# THE ASYMMETRY OF AN ANTI-AUTOMORPHISM

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Abstract. The asymmetry of a nonsingular pairing on a vector space is an endomorphism of the space on which the classification of arbitrary pairings (not necessarily symmetric or skew-symmetric) is based. A general notion of asymmetry is defined for arbitrary anti-automorphisms on a central simple algebra, and conditions are given to characterize the elements which are the asymmetries of some anti-automorphism. The asymmetry is used to define the determinant of an anti-automorphism.

## Introduction

The asymmetry of an arbitrary nonsingular pairing (not necessarily symmetric or skew-symmetric) on a finite-dimensional vector space $V$ is an invertible endomorphism of $V$ which is an important invariant of the pairing. It is 1 if and only if the pairing is symmetric and $-1$ if and only if it is skew-symmetric. This invariant was first considered by Williamson [9], and more recently by Riehm [6].

In the present paper, we determine under which conditions a linear map $a \in \mathrm{GL}(V)$ is the asymmetry of some nonsingular pairing on $V$: the map $a$ must be conjugate to its inverse and satisfy some conditions on the generalized eigenspaces of eigenvalues $+1$ and $-1$, see Theorem 1. As pointed out by Ranicki, the property that $a$ is an asymmetry could be rephrased by saying that a certain asymmetric Poincaré complex of dimension 1 is round simple null-cobordant. (See [5, Ch. 20] for background information on Poincaré complexes.)

In section 2, we define the asymmetry of an anti-automorphism $\sigma$ on a central simple algebra $A$: it is an element $a_\sigma \in A^\times$ which is mapped, under scalar extension to a splitting field of $A$, to the asymmetry of any nonsingular pairing to which $\sigma$ is adjoint. It is defined up to sign by the properties that $\sigma^2(x) = a_\sigma x a_\sigma^{-1}$ for all $x \in A$ and that $\sigma(a_\sigma) = a_\sigma^{-1}$. This element was incidentally used by Saltman [7, Lemma 3.3, Theorem 4.4] to show that if an Azumaya algebra $A$ carries an anti-automorphism, then the ring of $2 \times 2$ matrices $M_2(A)$ carries an involution, and that Azumaya algebras over connected semilocal rings which are isomorphic to their opposite have an involution. We show that in a central simple algebra of exponent 2, an invertible element is the asymmetry of some anti-automorphism if and only if it is conjugate to its inverse (Theorem 2). Albert's theorem that every central simple algebra of exponent 2 has an involution is an immediate consequence, since involutions are the anti-automorphisms of asymmetry $\pm 1$. In the final section, the asymmetry is used to define the determinant of an anti-automorphism.

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## 1. The asymmetry of a nonsingular pairing

Throughout this section, $V$ denotes a finite-dimensional vector space over an arbitrary field $F$. We define the asymmetry and the adjoint anti-automorphism of a nonsingular pairing on $V$, and determine which linear transformations of $V$ are asymmetries.

1.1. **Definitions.** Let $V^* = \operatorname{Hom}_F(V,F)$ be the dual of $V$. Every pairing (or bilinear form) $b\colon V\times V\to F$ induces a linear map $\hat b\colon V\to V^*$ which carries $x\in V$ to $b(x,\bullet)\in V^*$. The transpose map $\hat b^t\colon V=V^{**}\to V^*$ carries $x\in V$ to $b(\bullet,x)\in V^*$.

**Proposition 1.** *For a pairing $b$ on $V$, the following conditions are equivalent:*

(a) *if $x\in V$ is such that $b(x,y)=0$ for all $y\in V$, then $x=0$;*
(b) *if $y\in V$ is such that $b(x,y)=0$ for all $x\in V$, then $y=0$;*
(c) *the map $\hat b$ is bijective.*

If these conditions hold, the pairing $b$ is called *nonsingular*.

*Proof.* Condition (a) is equivalent to injectivity of $\hat b$, and (b) to injectivity of $\hat b^t$, hence also to surjectivity of $\hat b$. Since $\dim V=\dim V^*$, each of these conditions implies that $\hat b$ is bijective. □

All the pairings considered in the sequel are nonsingular. To every nonsingular pairing $b$ on $V$ we attach an anti-automorphism $\sigma_b$ of $\operatorname{End}_F V$ and a linear transformation $a_b\in \mathrm{GL}(V)$ as follows:

**Proposition 2.** *Let $b$ be a nonsingular pairing on $V$. There is a unique map $\sigma_b\colon \operatorname{End}_F V\to\operatorname{End}_F V$ and a unique map $a_b\colon V\to V$ such that*

$$b\big(f(x),y\big)=b\big(x,\sigma_b(f)(y)\big)\qquad \text{for all } x,\ y\in V,\ f\in\operatorname{End}_F V \tag{1}$$

*and*

$$b(x,y)=b\big(y,a_b(x)\big)\qquad \text{for all } x,\ y\in V. \tag{2}$$

*The map $\sigma_b$ is an $F$-linear anti-automorphism of $\operatorname{End}_F V$ and the map $a_b$ is linear and invertible. These maps satisfy the following properties:*

(i) *$\sigma_b^2(f)=a_b\circ f\circ a_b^{-1}$ for all $f\in\operatorname{End}_F V$;*
(ii) *$\sigma_b(a_b)=a_b^{-1}$.*

*Proof.* For $f\in\operatorname{End}_F V$, let $\sigma_b(f)=(\hat b\circ f\circ\hat b^{-1})^t$. Equality (1) is easily checked, and the fact that $\sigma_b$ is an $F$-linear anti-automorphism of $\operatorname{End}_F V$ follows. Uniqueness of $\sigma_b$ follows from the hypothesis that $b$ is nonsingular.

On the other hand, let $a_b=(\hat b^t)^{-1}\circ\hat b$. This map is clearly linear and invertible, and it satisfies (2). Uniqueness of $a_b$ is clear. To check the additional properties, observe that for $f\in\operatorname{End}_F V$

$$\sigma_b^2(f)=\big(\hat b\circ(\hat b\circ f\circ\hat b^{-1})^t\circ\hat b^{-1}\big)^t=\big((\hat b^t)^{-1}\circ\hat b\big)\circ f\circ\big((\hat b^t)^{-1}\circ\hat b\big)^{-1}$$

and

$$\sigma_b\big((\hat b^t)^{-1}\circ\hat b\big)=\big(\hat b\circ\big((\hat b^t)^{-1}\circ\hat b\big)\circ\hat b^{-1}\big)^t=\big((\hat b^t)^{-1}\circ\hat b\big)^{-1}.$$

□

We call $\sigma_b$ the anti-automorphism *adjoint* to $b$. Using the Skolem-Noether theorem, it is easily seen that every $F$-linear anti-automorphism of $\operatorname{End}_F V$ is adjoint to some nonsingular pairing, see [4, p. 1]. The map $a_b$ is called the *asymmetry* of $b$. From the definition, it is clear that the adjoint anti-automorphism and the asymmetry of any scalar multiple of $b$ are the same as those of $b$. Moreover, the map $a_b$ is determined up to sign by properties (i) and (ii).

