SK_1-LIKE FUNCTORS FOR DIVISION ALGEBRAS

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Abstract. We introduce a Dieudonne/reduced Dieudonne functor for central simple algebras A, which embraces different groups which are associated to A in different contexts, including the reduced Whitehead group $SK_1(A)$. We show that a reduced Dieudonne functor has most important functorial properties of the reduced Whitehead group $SK_1(A)$. Specializing a Dieudonne functor to the group $G(A) = A^*/F^*A'$, where F is the center of A and A' its commutator subgroup, we establish a fundamental connection between this group, its residue version and relative valued group when A is a henselian division algebra. The structure of $G(A)$ turns out to carry significant information about the arithmetic of A. Along these lines, we employ $G(A)$ to compute the group $SK_1(A)$. As an application, we obtain theorems of reduced K-theory which requires heavy machinery, as simple examples of our method.

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1. Introduction

Let D be a division algebra with center F. The non-triviality of the important group $SK_1(D)$ is shown by V. P. Platonov who developed a so-called *Reduced K-Theory* to compute $SK_1(D)$ for certain division algebras. The group $SK_1(D)$ enjoys some interesting properties which makes it distinguish from the *K-Theory* functor $K_1(D)$. Among interesting characteristic of the torsion group $SK_1(D)$ is its behavior under extension of the ground field. Also in the case of valued division algebra, its stability under reduction, namely $SK_1(D) = SK_1(\overline{D})$, where D is unramified division algebra. Also the primary decomposition of a division algebra induces a corresponding decomposition of $SK_1(D)$.

On the other hand, there have been other groups which are associated with a division algebra D, in order to study the group and arithmetic structure of D, e.g, $D^*/Nrd(D^*)D'$ in [1], or D^*/D^{*2} in [8]. Also see [5]. It can be seen that these groups are in close connection with the group $SK_1(D)$.

On this note we introduce a "Dieudonne" functor for division algebras which embraces groups that have been already associated with division algebra in different contexts. We then show that a "reduced" Dieudonne functor (See Definition 2.2) shares most important functorial properties of the reduced Whitehead group. Notice that a reduced Dieudonne functor covers a wide range of groups of different nature. In section 2 we will show that a reduced Dieudonne functor is torsion of bounded exponent. we then obtain some SK_1-like properties for this functor. we show that a reduced Dieudonne functor may grow "pathologically" for algebraic extension of ground field whose degree is prime to the index of D. It is then shown

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that this functor satisfies a decomposition property analog to one for $SK_1(D)$ (Theorem 2.10).

We then specialize a Dieudonne functor to the group $G(D) = D^*/F^*D'$ where D is a division algebra over its center F and D' is its commutator subgroup. It turns out that there is a close connection between the group structure of $G(D)$ and algebraic structure of D. For example in section 3, after establishing a fundemental connection between $G(D)$, its residue version and relative valued group when D enjoys a henselian valuation, we show that if D is a totally ramified division algebra, then there is a one to one correspondence between isomorphism classes of F-subalgebra of D and the subgroups of $G(D)$. We then use $G(D)$ to compute $SK_1(D)$ for certain division algebras. We show that if $G(D)$ canonically coincides with the relative valued group, then there is a explicit formula for the group $SK_1(D)$ (Theorem 3.7). It turns out that many theorems and examples of reduced K-theory which require heavy machinery can all be viewed as simple examples of our case. Section 3 is devoted to the unitary version of the group $G(D)$.

We fix some notations. Let D be a division algebra over its center F with index $i(D) = n$. Then $Nrd_{D/F} : D^* \longrightarrow F^*$ is the reduced norm function and $SK_1(D) = D^{(1)}/D'$ is the reduced Whitehead group where $D^{(1)}$ is the kernel of $Nrd_{D/F}$. Put $SH^0(D)$ for the cokernel of $Nrd_{D/F}$. we take $\mu_n(F)$ for the group of $n-th$ roots of unity in F, and $Z(D')$ for the center of the group D'. Observe that $\mu_n(F) = F^* \cap D^{(1)}$ and $Z(D') = F^* \cap D'$. If G is a group, by G^n we denote the subgroup of G generated by all elements of $n-th$ power of G. Let $exp(G)$ stands for the exponent of the group G. Let also $det : GL_n(D)/SL_n(D) \longrightarrow D^*/D'$ denote the Dieudonne determinant, where $GL_n(D)$ is the general linear group and $SL_n(D)$ is its commutator subgroup.

2. Dieudonne Functor

Let $\mathcal{C}$ be the class of all central simple algebras and $\mathfrak{G} : \mathcal{C} \longrightarrow \mathcal{A}b$ be a functor from $\mathcal{C}$ to the category of abelian groups.

Definition 2.1. The functor $\mathfrak{G}$ is called a *Dieudonne functor* if for any division algebra D and a nonnegative integer n, there is a homomorphism $d : \mathfrak{G}(M_n(D)) \longrightarrow \mathfrak{G}(D)$ such that for any $x \in \mathfrak{G}(A)$, $di(x) = x^n$, where $i : \mathfrak{G}(D) \longrightarrow \mathfrak{G}(M_n(D))$ induced by the natural embedding $D \longrightarrow M_n(D)$.

The functor K_1 with the Dieudonne determinant det as d, clearly forms a Dieudonne functor.

Let A be a central simple algebra and $(A^*)^2$ be the subgroup of A^* generated by the squares of A^*. It is easy to see that $\mathfrak{G}(A) = A^*/(A^*)^2$ with det as above is a Dieudonne functor. This group is studied in [8] in connection with Witt rings of division algebras.

Definition 2.2. A Dieudonne functor $\mathfrak{G}$ is said to be *reduced* if for any field F, $\mathfrak{G}(F)$ is trivial and if $x \in Ker(\mathfrak{G}(M_n(D)) \longrightarrow \mathfrak{G}(D))$, where D is a division algebra and $n \in \mathbb{N}$, then $x^n = 1$.

Example 2.3. Let $A \in \mathcal{C}$ with center F, then the functors $\mathfrak{G}(A) = A^*/F^*A'$, $\mathfrak{G}(A) = (A^*)^2/(F^*)^2A'$, and $\mathfrak{G}(A) = A^*/F^*A'_r$ where $A'_r = \{x \in A^* | x^r \in A'\}$ and $r \in \mathbb{N}$ can be all viewed as reduced Dieudonne functors.

