On hermitian trace forms over hilbertian fields

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Abstract: Let k be a field of characteristic different from 2. Let E/k be a finite separable extension with a k-linear involution σ. For every σ-symmetric element $\mu \in E^*$, we define a *hermitian scaled trace form* by $x \in E \mapsto \mathrm{Tr}_{E/k}(\mu x x^\sigma)$. If $\mu = 1$, it is called a *hermitian trace form*. In the following, we show that every even-dimensional quadratic form over a hilbertian field, which is not isomorphic to the hyperbolic plane, is isomorphic to a hermitian scaled trace form. Then we give a characterization of Witt classes of hermitian trace forms over some hilbertian fields.

Introduction: In this paper, the words "quadratic form" are reserved to mean "non-degenerate quadratic form". Let k be a field of characteristic different from 2. If E/k is a finite separable extension and $\lambda \in k^*$, we can define a quadratic form $E \to k, x \mapsto \mathrm{Tr}_{E/k}(\lambda x^2)$, denoted by $\mathrm{Tr}_{E/k}(<\lambda>)$. Such a form is called a *scaled trace form.* If $\lambda = 1$ this form is called *the trace form of E/k.* Recall that a field k is hilbertian if Hilbert's irreducibility theorem holds. To simplify, k is hilbertian if for all $n, m \geq 1$ and for all irreducible polynomial $P \in k(Y_1, \cdots, Y_m)[X_1, \cdots, X_n]$, there exist infinitely many specializations $(y_1, \cdots, y_m) \in k^m$ such that $P(y_1, \cdots, y_m, X_1, \cdots, X_n)$ is still irreducible (for a precise statement of this theorem, see [La] for example). A natural problem is to know which quadratic forms over k are isomorphic, or more reasonably Witt-equivalent, to a (scaled) trace form. No answer has been given in general, but Scharlau and Waterhouse gave independently a characterization of scaled trace forms over a hilbertian field (see Theorem 1 below). The characterization of trace forms, which is much more difficult, has been initiated by Conner and Perlis, who were interested in the following question: which quadratic forms over $\mathbb{Q}$ are Witt-equivalent to a trace form ? In [C-P], they showed that such forms are precisely the positive quadratic forms (Recall that a quadratic form over a field k is called *positive* if all its signatures are non-negative). In [S], Scharlau showed that the result is always true when k is a number field. Finally, Krüskemper and Scharlau proved the validity of this result for some hilbertian fields (see Theorem 3 below). Note that a characterization of isometry classes of trace forms has been obtained by Epkenhans in [E] when k is a number field.

Assume now that there exists a k-linear non-trivial involution σ on E. For each σ-symmetric element $\mu \in E^*$, we can also define a quadratic form $E \to k, x \mapsto \mathrm{Tr}_{E/k}(\mu x x^\sigma)$, denoted by $\mathrm{Tr}_{E/k}(< \mu >_\sigma)$. Such a form is called a *hermitian scaled trace form.* If $\mu = 1$, this form is called *the hermitian trace form of E/k relative to σ*. Note that the existence of σ implies that $[E : k]$ is even because the subfield E_0 fixed by the involution verifies $[E : E_0] = 2$. In this paper, we give similar characterizations of hermitian trace forms and hermitian scaled trace forms over hilbertian fields. A complete characterization of hermitian trace forms of number fields can be found in [B1].

A. Scaled trace forms and hermitian scaled trace forms over hilbertian fields.

1. Scaled trace forms over hilbertian fields.

In [S] and [W], Scharlau and Waterhouse respectively proved the following theorem:

Theorem 1: *Let k be a hilbertian field of characteristic different from 2. Then every quadratic form over k is isomorphic to a scaled trace form.*

2. Hermitian scaled trace forms over hilbertian fields.

In the following, we prove a similar theorem concerning hermitian scaled trace forms. In fact, we show:

Theorem 2: *Let k be a hilbertian field of characteristic different from 2. Then every even-dimensional quadratic form over k, which is not isomorphic to the hyperbolic plane, is isomorphic to a hermitian scaled trace form.*