We combine $a_b$ and $\sigma_b$ into a linear involution of $\operatorname{End}_F V$ as follows:

**Proposition 3.** *Let $b$ be a nonsingular pairing on $V$. There is a unique linear map $\gamma_b\colon \operatorname{End}_F V \to \operatorname{End}_F V$ such that*

$$b\big(x, f(y)\big) = b\big(y, \gamma_b(f)(x)\big) \qquad \text{for all } x,\ y \in V,\ f \in \operatorname{End}_F V. \tag{3}$$

*This map satisfies the following additional properties:*

(i) $\gamma_b\big(f \circ g \circ h\big) = \sigma_b(h) \circ \gamma_b(g) \circ \sigma_b^{-1}(f)$ *for* $f,\ g,\ h \in \operatorname{End}_F V$*;*
(ii) $\gamma_b^2 = \operatorname{Id}_{\operatorname{End} V}$*;*
(iii) $\gamma_b(\operatorname{Id}_V) = a_b$*.*

*Proof.* Set $\gamma_b(f) = \sigma_b(f) \circ a_b$ $(= a_b \circ \sigma_b^{-1}(f))$ for $f \in \operatorname{End}_F V$; then (iii) is clear and (3), (i), (ii) follow from the properties of $\sigma_b$ and $a_b$. □

We call $\gamma_b$ the *linear involution* of $\operatorname{End}_F V$ associated to $b$. As for the adjoint anti-automorphism $\sigma_b$ and the asymmetry $a_b$, it is clear that $\gamma_b$ is also the linear involution associated to any scalar multiple of $b$.

*Remark.* There are corresponding notions for pairings on faithfully projective modules with values in invertible modules (over an arbitrary commutative ring $R$): see [3, Chap. III, (8.2)].

1.2. **Characterization of asymmetries.** The goal of this subsection is to answer the following question: Under which conditions on a map $a \in \operatorname{GL}(V)$ does there exist a nonsingular pairing $b$ on $V$ whose asymmetry is $a$, i.e., such that $a_b = a$? Identifying $\operatorname{End}_F V$ with a matrix algebra $M_n(F)$ through the choice of a basis of $V$, this amounts to asking for which invertible matrices $a \in \operatorname{GL}_n(F)$ the equation $a = (x^t)^{-1}x$ has a solution $x \in \operatorname{GL}_n(F)$, in view of the definition of $a$ in terms of $\hat{b}$ in the proof of Proposition 2.

The conditions involve the following vector spaces: for an arbitrary integer $m \geq 1$ and $\varepsilon = \pm 1$, we let

$$V_m^{\varepsilon} = \frac{\ker(a - \varepsilon \operatorname{Id}_V)^m}{\ker(a - \varepsilon \operatorname{Id}_V)^{m-1} + (a - \varepsilon \operatorname{Id}_V)\big(\ker(a - \varepsilon \operatorname{Id}_V)^{m+1}\big)}.$$

**Theorem 1.** *Suppose* $\operatorname{char} F \neq 2$*. A map $a \in \operatorname{GL}(V)$ is the asymmetry of some nonsingular pairing on $V$ if and only if the following conditions hold:*

(1) *$a$ is conjugate to $a^{-1}$ in $\operatorname{GL}(V)$;*
(2) *for every even integer $m$, $\dim V_m^{+1}$ is even;*
(3) *for every odd integer $m$, $\dim V_m^{-1}$ is even.*

*If* $\operatorname{char} F = 2$*, a map $a \in \operatorname{GL}(V)$ is the asymmetry of some nonsingular pairing on $V$ if and only if conditions* (1) *and* (2) *hold.*

*Proof.* We first show that the conditions are necessary. Suppose $b$ is a nonsingular pairing on $V$ such that $a_b = a$. Proposition 2 shows that $\sigma_b(a) = a^{-1}$. To see how this equality implies condition (1), we argue in terms of matrices. Using a basis of $V$, we identify $\operatorname{End}_F V$ with the matrix algebra $M_n(F)$. Since the transpose

map $t$ is an anti-automorphism, $\sigma_b \circ t$ is a linear automorphism of $M_n(F)$, hence the Skolem-Noether theorem yields an invertible matrix $u$ such that $\sigma_b \circ t$ is the conjugation by $u$. Then $\sigma_b(x) = ux^t u^{-1}$ for all $x \in M_n(F)$. In particular, since $\sigma_b(a) = a^{-1}$ it follows that $a^{-1}$ is conjugate to $a^t$. But it is well-known that every matrix is conjugate to its transpose, hence condition (1) is proved.

To show that conditions (2) and (3) are necessary if $\operatorname{char} F \neq 2$, we show that the nonsingular pairing $b$ induces a nonsingular skew-symmetric pairing on $V_m^{+1}$ if $m$ is even and on $V_m^{-1}$ if $m$ is odd. Conditions (2) and (3) follow because only even-dimensional vector spaces carry nonsingular skew-symmetric pairings if the characteristic of the base field is different from 2.

Fix some integer $m$ and $\varepsilon = \pm 1$. For the convenience of notation, we let

$$U_m^\varepsilon = \ker(a - \varepsilon \operatorname{Id}_V)^m,$$

so $V_m^\varepsilon = U_m^\varepsilon / \big(U_{m-1}^\varepsilon + (a - \varepsilon \operatorname{Id}_V)(U_{m+1}^\varepsilon)\big)$. For $x$, $y \in U_m^\varepsilon$, define

$$b_m^\varepsilon(x, y) = b\big(x, (a - \varepsilon \operatorname{Id}_V)^{m-1}(y)\big).$$

Since $y \in U_m^\varepsilon$, we have

$$a \circ (a - \varepsilon \operatorname{Id}_V)^{m-1}(y) = \varepsilon (a - \varepsilon \operatorname{Id}_V)^{m-1}(y), \tag{4}$$

hence

$$\begin{aligned} b\big(y, (a - \varepsilon \operatorname{Id}_V)^{m-1}(x)\big) &= \varepsilon b\big(y, a \circ (a - \varepsilon \operatorname{Id}_V)^{m-1}(x)\big) \\ &= \varepsilon b\big((a - \varepsilon \operatorname{Id}_V)^{m-1}(x), y\big). \end{aligned} \tag{5}$$

On the other hand, equality (4) yields

$$(a - \varepsilon \operatorname{Id}_V)^{m-1}(y) = (\varepsilon a^{-1})^{m-1}(a - \varepsilon \operatorname{Id}_V)^{m-1}(y) = (-1)^{m-1}\sigma_b(a - \varepsilon \operatorname{Id}_V)^{m-1}(y),$$

hence

$$b\big((a - \varepsilon \operatorname{Id}_V)^{m-1}(x), y\big) = (-1)^{m-1} b\big(x, (a - \varepsilon \operatorname{Id}_V)^{m-1}(y)\big). \tag{6}$$

Comparing (5) and (6), we obtain

$$b_m^\varepsilon(y, x) = (-1)^{m-1} \varepsilon b_m^\varepsilon(x, y).$$

Therefore, $b_m^\varepsilon$ is a skew-symmetric bilinear form on $U_m^\varepsilon$ if $\varepsilon = +1$ and $m$ is even, and also if $\varepsilon = -1$ and $m$ is odd.