Example 2.4. Let $A \in \mathcal{C}$ be a central simple algebra finite over its center. In order to consider the reduced Whitehead group $SK_1(A)$ and $SH^0(A)$ as a reduced Dieudonne functors, we shall limit the morphisms of our category (See chapters 22 and 23 in [3]). The following commutative diagram shows that SK_1 and SH^0 are Dieudonne functors.

$$\begin{tikzcd}
1 \arrow[r] & SK_1(D) \arrow[r] \arrow[d] & K_1(D) \arrow[r, "Nrd_{D/F}"] \arrow[d] & F^* \arrow[r] \arrow[d, "\eta_n"] & SH^0(D) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & SK_1(M_n(D)) \arrow[r] \arrow[d] & K_1(M_n(D)) \arrow[r, "Nrd_{D/F}"] \arrow[d, "det"'] & F^* \arrow[r] \arrow[d, "1"] & SH^0(M_n(D)) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & SK_1(D) \arrow[r] & K_1(D) \arrow[r, "Nrd_{D/F}"] & F^* \arrow[r] & SH^0(D) \arrow[r] & 1
\end{tikzcd}$$

where $\eta_n(x) = x^n$ for any $x \in F^*$ and D is a division algebra with center F. Now it is easy to see that $SK_1(A) = A^{(1)}/A'$ and $SH^0(A) = F^*/Nrd_{A/F}(A^*)$ satisfy conditions of being reduced Dieudonne functor.

In the same way, it can be seen that $\mathfrak{G}(A) = A^*/Nrd_{A/F}(A^*)A'$ and $\mathfrak{G}(A) = A^*/F^*A^{(1)} \simeq Nrd(A^*)/F^{*DegA}$ are also reduced Dieudonne functors.

Our primary aim in this section is to show that a reduced Dieudonne functor $\mathfrak{G}$ shares almost all important functorial properties of SK_1. Note that a reduced Dieudonne functor covers groups of different nature which are associated to a division algebra. We will show that this functor is torsion of bounded exponent.

We begin by showing that a reduced Dieudonne functor may grow pathologically for algebraic extension of ground field whose degree is prime to the index of division algebra D.

Clearly the natural embedding of D in $D \otimes_F L$ where L is a finite field extension of F, induce a group homomorphism $\mathcal{I} : \mathfrak{G}(D) \longrightarrow \mathfrak{G}(D \otimes_F L)$. The following proposition provides us with a homomorphism in the opposite direction.

Proposition 2.5. *Let $\mathfrak{G}$ be a Dieudonne functor and D be a division ring with center F. If L is a finite extension of F then there is a homomorphism $\mathcal{P} : \mathfrak{G}(D \otimes_F L) \longrightarrow \mathfrak{G}(D)$ such that $\mathcal{PI} = \eta_{[L:F]}$, where $\eta_m(x) = x^m$.*

Proof. Let $[L : F] = m$. Consider the regular representation $L \xrightarrow{\iota} M_m(F)$ and the corresponding sequence when we tensor over F with D:

$$D \longrightarrow D \otimes_F L \xrightarrow{1\otimes\iota} D \otimes_F M_m(F) \longrightarrow M_m(D)$$

$$a \longmapsto a \otimes 1 \longmapsto a \otimes 1 \longmapsto aI_m \tag{2.1}$$

$$1 \otimes \ell \longmapsto 1 \otimes \iota(\ell) \longmapsto \iota(\ell).$$

Since $\mathfrak{G}$ is a Dieudonne functor, we obtain a homomorphism $\mathcal{P} : \mathfrak{G}(D \otimes_F L) \longrightarrow \mathfrak{G}(M_m(D)) \xrightarrow{d} \mathfrak{G}(D)$. Again the sequence (2.1) shows that, thanks to definition of a Dieudonne functor, $\mathcal{PI}(x) = x^m$. $\square$

Note that in the above proposition D could be an infinite dimensional division algebra. If D is finite dimension over its center F, then it turns out that $\mathfrak{G}(D)$, where $\mathfrak{G}$ is reduced Dieudonne functor, is a torsion group.

Corollary 2.6. *Let $\mathfrak{G}$ be a reduced Dieudonne functor. Then for any division algebra D of index n, $\mathfrak{G}(D)$ is a torsion group of bounded exponent $n^2 = [D : Z(D)]$.*

Proof. Thanks to Proposition 2.5, for any finite field extension L of $F = Z(D)$, we have the sequence of homomorphisms $\mathfrak{G}(D) \xrightarrow{\mathcal{I}} \mathfrak{G}(D \otimes_F L) \xrightarrow{\mathcal{P}} \mathfrak{G}(D)$, such that $\mathcal{P}\mathcal{I}(x) = x^m$, where $x \in \mathfrak{G}(D)$ and $[L : F] = m$. Now let L be a maximal subfield of D. Since L is a splitting field for D, we get the sequence of homomorphisms $\mathfrak{G}(D) \xrightarrow{\mathcal{I}} \mathfrak{G}(M_n(L)) \xrightarrow{\mathcal{P}} \mathfrak{G}(D)$. Since $\mathfrak{G}$ is a reduced Dieudonne functor, it follows that $\mathfrak{G}(M_n(L))$ is a torsion group of bounded exponent n (See Definition 2.2). Now the fact that for any $x \in \mathfrak{G}(D)$, $\mathcal{P}\mathcal{I}(x) = x^n$, shows that $\mathfrak{G}(D)$ is a torsion group of bounded exponent $n^2 = [D : Z(D)]$. $\square$

Now it is immediate that if A is a central simple algebra, then $\mathfrak{G}(A)$ is also torsion.

We later specialize a Dieudonne functor and show that we can reduce this bound for certain functors. The following corollary shows that the analog result of the behavior of $SK_1(D)$ under extension of the ground field holds for a reduced Dieudonne functor. Namely, we show that $\mathfrak{G}(D)$ embeds in $\mathfrak{G}(D \otimes_F L)$ when the index of D and $[L : F]$ are relatively prime.

