A LOCAL-GLOBAL PRINCIPLE FOR ALGEBRAS WITH INVOLUTION AND HERMITIAN FORMS

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Abstract. Weakly hyperbolic involutions are introduced and a proof is given of the following local-global principle: a central simple algebra with involution of any kind is weakly hyperbolic if and only if its signature is zero for all orderings of the ground field. Also, the order of a weakly hyperbolic algebra with involution is a power of two, this being a direct consequence of a result of Scharlau. As a corollary an analogue of Pfister's local-global principle is obtained for the Witt group of hermitian forms over an algebra with involution.

1. Introduction

Pfister's well-known local-global principle states that a nonsingular quadratic form q over a field k (which we assume to be of characteristic different from 2) is a torsion element in the Witt ring $W(k)$ of k if and only if the signature of q is zero for all orderings of k. Furthermore, every torsion element of $W(k)$ has 2-power order.

If $W(A,\sigma)$ is the Witt group of hermitian forms over some central simple k-algebra with involution of any kind (A,σ), then Scharlau showed that the torsion elements of $W(A,\sigma)$ have 2-power order.

In this paper, we complement Scharlau's result by showing that $h \in W(A,\sigma)$ is a torsion element if and only if h has signature zero for all orderings of the ground field, thus obtaining an analogue of Pfister's local-global principle for hermitian forms. In fact, this will follow from our main theorem which states that if (A,σ) is a central simple k-algebra with involution of any kind, then the signature of σ is zero for all orderings of k if and only if (A,σ) is weakly hyperbolic. As a consequence of Scharlau's theorem we also get that the order of (A,σ) is a power of 2 when (A,σ) is weakly hyperbolic.

Weakly hyperbolic involutions are a natural generalization of torsion forms and are a special case of weakly isotropic involutions. The latter were studied in [6] as an ingredient of a different local-global principle for algebras with an involution of the first kind.

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All involutions on central simple algebras considered in this paper are arbitrary and all forms (quadratic, hermitian, *etc.*) are assumed to be nonsingular. Standard references are [5] and [10] for the theory of quadratic forms, [4] for central simple algebras with an involution and [8] for real fields.

2. Preliminaries

Let K be a field of characteristic different from two and let A be a central simple K-algebra, equipped with an involution σ. We say that σ is of the *first kind* if $\sigma|_K = \mathbf{1}_K$ and of the *second kind* otherwise. If σ is of the second kind, let k denote its fixed field. Then K is a quadratic étale extension field of k, *i.e.* either $K = k(\sqrt{\alpha})$ for some $\alpha \in k^\times$ or $K = k \times k$. In the latter case, A is no longer simple. Unless stated otherwise, we will always assume that K is a field, for reasons which will become apparent later. To cater for both kinds of involution, we take k as the base field and simply say that (A, σ) is a central simple k-algebra with involution, it being understood that the center is k or K, depending on the situation.

By Wedderburn's theorem, we can write

$$A \cong \mathrm{End}_D(V),$$

where D is a central division algebra over k (with center K) and V is some finite dimensional right D-vector space. The *degree* of A is $\deg A := \sqrt{\dim_K A}$ and the (*Schur*) *index* of A is $\operatorname{ind} A := \deg D$. We say that A is *split* if $D = K$. Any field extension $L \supset k$ such that $A \otimes_k L$ is split, is called a *splitting field* of A. In particular, the separable closure k_s of k is a splitting field of A.

Let $n = \deg A$. If σ is of the first kind, σ is called *orthogonal* (resp. *symplectic*) if the extended involution $\sigma \otimes \mathbf{1}_{k_s}$ on $A \otimes_k k_s \cong M_n(k_s)$ is adjoint to a symmetric (resp. skew-symmetric) bilinear form. Involutions of the second kind are also called *unitary involutions*.

It is well-known that σ is the adjoint involution of some non-singular ε-hermitian form ($\varepsilon = \pm 1$) $h : V \times V \to D$, with respect to some involution ϑ on D which is of the same kind as σ. (If σ is of the second kind, then $\varepsilon = +1$ and $\vartheta(\alpha) = \sigma(\alpha) \forall \alpha \in K$.) So σ is of the form σ_h, where σ_h is implicitly defined by

$$h\big(x, f(y)\big) = h\big(\sigma_h(f)(x), y\big) \qquad \text{for } x, y \in V \text{ and } f \in \mathrm{End}_D(V).$$

The form h is uniquely determined up to multiplication with a ϑ-invariant factor in $K^\times$. (More generally, we can substitute a central simple algebra E for D and a finitely generated right E-module M for V, see [4, 4.A].)