To see that $b_m^\varepsilon$ induces a nonsingular pairing on $V_m^\varepsilon$, we consider the radical of $b_m^\varepsilon$, which is

$$\operatorname{rad} b_m^\varepsilon = \big\{x \in U_m^\varepsilon \mid b(x, z) = 0 \text{ for all } z \in (a - \varepsilon \operatorname{Id}_V)^{m-1}(U_m^\varepsilon)\big\}.$$

Thus, $\operatorname{rad} b_m^\varepsilon$ is the intersection of $U_m^\varepsilon$ with the orthogonal[1] complement for the form $b$ of

$$(a - \varepsilon \operatorname{Id}_V)^{m-1}(U_m^\varepsilon) = \operatorname{im}(a - \varepsilon \operatorname{Id}_V)^{m-1} \cap \ker(a - \varepsilon \operatorname{Id}_V),$$

which is $\ker \sigma_b(a - \varepsilon \operatorname{Id}_V)^{m-1} + \operatorname{im} \sigma_b(a - \varepsilon \operatorname{Id}_V)$. Since $\sigma_b(a) = a^{-1}$, we have $\ker \sigma_b(a - \varepsilon \operatorname{Id}_V)^{m-1} = \ker(a - \varepsilon \operatorname{Id}_V)^{m-1}$ and $\operatorname{im} \sigma_b(a - \varepsilon \operatorname{Id}_V) = \operatorname{im}(a - \varepsilon \operatorname{Id}_V)$,

[1] If $b$ is not symmetric nor skew-symmetric, one has to distinguish orthogonality on the left and on the right; the orthogonal complements of $a$-invariant subspaces coincide, however.

hence

$$\begin{aligned}\operatorname{rad} b_m^\varepsilon &= \big(U_{m-1}^\varepsilon + \operatorname{im}(a - \varepsilon \operatorname{Id}_V)\big) \cap U_m^\varepsilon \\ &= U_{m-1}^\varepsilon + \big(\operatorname{im}(a - \varepsilon \operatorname{Id}_V) \cap U_m^\varepsilon\big) \\ &= U_{m-1}^\varepsilon + (a - \varepsilon \operatorname{Id}_V)(U_{m+1}^\varepsilon).\end{aligned}$$

Therefore, $b_m^\varepsilon$ induces a nonsingular pairing on $U_m^\varepsilon / \big(U_{m-1}^\varepsilon + (a - \varepsilon \operatorname{Id}_V)(U_{m+1}^\varepsilon)\big) = V_m^\varepsilon$.

Suppose now $\operatorname{char} F = 2$. The arguments above still show that $b_m^\varepsilon$ induces a nonsingular bilinear pairing on $V_m^\varepsilon$, but in characteristic 2 skew-symmetric pairings are symmetric, hence we cannot conclude that $\dim V_m^\varepsilon$ is even. To show that $\dim V_m^{+1}$ is even if $m$ is even, we show that $b_m^{+1}$ is in fact alternating if $m$ is even. For $x \in U_m^{+1}$ we have

$$(a - \operatorname{Id}_V)^{m-2}(x) \in \ker(a - \operatorname{Id}_V)^2 = \ker(a^2 - \operatorname{Id}_V),$$

hence $a^2 \circ (a - \operatorname{Id}_V)^{m-2}(x) = (a - \operatorname{Id}_V)^{m-2}(x)$. Since $m$ is even, we obtain by induction

$$a^{m-2} \circ (a - \operatorname{Id}_V)^{m-2}(x) = (a - \operatorname{Id}_V)^{m-2}(x),$$

hence

$$(a - \operatorname{Id}_V)^{m-2}(x) = a^{2-m} \circ (a - \operatorname{Id}_V)^{m-2}(x) = \sigma(a - \operatorname{Id}_V)^{m-2}(x).$$

Therefore,

$$b\big(x, (a - \operatorname{Id}_V)^{m-2}(x)\big) = b\big((a - \operatorname{Id}_V)^{m-2}(x), x\big) = b\big(x, a \circ (a - \operatorname{Id}_V)^{m-2}(x)\big).$$

It follows that $b\big(x, (a - \operatorname{Id}_V)^{m-1}(x)\big) = 0$, hence $b_m^{+1}$ is alternating. This completes the proof that the conditions are necessary.

To prove that the conditions are sufficient, we shall make $V$ into a module over the ring $F[X, X^{-1}]$ of Laurent polynomials in one indeterminate $X$. As a preparation, we make some observations on the prime ideals of this principal ideal domain.

Let $J$ be the automorphism of $F[X, X^{-1}]$ which maps $X$ to $X^{-1}$. We also denote by $J$ the extension of this automorphism to the field of fractions $F(X)$ and to the factor module $E = F(X)/F[X, X^{-1}]$. Every prime ideal $P \subset F[X, X^{-1}]$ is generated by an irreducible polynomial of the form

$$\pi = a_0 + a_1 X + \cdots + a_d X^d \in F[X]$$

such that $a_0$, $a_d \neq 0$. If $P^J = P$, the Laurent polynomials $\pi$, $\pi^J$ differ by a factor which is invertible in $F[X, X^{-1}]$, hence $\pi = \alpha X^d \pi^J$ for some $\alpha \in F^\times$. Comparing coefficients, we have

$$a_i = \alpha a_{d-i} \qquad \text{for } i = 0, \dots, d,$$

hence $a_d = \alpha a_0 = \alpha^2 a_d$ and therefore $\alpha = \pm 1$. If $d$ is odd, then

$$\pi = \sum_{i=0}^{(d-1)/2} a_i (X^i + \alpha X^{d-i}),$$

hence $\pi$ is divisible by $1 + \alpha X$. As $\pi$ is irreducible, we may then choose $\pi = X + 1$ if $\alpha = 1$, and $\pi = X - 1$ if $\alpha = -1$. Suppose next $d$ is even. If $\alpha = -1$ and $\operatorname{char} F \neq 2$,

then $a_{d/2} = -a_{d/2}$ implies $a_{d/2} = 0$. In that case, we have

$$\pi = \sum_{i=0}^{d/2-1} a_i(X^i - X^{d-i}),$$

hence $\pi$ is divisible by $1 - X$. This is a contradiction, since $\pi$ is assumed to be irreducible. Therefore, $\alpha = 1$ and $(X^{d/2}\pi^{-1})^J = X^{d/2}\pi^{-1}$. We may then choose $\pi$ of the form

$$\pi = 1 + a_1X + a_2X^2 + \cdots + a_2X^{d-2} + a_1X^{d-1} + X^d.$$

Let $\mathcal{R}_1$ be the set of irreducible polynomials of this form.

For each pair of prime ideals $\{P, P^J\}$ with $P^J \neq P$, we arbitrarily choose a generator $\pi \in F[X]$ of one of $P$, $P^J$ and denote by $\mathcal{R}_2$ the set of irreducible polynomials thus chosen. Thus, the set of prime ideals of $F[X, X^{-1}]$ is $\{\pi F[X, X^{-1}]\}$ where $\pi$ runs over the set $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_2^J \cup \{X - 1, X + 1\}$, and we have $\pi^J F[X, X^{-1}] \neq \pi F[X, X^{-1}]$ if and only if $\pi \in \mathcal{R}_2 \cup \mathcal{R}_2^J$.

Returning to the proof of Theorem 1, we define a structure of $F[X, X^{-1}]$-module on $V$ by letting

$$X \cdot v = a(v) \qquad \text{for all } v \in V.$$

Since $F[X, X^{-1}]$ is a principal ideal domain, the $F[X, X^{-1}]$-module $V$ decomposes as a (finite) direct sum of quotients of $F[X, X^{-1}]$, as follows:

$$V \simeq \bigoplus_{\pi,m} \big(F[X, X^{-1}]/\pi^m\big)^{\mu(\pi,m)}$$

for some integers $\mu(\pi, m)$ which all vanish except a finite number, where $\pi$ runs over $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_2^J \cup \{X - 1, X + 1\}$, and $m$ over the positive integers.

