5], [6], [7], and the references thereof for more recent results on multiplicative subgroups of $GL_n(D)$. For convenience we shall deal with the case $n = 1$ separately. Our key result is the following

THEOREM 1. *Let D be a finite dimensional division algebra with centre F. Then any finitely generated subnormal subgroup of D^* is central.*

PROOF. We first claim that if D^* contains a non-central finitely generated subgroup N, which is subnormal, then F is finitely generated over its prime subfield.

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To see this, assume that E is the division algebra generated by all elements u of N. Since E is invariant under all inner automorphisms $x \to uxu^{-1}$, it follows from a result of Stuth (cf. [8, p. 439]) that either E is central or $E = D$. If E is central, then N is central, a contradiction. Thus, we may assume that $E = D$. Suppose that $[D : F] = n$ and consider the regular matrix representation of D^* in $GL_n(F)$. Since N is finitely generated, there exist matrices $A_1, \dots, A_k \in GL_n(F)$ such that $N =< A_1, \dots, A_k >$. Let Λ be the set of all elements in F occurring as the entries of A_i and $A_i^{-1}, i = 1, \dots, k$. If H is the subring generated by N, then we have $H \subset GL_n(P(\Lambda))$, where $P(\Lambda)$ is the subfield of F generated by Λ over the prime subfield P. Now, since $E \subset GL_n(P(\Lambda))$ we have $aI \in GL_n(P(\Lambda))$, for any $a \in F^*$ and so $a \in P(\Lambda)$. Hence, $F = P(\Lambda)$ and the claim is established. To proceed the proof, let N be a non-central finitely generated subgroup which is subnormal in D^*. Since F is finitely generated over the prime subfield P, by Noether Normalization Lemma, there exist elements $r_1, \dots, r_s \in F$ which are algebraically independent over P such that $[F : P(r_1, \dots, r_s)] < \infty$, where s denotes the transcendency degree of F over P. For convenience set $r_s = y$, $L = P(r_1, \dots, r_s)$, and $K = P(r_1, \dots, r_{s-1})$ with $K = P$ if $s = 0$. Since $[F : L] < \infty$ we obtain $k = [D : L] < \infty$ and so D^* has a matrix representation in $GL_k(L)$. We may now consider two cases:

Case (1): $s = 0$. If $CharD = p > 0$, then D is algebraic over a finite field and consequently , by a result of Jacobson (cf. [4]), we conclude that D is commutative which is in contradiction with the fact that N is non-central. Thus, we may assume that $P = Q$, the field of rational numbers. For each $a, b \in N$ and $x \in L$, set $c_1 = c_1(a, b, x) = (b + x)a(b + x)^{-1}$, and for $m > 1$ define c_m inductively by $c_m = c_{m-1}bc_{m-1}^{-1}$. Since N is subnormal in D^* there exists a natural number r such that $N = N_r \lhd N_{r-1} \lhd \cdots \lhd N_1 \lhd D^*$. Thus, $c_1 \in N_1$ and by induction we conclude that $c_r \in N_r$. We now claim that for each i we have $c_i = (b+x)w_i(a, b)(b+x)^{-1}$, where $w_i(a, b)$ is a reduced word in a, a^{-1}, b, b^{-1} whose first and last alphabets are a or a^{-1}, respectively. In fact, for $i = 1$ we have $c_1 = (b + x)a(b + x)^{-1}$ and if $c_i = (b + x)w_i(a, b)(b + x)^{-1}$, then, by induction, we conclude that $c_{i+1} = c_ibc_i^{-1} = [(b+x)w_i(a, b)(b+x)^{-1}]b[(b+x)w_i(a, b)^{-1}(b+x)^{-1}] = (b + x)[w_i(a, b)bw_i(a, b)^{-1}](b + x)^{-1}$, and since the first and the last alphabets of $w_i(a, b)bw_i(a, b)^{-1}$ are a or a^{-1}, the claim is established. Since N is subnormal in D^* and $[D : F] < \infty$, by a result of Goncalves (cf. [3]), N contains a non-cyclic free subgroup G, say. Take a, b to be the generators of G, and denote their matrix representations in $GL_k(L)$ by A and B, respectively. Since N is finitely generated, by the argument used above, we conclude that there is a set $\Lambda = \{m_1/n_1, \ldots, m_t/n_t\} \subset Q$ such that each element of N has a matrix representation in $GL_k(Z[\Lambda])$, where Z is the ring of integers. As observed above, since N is subnormal in D^*, for each element $x \in Q$ we have $c_r = c_r(a, b, x) \in N$. Thus, $c_r(A, B, xI) = (B + xI)w_r(A, B)(BI + x)^{-1} \in N$. Since $det(B + xI)$ is a polynomial in x of degree k, and for each $1 \leq i, j \leq k$, the (i, j)-th entry of $(B + xI)^{-1}$ is of the form $f_{ij}(x)/g(x) \in Q(x)$, where $deg\, g(x) = k$, $deg\, f_{ij}(x) \leq k - 1$, we conclude that the (i, j)-th entry of the matrix $c_r(A, B, xI)$ is of the form $f_{ij}(x)/g(x)$, where for each $1 \leq i, j \leq k$, we have $deg\, f_{ij}(x) \leq k$. If for each $1 \leq i, j \leq k$, there are rational numbers q_{ij} such that for any $x \in Q$, $f_{ij}(x)/g(x) = q_{ij}$, then $c_r(a, b, x)$ is independent of x and so putting $x = 0$, and $x = 1$ we obtain $c_r(a, b, 0) = c_r(a, b, 1)$. This implies that $bw_r(a, b)b^{-1} = (b+1)w_r(a, b)(b+1)^{-1}$, and consequently, $bw_r(a, b) = w_r(a, b)b$. Since the first and the last alphabets of $w_r(a, b)$ are a or a^{-1}, respectively, this gives us a non-trivial relation between a and b which is a contradiction to the fact that G is free. Thus there exists an entry of $c_r(A, B, xI)$, say (i, j)-th which depends on x. Put $f_{ij}(x) = \Sigma_{i=0}^{k} a_i x^i$, $g(x) = x^k + \Sigma_{i=0}^{k-1} b_i x^i$. Thus for each $x \in Q$ we have $f_{ij}(x)/g(x) \in Z[\Lambda]$. If $a_k = m_{t+1}/n_{t+1}$, then for each $x \in Q$ we obtain $f_{ij}(x)/g(x) - a_k \in Z[\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. So there exists a polynomial $f(x) \in Q[x]$ such that $deg\, f(x) \leq k - 1$ and for each $x \in Q$ we have $f(x)/g(x) \in Z[\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. Multiplying $f(x)$ and $g(x)$ by suitable scalars, we may assume that $f(x), g(x) \in Z[x]$. Put $f(x) = \Sigma_{i=0}^{k-1} a'_i x^i \in Z[x]$, $g(x) = \Sigma_{i=0}^{k} b'_i x^i$. Since $det\, B \neq 0$, we may assume that $b'_0 \neq 0$. Now, change the variable x to $b'_0 x$ to obtain $f_1(x), g_1(x) \in Z[x]$, such that $deg\, g_1 = k$, $deg f_1 \leq k - 1$, where the constant term of $g_1(x)$ is 1, and for each $x \in Q$ we have, $f_1(x)/g_1(x) \in Z[\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. Assume that $S = \{p_1, \ldots, p_l\}$ be the set of all primes occurring in the factorizations of $n_1, \ldots, n_{t+1}$ into prime numbers. For each natural number r, put $x_r = (p_1 p_2 \ldots p_l)^r$. Since $deg\, f_1 < deg\, g_1$, for a large enough number r, we obtain that $f_1(x_r)/g_1(x_r) < 1$. On the other hand, for each $r \geq 1$, and each $1 \leq i \leq l$, $g_1(x_r)$ and p_i are coprime, that is, $(g_1(x_r), p_i) = 1$. It is not hard to see that if $u/v \in Z[m_1/n_1, \ldots, m_{t+1}/n_{t+1}]$ with $(u, v) = 1$, then each prime factor of v belongs to S. Now since $f_1(x_r)/g_1(x_r) \in Z[m_1/n_1, \ldots, m_{t+1}/n_{t+1}]$ and for each $1 \leq i \leq l$, $r \geq 1$, $(g_1(x_r), p_i) = 1$, we reach a contradiction, and so the result follows in this case.

Case (2): $s > 0$. Since N is finitely generated, by the argument used in the first case, we conclude that there is a set $\Lambda = \{m_1/n_1, \ldots, m_t/n_t\} \subset L$, where $m_i, n_i \in K[y]$ such that each element of N has a matrix representation in $GL_k(K[y][\Lambda])$, and