2.1. Hyperbolic Involutions.

A right ideal I of A is called *isotropic* if $\sigma(I)I = 0$. The algebra with involution (A, σ), or the involution σ itself, is called *isotropic* if A contains an isotropic right ideal and *anisotropic* if $\sigma(x)x = 0$ implies $x = 0$, $\forall x \in A$. We also say that (A, σ) is *weakly isotropic* if there exist nonzero

$x_1, \ldots, x_n \in A$ such that $\sigma(x_1)x_1 + \cdots + \sigma(x_n)x_n = 0$ and *strongly anisotropic* otherwise.

The notion of a hyperbolic involution was first defined by Bayer-Fluckiger *et al.* [2] (see also [4, 6.B]): an algebra with involution (A, σ), or the involution σ itself, is called *hyperbolic* if either the center of A is isomorphic to $k \times k$, or σ is the adjoint involution of some hyperbolic ε-hermitian form. In both cases (A, σ) is hyperbolic if and only if A contains an idempotent e such that $\sigma(e) = 1 - e$. This is also equivalent with the existence of an isotropic right ideal I of dimension $\dim_k I = \frac{1}{2} \dim_k A$.

Note that hyperbolic involutions remain hyperbolic under arbitrary scalar extensions and if (A, σ) is hyperbolic and (B, τ) is arbitary, then $(A \otimes_k B, \sigma \otimes \tau)$ is hyperbolic.

2.2. **Signatures of Involutions.** Suppose that k is a real field. We fix some notation for the rest of this paper. We denote the space of orderings of k by X_k. If $P \in X_k$, k_P will denote the real closure of k with respect to P. Furthermore, we set $A_P := A \otimes_k k_P$, $\sigma_P = \sigma \otimes \mathbf{1}_{k_P}$, *etc.*

Let $P \in X_k$. The signature of an involution was defined by Lewis and Tignol [7] for involutions of the first kind and by Quéguiner [9] for involutions of the second kind as

$$\operatorname{sig}_P \sigma = \begin{cases} \sqrt{\operatorname{sig}_P T_\sigma} & \text{if } \sigma \text{ is of the first kind,} \\ \sqrt{\frac{1}{2} \operatorname{sig}_P T_\sigma} & \text{if } \sigma \text{ is of the second kind.} \end{cases}$$

Here T_σ denotes the *trace quadratic form* $T_\sigma(x) := \operatorname{Trd}_A(\sigma(x)x), \forall x \in A$, which takes values in k.

If (A, σ) is split with orthogonal involution, $(A, \sigma) \cong (\operatorname{End}_k(V), \sigma_q)$, q being a quadratic form over k, then Lewis and Tignol showed that

$$\operatorname{sig}_P \sigma_q = |\operatorname{sig}_P q|.$$

Likewise, if (A, σ) is split with unitary involution, $(A, \sigma) \cong (\operatorname{End}_K(V), \sigma_h)$, h being a K/k-hermitian form, then Quéguiner showed that

$$\operatorname{sig}_P \sigma_h = |\operatorname{sig}_P h|.$$

Clearly, the signature of σ will be zero in the split-symplectic case.

2.3. **The n-fold Orthogonal Sum.** Let f be a quadratic or K/k-hermitian form, then the notion of n-fold orthogonal sum of f,

$$\perp^n f = n \times f = \langle \underbrace{1, \ldots, 1}_{n} \rangle \otimes f,$$

can be extended to the realm of algebras with an involution in a straightforward way:

Definition 2.1. Let (A, σ) be a central simple k-algebra with involution of any kind. The *n-fold orthogonal sum* $\boxplus^n(A, \sigma)$ is defined by

$$\overset{n}{\boxplus}(A, \sigma) := (M_n(K), *) \otimes_k (A, \sigma),$$

with $*$ the conjugate transpose involution, defined by $(a_{ij})^* = (\iota(a_{ij}))^t, \forall (a_{ij}) \in M_n(K)$, where ι is either the nontrivial automorphism of K if σ is of the second kind or the identity otherwise. It is clearly again a central simple k-algebra of degree $n \cdot \deg A$.