Condition (1) shows that the elementary divisors of $a$ are the same as those of $a^{-1}$, hence

$$V \simeq \bigoplus_{\pi,m} \big(F[X, X^{-1}]/(\pi^J)^m\big)^{\mu(\pi,m)}.$$

Therefore, we have $\mu(\pi, m) = \mu(\pi^J, m)$ for all $m$ if $\pi \in \mathcal{R}_2$.

For all integers $m$ and for $\varepsilon = \pm 1$ we have

$$\dim V_m^\varepsilon = \mu(X - \varepsilon, m).$$

Therefore, condition (2) says that $\mu(X - 1, m)$ is even for all $m$ even, and condition (3) says that $\mu(X + 1, m)$ is even for all $m$ odd. Assuming $\operatorname{char} F \neq 2$ and conditions (1), (2) and (3) hold, we may decompose $V$ into a direct sum of six $F[X, X^{-1}]$-submodules

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5 \oplus V_6$$

where

$$V_1 \simeq \bigoplus_{\pi\in\mathcal{R}_1}\bigoplus_m \big(F[X,X^{-1}]/\pi^m\big)^{\mu(\pi,m)},$$

$$V_2 \simeq \bigoplus_{\pi\in\mathcal{R}_2}\bigoplus_m \big(F[X,X^{-1}]/\pi^m \oplus F[X,X^{-1}]/(\pi^J)^m\big)^{\mu(\pi,m)},$$

$$V_3 \simeq \bigoplus_{m \text{ odd}} \big(F[X,X^{-1}]/(X-1)^m\big)^{\mu(X-1,m)},$$

$$V_4 \simeq \bigoplus_{m \text{ even}} \big(F[X,X^{-1}]/(X-1)^m \oplus F[X,X^{-1}]/(X-1)^m\big)^{\mu(X-1,m)/2},$$

$$V_5 \simeq \bigoplus_{m \text{ even}} \big(F[X,X^{-1}]/(X+1)^m\big)^{\mu(X+1,m)},$$

$$V_6 \simeq \bigoplus_{m \text{ odd}} \big(F[X,X^{-1}]/(X+1)^m \oplus F[X,X^{-1}]/(X+1)^m\big)^{\mu(X+1,m)/2}.$$

If $\operatorname{char} F = 2$ and conditions (1), (2) hold, there is a similar decomposition

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

where $V_1, \dots, V_4$ are as above. We shall show below (see Lemma 1) that there are nonsingular $(-X)$-hermitian forms with values in $E$ (with respect to $J$) on

$$\begin{array}{cl} F[X,X^{-1}]/\pi^m & \text{if } \pi\in\mathcal{R}_1,\\ F[X,X^{-1}]/\pi^m \oplus F[X,X^{-1}]/(\pi^J)^m & \text{if } \pi\in\mathcal{R}_2,\\ F[X,X^{-1}]/(X-1)^m & \text{if } m \text{ is odd},\\ \big(F[X,X^{-1}]/(X-1)^m\big)^2 & \text{if } m \text{ is even},\\ F[X,X^{-1}]/(X+1)^m & \text{if } m \text{ is even and } \operatorname{char} F\neq 2,\\ \big(F[X,X^{-1}]/(X+1)^m\big)^2 & \text{if } m \text{ is odd and } \operatorname{char} F\neq 2. \end{array} \tag{7}$$

The orthogonal sum of these forms yields a nonsingular $(-X)$-hermitian form

$$h\colon V\times V \to E$$

with respect to $J$. As Ischebeck-Scharlau [2] or Waterhouse [8], define an $F$-linear map $T\colon E\to F$ by observing that every element in $E$ is represented by a unique rational fraction $f$ which has a zero at $\infty$ and does not have a pole at 0, and letting

$$T\big(f + F[X,X^{-1}]\big) = f(0).$$

It is easily verified that $T(r^J) = -T(r)$ for all $r\in E$. Moreover, for every nonzero $r\in E$ there exists an integer $k$ such that $T(X^{-k}r)\neq 0$, hence $T$ does not vanish on any nonzero $F[X,X^{-1}]$-submodule of $E$.

Let $T_*(h)\colon V\times V\to F$ be the transfer bilinear map, defined by

$$T_*(h)(x,y) = T\big(h(x,y)\big) \qquad \text{for } x,\, y\in V.$$

If $x\in V$ is such that $T_*(h)(x,y)=0$ for all $y\in V$, then $T$ vanishes on the $F[X,X^{-1}]$-submodule $h(x,V)$, hence $h(x,V)=\{0\}$ and therefore $x=0$ since $h$ is nonsingular. This shows that $T_*(h)$ is nonsingular.

Moreover, since $h$ is $(-X)$-hermitian we have

$$\begin{aligned} T_*(h)(y,x) &= T\big((-X)h(x,y)^J\big) = -T\big(Xh(x,y)^J\big) = \\ &= T\big(X^J h(x,y)\big) = T\big(h(x,Xy)\big) = T_*(h)\big(x,a(y)\big) \end{aligned}$$

for all $x,\, y\in V$. Therefore, $a$ is the asymmetry of $T_*(h)$.

To complete the proof, we prove the existence of nonsingular $(-X)$-hermitian forms as asserted above.

**Lemma 1.** *There are nonsingular $(-X)$-hermitian forms with values in $E$ (with respect to $J$) on the modules listed in* (7).

*Proof.* Suppose first $\pi \in \mathcal{R}_1$, hence $(X^{d/2}\pi^{-1})^J = X^{d/2}\pi^{-1}$, where $d$ is the degree of $\pi$. For $u$, $v \in F[X, X^{-1}]$, let

$$h(u, v) = (X-1)(X^{d/2}\pi^{-1})^m u^J v + F[X, X^{-1}] \in E.$$

This map induces a sesquilinear form on $F[X, X^{-1}]/\pi^m$. The induced form is $(-X)$-hermitian since $(X-1)^J = -X^{-1}(X-1)$; it is nonsingular since $h(1, v) = 0$ implies $\pi^m$ divides $(X-1)v$ in $F[X, X^{-1}]$, hence $v = 0$ in $F[X, X^{-1}]/\pi^m$ since $\pi$ is prime to $X-1$.

Next, suppose $\pi \in \mathcal{R}_2$. For $u_1$, $u_2$, $v_1$, $v_2 \in F[X, X^{-1}]$, we let

$$h\big((u_1, u_2), (v_1, v_2)\big) = \pi^{-m} u_1^J v_2 - X(\pi^J)^{-m} u_2^J v_1 + F[X, X^{-1}] \in E.$$

Computation shows that this map induces a nonsingular $(-X)$-hermitian form on $\big(F[X, X^{-1}]/\pi^m\big) \times \big(F[X, X^{-1}]/(\pi^J)^m\big)$.