$K[y][\Lambda]$ is the subring of L generated by Λ over $K[y]$. On the other hand, N is subnormal in D^* and thus by the same argument used in the first case, there exists an entry of $c_r(A, B, xI)$, say (i, j)-th which depends on x. Put $f_{ij}(x) = \Sigma_{i=0}^{k} a_i x^i$, $g(x) = x^k + \Sigma_{i=0}^{k-1} b_i x^i$. Thus for each $x \in L$ we have $f_{ij}(x)/g(x) \in K[y][\Lambda]$. If $a_k = m_{t+1}/n_{t+1}$, then for each $x \in L$ we obtain $f_{ij}(x)/g(x) - a_k \in K[y][\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. So there exists a non-zero polynomial $f(x) = f_{ij}(x) - a_k g(x) \in L[x]$ such that $deg\, f(x) \leq k - 1$ and for each $x \in L$ we have $f(x)/g(x) \in K[y][\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. Multiplying $f(x)$ and $g(x)$ by suitable elements of $K[y]$, we may assume that $f(x), g(x) \in K[y][x]$. Put $f(x) = \Sigma_{i=0}^{k-1} a'_i x^i$, $g(x) = \Sigma_{i=0}^{k} b'_i x^i$. Since $det\, B \neq 0$, we may assume that $b'_0 \neq 0$. Now, change the variable x to $b'_0 x$ to obtain $f_1(x), g_1(x) \in K[y][x]$, such that $deg\, g_1 = k$, $deg f_1 \leq k - 1$, and the constant term of $g_1(x)$ is 1. Further, for each $x \in L$ we have $f_1(x)/g_1(x) \in K[y][\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. Assume that $S = \{p_1, \ldots, p_l\}$ is the set of all irreducible polynomials occurring in the factorizations of $n_1, \ldots, n_{t+1}$ into irreducible polynomials. For each natural number r, put $x_r = (p_1 p_2 \ldots p_l)^r$. Since $deg\, f_1 < deg\, g_1$, for a large enough number r, the degree of the denominator of $f_1(x_r)/g_1(x_r)$ with respect to y is greater than that of the nominator. On the other hand, for each $r \geq 1$, and each $1 \leq i \leq l$, $g_1(x_r)$ and p_i are coprime, that is, $(g_1(x_r), p_i) = 1$. It is not hard to see that if $u/v \in K[y][m_1/n_1, \ldots, m_{t+1}/n_{t+1}]$ with $(u, v) = 1$, then each irreducible factor of v belongs to S. Now since $f_1(x_r)/g_1(x_r) \in K[y][m_1/n_1, \ldots, m_{t+1}/n_{t+1}]$ and for each $1 \leq i \leq l$, $r \geq 1$, $(g_1(x_r), p_i) = 1$, we arrive at a contradiction, and so the result follows.

As a consequence of the above theorem, we have the following

Corollary 1. *Let D be an infinite division algebra of finite dimension over its centre F. Assume that N is a subnormal subgroup of $GL_n(D)$ with $n \geq 1$. If N is finitely generated, then $N \subset F^*$.*

Proof. The case $n = 1$ follows from Theorem 1. Now, consider the case $n > 1$. By Theorem 11 of [5], we have $SL_n(D) \subset N$. Thus N is normal in $GL_n(D)$. Finally, using Theorem 5 of [1], we obtain the result for $n \geq 2$.

Corollary 2. *Let D be an infinite division algebra of finite dimension over its centre F and $n \geq 1$. Then $GL_n(D)/Z(GL_n(D))$ contains no non-trivial finitely generated subnormal subgroups.*

Proof. Identify $Z(GL_n(D))$ with F^* and let N/F^* be a finitely generated subnormal subgroup of $GL_n(D)/F^*$. Let $x_1F^*, \ldots, x_rF^*$ be a set of generators of N/F^*. If we set $G =< x_1, \ldots, x_r >$, the group generated by $x_1, \ldots, x_r$, we conclude that $N = GF^*$. Thus, $N' = G'$ and consequently G is normal in N. This implies that G is subnormal in $GL_n(D)$. Now, using Corollary 1, we conclude that $N = F^*$ which completes the proof.

Corollary 3. *Let D be a division algebra of finite dimension over its centre F and $n \geq 1$. Assume that N is an infinite subnormal subgroup of $GL_n(D)$. Then N contains no finitely generated maximal subgroups. In particular, if D is infinite, then $GL_n(D)$ contains no finitely generated maximal subgroups.*

Proof. Assume that M is a maximal subgroup of N which is finitely generated. Then for any $x \in N \backslash M$ we have $< M, x >= N$. This implies that N is finitely generated which contradicts Corollary 1.

Bearing in mind the fact that $Z(SL_n(D)) = SL_n(D) \cap Z(GL_n(D))$, we may conclude

Corollary 4. *Let D be a division algebra of finite dimension over its centre F and $n \geq 1$. Assume that N is an infinite subnormal subgroup of $SL_n(D)$. If N is finitely generated, then $N \subset Z(SL_n(D))$. Furthermore, if D is infinite, then $PSL_n(D) = SL_n(D)/Z(SL_n(D))$ contains no non-trivial finitely generated subnormal subgroups.*

Proof. The first part follows from Corollary 1 and the fact that $Z(SL_n(D)) = SL_n(D) \cap Z(GL_n(D))$. The proof of the final part is similar to that of Corollary 2.

It is believed that the condition on D being of finite dimension over its centre is superfluous in all the above results.

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