Corollary 2.7. *Let D be a division ring over its center F and L/F be a finite field extension such that $[L : F]$ is relatively prime to the index of D. Then the canonical homomorphism $\mathfrak{G}(D) \xrightarrow{\mathcal{I}} \mathfrak{G}(D \otimes_F L)$ is injective.*

Proof. Let $i(D) = n$ and $[L : F] = m$. Suppose $\mathcal{I}(x) = 1$ for some $x \in \mathfrak{G}(D)$. By Proposition 2.5, $\mathcal{P}\mathcal{I}(x) = x^m = 1$. But $\mathfrak{G}(D)$ is a torsion of bounded exponent n^2. Hence $x^{n^2} = 1$. Since m and n are relatively prime, $x = 1$ and the proof is complete. $\square$

Computing a Dieudonne functor in general is a difficult task, as it covers groups of different nature like $SK_1(D)$, or the group $G(D) = D^*/F^*D'$. In the next section we specialize a Dieudonne functor to $\mathfrak{G}(D) = D^*/F^*D'$ and compute it for certain division algebras. But before we continue with the functorial properties of $\mathfrak{G}$, let us consider a class of reduced Dieudonne functors which enjoy an additional property. Namely let τ be a natural transformation $\tau : K_1 \longrightarrow \mathfrak{G}$ such that,

(1) For any object A in $\mathcal{C}$, $\tau_A : K_1(A) \longrightarrow \mathfrak{G}(A)$ is an epimorphism.

(2) For any division algebra D and a nonnegative integer n, the following diagram commutes,

$$\begin{array}{ccc} K_1(M_n(D)) & \xrightarrow{\tau} & \mathfrak{G}(M_n(D)) \\ \Big\downarrow{\scriptstyle det} & & \Big\downarrow{\scriptstyle d} \\ K_1(D) & \xrightarrow{\tau} & \mathfrak{G}(D). \end{array}$$

For example, the functors of Example 2.3, or $\mathfrak{G}(D) = Nrd_{D/F}(D^*)/F^{*i(D)}$ and $\mathfrak{G}(A) = A^*/Nrd_{A/F}(A^*)A'$ enjoy the above property.

Even in this case, it is not clear when a reduced Dieudonne functor is trivial. The following theorem is almost the only known example where $\mathfrak{G}(D)$ with above property is trivial.

Corollary 2.8. *Let D be a division ring of quaternions over real-closed field. Let $\mathfrak{G}$ be a reduced Dieudonne functor with a natural epimorphism $\tau : K_1 \longrightarrow \mathfrak{G}$ as above. Then $\mathfrak{G}(D) = 1$.*

Proof. For any finite field extension L of $F = Z(D)$, the following diagram is commutative,

$$\begin{array}{ccc} K_1(D \otimes_F L) & \xrightarrow{\mathcal{P}} & K_1(D) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ \mathfrak{G}(D \otimes_F L) & \xrightarrow{\mathcal{P}} & \mathfrak{G}(D). \end{array}$$

Now since D is algebraically closed (See [7], Section 16), thanks to Proposition 2.5 and above diagram, $\mathfrak{G}(D \otimes_F L) \xrightarrow{\mathcal{P}} \mathfrak{G}(D)$ is an epimorphism. Replace L by $\overline{F}$, the algebraic closure of F. Because $\overline{F}$ is a splitting field for D, $\mathfrak{G}(D \otimes_F \overline{F}) = \mathfrak{G}(M_2(\overline{F}))$. We show that $\mathfrak{G}(M_2(\overline{F}))$ is a trivial group and hence the corollary follows. Since $\tau : K_1 \longrightarrow \mathfrak{G}$ is a natural epimorphism, there is a sequence of homomorphism

$$\psi : K_1(\overline{F}) = \overline{F}^* \xrightarrow{\simeq} K_1(M_2(\overline{F})) \xrightarrow{epi.} \mathfrak{G}(M_2(\overline{F})).$$

Take $x \in \mathfrak{G}(M_2(\overline{F}))$. Since $\overline{F}$ is algebraically closed, there exist $y \in K_1(\overline{F}) = \overline{F}^*$ such that $\psi(y^2) = x$. But $\mathfrak{G}(M_2(\overline{F}))$ is a torsion group of bounded exponent 2, hence $x = 1$. This shows that $\mathfrak{G}(M_2(\overline{F}))$ is trivial and the proof is complete. $\square$

Back to the functorial properties of $\mathfrak{G}$, the next step is to replace the field L in Proposition 2.5 by a division ring. The following proposition shows that the same result holds here too.

Proposition 2.9. *Let $\mathfrak{G}$ be a Dieudonne functor. Let A and B be division rings with center F such that $[B : F]$ is finite. Then there is a homomorphism $\mathcal{P}$: $\mathfrak{G}(A \otimes_F B) \longrightarrow \mathfrak{G}(A)$ such that $\mathcal{P}\mathcal{I} = \eta_{[B:F]}$.*

Proof. Let $[B : F] = m$. We have the following sequence of F-algebra homomorphisms,

$$A \longrightarrow A \otimes_F B \longrightarrow A \otimes_F B \otimes_F B^{op} \longrightarrow A \otimes_F M_m(F) \longrightarrow M_m(A).$$

This implies the group homomorphism $\mathcal{P} : \mathfrak{G}(A \otimes_F B) \longrightarrow \mathfrak{G}(M_m(A)) \xrightarrow{d} \mathfrak{G}(A)$. The rest of the proof is similar to one in Proposition 2.5. $\square$

Note that in the above proposition A could be of infinite dimension over its center F. A same statement as Corollary 2.7 could be obtained here too. In particular if $(i(A) : i(B)) = 1$ then $\mathfrak{G}(A)$ embeds in $\mathfrak{G}(A \otimes_F B)$ and similarly for B. Employing torsion theory of groups and sequences which are appeared in the above propositions, we can write the primary decomposition for $\mathfrak{G}(D)$. The proof follows more or less the same pattern as for $SK_1(D)$.