3. Proof of theorem 2.

The underlying idea of the proof is the following principle from the general theory of hermitian forms used by Scharlau in [S] in order to prove the theorem 1 above. An hermitian operator over an arbitrary field k is a triple (V, b, u), where (V, b) is a regular symmetric bilinear space and $u : V \to V$ is a self-adjoint operator, that is, $b(ux, y) = b(x, uy)$ for all x, y in V. Let us assume that the characteristic polynomial p of u is separable and irreducible, and let $L = k[X]/(p)$. Thus, L is a field and $[L : k] = \dim_k(V)$. Then the law

$L\times V \to V, (\overline{R}, v) \mapsto R(u)(v)$ endows V with a structure of an L-vector space. Moreover, $\dim_L(V) = \dfrac{\dim_k(V)}{[L:k]} = 1$, so we can identify V and L. Since L/k is separable, the k-bilinear form $(x,y) \mapsto \mathrm{Tr}_{L/k}(xy)$ is non-degenerate, so we get a k-isomorphism $L \to \mathrm{End}_k(L,k), s \mapsto (a \in L \mapsto \mathrm{Tr}_{L/k}(as))$. Since $L \to k, a \mapsto b(ax,y)$ is k-linear, there exists a unique element $B(x,y)$ of L such that $\mathrm{Tr}_{L/k}(aB(x,y)) = b(ax,y)$ for all $a \in L$. It is easy to see that B is symmetric and bilinear. Writing $(V,B) =< \lambda >$, we get $\mathrm{Tr}_{L/k}(< \lambda >) \simeq (V,b)$.
Assume that $\dim_k(V) = 2n$ and that p is even. Then $L = k(\alpha)$ has a subfield of codimension 2, namely $k(\alpha^2)$. Suppose that we have
$\alpha^2 \not\equiv -1 \pmod{k(\alpha^2)^{*2}}$, and that $\lambda \in k(\alpha^2)$. One can easily check that $\mathrm{Tr}_{L/k}(< \lambda >) \simeq \mathrm{Tr}_{k(\alpha^2)/k}(< 2\lambda >) \perp \mathrm{Tr}_{k(\alpha^2)/k}(< 2\lambda\alpha^2 >) \simeq$
$\mathrm{Tr}_{k(\sqrt{-\alpha^2})/k}(< \lambda >_\sigma)$, where σ is the k-linear involution of L defined by
$\sigma(\alpha) = -\alpha$ and $\sigma|_{k(\alpha^2)} = \mathrm{Id}$.
Let $q \simeq< s_1, \cdots, s_{2n} >$ be a non-degenerate quadratic form over k, and D the corresponding diagonal matrix. Notice that $U = BD$ is hermitian if and only if B is symmetric, that is $U^tD = DU$. In order to keep the notations of [B2], let $A = U^t$. So we have $D^{-1}AD = A^t$. By the above arguments, it suffices to find a matrix A with an even separable and irreducible characteristic polynomial p with a root α such that $\alpha^2 \not\equiv -1 \pmod{k(\alpha^2)^{*2}}$ and which verifies the above equality, and then to show that we have $\lambda \in k(\alpha^2)$. Unfortunately, we are unable to show that $\lambda \in k(\alpha^2)$ using this point of view, so the proof of the proposition below, which is sufficient to prove theorem 2, uses another method to get the existence of λ, though the underlying idea is the same.

Before starting the proof, we recall the following lemma, proved by O.Taussky in [T], first proof of theorem 1:

Lemma 1: *Let k be a field of characteristic different from 2. Let $A \in M_n(k)$ with irreducible separable characteristic polynomial. If α is an eigenvalue of A, and $\mathbf{v}_\alpha = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ is a corresponding eigenvector, then $(v_1, \cdots, v_n)$ is a k-basis of $k(\alpha)$ and their exists an eigenvector $\mathbf{v}'_\alpha = \begin{pmatrix} v'_1 \\ \vdots \\ v'_n \end{pmatrix}$ of A^t corresponding to α such that $(v'_1, \cdots, v'_n)$ is the dual basis of $(v_1, \cdots, v_n)$ relative to $\mathrm{Tr}_{k(\alpha)/k}$.*