Remark 2.2. When A is split, the involution $* \otimes \sigma$ will be the adjoint of $n \times f$, for an appropriate form f.

Remark 2.3. If (A, σ) is a central simple k-algebra with involution of the first kind, the n-fold orthogonal sum $\boxplus^n(A, \sigma)$ was defined in a more intrinsic way in [12], so as to conform with Dejaiffe's [3] construction of an orthogonal sum of two Morita equivalent algebras with involution of the first kind.

3. The Local-Global Principle for Algebras with Involution

Definition 3.1. The algebra with involution (A, σ) is called *weakly hyperbolic* if there exists an $n \in \mathbb{N}$ such that $\boxplus^n(A, \sigma)$ is hyperbolic. The *order* of (A, σ) is the least integer n such that $\boxplus^n(A, \sigma)$ is hyperbolic.

The main theorem of this paper reads:

Theorem 3.2. *Let (A, σ) be a central simple k-algebra with involution of any kind. Then $\operatorname{sig}_P \sigma = 0, \forall P \in X_k$ if and only if (A, σ) is weakly hyperbolic. Furthermore, the order of (A, σ) is a power of two when (A, σ) is weakly hyperbolic.*

Note that when k is not real, there is no notion of signature and $(M_n(k), t)$ will be hyperbolic for some positive integer n, so that (A, σ) will be weakly hyperbolic. The main theorem will thus hold trivially. Hence, we assume from now on that k is a real field. In order to prove this theorem, we first need to establish a few lemmas.

Lemma 3.3. *The main theorem holds when A is split; $A \cong \operatorname{End}_k(V)$.*

Proof. When A is split, σ will be the adjoint of a quadratic, a skew-symmetric bilinear, or a K/k-hermitian form f. If f is quadratic, we have proof by Pfister's local-global principle. If f is skew-symmetric, the statement is trivial. If f is K/k-hermitian, f is completely determined by its trace form $q_f(x) := f(x, x) \in k, \forall x \in V$ (Jacobson's theorem, see [10, Theorem 10.1.1]), in which case we have proof again by Pfister's local-global principle. ∎

Lemma 3.4. *Let (A, σ) be a central simple k-algebra with involution of any kind and let L be a real closed extension field of k such that $A_L \cong M_n\big((-1, -1)_L\big)$ for some positive integer n. Then the main theorem holds for (A_L, σ_L).*

Proof. If σ_L is symplectic, then σ_L is the conjugate transpose involution ^{-}t, where $^{-}$ is quaternion conjugation. In this case, (A_L, σ_L) is totally anisotropic and σ_L is positive definite, so the lemma holds.

If σ_L is orthogonal, then σ_L is the conjugate transpose involution $\hat{}\,t$, where $\hat{}$ is the orthogonal involution on $(-1,-1)_L$, determined by $\hat{1} = 1$, $\hat{i} = -i$, $\hat{j} = j$, $\hat{k} = k$. Now $\operatorname{sig} \sigma_L = 0$ and $\boxplus^2(A_L, \sigma_L)$ is hyperbolic, since $\boxplus^2\big((-1,-1)_L, \hat{}\,\big)$ is hyperbolic (the element $e := \frac{1}{2}\left(\begin{smallmatrix} 1 & j^{-1} \\ j & 1 \end{smallmatrix}\right)$ satisfies $e^2 = e$ and $\hat{e}^t = 1 - e$.)

If σ_L is unitary, then the center of A_L is $L(\sqrt{-1})$, which is algebraically closed, since L is real. Hence $A_L \cong M_{2n}\big(L(\sqrt{-1})\big)$ and σ_L is the conjugate transpose involution $*$, defined earlier. Now (A_L, σ_L) is totally anisotropic and σ_L is positive definite, so that we are done. ∎

Lemma 3.5. *Suppose σ is an anisotropic involution of any kind on a central simple k-algebra A and let $L = k(\sqrt{\delta})$ be a quadratic field extension of k. If the involution $\sigma_L = \sigma \otimes \mathbf{1}_L$ on $A \otimes_k L$ is hyperbolic, then there exists $r \in A$ such that $r^2 = \delta$ and $\sigma(r) = -r$.*