Theorem 2.10. *Let $\mathfrak{G}$ be a reduced Dieudonne functor. Let A and B be division algebras with center F such that $(i(A), i(B)) = 1$. Then $\mathfrak{G}(A \otimes_F B) = \mathfrak{G}(A) \times \mathfrak{G}(B)$.*

Proof. By Corollary 2.6, $\mathfrak{G}(A \otimes_F B)$ is a torsion group of bounded exponent m^2n^2 where $m = i(A)$ and $n = i(B)$. Therefore $\mathfrak{G}(A \otimes_F B) \simeq \mathcal{G} \times \mathcal{H}$, where $exp(\mathcal{G})|m^2$ and $exp(\mathcal{H})|n^2$. By Proposition 2.9, we have the sequence:

$$\mathfrak{G}(A) \xrightarrow{\phi} \mathfrak{G}(A \otimes_F B) \xrightarrow{\psi} \mathfrak{G}(A \otimes B \otimes B^{op}) \xrightarrow{\theta} \mathfrak{G}(A) \tag{2.2}$$

such that $\theta\psi\phi = \eta_{n^2}$. Hence $\mathfrak{G}(A) = \eta_{n^2}\eta_{n^2}(\mathfrak{G}(A)) = \eta_{n^2}\theta\psi\phi(\mathfrak{G}(A)) \subseteq \theta\psi\eta_{n^2}(\mathcal{G} \times \mathcal{H}) = \theta\psi(\mathcal{G}) \subseteq \mathfrak{G}(A)$. This shows that $\theta\psi|_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathfrak{G}(A)$ is surjective. Next we show that $\theta\psi|_{\mathcal{G}}$ is injective. Considering the regular representation $B^{op} \longrightarrow M_{n^2}(F)$, by Proposition 2.5, we have the following sequence

$$\mathfrak{G}(A \otimes_F B) \xrightarrow{\psi} \mathfrak{G}(A \otimes B \otimes B^{op}) \xrightarrow{\psi'} \mathfrak{G}(A \otimes B \otimes M_{n^2}(F)) \xrightarrow{\theta'} \mathfrak{G}(A \otimes_F B)$$

such that $\theta'\psi'\psi = \eta_{n^2}$. Now if $w \in \mathcal{G} - 1$, then $\theta'\psi'\psi(w) = \eta_{n^2}(w) = w^{n^2} \neq 1$. Therefore $\psi|_{\mathcal{G}}$ is injective. Rewrite the sequence (2.2) as follows:

$$\mathfrak{G}(A \otimes_F B) \xrightarrow{\psi} \mathfrak{G}(A \otimes B \otimes B^{op}) \xrightarrow{iso.} \mathfrak{G}(M_{n^2}(A)) \xrightarrow{d} \mathfrak{G}(A).$$

Suppose $x \in \mathcal{G}$ such that $\theta\psi(x) = 1$. The above sequence and the definition of reduced Dieudonne functor shows that $\psi(x)^{n^2} = 1$. Since $\psi|_{\mathcal{G}}$ is injective, $x^{n^2} = 1$. On the other hand because $exp(\mathcal{G})|m^2$ then $x^{m^2} = 1$. Since m and n are relatively prime, $x = 1$. This shows that $\theta\psi$ is an isomorphism and so $\mathfrak{G}(A) \simeq \mathcal{G}$. In the similar way it can be shown that $\mathfrak{G}(B) \simeq \mathcal{H}$. Therefore the proof is complete. ☐

Now we specialize a Dieudonne functor to $\mathfrak{G}(D) = D^*/F^*D'$ where D is a division ring with center F. It turns out that the structure of this group carries significant information about the arithmetic of D. Let us start with the following definition.

Definition 2.11. Let A be a central simple algebra with center F. For each integer $s \geq 0$ define $G_s(A) = A^*/F^{*s}A'$ and $\hat{G}_s(A) = \lim\limits_{\leftarrow k} G_{s^k}(A)$ where $k \geq 1$.

Clearly $G_0(A) = K_1(A)$ and $G_1(A) = A^*/F^*A'$. It is easy to see that $G_s(A)$ and $\hat{G}_s(A)$ are Dieudonne functors. Also $\mathfrak{G}(A) = G_1(A)$ is a reduced Dieudonne functor. It is immediate from Corollary 2.6 that $G_1(A)$ is a torsion group of bounded exponent $m[D : Z(D)]$ where $A = M_m(D)$. So it follows that $G_s(A)$ are all torsions. Note that $\hat{G}_s(A)$ is not torsion in general. As we mentioned above we specialize $\mathfrak{G}$ to the group $G(A) = A^*/F^*A'$. In this case we can reduce the bound of the group $G(A)$ as follows.

Let D be a division algebra over its center F. If $a \in D$ is algebraic over F of degree m, then by Wedderburn's factorization theorem, we can associate m conjugates to a such that the sum and the product of them are in F. The first observation is used, for instance, in [2] to compute the center of the group $D^*/1 + M_D$ (which we will use it in this note) where M_D is the maximal ideal of the valuation ring of D (See Section 3). The fact that the product of the conjugates is in F, shows that $a^{[F(a):F]}$ is in F^*D'. Therefore if D is an algebraic division algebra over its center F, then $G(D) = D^*/F^*D'$ is a torsion group. This is used in [10] to investigate the role of D' in the structure of a division ring.

Now let D be division algebra over its center F with index n. Let N be a normal subgroup of D^*. For $x \in N$, it follows that $Nrd_{D/F}(x)$ is the product of n conjugates of x. This shows that $Nrd|_N : N \longrightarrow Z(N)$ is well defined, where $Z(N) = F^* \cap N$ is the center of the group N. We will use the following lemma in Section 3 for normal subgroups of D^* which arise from a valuation on D.

Lemma 2.12. *Let D be a division ring over its center F with index n. Let N be a normal subgroup of D^*. Then $N^n \subseteq Z(N)[D^*, N]$.*

Proof. Let $x \in N$. As stated above, $Nrd_{D/F}(x) = d_1 x {d_1}^{-1} \cdots d_n x {d_n}^{-1}$. This implies that $x^n = Nrd_{D/F}(x) d_x$ where $d_x \in [D^*, N]$ and $Nrd_{D/F}(x) \in F^* \cap N$. This shows that $N^n \subseteq (F^* \cap N)[D^*, N]$. $\square$

If in the above lemma we take $N = D^*$, then for any $x \in D$, $x^n = Nrd_{D/F}(x) d_x$ where $d_x \in D'$. This in effect shows that $G(D) = D^*/F^*D'$ is a torsion group of bounded exponent n.

In the next section we will show yet another SK_1-like property for the group $G(D)$. Namely $G(D)$ satisfy the following stability, $G(D) \simeq G\big(D((x))\big)$ where $G\big(D((x))\big)$ is the division ring of Laurent series (Corollary 3.6). We close this section by the following theorem, which shows that the group $G(D) = D^*/F^*D'$ does not always follow the same pattern as the reduced Whitehead group $SK_1(D)$.