For the reader's convenience, we recall the sketch of the proof. Denote by $v_j^{(i)}$ the i^{th} conjugate of v_j. Let $M = (v_j^{(i)})$ and let c_{i1} be the cofactor of $v_1^{(i)}$. Then let $v'_i = \dfrac{c_{i1}}{\det(M)}$. To be a little more concrete, consider M^t. The columns of this matrix are the eigenvectors for A, with v_α as the first column. Then the entries of the corresponding v'_α lie in the first row of the inverse $(M^t)^{-1}$. It can be shown that the coordinates of this vector belong to $k(\alpha)$. Using the properties of the determinant, one can easily show that the vector defined by these n elements is an eigenvector of A^t corresponding to α. Moreover, the j^{th} conjugate of v'_i is $\dfrac{c_{ij}}{\det(M)}$. Thus, $\mathrm{Tr}_{k(\alpha)/k}(v_i v'_k) = \sum_{j=1}^{n} v_i^{(j)} \dfrac{c_{kj}}{\det(M)}$, and the relation $\det(M) I_n = (\mathrm{com}(M))^t M$ (where $\mathrm{com}(M)$ is the matrix of cofactors associated to M) shows that the last sum is equal to δ_{ik}.

In fact, O.Taussky proved this lemma for $k = \mathbb{Q}$ and $A \in M_n(\mathbb{Z})$, but her proof is also valid under our hypotheses. Indeed, during this part of the proof of Theorem 1, she did not use the fact that the entries of A are integers, and she only used the separability of $\mathbb{Q}(\alpha)/\mathbb{Q}$.

Proof of theorem 2: An easy computation shows that any hermitian scaled trace form of a quadratic extension of k is isomorphic to $< 2\lambda, -2\lambda D >$, with $\lambda, D \in k^*$, $D \notin k^{*2}$. Comparing the discriminants shows that the hyperbolic plane can't be realized as a hermitian scaled trace form. Moreover, it is clear that any binary quadratic form, which not isomorphic to the hyperbolic plane, is isomorphic to a hermitian scaled trace form, so the case $n = 1$ is completely handled. For the other cases, the comments given at the beginning imply that it suffices to prove the following proposition:

Proposition 1: *Let k be an hilbertian field of characteristic different from 2. Let D be a diagonal matrix of $M_{2n}(k), n \geq 2$ with $\det(D) \neq 0$. Then there exist an element α which is separable algebraic of degree $2n$ over k, such that $\alpha^2 \not\equiv -1 \pmod{k(\alpha^2)^{*2}}$, $\lambda \in k(\alpha^2)$ and a basis of $(v_1, \cdots, v_{2n})$ of $k(\alpha)$ such that $D = (\mathrm{Tr}_{k(\alpha)/k}(\lambda v_i v_j))$.*

Proof of the proposition:

• First, we show the existence of a matrix $A_{2n} \in M_{2n}(k)$ with an even irreducible and separable characteristic polynomial $\chi_{A_{2n}}$ such that $D^{-1} A_{2n} D = A_{2n}^t$. We construct a tridiagonal matrix as follows.
Let $T_4, T_6, \cdots, T_{2n}$ and X be independent indeterminates over k.
For $2 \leq m \leq n$, let

$$B_{2m}(T_4,T_6,\cdots,T_{2m}) = \begin{pmatrix} 0 & T_{2m} & & & & & & & & \\ T_{2m} & 0 & 1 & & & & & & & \\ & 1 & 0 & T_{2m-2} & & & & & & \\ & & T_{2m-2} & 0 & 1 & & & & & \\ & & & 1 & 0 & \ddots & & & & \\ & & & & \ddots & \ddots & T_4 & & & \\ & & & & & T_4 & 0 & 1 & & \\ & & & & & & 1 & 0 & 1 \\ & & & & & & & 1 & 0 \end{pmatrix}.$$