Similarly, the following maps induce nonsingular $(-X)$-hermitian forms on the corresponding modules (where $e$ is an arbitrary non-negative integer):

$$h(u, v) = X^{e-1}(X-1)^{-2e-1} u^J v + F[X, X^{-1}] \in E \quad \text{on } F[X, X^{-1}]/(X-1)^{2e+1};$$

$$h\big((u_1, u_2), (v_1, v_2)\big) = X^e (X-1)^{-2e}(u_1^J v_2 - X u_2^J v_1) + F[X, X^{-1}] \in E \quad \text{on } \big(F[X, X^{-1}]/(X-1)^{2e}\big)^2;$$

and if $\operatorname{char} F \neq 2$,

$$h(u, v) = (X-1)X^e (X+1)^{-2e} u^J v + F[X, X^{-1}] \in E \quad \text{on } F[X, X^{-1}]/(X+1)^{2e};$$

$$h\big((u_1, u_2), (v_1, v_2)\big) = (X-1)^{2e+1}(X+1)^{-2e-1}(u_1^J v_2 + X u_2^J v_1) + F[X, X^{-1}] \in E \quad \text{on } \big(F[X, X^{-1}]/(X+1)^{2e+1}\big)^2.$$

We omit the straightforward verifications. □

*Remark.* The theory of hermitian forms over principal ideal domains can also be used to show that the conditions in Theorem 1 are necessary.

## 2. The asymmetry of an anti-automorphism

2.1. **Definition.** Let $A$ be a (finite-dimensional) central simple algebra over an arbitrary field $F$, and let $\sigma\colon A \to A$ be an $F$-linear anti-automorphism of $A$. Our goal is to attach to $\sigma$ a unit $a_\sigma \in A^\times$ which plays the same rôle as the asymmetry $a_b$ of a nonsingular pairing $b$ with respect to the adjoint anti-automorphism $\sigma_b$. The key to the definition is an analogue of the linear involution $\gamma_b$, which we now define.

**Proposition 4.** *There is a unique linear map $\gamma_\sigma\colon A \to A$ which satisfies the following property: for any splitting field $K$ of $A$, any isomorphism*

$$\theta\colon A_K = A \otimes_F K \to \operatorname{End}_K V$$

*and any nonsingular pairing $b$ on $V$ such that $\sigma_b = \theta \circ (\sigma \otimes \mathrm{Id}_K) \circ \theta^{-1}$,*

$$\theta \circ (\gamma_\sigma \otimes \mathrm{Id}_K) \circ \theta^{-1} = \gamma_b.$$

*This map satisfies the following additional properties:*

(i) $\gamma_\sigma\big(xyz\big) = \sigma(z)\gamma_\sigma(y)\sigma^{-1}(x)$ *for $x$, $y$, $z \in A$;*
(ii) $\gamma_\sigma^2 = \mathrm{Id}_A$.

*Proof.* It suffices to prove the existence of $\gamma_\sigma$. Uniqueness is then clear, and the additional properties follow from those of $\gamma_b$ in Proposition 3.

Let $T_\sigma\colon A \times A \to F$ be the nonsingular pairing defined by

$$T_\sigma(x, y) = \mathrm{Trd}_A\big(\sigma(x)y\big) \qquad \text{for } x,\, y \in A,$$

where $\mathrm{Trd}_A$ is the reduced trace. Let $(e_i)_{i\in I}$ be a basis of $A$ and let $(e_i^\sharp)_{i\in I}$ be the dual basis with respect to the pairing $T_\sigma$, so that

$$T_\sigma(e_i^\sharp, e_j) = \delta_{ij} \qquad \text{for } i,\, j \in I.$$

We let

$$\gamma_\sigma(x) = \sum_{i\in I} e_i x e_i^\sharp \qquad \text{for } x \in A.$$

In other words, $\gamma_\sigma$ is the image of $\sum_{i\in I} e_i \otimes e_i^\sharp \in A \otimes_F A$ under the "sandwich" map $\mathrm{Sand}\colon A \otimes_F A \to \mathrm{End}_F A$ defined by $\mathrm{Sand}(x \otimes y)(z) = xzy$. Observe that $\gamma_\sigma$ does not depend on the choice of the basis $(e_i)_{i\in I}$ since $\sum_{i\in I} e_i \otimes e_i^\sharp$ is the element which corresponds to $\mathrm{Id}_A$ under the bijection $\mathrm{Id}_A \otimes \hat{T}_\sigma\colon A \otimes_F A \to A \otimes_F A^* = \mathrm{End}_F A$.

As a consequence, for every field extension $K/F$, the map $\gamma_{\sigma\otimes\mathrm{Id}_K}\colon A \otimes K \to A \otimes K$ satisfies

$$\gamma_{\sigma\otimes\mathrm{Id}_K} = \gamma_\sigma \otimes \mathrm{Id}_K$$

since for $x \in A \otimes K$,

$$\gamma_{\sigma\otimes\mathrm{Id}_K}(x) = \sum_{i\in I}(e_i \otimes 1)x(e_i^\sharp \otimes 1) = (\gamma_\sigma \otimes \mathrm{Id}_K)(x).$$

To show that $\gamma_\sigma$ is as required, assume that $A$ is split: let $A = \mathrm{End}_F V$ and let $b$ be a nonsingular pairing on $V$ such that $\sigma = \sigma_b$. We have to show that $\gamma_\sigma = \gamma_b$. To prove this equality, we use the identification $V \otimes_F V = \mathrm{End}_F V$ defined by the linear isomorphism $\mathrm{Id}_V \otimes \hat{b}\colon V \otimes_F V \to V \otimes_F V^* = \mathrm{End}_F V$. Then $(v \otimes w)(x) = vb(w, x)$ for $v$, $w$, $x \in V$ and moreover

$$f \circ (v \otimes w) = f(v) \otimes w, \quad \sigma(v \otimes w) = a_b(w) \otimes v \quad \text{and} \quad \mathrm{Trd}(v \otimes w) = b(w, v)$$

for $v$, $w \in V$ and $f \in \mathrm{End}_F V$. Let $(v_i)_{1\le i\le n}$ be a basis of $V$ and let $(v_i')_{1\le i\le n}$ be the dual basis for the pairing $b$, so that

$$b(v_i', v_j) = \delta_{ij} \qquad \text{for } i,\, j = 1, \dots, n. \tag{8}$$

Then $(v_i \otimes v_j)_{1\le i,j\le n}$ is a basis of $\mathrm{End}_F V$, and the dual basis with respect to $T_\sigma$ is given by

$$(v_i \otimes v_j)^\sharp = v_i' \otimes v_j'.$$

Therefore, we have for $f \in \operatorname{End}_F V$

$$\begin{aligned}\gamma_\sigma(f) &= \sum_{i,j=1}^n (v_i \otimes v_j) \circ f \circ (v_i' \otimes v_j') \\ &= \sum_{i,j=1}^n v_i \otimes v_j'\, b\big(v_j, f(v_i')\big) \\ &= \sum_{i,j=1}^n v_i \otimes v_j'\, b\big(v_i', \gamma_b(f)(v_j)\big).\end{aligned}$$

For all $x \in V$ we have $x = \sum_{i=1}^n v_i b(v_i', x)$, hence $\sum_{i=1}^n v_i\, b\big(v_i', \gamma_b(f)(v_j)\big) = \gamma_b(f)(v_j)$ for all $j$, and the last equality above simplifies to

$$\gamma_\sigma(f) = \sum_{j=1}^n \gamma_b(f)(v_j) \otimes v_j' = \gamma_b(f) \circ \Big(\sum_{j=1}^n v_j \otimes v_j'\Big).$$

Since $\sum_{j=1}^n v_j \otimes v_j' = \operatorname{Id}_V$, it follows that $\gamma_\sigma(f) = \gamma_b(f)$. □

In view of property (i), we have

$$\gamma_\sigma(x) = \sigma(x)\gamma_\sigma(1) = \gamma_\sigma(1)\sigma^{-1}(x) \qquad \text{for all } x \in A. \tag{9}$$

Therefore, $\gamma_\sigma$ is completely determined by the element $\gamma_\sigma(1) \in A^\times$.