This is in fact [2, Lemma 3.2] which holds for involutions of any kind, not just for those of the first kind (as it is stated in [2]). Another result from [2] which holds for arbitrary involutions, is their Theorem 3.3:

Theorem 3.6. *Let (A, σ) be a central simple k-algebra with involution of any kind and let $L = k(\sqrt{\delta})$ be a quadratic field extension of k. If there exists $r \in A$ such that $r^2 = \delta$ and $\sigma(r) = -r$, then $(A \otimes_k L, \sigma_L)$ is hyperbolic. The converse holds except in the case where A is split, σ is orthogonal and its associated quadratic form has odd Witt index.*

Proof. If σ is an involution of the first kind, we refer to [2, Theorem 3.3] for a proof, which can be made to work for involutions of the second kind quite easily, as we will proceed to do now. (The crucial ingredients, Witt cancellation and Witt decomposition, also work in this situation.) So, suppose that σ is unitary and note that the exceptional case does not occur now. Recall that $K = k(\sqrt{\alpha})$ for some $\alpha \in k^\times$.

Given r, let $t = \delta^{-1} r \otimes \sqrt{\delta} \in A \otimes_k L$. Note that $t^2 = 1$ and $\sigma_L(t) = -t$. Therefore, the element $e = \frac{1}{2}(1 + t)$ satisfies $e^2 = e$ and $\sigma_L(e) = 1 - e$, so that $(A \otimes_k L, \sigma_L)$ is hyperbolic.

Conversely, let V be an irreducible left A-module and $D = \operatorname{End}_A(V)$, so that $A = \operatorname{End}_D(V)$. Choose an involution of the second kind $^{-}$ on D and let h be a hermitian form on V with respect to which σ is the adjoint involution.

Consider a Witt decomposition

$$(V, h) \cong (V_1, h_1) \perp (V_0, h_0),$$

where (V_1, h_1) is hyperbolic and (V_0, h_0) is anisotropic. Let σ_i denote the adjoint involution with respect to h_i on $\operatorname{End}_D(V_i)$ for $i = 0, 1$. By Witt cancellation,

(V_0, h_0) becomes hyperbolic over L, hence Lemma 3.5 yields an element $r_0 \in \mathrm{End}_D(V_0)$ such that $r_0^2 = \delta$ and $\sigma_0(r_0) = -r_0$.

By [2, Theorem 2.2], $\mathrm{End}_D(V_1)$ contains a σ_1-invariant subalgebra M such that $(M, \sigma_1|_M) \cong (M_2(K), \Theta)$, where Θ is defined by

$$\Theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{d} & \overline{b} \\ \overline{c} & \overline{a} \end{pmatrix}, \quad \forall a, b, c, d \in K.$$

Let $r_1 \in M \subset \mathrm{End}_D(V_1)$ denote the image of $\begin{pmatrix} 0 & \delta\sqrt{\alpha} \\ \frac{1}{\sqrt{\alpha}} & 0 \end{pmatrix}$, then $\sigma_1(r_1) = -r_1$ and $r_1^2 = \delta$.

Finally, regarding $V = V_1 \oplus V_0$, define $r \in A = \mathrm{End}_D(V)$ by

$$r(x, y) = \big(r_1(x), r_0(y)\big).$$

It is straightforward to check that $\sigma(r) = -r$ and $r^2 = \delta$. ∎

Proof of Theorem 3.2. Suppose first that $(B, \tau) := \boxplus^n (A, \sigma)$ is hyperbolic for some n. Since hyperbolic involutions remain hyperbolic under arbitrary scalar extensions, (B_P, τ_P) is hyperbolic for every ordering P of k. Since (B_P, τ_P) is either split or a matrix algebra over $(-1,-1)_{k_P}$ and since signatures do not change under scalar extension, we obtain $\mathrm{sig}_P\, \tau = 0$, $\forall P \in X_k$ from Lemmas 3.3 and 3.4. This implies $\mathrm{sig}_P\, \sigma = 0$, $\forall P \in X_k$.

Conversely, suppose for the sake of contradiction that (A, σ) has zero signature at each ordering of the field k, but that (A, σ) is not weakly hyperbolic. By Zorn's Lemma, there exists a maximal algebraic extension field L of k for which (A_L, σ_L) is not weakly hyperbolic, see Remark 3.8.