If $D = diag < s_{2n},\cdots,s_1 >$, then set $D_{2m} = diag < s_{2m},\cdots,s_1 >$ for $2 \leq m \leq n$. Now define $\Delta_{2m}(T_4,\cdots,T_{2m},X) = \det(B_{2m} - XD_{2m}^{-1})$. Since this tridiagonal matrix has zeros on the diagonal, its characteristic polynomial is even (and so is Δ_{2m}). Indeed, for any matrix $C \in M_{2m}(k)$ with the previous properties, it is easy to see that $Q^{-1}CQ = -C$, where $Q = diag < 1,-1,1,-1,\cdots,1,-1 >$. Then we get $\chi_{Q^{-1}CQ}(X) = \chi_C(X) = \chi_{-C}(X) = \det(-C - XI_{2m}) = \det(C + XI_{2m}) = \chi_C(-X)$. It is shown in [B2] that Δ_{2n} is an irreducible polynomial of $k(T_4,T_6,\cdots,T_{2n})[X]$ (In [B2], it is shown for $k = \mathbb{Q}$ but it is easy to see that this proof is still valid when k is hilbertian).

Now, we prove that it is separable in X. It is well known that an irreducible polynomial is separable if and only if the derived polynomial is not zero. As a polynomial in X, $\Delta_4(T_4,X) = (s_1s_2s_3s_4)^{-1}X^4+$ (lower degree terms). Since all $s_i \neq 0$ and $char(k) \neq 2$, the polynomial $\dfrac{\partial\Delta_4}{\partial X}(T_4,X)$ has nonzero X^3-coefficient, so Δ_4 is separable in X. If $m > 2$, let P_{2m} be the determinant of the matrix obtained by cancellation of the first row and the first column of $B_{2m} - XD_{2m}^{-1}$. Thus, we have $\Delta_{2m} = -s_1^{-1}XP_{2m} - T_{2m}^2\Delta_{2m-2}$. Since P_{2m} does not depend on T_{2m}, we get $\dfrac{\partial\Delta_{2m}}{\partial X} = U + \dfrac{\partial\Delta_{2m-2}}{\partial X}T_{2m}^2$, where U is a polynomial which does not depend on T_{2m}. The separability follows by induction.

By Hilbert's irreducibility theorem, we can find $t_4,\cdots,t_{2m} \in k$ such that $\Delta_{2n}(t_4,\cdots,t_{2n},X)$ is a separable irreducible polynomial. Now $A_{2n} = DB_{2n}$ satisfies the required conditions, because the characteristic polynomial of this matrix is a scalar multiple of $\Delta_{2n}(t_4,\cdots,t_{2n},X)$.

• Let α be an eigenvalue of A_{2n}. Since the characteristic polynomial of this matrix has distinct roots, the matrix $A_{2n} - \alpha I_{2n}$ has rank $2n-1$, so there exists some non-zero cofactor. Let v_j be the j^{th} cofactor of $A - \alpha I_{2n}$

in this row and set $\mathbf{v}_\alpha = \begin{pmatrix} v_1 \\ \vdots \\ v_{2n} \end{pmatrix}$. Then $\mathbf{v}_\alpha \neq 0$ and the properties of the determinant show easily that $\mathbf{v}_\alpha$ is an eigenvector corresponding to α. By the lemma, there exists an eigenvector $\mathbf{v}'_\alpha = \begin{pmatrix} v'_1 \\ \vdots \\ v'_{2n} \end{pmatrix}$ of A^t_{2n} which corresponds to α such that $(v_1, \cdots, v_{2n})$ and $(v'_1, \cdots, v'_{2n})$ are dual bases with respect to $\mathrm{Tr}_{k(\alpha)/k}$. The relation $D^{-1}A_{2n}D = A^t_{2n}$ implies that $D^{-1}\mathbf{v}_\alpha$ is an eigenvector of A^t_{2n} corresponding to α. Since the irreducible polynomials of A_{2n} and its transpose are the same, and since $\chi_{A_{2n}}$ is separable, the characteristic spaces associated to A^t_{2n} have dimension one. So there exists $\lambda \in k(\alpha)$ such that $\lambda D^{-1}\mathbf{v}_\alpha = \mathbf{v}'_\alpha$. It is shown in [B2] that $\lambda \in k(\alpha^2)$ for this particular choice of $\mathbf{v}_\alpha$. Since we have $\lambda v_j = s_{2n-j+1}v'_j$ for all j, we get $\mathrm{Tr}_{k(\alpha)/k}(\lambda v_i v_j) = \mathrm{Tr}_{k(\alpha)/k}(s_{2n_j+1}v_i v'_j) = \delta_{ij}s_{2n-j+1}$, that is $D = (\mathrm{Tr}_{k(\alpha)/k}(\lambda v_i v_j))$.