**Definition.** The *asymmetry* of the anti-automorphism $\sigma$ is the element $a_\sigma = \gamma_\sigma(1) \in A^\times$, where $\gamma_\sigma$ is the linear involution defined in Proposition 4.

If $A = \operatorname{End}_F V$ and $\sigma = \sigma_b$ is the anti-automorphism adjoint to some nonsingular pairing $b$ on $V$, it follows from Proposition 4 and property (iii) of Proposition 3 that $a_\sigma$ is the asymmetry of the nonsingular form $b$, i.e.,

$$a_\sigma = a_b.$$

In the general case, equation (9) shows that

$$\sigma^2(x) = a_\sigma x a_\sigma^{-1} \qquad \text{for all } x \in A. \tag{10}$$

Moreover, since $\gamma_\sigma^2 = \operatorname{Id}_A$ we have

$$1 = \gamma_\sigma(a_\sigma) = \sigma(a_\sigma)a_\sigma. \tag{11}$$

The element $a_\sigma$ is uniquely determined up to sign by (10) and (11).

Recall that an anti-automorphism $\sigma$ is called an *involution* if $\sigma^2 = \operatorname{Id}_A$.

**Proposition 5.** *A linear anti-automorphism is an involution if and only if its asymmetry is* $+1$ *or* $-1$.

*Proof.* If $a_\sigma = \pm 1$, equation (10) shows that $\sigma^2 = \operatorname{Id}_A$. Conversely, if $\sigma$ is an involution, (10) shows that $a_\sigma \in F^\times$. It then follows from (11) that $a_\sigma^2 = 1$, hence $a_\sigma = \pm 1$. □

If $\operatorname{char} F \neq 2$, a linear involution $\sigma$ is called *orthogonal* (resp. *symplectic*) if after scalar extension to a splitting field it is adjoint to a symmetric (resp. skew-symmetric) bilinear pairing. Therefore, orthogonal involutions are exactly the linear anti-automorphisms with asymmetry $+1$, and symplectic involutions are those with asymmetry $-1$. Therefore, equations (10) and (11) are not sufficient to determine the type of the involution. This observation suggests that the sign of $a_\sigma$ is meaningful for arbitrary anti-automorphisms.

The following proposition yields an alternative definition of the asymmetry $a_\sigma$, without reference to the linear involution $\gamma_\sigma$ and without scalar extension to a splitting field.

Let $\sigma_*\colon A \otimes_F A \to \operatorname{End}_F A$ be the $F$-algebra homomorphism defined by

$$\sigma_*(a \otimes b)(x) = ax\sigma(b) \qquad \text{for } a,\, b,\, x \in A,$$

and recall (from [4, (3.5)], for instance) the *Goldman element* of $A$: this is the element $g \in A \otimes_F A$ such that $\operatorname{Sand}(g)(x) = \operatorname{Trd}_A(x)$ for all $x \in A$. Thus, there is a well-defined linear endomorphism $\sigma_*(g)\colon A \to A$.

**Proposition 6.** *The asymmetry of $\sigma$ is the unique element $a_\sigma \in A^\times$ such that*

$$\sigma\big(\sigma_*(g)(f)\big) = a_\sigma f$$

*for all $f \in A$.*

*Proof.* It suffices to prove that $a_\sigma$ satisfies the property above, since uniqueness is clear. To do this, we may extend scalars to a splitting field. Therefore, we may assume $A = \operatorname{End}_F V$ for some $F$-vector space $V$, and $\sigma = \sigma_b$ is the anti-automorphism adjoint to some nonsingular pairing $b$ on $V$.

For all $f \in A$ and all $x,\, y \in V$ we have

$$b\big(f(x), y\big) = b\big(y, a_\sigma \circ f(x)\big),$$

by definition of the asymmetry (see (2)), hence we have to show

$$b\big(f(x), y\big) = b\big(y, \sigma\big(\sigma_*(g)(f)\big)(x)\big)$$

or, equivalently (by definition of $\sigma = \sigma_b$),

$$b\big(f(x), y\big) = b\big(\sigma_*(g)(f)(y), x\big) \tag{12}$$

for all $f \in A$ and all $x,\, y \in V$.

In order to compute the right-hand side, we identify $A = \operatorname{End}_F V$ to $V \otimes_F V$ via the linear isomorphism $\operatorname{Id}_V \otimes \hat{b}\colon V \otimes_F V \to V \otimes_F V^* = \operatorname{End}_F V$, as in the proof of Proposition 4. If $(v_i)_{1\le i\le n}$ is a basis of $V$ and $(v_i')_{1\le i\le n}$ is the dual basis for the pairing $b$ (see (8)), then the Goldman element is

$$g = \sum_{i,j} (v_i \otimes v_j') \otimes (v_j \otimes v_i')$$

since it is easily computed that for all $u,\, w \in V$

$$\begin{aligned}\operatorname{Sand}(g)(u \otimes w) &= \sum_{i,j} (v_i \otimes v_j') \circ (u \otimes w) \circ (v_j \otimes v_i') = \\ &= \Big(\sum_i v_i \otimes v_i'\Big)\Big(\sum_j b(v_j', u) b(w, v_j)\Big) = b(w, u) \sum_i v_i \otimes v_i' = \operatorname{Trd}(u \otimes w)\operatorname{Id}_V .\end{aligned}$$

Now, for $u,\, w \in V$,

$$\sigma_*(g)(u \otimes w) = \sum_{i,j} (v_i \otimes v_j') \circ (u \otimes w) \circ \sigma(v_j \otimes v_i').$$

Since $(u \otimes w) \circ \sigma(f) = u \otimes f(w)$ for $f \in \operatorname{End}_F V$, the right-hand side of the last equality simplifies to

$$\sum_{i,j} \big((v_i \otimes v_j'(u)\big) \otimes \big((v_j \otimes v_i')(w)\big) = \sum_{i,j} v_i \otimes v_j b(v_j', u) b(v_i', w),$$

hence

$$\sigma_*(g)(u \otimes w) = w \otimes u.$$

Therefore, for $u$, $w$, $x$, $y \in V$,

$$b\big(\sigma_*(g)(u \otimes w)(y), x\big) = b\big((w \otimes u)(y), x\big) = b(w, x)b(u, y).$$

Since we also have $b\big((u \otimes w)(x), y\big) = b(u, y)b(w, x)$, equation (12) holds for $f = u \otimes w$. Since $\operatorname{End}_F V = V \otimes_F V$, it follows that (12) holds for all $f \in A$, and the proof is complete. □

*Remark.* Asymmetries can be defined on the same model for anti-automorphisms of Azumaya algebras; one may avoid the use of a basis of $A$ in Proposition 4 by defining $\gamma_\sigma = \operatorname{Sand}(\xi_\sigma)$ where $\xi_\sigma \in A \otimes A$ is the element mapped to $\operatorname{Id}_A$ by $\operatorname{Id}_A \otimes \hat{T}_\sigma$. Alternatively, we may set $\xi_\sigma = (\operatorname{Id}_A \otimes \sigma^{-1})(g)$ where $g \in A \otimes A$ is the Goldman element. This is the approach taken by Saltman in [7] (see also [3, Chap. III, §8]).