Theorem 2.13. **(J. -P. Tignol)** *Let D be a division algebra over its center F with index n. Then the following sequence where $\wp$ runs over the irreducible monic polynomials of $F[x]$ and $n_\wp$ is the index of $D \otimes_F F[x]/\wp$, is split exact.*

$$1 \longrightarrow G(D) \longrightarrow G(D(x)) \longrightarrow \bigoplus_{\wp} \frac{\mathbb{Z}}{n/n_\wp \mathbb{Z}} \longrightarrow 1.$$

Proof. By Proposition 7 in [8], the sequence

$$1 \longrightarrow K_1(D) \longrightarrow K_1(D(x)) \longrightarrow \bigoplus_{\wp} n_\wp/n\mathbb{Z} \longrightarrow 1$$

which is obtained from the localization exact sequence of algebraic K-theory is split exact. Now since the group $G(D)$ is the cokernel of the natural map $K_1(F) \longrightarrow K_1(D)$, applying the snake lemma to the commutative diagram,

$$\begin{tikzcd}
1 \arrow[r] & K_1(F) \arrow[r] \arrow[d] & K_1(F(x)) \arrow[r] \arrow[d] & \bigoplus_\wp \mathbb{Z} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_1(D) \arrow[r] & K_1(D(x)) \arrow[r] & \bigoplus_\wp n_\wp/n\mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

the result follows. $\square$

3. On The Group $G(D)$ Over Henselian Division Algebra

In this section we assume that D is a finite dimensional division algebra over a henselian field $F = Z(D)$. Recall that a valuation v on F is *Henselian* if and only if v has a unique extension to each field algebraic over F. Therefore v has a unique extension to D (see [6] and [15]). Denote by V_D, V_F the valuation rings, M_D, M_F their maximal ideals, and $\overline{D}, \overline{F}$ the residue division algebra and the residue field of D and F respectively. We also take Γ_D, Γ_F for the value groups, U_D, U_F for the groups of units of V_D, V_F respectively. Furthermore, we assume that D is a *tame* division algebra, i.e., $Z(\overline{D})$ is separable over $\overline{F}$ and $Char\overline{F}$ does not divide $i(D)$. The quotient group Γ_D/Γ_F is called the *relative valued group* of the valuation. In this setting it turns out that D is *defectless*, namely we have $[\overline{D}:\overline{F}][\Gamma_D:\Gamma_F]=[D:F]$. D is said to be *unramified* over F if $[\Gamma_D:\Gamma_F]=1$. At the other extreme D is said to be *totally ramified* if $[D:F]=[\Gamma_D:\Gamma_F]$. D is called *semiramified* if $\overline{D}$ is a field and $[\overline{D}:\overline{F}]=[\Gamma_D:\Gamma_F]=i(D)$. For a recent account of the theory of henselian valued division algebras see [6].

We start with the following theorem which describes a fundemental connection between the group $G(D)$ and its residue version.

Theorem 3.1. *Let D be a tame division algebra over a henselian field $F=Z(D)$ with index n. Let L/F be a subfield of D. Then the following sequence is exact.*

$$1\longrightarrow \overline{D}^*/\overline{L}^*\overline{D'}\longrightarrow D^*/L^*D'\longrightarrow \Gamma_D/\Gamma_L\longrightarrow 1. \tag{3.1}$$

Proof. Consider the normal subgroup $1+M_D$ of D^*. Thanks to Lemma 2.12, we have $(1+M_D)^n\subseteq((1+M_D)\cap F^*)[D^*,1+M_D]$. But applying the Hensel lemma to the elements of the group $1+M_D$, shows that, this group is *n-divisible*. Therefore $1+M_D=(1+M_D)^n$. Hence $1+M_D\subseteq(1+M_F)D'$. Now consider the reduction map $U_D\longrightarrow\overline{D}^*$. We have the following sequence:

$$\overline{D}^*\xrightarrow{\simeq}U_D/1+M_D\xrightarrow{nat.}U_D/(1+M_F)D'\xrightarrow{nat.}U_D/(1+M_L)D'\xrightarrow{nat.}$$
$$\xrightarrow{nat.}U_D/U_LD'\xrightarrow{\simeq}L^*U_D/L^*D'.$$

Therefore $\psi:\overline{D}^*/(\overline{L}^*)\overline{D'}\longrightarrow L^*U_D/L^*D'$ is an isomorphism. Considering the fact that $D^*/L^*U_D\simeq\Gamma_D/\Gamma_L$, the theorem follows. $\square$

Now we are ready to compute $G(D)$ for some certain cases. The statements *i.* and *ii.* of the following theorem were first appeared in [5] using results from reduced K-theory.

Theorem 3.2. *Let D be a henselian division algebra tame over its center F with index n. Then*

i. If D is unramified over F then $G(D)\simeq G(\overline{D})$.
ii. If D is totally ramified over F then $G(D)=\Gamma_D/\Gamma_F$.
iii. If D is semiramified and $\overline{D}$ is cyclic over $\overline{F}$ then the following sequence where $N_{\overline{D}/\overline{F}}$ is the norm function, is exact.

$$1\longrightarrow N_{\overline{D}/\overline{F}}(\overline{D})/\overline{F}^n\longrightarrow G(D)\longrightarrow\Gamma_D/\Gamma_F\longrightarrow 1.$$

iv. If D is unramified and s is an integer relatively prime to $Char(\overline{F})$, then the sequence $1\longrightarrow\hat{G}_s(\overline{D})\longrightarrow\hat{G}_s(D)\longrightarrow\hat{\Gamma}_{D_{(s)}}$ is exact.