• Now we show that we can choose α such that $\alpha^2 \not\equiv -1 \pmod{k(\alpha^2)^{*2}}$. Since Δ_{2m} is even, we can write $\Delta_{2m}(T_4, \cdots, T_{2m}, X) = U_{2m}(T_4, \cdots, T_{2m}, X^2)$. We can show (as for Δ_{2m}) by an inductive argument that the polynomial $F_{2m}(T_4, \cdots, T_{2m}, X) = U_{2m}(T_4, \cdots, T_{2m}, -X^2)$ is irreducible in $k(T_4, \cdots, T_{2m})[X]$ and separable for all m. Indeed, if we replace X^2 by $-X^2$ in the proof of the proposition in [Be2], we obtain a relation between F_{2m} and F_{2m-1}. Then we can conclude using suitable valuations as in [Be2]. Then, by Hilbert's irreducibility theorem, we can find a specialization of the T_i such that the specialized polynomials $\Delta_{2n}(X)$ and $F_{2n}(X)$ are irreducible and separable. Let α be a root of Δ_{2n}. By definition, $F_{2n}(\sqrt{-\alpha^2}) = \Delta_{2n}(\alpha) = 0$. Since $(-1)^n F_{2n}$ is monic and irreducible, we get $Irr(\sqrt{-\alpha^2}, k) = (-1)^n F_{2n}$. So we have $[k(\sqrt{-\alpha^2}) : k] = 2n$ and $[k(\sqrt{-\alpha^2}) : k(\alpha^2)] = 2$, which means that $-\alpha^2$ is not a square in $k(\alpha^2)$. This concludes the proof of the proposition.

B. Witt classes of trace forms and hermitian trace forms over hilbertian fields.

1. Definitions and notation.

We denote by $\sim$ the equivalence in $W(k)$ and by $\mathbb{H}$ the hyperbolic plane $< 1, -1 >$. *The stability index* $st(k)$ of a field k is the least integer s such that, for all $n > s$, and all $a_1, \cdots, a_n \in k^*$, there exist $b_1, \cdots, b_s \in k^*$ such that the n-fold Pfister forms $<< a_1, \cdots, a_n >>$ and $<< b_1, \cdots, b_s, 1, \cdots, 1 >>$ differ only by a torsion form in $W(k)$ (cf.[K-S2]). We say that a quadratic form ϕ is *totally positive definite* if $\mathrm{sign}_P \phi = \dim(\phi)$ for each ordering P of k. This is equivalent to say that there exists an isometry $\phi \simeq < a_1, \cdots, a_n >$ where every a_i is totally positive. Finally, a quadratic form is called *algebraic* (respectively *h-algebraic*) if it is Witt-equivalent to a trace form (respectively a hermitian trace form).

2. Witt classes of trace forms over hilbertian fields.

Krüskemper and Scharlau proved the following result:

Theorem 3:

0. *Let k be a hilbertian field. Then every 2-dimensional quadratic form is algebraic.*

1. *Let k be a hilbertian field, and let ϕ be a totally positive definite quadratic form. Then ϕ is algebraic.*

2. *Let k be a non formally real hilbertian field. Then every quadratic form is algebraic.*

3. *Let $k = R(X)$, where R is a real closed field. Then every positive quadratic form is algebraic.*

4. *More generally, let k be a hilbertian field such that $st(k) \leq 2$. Then every positive quadratic form is algebraic.*

The fourth first points can be found in [K-S1], and the last one can be obtained using theorems 2.1 and 2.3 of [K-S2].