Observe that L must be real, because if not, then $(M_n(L), t)$ will be hyperbolic for some positive integer n, so that $\boxplus^n (A_L, \sigma_L)$ is hyperbolic. Also, L cannot be a splitting field for A, because the local-global principle works in the split case (Lemma 3.3).

Next we will show that L is euclidean by showing that ± 1 are the only two square classes in L. Suppose for the sake of contradiction that there exists $\alpha \in L$ such that $\pm\alpha$ is not a square in L. Then $L(\sqrt{\alpha})$ and $L(\sqrt{-\alpha})$ are each quadratic extension fields of L. By maximality of L we have that (A, σ) must become weakly hyperbolic on extension to each of these two fields. Thus there exists an integer n for which the n-fold orthogonal sum of (A, σ) becomes hyperbolic over both $L(\sqrt{\alpha})$ and $L(\sqrt{-\alpha})$. Let us write $(B, \tau) := \boxplus^n (A_L, \sigma_L)$.

Applying Theorem 3.6 to the two quadratic extensions above, yields two elements $r, s \in B$ such that $r^2 = \alpha$, $\tau(r) = -r$, $s^2 = -\alpha$ and $\tau(s) = -s$. Now we will show that the sum of two copies of (B, τ) is hyperbolic (which is a contradiction and hence L must be euclidean). To show this, note that the sum of two copies of (B, τ) is $(M_2(B), t \otimes \tau)$ and let $e = \frac{1}{2} \begin{pmatrix} 1 & rs^{-1} \\ sr^{-1} & 1 \end{pmatrix}$. Then it is easy to check that $e^2 = e$ and $t \otimes \tau(e) = 1 - e$. Therefore, the involution $t \otimes \tau$ is hyperbolic. Thus we have proved that L is euclidean.

In fact, L is real closed, since L cannot have any proper odd degree extensions by [4, Corollary 6.16]. But then $Q := (-1,-1)_L$ is the only division algebra over L. Hence, A_L is an endomorphism algebra over Q (since L is not a splitting field of A). Since the local-global principle holds for such algebras (Lemma 3.4), we have a contradiction and conclude that (A, σ) has to be weakly hyperbolic.

The second statement is a direct consequence of Scharlau's result [11] that the torsion in the Witt group of central simple algebras with involution is 2-primary. ∎

Remark 3.7. Whenever there exists an invertible element $y \in B$ with $\tau(y) = -y^{-1}$ one can get that the sum of two copies of (B, τ) is hyperbolic by defining $e = \frac{1}{2}\begin{pmatrix} 1 & y^{-1} \\ y & 1 \end{pmatrix}$. When B is split and τ is the adjoint of some quadratic form ϕ, the condition $\tau(y) = -y^{-1}$ reduces to the fact that ϕ is isometric to $-\phi$, which of course implies that the sum of two copies of ϕ is hyperbolic.

Remark 3.8. The reasoning behind the use of Zorn's Lemma is analogous to what happens in the proof of the "(iv) $\Longrightarrow$ (v)" direction of [10, Theorem 2.7.1]: Let $\overline{k}$ be an algebraic closure of k and let $\mathcal{M}$ be the set of all intermediate fields M such that $A_M := A \otimes_k M$ is not weakly hyperbolic. The set $\mathcal{M}$ is ordered by inclusion. For a totally ordered chain of subfields M_i in $\mathcal{M}$, we let $A_i := A \otimes_k M_i$ and $M := \bigcup M_i$.

Assume for the sake of contradiction that (A_M, σ_M) is weakly hyperbolic. This means that there exists a square matrix X with entries in A_M such that $X^2 = X$ and $\sigma_M \otimes *(X) = I - X$. These are matrix equations with finitely many coefficients from A_M. So all coefficients must lie in A_i for some i. Thus M_i would not belong to $\mathcal{M}$, a contradiction. Therefore $M \in \mathcal{M}$ and we have found an upper bound. By Zorn's Lemma, $\mathcal{M}$ has a maximal element.