2.2. **Characterization of asymmetries.** In this subsection, we show that in a central simple algebra of exponent 2, every unit which is conjugate to its inverse is the asymmetry of some anti-automorphism.

We first compare the asymmetries of two anti-automorphisms $\sigma$, $\tau$ on a central simple algebra $A$. The Skolem-Noether theorem shows that the automorphism $\tau \circ \sigma^{-1}$ is the conjugation by some unit $u \in A^\times$, i.e.,

$$\tau(x) = u\sigma(x)u^{-1} \qquad \text{for all } x \in A. \tag{13}$$

**Proposition 7.** *Let $\sigma$, $\tau$ be anti-automorphisms of a central simple algebra $A$, and let $u \in A^\times$ be such that* (13) *holds. The asymmetries $a_\sigma$, $a_\tau$ of $\sigma$ and $\tau$ are related by*

$$a_\tau = u\sigma(u)^{-1}a_\sigma.$$

*Proof.* We use the definition of asymmetry provided by Proposition 6. For $a$, $b$, $x \in A$, we have

$$\tau_*(a \otimes b)(x) = ax\tau(b) = axu\sigma(b)u^{-1}$$

hence

$$\tau_*(a \otimes b)(x) = \sigma_*(a \otimes b)(xu)u^{-1}.$$

Therefore, denoting by $r_u\colon A \to A$ the linear map of multiplication on the right by $u$, we have

$$\tau_*(a \otimes b) = (r_u)^{-1} \circ \sigma_*(a \otimes b) \circ r_u$$

for all $a$, $b \in A$, hence also

$$\tau_*(g) = (r_u)^{-1} \circ \sigma_*(g) \circ r_u$$

for $g$ the Goldman element of $A$. It follows that for all $f \in A$,

$$\tau_*(g)(f) = \sigma_*(fu)u^{-1}. \tag{14}$$

By Proposition 6, the asymmetry $a_\tau$ satisfies

$$a_\tau f = \tau\big(\tau_*(g)(f)\big) \qquad \text{for all } f \in A.$$

Using (14), we obtain

$$a_\tau f = \tau\big(\sigma_*(g)(fu)u^{-1}\big) = u\sigma\big(\sigma_*(g)(fu)u^{-1}\big)u^{-1} = u\sigma(u)^{-1}\sigma\big(\sigma_*(g)(fu)\big)u^{-1}.$$

Proposition 6 also yields $\sigma\big(\sigma_*(g)(fu)\big) = a_\sigma fu$, hence

$$a_\tau f = u\sigma(u)^{-1} a_\sigma f \qquad \text{for all } f \in A.$$

The proposition follows. □

**Theorem 2.** *Let $A$ be a central simple algebra of exponent 2 over an arbitrary field $F$. A unit is the asymmetry of some anti-automorphism of $A$ if and only if it is conjugate to its inverse.*

*Proof.* Suppose $a \in A^\times$ is the asymmetry of some anti-automorphism $\sigma$. We have to show that the $F$-vector space

$$U = \{x \in A \mid xa = a^{-1}x\}$$

contains an invertible element. This amounts to proving that the restriction of the reduced norm polynomial $\operatorname{Nrd}_A$ does not vanish on $U$. Theorem 1 shows that this polynomial does not vanish on $U \otimes K$, for any splitting field $K$ of $A$, since $a$ is the asymmetry of $\sigma \otimes \operatorname{Id}_K$. Therefore, the reduced norm does not vanish on $U$, since $F$ is an infinite field. (Note that every central simple algebra over a finite field is split, hence of exponent 1.)

For the converse, suppose $a \in A^\times$ is conjugate to $a^{-1}$. Let $K$ be a splitting field of $A$; identify $A \otimes K = \operatorname{End}_K V$ for some $K$-vector space $V$. We first show, by using Theorem 1, that $a$ $(= a \otimes 1)$ is the asymmetry of some anti-automorphism of $\operatorname{End}_K V$. With the same notation as in Theorem 1, we have to prove that $\dim_K V_m^{+1}$ is even if $m$ is even, and moreover that $\dim_K V_m^{-1}$ is even if $m$ is odd and $\operatorname{char} F \neq 2$. For every integer $m \geq 1$ and $\varepsilon = \pm 1$, we have an exact sequence of $K$-vector spaces

$$0 \to \frac{\ker(a - \varepsilon \operatorname{Id}_V)^{m+1}}{\ker(a - \varepsilon \operatorname{Id}_V)^m} \xrightarrow{a - \varepsilon \operatorname{Id}_V} \frac{\ker(a - \varepsilon \operatorname{Id}_V)^m}{\ker(a - \varepsilon \operatorname{Id}_V)^{m-1}} \to V_m^\varepsilon \to 0,$$

hence

$$\dim V_m^\varepsilon = \operatorname{rk}(a - \varepsilon \operatorname{Id}_V)^{m-1} - 2\operatorname{rk}(a - \varepsilon \operatorname{Id}_V)^m + \operatorname{rk}(a - \varepsilon \operatorname{Id}_V)^{m+1}, \tag{15}$$

where rk denotes the rank.

For all $b \in A$ we have

$$\operatorname{rk} b = \frac{\dim_K b(A \otimes K)}{\deg(A \otimes K)} = \frac{\dim_F bA}{\deg A},$$

hence $\operatorname{rk} b$ is divisible by the Schur index $\operatorname{ind} A$ (see [4, (1.9)]). Since $A$ has exponent 2, $\operatorname{ind} A$ is even, by [1, Theorem 5.17]. Therefore, $\operatorname{rk} b$ is even for all $b \in A$, and equation (15) shows that $\dim V_m^\varepsilon$ is even for every integer $m$ and for $\varepsilon = \pm 1$. By Theorem 1, it follows that $a$ is the asymmetry of some anti-automorphism $\theta$ of $A \otimes K$.

Now, fix some anti-automorphism $\sigma$ of $A$. Let $a_\sigma$ be its asymmetry and consider the $F$-vector space

$$W = \{x \in A \mid xa = \sigma(x)a_\sigma\}.$$

If $u \in (A \otimes K)^\times$ is such that $\theta(x) = u(\sigma \otimes \operatorname{Id}_K)(x)u^{-1}$ for all $x \in A \otimes K$, then $u^{-1} \in W \otimes K$, by Proposition 7. Therefore, the same arguments as in the first part of the proof show that $W$ contains an invertible element $w$. Using Proposition 7 again, we see that $a$ is the asymmetry of the anti-automorphism $x \mapsto w^{-1}\sigma(x)w$. □

**Corollary 1** (Albert)**.** *Every central simple algebra of exponent* 2 *carries an involution. Moreover, if the characteristic of the base field is different from* 2*, every central simple algebra of exponent* 2 *carries involutions of both orthogonal and symplectic types.*

*Proof.* It readily follows from Theorem 2 that $+1$ and $-1$ are asymmetries of some anti-automorphisms. These anti-automorphisms are involutions, by Proposition 5. □

2.3. **The determinant of an anti-automorphism.** Let $\sigma$ be a linear anti-automorphism of a central simple algebra $A$ over an arbitrary field $F$. Let $a_\sigma \in A^\times$ be the asymmetry of $A$ and $\gamma_\sigma$ the linear involution of Proposition 4. Consider the vector spaces

$$\operatorname{Alt}(A,\sigma) = \{x - \sigma(x)a_\sigma \mid x \in A\} = \{x - \gamma_\sigma(x) \mid x \in A\}$$

and

$$\operatorname{Sk}(A,\sigma) = \{x \in A \mid \sigma(x) + xa_\sigma^{-1} = 0\} = \{x \in A \mid \gamma_\sigma(x) = -x\}.$$

From equations (10) and (11), it follows that $\operatorname{Alt}(A,\sigma) \subset \operatorname{Sk}(A,\sigma)$. Moreover, we have $x - \gamma_\sigma(x) = 2x$ for all $x \in \operatorname{Sk}(A,\sigma)$, hence $\operatorname{Alt}(A,\sigma) = \operatorname{Sk}(A,\sigma)$ if $\operatorname{char} F \neq 2$.