Proof. i. Writing (3.1) for $L=F$, we have:

$$1\longrightarrow\overline{D}^*/\overline{F}^*\overline{D'}\longrightarrow G(D)\longrightarrow\Gamma_D/\Gamma_F\longrightarrow 1.$$

Now if (D,v) is unramified, namely $[\Gamma_D:\Gamma_F]=1$, then $\overline{D}^*/\overline{F}^*\overline{D'}\simeq D^*/F^*D'$. On the other hand $Z(\overline{D})=\overline{F}$ and $D^*=F^*U_D$. Therefore, for $a,b\in D^*$, the element $c=aba^{-1}b^{-1}$ may be written in the form $c=\alpha\beta\alpha^{-1}\beta^{-1}$, where α and $\beta\in U_D$. This shows $\overline{D'}=\overline{D}'$, so $G(D)\simeq G(\overline{D})$.

ii. If D is totally ramified over F then $\overline{D} = \overline{F}$. Writing (3.1) for $L = F$, since the group $\overline{D}^*/\overline{F}^*\overline{D'}$ is trivial then $G(D) = \Gamma_D/\Gamma_F$.

iii. Let D be semiramified and $\overline{D}$ be cyclic over $\overline{F}$. Consider the norm function $N_{\overline{D}/\overline{F}} : \overline{D}^* \longrightarrow \overline{F}^*$. Moreover for any $x \in U_D$ we have $\overline{Nrd_{D/F}(x)} = N_{\overline{D}/\overline{F}}(\overline{x})$ (See [4]). This shows that $\overline{D'} \subseteq KerN_{\overline{D}/\overline{F}}$. But if $x \in KerN_{\overline{D}/\overline{F}}$ then by Hilbert theorem 90, there is a $\overline{a}$ such that $x = \overline{a}\sigma(\overline{a})^{-1}$, where σ is the generator of $Gal(\overline{D}/\overline{F})$. It is well known that the *fundemental homomorphism* $D^* \longrightarrow Gal(Z(\overline{D})/\overline{F})$ is surjective. Therefore $\sigma : \overline{D} \longrightarrow \overline{D}$ is of the form $\sigma(\overline{a}) = \overline{cac^{-1}}$, for some $c \in D^*$. This shows that $x \in \overline{D'}$. Therefore $KerN_{\overline{D}/\overline{F}} = \overline{D'}$. Now it is easy to see that $\overline{D}^*/\overline{F}^*\overline{D'} \simeq N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*n}$. So thanks to (3.1), $1 \longrightarrow N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*n} \longrightarrow G(D) \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 1$ is exact.

iv. We can write the similar statement as Theorem 3.1 for $L = F^s$, where s is an integer relatively prime to $Char\overline{F}$. In this case we only have to change Γ_L to $s\Gamma_F$. Hence we obtain the exact sequence:

$$1 \longrightarrow \overline{D}^*/(\overline{F}^*)^s\overline{D'} \longrightarrow D^*/F^{*s}D' \longrightarrow \Gamma_D/s\Gamma_F \longrightarrow 1.$$

Now because inverse limit is a left exact functor, we obtain the following exact sequence:

$$1 \longrightarrow \lim_{\longleftarrow_k} \overline{D}^*/(\overline{F}^*)^{s^k}\overline{D'} \longrightarrow \hat{G}_s(D) \longrightarrow \lim_{\longleftarrow_k} \Gamma_D/s^k\Gamma_F.$$

If D is unramified, thanks to *i.* $\overline{F} = Z(\overline{D})$ and $\overline{D'} = \overline{D}'$. therefore the result follows. $\square$

Remark 3.3. If $\overline{D}$ is a cyclic field extension of $\overline{F}$, a modification of the proof of *iii.* above shows that $N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*f} \longrightarrow \overline{D}^*/\overline{F}^*\overline{D'}$, where $[\overline{D} : \overline{F}] = f$, is always surjective. Therefore if $N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*f} = 1$ then $G(D) = \Gamma_D/\Gamma_F$.

Example 3.4. Let $\mathbb{C}$ be a field of complex numbers. Let $1 \neq \sigma \in Gal(\mathbb{C}/\mathbb{R})$ where $\mathbb{R}$ is real numbers. Then by Hilbert construction, $D = \mathbb{C}((x, \sigma))$ is a division ring with center $F = \mathbb{R}((x^2))$. We show that $G(D) = \mathbb{Z}_2$. D has a natural valuation such that $\Gamma_D/\Gamma_F = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. Clearly $\overline{D} = \mathbb{C}$ and $\overline{F} = \mathbb{R}$. Since $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}) = \mathbb{R}^2$ by Theorem 3.2 *iii.*, $G(D) = \Gamma_D/\Gamma_F = \mathbb{Z}_2$.

There have been significant results on the structure of relative valued group in the case of totally ramified algebra. Using Theorem 3.2 we can write interesting statements relate group structure of $G(D)$ to algebraic structure of D. Recall that the group $G(D)$ is torsion of bounded exponent n.

Theorem 3.5. *Let D be a valued division algebra tame and totally ramified over henselian field $F = Z(D)$ of index n. Then,*

i. There is a one to one correspondence between isomorphism classes of F-subalgebra of D and the subgroups of $G(D)$.

ii. $exp(G(D))$ divides the exponent of D, i.e., the order of $[D]$ in $Br(F)$, the Brauer group of F.

iii. D is cyclic division algebra if and only if $exp(G(D)) = n$.

Proof. The theorem follows by comparing Theorem 3.2 *ii.*, with the results on relative valued group in the case of totally ramified valuation (See for example [15]). $\square$

Corollary 3.6. *Let D be a finite dimensional division algebra over its center F such that $Char F \nmid i(D)s$. Then there is a monomorphism $\hat{G}_s(D) \longrightarrow \hat{G}_s(D((x)))$. In particular if $Char F \nmid i(D)$ then $G(D) \backsimeq G(D((x)))$.* $\square$

Now we are in a position to use the group $G(D)$ to compute $SK_1(D)$. The following theorem enables us to compute $SK_1(D)$ when, roughly speaking, $G(\overline{D})$ is trivial. Note that we do not use any results from reduced K-theory.

Theorem 3.7. *Let D be a tame division algebra over henselian field $F = Z(D)$ of index n.*

i. If $\overline{D}^*/\overline{F}^*\overline{D'} = 1$ then $SK_1(D) = \mu_n(F)/Z(D')$.

ii. If D is a cyclic division algebra with a maximal cyclic extension L/F such that $\overline{D}^*/\overline{L}^*\overline{D'} = 1$ then $SK_1(D) = 1$.