We could hope that every positive quadratic form over a hilbertian field is algebraic, as it has been conjectured by Krüskemper and Scharlau in [K-S1]. In fact, this conjecture is false. Indeed, it becomes false as soon as $st(k) \geq 4$ (cf.[K-S2], theorem 2.4).

3. Witt classes of hermitian trace forms over hilbertian fields.

In the following, we prove

Theorem 4:

1. *Let k be a hilbertian field, and let ϕ be an even-dimensional quadratic form, which is totally positive definite. Then ϕ is h-algebraic.*
2. *Let k be a non formally real hilbertian field. Then every even-dimensional quadratic form is h-algebraic.*
3. *Let $k = R(X)$, where R is a real closed field. Then every even-dimensional positive quadratic form is h-algebraic.*
4. *More generally, let k be a hilbertian field such that $st(k) \leq 2$. Then every even-dimensional positive quadratic form is h-algebraic.*

4. Some useful results.

Proposition 2 (cf.[K-S2], Theorem 2.1): *Let k a hilbertian field. Then a sum of algebraic forms is algebraic.*

Proposition 3 (cf.[K-S1], Corollary 4): *Let k be a field. If every odd-dimensional positive quadratic form over k is algebraic, then every even-dimensional positive quadratic form $\phi \not\sim < -1, -1 >$ is Witt-equivalent to a form $\mathrm{Tr}_{L(\sqrt{\lambda})/k}(< 1 >), \lambda \notin L^{*2}$.*

For the reader's convenience, we recall the sketch of the proof:

By assumption, $\psi =< 2 > \phi \perp < 1 >$ is algebraic.
Thus we have $\psi \sim \mathrm{Tr}_{L/k}(< 1 >)$. Then there exists $\lambda \in L^*$ such that $\mathrm{Tr}_{\mathrm{L/k}}(< \lambda >) \sim < -1 >$ (cf.[K-S1], Lemma 5). λ is not a square, otherwise an easy computation shows that $\phi \sim < -1, -1 >$.
Then, $\mathrm{Tr}_{L(\sqrt{\lambda})/k}(< 1 >) \simeq \mathrm{Tr}_{L/k}(< 2, 2\lambda >) \sim < 2 > \psi \perp < -2 > \sim \phi$.

If we apply the previous proof to odd-dimensional quadratic forms, which are totally positive definite, we get:

Proposition 4: *Let k be a field. If every odd-dimensional and totally positive definite quadratic form is algebraic, then every even-dimensional and* *totally positive definite quadratic form* $\phi \not\sim < -1, -1 >$ *is Witt-equivalent to a form* $\mathrm{Tr}_{L(\sqrt{b})/k}(< 1 >)$. *In particular, it is true over any hilbertian field.*

The last statement can be proved using proposition 2 and the fact that every 1-dimensional positive form over a hilbertian field is algebraic (cf.[K-S1], theorem 2).

We finish this section by a well-known lemma (cf. [K-S1], lemma 5 for example):

Lemma 2: *Let* L/k *be a separable extension of odd degree. For any* $\delta \in k^*$ *there exists a* $\lambda \in L$ *such that* $\mathrm{Tr}_{L/k}(< \lambda >) \sim < \delta >$.