**Lemma 2.** *Suppose $\sigma$, $\tau$ are anti-automorphisms of $A$, and let $u \in A^\times$ be such that*

$$\tau(x) = u\sigma(x)u^{-1} \qquad \textit{for all } x \in A.$$

*Then*

$$\operatorname{Alt}(A,\tau) = u\operatorname{Alt}(A,\sigma) \quad \textit{and} \quad \operatorname{Sk}(A,\tau) = u\operatorname{Sk}(A,\sigma).$$

*Proof.* Proposition 7 yields $a_\tau = u\sigma(u)^{-1}a_\sigma$ and $a_\sigma = u^{-1}\tau(u)a_\tau$. Therefore, for all $x \in A$ we have

$$x - \tau(x)a_\tau = u\big(u^{-1}x - \sigma(u^{-1}x)a_\sigma\big) \quad \text{and} \quad u\big(x - \sigma(x)a_\sigma\big) = ux - \tau(ux)a_\tau,$$

proving that $\operatorname{Alt}(A,\tau) = u\operatorname{Alt}(A,\sigma)$. The proof that $\operatorname{Sk}(A,\tau) = u\operatorname{Sk}(A,\sigma)$ is along the same lines. □

**Lemma 3.** *If* $\deg A$ *is even,* $\operatorname{Alt}(A,\sigma)$ *contains invertible elements. Moreover, the square class* $\operatorname{Nrd}_A(x)\cdot F^{\times 2} \in F^\times/F^{\times 2}$ *does not depend on the choice of* $x \in A^\times \cap \operatorname{Alt}(A,\sigma)$.

*Proof.* Let $\tau$ be an anti-automorphism of $A$ with asymmetry $+1$ and let $u \in A^\times$ be such that

$$\tau(x) = u\sigma(x)u^{-1} \qquad \text{for all } x \in A.$$

By Lemma 2, we have

$$\operatorname{Alt}(A,\sigma) = u^{-1}\operatorname{Alt}(A,\tau). \tag{16}$$

Since $\tau$ is an involution, Corollary (2.8) of [4] shows that $\operatorname{Alt}(A,\tau)$ contains invertible elements if $\deg A$ is even, hence $\operatorname{Alt}(A,\sigma)$ also contains invertible elements. Moreover, from [4, (7.1)], it follows that all the invertible elements have the same reduced norm up to a square of $F$; therefore, if $v \in A^\times \cap \operatorname{Alt}(A,\tau)$ it follows from (16) that $\operatorname{Nrd}_A(x) \in \operatorname{Nrd}_A(u^{-1}v)\cdot F^{\times 2}$ for all $x \in A^\times \cap \operatorname{Alt}(A,\sigma)$. □

This last lemma allows us to define the *determinant* of an anti-automorphism $\sigma$ of a central simple algebra $A$ of even degree, as follows:

$$\det\sigma = \operatorname{Nrd}_A(x)\cdot F^{\times 2} \in F^\times/F^{\times 2}$$

for any $x \in A^\times \cap \operatorname{Alt}(A,\sigma)$.

This definition is consistent with [4, (7.2)], where the determinant of an orthogonal involution is defined.

*Example* 1. Since clearly $1 - a_\sigma \in \operatorname{Alt}(A,\sigma)$, we have

$$\det\sigma = \operatorname{Nrd}_A(1-a_\sigma)\cdot F^{\times 2}$$

if $1-a_\sigma$ is invertible. Therefore, the determinant of $\sigma$ is entirely determined by its asymmetry in this particular case.

*Example* 2. The transpose involution on a matrix algebra $M_n(F)$ (with $n$ even) has trivial determinant. Indeed, the matrix

$$\begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_{n/2} \end{pmatrix} \qquad \text{where } m_1 = \cdots = m_{n/2} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

is in $\operatorname{Alt}\bigl(M_n(F),t\bigr)$ and has determinant 1.

**Proposition 8.** *Let $\sigma$, $\tau$ be anti-automorphisms of a central simple algebra $A$ of even degree, and let $u \in A^\times$ be such that*

$$\tau(x) = u\sigma(x)u^{-1} \qquad \textit{for all } x \in A.$$

*Then*

$$\det\tau = \operatorname{Nrd}_A(u)\det\sigma.$$

*Proof.* This readily follows from Lemma 2. □

**Proposition 9.** *Let $V$ be an even-dimensional vector space over an arbitrary field $F$ and let $b$ be a nonsingular pairing on $V$. For every basis $(v_i)_{1\le i\le n}$ of $V$,*

$$\det\sigma_b = \det\bigl(b(v_i,v_j)\bigr)_{1\le i,j\le n}\cdot F^{\times 2}.$$

*Proof.* Identify $\operatorname{End}_F V$ with the matrix algebra $M_n(F)$ by means of the basis $(v_i)_{1\le i\le n}$. The anti-automorphism $\sigma_b$ is then given by

$$\sigma_b(m) = u^{-1}m^t u \qquad \text{for all } m \in M_n(F),$$

where $u = \bigl(b(v_i,v_j)\bigr)_{1\le i,j\le n} \in M_n(F)$. Therefore, Proposition 8 yields

$$\det\sigma_b = \det u^{-1}\det t.$$

Since it was observed in Example 2 above that $\det t$ is trivial, the proposition follows. □

## References

[1] A.A. Albert, *Structure of Algebras,* Coll. Pub. **24**, Amer. Math. Soc., Providence, RI, 1939.
[2] F. Ischebeck, W. Scharlau, *Hermitesche und orthogonale Operatoren über kommutativen Ringen,* Math. Ann. **200** (1973), 327–334.
[3] M.-A. Knus, *Quadratic and Hermitian Forms over Rings,* Grundlehren der Mathematischen Wissenschaften, vol. 294, Springer-Verlag, Berlin, 1991.
[4] M.-A. Knus, A.S. Merkurjev, M. Rost and J.-P. Tignol, *The Book of Involutions,* Coll. Pub. **44**, Amer. Math. Soc., Providence, RI, 1998.
[5] A. Ranicki, *High-dimensional Knot Theory,* Springer Monographs in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1998.
[6] C. Riehm, *The equivalence of bilinear forms,* J. Algebra **31** (1974), 45–66.
[7] D.J. Saltman, *Azumaya algebras with involution,* J. Algebra **52** (1978), 526–539.
[8] W.C. Waterhouse, *A nonsymmetric Hasse-Minkowski theorem,* Amer. J. Math. **99** (1977), 755–759.
[9] J. Williamson, *On the algebraic problem concerning the normal form of linear dynamical systems,* Amer. J. Math. **58** (1936), 141–163.

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