Proof. i. As the proof of Theorem 3.1 shows, we have a natural isomorphism,

$$\psi : \overline{D}^*/(\overline{F}^*)\overline{D'} \longrightarrow U_D/U_F D'.$$

Now if $\overline{D}^*/\overline{F}^*\overline{D'} = 1$ then $U_D = U_F D'$. But $D^{(1)} \subseteq U_D$. This shows that $D^{(1)} = \mu_n(F)D'$. Using the fact that $\mu_n(F) \cap D' = Z(D')$ the theorem follows.

ii. The same proof as *i.* shows that if $\overline{D}^*/\overline{L}^*\overline{D'} = 1$ then $U_D = U_L D'$. Therefore $D^{(1)} \subseteq U_L D'$. Let $x \in D^{(1)}$. Then $x = ld$ where $l \in L$ and $d \in D'$. So $Nrd_{D/F}(x) = N_{L/F}(l) = 1$. Hilbert theorem 90 for the cyclic extension L/F guarantee that $l = a\sigma(a)^{-1}$, where σ is a generator of $Gal(L/F)$. Now Skolem-Noether theorem implies that $\sigma(a) = cac^{-1}$ where $c \in D^*$. Therefore $l = aca^{-1}c^{-1}$. This shows that $D^{(1)} = D'$. $\square$

The part *i.* of the above theorem shows that if D is totally ramified, then $SK_1(D) = \mu_n(F)/Z(D')$.

We conclude both theorems of [9] which are obtained by using a heavy machinery of reduced K- theory, as natural examples of the above theorem.

Example 3.8. *For any division algebra D with center $F = \mathbb{R}((x_1, \cdots, x_m))$ where $\mathbb{R}$ is the real numbers, $SK_1(D)$ is trivial.*

Proof. From number theory, it is well known that $[D : F] = 2^s$ where $s \leq m$. Since the complete field $F = \mathbb{R}((x_1, \cdots, x_m))$ has a natural valuation, then D enjoys a valuation which is obviously tame. It is clear that $\overline{F} = \mathbb{R}$. Because the only division algebras over real numbers are either the quaternion $\mathbb{H}_{\mathbb{R}}$ or the field $\mathbb{C}$ of complex numbers, therefore $\overline{D} = \mathbb{H}_{\mathbb{R}}$ or $\overline{D} = \mathbb{C}$. Now Corollary 2.6 and Remark 3.3 show that in either case $\overline{D}^*/\overline{F}^*\overline{D'} = 1$. Now by Theorem 3.7, $SK_1(D) \simeq \mu_{i(D)}(F)/Z(D')$. But clearly $\mu_{i(D)}(F) = \{1, -1\}$ and because the index of D is even, $-1 \in D'$. This implies that $SK_1(D) = 1$.

Example 3.9. *For any division algebra with center $F = C((x_1, \cdots, x_m))$ where C is an algebraically closed field, $SK_1(D)$ is cyclic.*

Example 3.10. *Hilbert classical construction of division algebras.* Let L be a field and $\sigma \in Aut(L)$ such that $o(\sigma) = n$. Let $F = Fix(\sigma)$ be the fixed field of σ. Hence $Gal(L/F)$ is a cyclic group with the generator σ. Let $D = L((x, \sigma))$ be the division ring of formal Laurent series. It follows that $Z(D) = F((x^n))$ and $i(D) = n$. D has a natural valuation, and it is easy to see that with this valuation D is semiramified and $L((x^n))$ is a maximal subfield of D. Now by Theorem 2.7 *ii.*, $SK_1(D)$ is trivial.

Example 3.11. From Theorem 3.7 *ii.*, it is immediate that reduced Whitehead group of division algebra over a local field is trivial.

Because most of the interesting valued division algebras arise from the iterated formal power series fields, we may consider *r-iterated Henselian division algebras.* Following Platonov in [13], we define inductively an r-iterated henselian field F if its residue field $\overline{F}$ is an $(r-1)$-iterated henselian field.[1] Let $(D_i, v_i), 0 \leq i \leq r-1$ be an r iterated henselian division algebra ($\overline{D_i} = D_{i+1}$). Let $\Phi_i : U_{D_{i-1}} \longrightarrow D_i$ be the i-th natural reduction map. Then $\Phi_i(\Phi_{i-1}(\cdots(\Phi_1(a))\cdots))$ is called i-iterated reduction, if it is defined. Denote the r iterated henselian division algebra by D, $(D = D_0, Z(D) = F = F_0)$. We also need the following notations in order to state the following lemma. By $[U_D]_i$ and $[U_F]_i$ we denote the set of all elements of D and F respectively, such that i iterated reduction defines. Also by $[1 + M_D]_i$ and $[1 + M_F]_i$, we denote the subsets of $[U_D]_i$ and $[U_F]_i$ such that i iterated reduction equals one. Clearly $[1 + M_F]_1 = 1 + M_F$. we can write the main lemma of [5] in this setting.

Lemma 3.12. *Let D be a i iterated tame division algebra of finite dimension over a henselian field $F = Z(D)$ with index n.*

i. For each $a \in [1 + M_D]_i$ there is $b \in [1 + M_F]_i$ such that $ab \in D'$.
ii. $[1 + M_D]_i \subsetneqq [1 + M_F]_i D'$.

Proof. i. Let $a \in [1+M_D]_i$. Then a is contained in a maximal subfield of D, say L. Therefore $a \in [1+M_L]_i$. By lemma 3 [13], we have $N_{L/F}([1+M_L]_i) = [1+M_F]_i$. So $Nrd_{D/F}(a) = N_{L/F}(a) \in [1+M_F]_i$. Let $t = Nrd_{D/F}(a)$. Using inductive argument for Hensel lemma, we will show that there exist $c \in [1+M_F]_i$ such that $c^n = t$. Let $s \in 1+M_F = [1+M_F]_1$. Applying Hensel lemma for $f(x) = x^n - s$ gives $c \in 1+M_F$ such that $c^n = s$. Now it is not hard to see that $\Phi_1([1 + M_F]_i) = [1 + M_{\overline{F}}]_{i-1}$. Therefore $[1 + M_F]_i/[1 + M_F]_1 \simeq [1 + M_{\overline{F}}]_{i-1}$. Now by induction, we conclude that $[1 + M_F]_i$ is n-divisible. Therefore exist $c \in [1 + M_F]_i$ such that $c^n = t$. Now $Nrd_{D/F}(a) = c^n$. So $Nrd_{D/F}(ac^{-1}) = 1$. Hence $ac^{-1} \in D^{(1)} \cap [1 + M_D]_i$. Applying the Platonov's generalized congruence theorem (cf. [13] and [4]) , we obtain $ac^{-1} \in D'$. Take $b = c^{-1}$ and the proof is complete.