5. Proof of theorem 4.

1. Let ϕ be a form which verifies the hypotheses.
   Suppose first that $\phi \sim 0$ (this case can occur if k is non formally real). Let X, T be two indeterminates over k, and let
   $F(X,T) = X^4 - (2X^2 - 1)T^2$. Clearly, F is an irreducible polynomial of $k(X)[T]$ (it suffices to show that it has no roots in $k(X)$), so F is irreducible in $k[T][X]$. As a polynomial in X, F is monic so F is irreducible in $k(T)[X]$. Since k is hilbertian, there exists $t \in k$ such that $F(X,t)$ is still irreducible. Let θ be a root of $X^4 - 2t^2X^2 + t^2$ in an algebraic closure of k. Then $k(\theta)/k$ is of degree 4, and $k(\theta^2)/k$ is of degree 2. Let σ the k-linear involution of $k(\theta)$ defined by $\sigma(\theta) = -\theta$. It is easy to see that the trace of θ, θ^3 and θ^5 is equal to zero. Then, one can verify that $1, \theta, t^2 - \theta^2, (2t^2 - 1)\theta - \theta^3$ is an orthogonal basis for $\mathrm{Tr}_{k(\theta)/k}(< 1 >_\sigma)$, and that the corresponding diagonalization is
   $< 4, -4t^2, 4t^2(t^2 - 1), (1 - t^2) >$. Since the last quadratic form is isomorphic to $2\mathbb{H}$, we get $\phi \sim \mathrm{Tr}_{k(\theta)/k}(< 1 >_\sigma)$.
   Assume now that $\phi \not\sim < -1, -1 >$. Using proposition 4, we get
   $\phi \sim \mathrm{Tr}_{L(\sqrt{b})/k}(< 1 >)$. By the previous case, we can assume that ϕ is not hyperbolic. Then it is easy to show that $-b \notin L^{*2}$, so we get
   $\mathrm{Tr}_{L(\sqrt{-b})/k}(< 1 >_\sigma) \simeq \mathrm{Tr}_{L(\sqrt{b})/k}(< 1 >) \sim \phi$, where σ is the k-linear involution of $L(\sqrt{-b})$ defined by $\sigma(\sqrt{-b}) = -\sqrt{-b}$.
   Finally suppose that $\phi \sim < -1, -1 >$. This implies that k is not formally real. If $-1 \in k^{*2}$, then $\phi \sim 0$ and we have finished. So we can suppose that $-1 \notin k^{*2}$. Since k is non formally real, $< -2 >$ is algebraic by theorem 3. So we have $\mathrm{Tr}_{L/k}(< 1 >) \sim < -2 >$. Notice

that $[L:k]$ is odd. By lemma 2, there exists a $\lambda \in L$ such that $\mathrm{Tr}_{L/k}(<\lambda>) \sim <2>$. Then λ is not a square in L, otherwise we get $<2> \sim \mathrm{Tr}_{L/k}(<\lambda>) \sim \mathrm{Tr}_{L/k}(<1>) \sim <-2>$, that is $<-2> \simeq <2>$, which implies that $-1 \in k^{*2}$. Now set $E = L(\sqrt{\lambda})$, and let σ be the k-linear involution of E defined by $\sigma(\sqrt{\lambda}) = -\sqrt{\lambda}$ and $\sigma|_L = \mathrm{Id}$. Then $\mathrm{Tr}_{E/k}(<1>_\sigma) \simeq \mathrm{Tr}_{L/k}(<2, -2\lambda>) \sim <-1, -1>$, and this concludes the proof.

2. It is a particular case of the previous point.

3. In the proof of [K-S1], Corollary 5, it is shown that every odd-dimensional positive quadratic form is algebraic. Let ϕ be an even-dimensional positive form. Since k is an ordered field in this case, $<-1,-1>$ is not positive, so $\phi \not\sim <-1,-1>$. By proposition 3, we have $\phi \sim \mathrm{Tr}_{L(\sqrt{\lambda})/k}(<1>)$. If $-\lambda \in k^{*2}$, ϕ is hyperbolic, and the argument used in the first point is still valid, since $R(X)$ is hilbertian. Otherwise, the isomorphism $\mathrm{Tr}_{L(\sqrt{\lambda})/k}(<1>) \simeq \mathrm{Tr}_{L(\sqrt{-\lambda})/k}(<1>_\sigma)$ gives the conclusion.

4. By the second point, we can assume that k is formally real. The fifth point of theorem 3 gives in particular that every odd-dimensional positive quadratic form is algebraic. Now conclude as previously.

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