4. The Local-Global Principle for Hermitian Forms

Let k be a real closed field and let K be either $k(\sqrt{-1})$ or $(-1,-1)_k$. Let $W(K, {}^-)$ denote the Witt group of hermitian forms over K with respect to the canonical involution on K (*i.e.* the nontrivial automorphism of $k(\sqrt{-1})$ or quaternion conjugation). Given $h \in W(K, {}^-)$, let q_h denote the trace form of h, defined by $q_h(x) := h(x,x)$ for all $x \in K$. In both cases q_h takes values in the ground field k. Let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be a diagonalization of h. Then $\alpha_i \in k^\times$ $(1 \le i \le n)$ in both cases and

$$q_h = \begin{cases} 2 \times \langle \alpha_1, \ldots, \alpha_n \rangle & \text{if } K = k(\sqrt{-1}), \\ 4 \times \langle \alpha_1, \ldots, \alpha_n \rangle & \text{if } K = (-1,-1)_k. \end{cases}$$

As is well-known (see *e.g.* [10, 10.1.6]), the signature of $\langle \alpha_1, \dots, \alpha_n \rangle$ is an invariant of the hermitian form h in both cases, aptly called the *signature* of h. To summarize,

$$\operatorname{sig} h = \begin{cases} \frac{1}{2} \operatorname{sig} q_h & \text{if } K = k(\sqrt{-1}), \\ \frac{1}{4} \operatorname{sig} q_h & \text{if } K = (-1,-1)_k. \end{cases}$$

Now let k be a real field and let (A, σ) be a central simple k-algebra (with involution of any kind). Let $W(A, \sigma)$ denote the Witt group of hermitian forms over A with respect to σ. The previous observations allow us to define the signature of h with respect to the ordering P, $\operatorname{sig}_P h$, by going over to the real closure k_P of k with respect to P and using Morita theory. We will not elaborate this here, but refer instead to [1] for a recent exposition. It turns out that $\operatorname{sig}_P h$ can be defined in a meaningful way in the following situations:

- σ unitary;
- σ orthogonal and A_P split;
- σ symplectic and $A_P \cong$ matrices over $(-1,-1)_{k_P}$.

In the other cases, $\operatorname{sig}_P h$ is defined to be equal to zero. (This includes the situation where $K_P = k_P \times k_P$, which, together with the fact that algebras with involution with center $k \times k$ are by definition hyperbolic, prompted us to exclude this possibility from our considerations.)

The main point for us is the observation that

$$\operatorname{sig}_P h = 0 \qquad \Longleftrightarrow \qquad \operatorname{sig}_P \sigma_h = 0,$$

where σ_h is the adjoint involution of h. This can be deduced quite easily from the work of Lewis–Tignol and Quéguiner mentioned earlier.

Given $h \in W(A, \sigma)$, it is now a simple matter of going over to the adjoint involution σ_h on $\operatorname{End}_A(V)$, in order to obtain the following consequence of our main theorem:

Theorem 4.1. *Let $h \in W(A, \sigma)$. Then h is a torsion element if and only if $\operatorname{sig}_P h = 0$ for all $P \in X_k$. Every torsion element of $W(A, \sigma)$ has 2-power order.*

The first statement of the theorem can be translated into a familiar looking exact sequence:

$$0 \longrightarrow W_t(A, \sigma) \longrightarrow W(A, \sigma) \xrightarrow{(\operatorname{sig}_P)} \prod_{P \in X_k} W(A_P, \sigma_P),$$

where $W_t(A, \sigma)$ denotes the torsion subgroup of $W(A, \sigma)$ and $W(A_P, \sigma_P)$ is either 0, $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$, depending on what happens at the ordering P.

We note again that the second statement of the theorem is nothing new. It is due to Scharlau [11], a fact which we already acknowledged in the proof of our main theorem. To be more precise, Scharlau proved that if A is a finite-dimensional *semisimple* k-algebra with involution σ such that $\sigma|_k = \mathbf{1}_k$, then $W_t(A, \sigma)$ is 2-primary [11, Theorem 5.1(i)]. If we denote the Grothendieck group of hermitian forms over (A,σ) by $\widehat{W}(A,\sigma)$ in this situation, then the following exact sequence can be deduced from [11, Theorem 5.2]:

$$0 \longrightarrow \widehat{W}_t(A,\sigma) \longrightarrow \widehat{W}(A,\sigma) \xrightarrow{r^*} \prod_{P\in X_k} \widehat{W}(A_P,\sigma_P),$$

where r^* is the canonical extension homomorphism.

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