ii. Applying the first part of the lemma for $i = 1$, in each step of reduction we have, $1 + M_{D_i} \subseteq (1 + M_{K_i}){D_i}'$ where $K_i = Z(D_i)$. First we show that in each step of reduction, ${D_i}' \nsubseteq 1 + M_{D_i}$. Consider the groups $\Delta = {D_i}^*/1 + M_{D_i}$ and $P(D_i) = (1 + M_{K_i}){D_i}'/(1 + M_{D_i})$. One can easily observe that $P(D_i) = \Delta'$ and as Theorem 2.11 of [2] shows, the center of Δ is ${K_i}^*(1 + M_{D_i})/(1 + M_{D_i})$. We claim that Δ is not an abelian group, for otherwise $U_{D_i} = U_{K_i}(1 + M_{D_i})$ which implies that D_i is totally ramified. Thus $\overline{{D_i}'} = \mu_e(\overline{K_i})$, where $e = exp(\Gamma_{D_i}/\Gamma_{K_i})$,

[1] See Ershov's comment in [3] on iterated valued field. Among other things, Considering iterated valued field, enables us to have more insight in each step of reduction.

(cf. the proof of Theorem 3.1 of [15]) which leads us to a contradiction. Therefore ${D_i}'$ is not in $1+M_{D_i}$ and Δ is not abelian. But $\Phi^{-1}(1+M_{D_i})=[1+M_D]_{i+1}$. If $D'\subseteq[1+M_D]_{i+1}$ then $\Phi(D')\subseteq\Phi([1+M_D]_{i+1})=1+M_{D_i}$. But ${D_i}'\subseteq\Phi(D')$ so ${D_i}'\subseteq 1+M_{D_i}$ a contradiction. $\square$

Remark 3.13. In the proof of *i.* above, we could use Lemma 2.12 and avoid the Platonov congruence theorem.

Theorem 3.14. *Let D be a tame division algebra of r iterated over henselian field F of index n. If there is an $0\leq\ell\leq r-1$ such that $\overline{D}_\ell/\overline{F}_\ell\overline{D'_\ell}=1$ then $SK_1(D)\simeq\mu_n(F)/Z(D')$.*

Proof. For any $0\leq k\leq r-1$, consider the $k+1-th$ reduction map

$$[U_D]_{k+1}\overset{\Phi_{k+1}\Phi_k\cdots\Phi_1}{\longrightarrow}\overline{D_k}^*.$$

Thanks to Lemma 3.12 *i.*, we have:

$$\overline{D_k}^*\overset{\simeq}{\to}[U_D]_{k+1}/[1+M_D]_{k+1}\overset{nat.}{\to}[U_D]_{k+1}/[1+M_F]_{k+1}D'\overset{nat.}{\to}[U_D]_{k+1}/[U_F]_{k+1}D'.$$

Therefore,

$$\overline{D_k}^*/\overline{F_k}^*\overline{D'_k}\overset{\simeq}{\longrightarrow}[U_D]_{k+1}/[U_F]_{k+1}D'.$$

Hence if there is a ℓ such that $\overline{D_\ell}^*/\overline{F_\ell}^*\overline{D'_\ell}^*=1$ then $[U_F]_{\ell+1}D'=[U_D]_{\ell+1}$. By lemma 1 in [13] $D^{(1)}\subseteq[U_D]_{\ell+1}$ so $D^{(1)}=\mu_n(F)D'$. Using the fact that $\mu_n(F)\cap D'=Z(D')$ the theorem follows. $\square$

Considering the fact that each henselian division algebra is 1 iterated division algebra, we recover Theorem 3.7 from the above theorem.

4. On The Unitary Setting

In this section we introduce the unitary version of the group $G(D)$ and obtain the similar results in the unitary setting. Let D be a division ring with an involution τ over its center F with index n. Let $S_\tau(D)=\{a\in D|a^\tau=a\}$ be the subspace of symmetric elements and $\Sigma_\tau(D)$ the subgroup of D^* generated by nonzero symmetric elements. Here we concentrate on the involution of the first kind, i.e. $\Sigma_\tau(D)\cap F^*=F^*$.

Definition 4.1. Let D be a division ring with an involution τ. Then the group $KU_1(D)=D^*/\Sigma_\tau(D)D'$ is called unitary Whitehead group and the $GU(D)=\Sigma_\tau(D)D'/F^*D'$ the unitary version of $G(D)$.

We will prove that there is a stability theorem for $GU(D)$ similar to one in Corollary 3.6. The first part of the following theorem was first proved by Platonov and Yanchevskii [14].

Theorem 4.2. *Let D be a finite dimensional tame and unramified division algebra with an involution of the first kind over a henselian field Z(D)=F. Then $KU_1(\overline{D})\backsimeq KU_1(D)$ and $GU(\overline{D})\backsimeq GU(D)$.*

Proof. Consider the following sequence:

$$\overline{D}^* \longrightarrow U_D/1+M_D \longrightarrow F^*U_D/F^*(1+M_D) \longrightarrow D^*/\Sigma_\tau(D)D'.$$

Because the valuation is unramified, we have $\Sigma_{\overline{\tau}}(\overline{D}) = \overline{\Sigma_\tau(D) \cap U_D}$ (See [14]), and $\overline{D}' = \overline{D'}$ (See Theorem 3.2 *i.*). Therefore we have the following isomorphism: $\overline{D}^*/\Sigma_{\overline{\tau}}(\overline{D})\overline{D}' \xrightarrow{\backsimeq} D^*/\Sigma\tau(D)D'$.

For the second part, consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & GU(\overline{D}) \arrow[r] \arrow[d] & G(\overline{D}) \arrow[r] \arrow[d, "iso."] & KU_1(\overline{D}) \arrow[r] \arrow[d, "iso."] & 1 \\
1 \arrow[r] & GU(D) \arrow[r] & G(D) \arrow[r] & KU_1(D) \arrow[r] & 1.
\end{tikzcd}$$

The two of the vertical arrows are isomorphisms, thanks to the first part of this theorem and Theorem 3.2 *i.*. Therefore the third one is also isomorphism which completes the proof. □

If D has an involution of the first kind, then $D((x))$ enjoys a natural involution which is induced by the one from D. Therefore if $Char F \nmid i(D)$ then thanks to the above theorem, we have $GU(D) \backsimeq GU(D((x)))$ which is a stability for $GU(D)$.

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