LINKAGE OF FIELDS IN CHARACTERISTIC 2

ROBERTO ARAVIRE AND RICARDO BAEZA

Abstract

Let F be a field with $2 = 0$ and $\varphi = \ll a_1, \dots, a_n \gg$ an n-fold anisotropic bilinear Pfister form over F with function field $F(\varphi)$. In this paper we compute $\ker[I_F^n/I_F^{n+1} \to I_{F(\varphi)}^n/I_{F(\varphi)}^{n+1}]$ where $I_F \subset W(F)$ is the maximal ideal in the Witt ring $W(F)$ of F. We use this computation to prove a n-linkage property of the subfields $F^2(a_1, \dots, a_n)$.

0. Introduction

In this paper F will denote throughout a field with $2 = 0$. Let $\Omega_F^* = \bigoplus_{n=0}^{\infty} \Omega_F^n$ be the F-algebra of differential forms over F and let $d : \Omega_F^n \to \Omega_F^{n+1}$ $(n \geq 0)$ be the differential operator (see [Ca], [Ka], [A-Ba 1]). In [Ka] (see also [Mi]) a homomorphism $\wp : \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1}$ is defined on generators as follows

$$\wp(x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}) = (x^2 - x)\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \mod d\Omega_F^{n-1}.$$

(if $n = 0$ we have the usual Artin-Schreier operator $\wp(x) = x^2 - x$). Let $\nu_F(n) = \ker(\wp)$ and $H^{n+1}(F) = \operatorname{coker}(\wp)$. In [Ka] it is shown that $\nu_F(n)$ is additively generated by the pure logarithmic differentials $\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ $(x_i \in F^* = F\backslash\{0\})$ and that there exists a natural isomorphism

$$(1) \qquad \alpha : \nu_F(n) \simeq \overline{I}_F^n := I_F^n / I_F^{n+1}$$

given on generators by $\alpha(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}) := \ll x_1, \dots, x_n \gg \mod I_F^{n+1}$. Here I_F denotes the maximal ideal of even dimensional forms in the Witt ring $W(F)$ of non-singular symmetric bilinear forms over F and $\ll x_1, \dots, x_n \gg$ is the n-fold Pfister form $< 1, x_1 > \otimes \cdots \otimes < 1, x_n >$

2000 Mathematics Subject Classification: 11E04, 11E81, 12E05, 12F20, 15A63.

Keywords and phrases: Bilinear forms, differential forms, Witt-groups, function fields.

First author was partially supported by Fondecyt 100 0391 and Programa Formas Cuadráticas, U. de Talca.

Second author was partially supported by Fondecyt 100 0392 and Programa Formas Cuadráticas, U. de Talca.

(see [Sa], [A-Ba 1] for basic definitions). If L/F is a field extension, let $\nu_{L/F}(n)$ denote $\ker(\nu_F(n) \to \nu_L(n))$ and $\overline{I}^n_{L/F} = \ker(\overline{I}^n_F \to \overline{I}^n_L)$. Thus α induces an isomorphism $\alpha : \nu_{L/F}(n) \simeq \overline{I}^n_{L/F}$.

The purpose of this note is to compute $\nu_{L/F}(n)$ when L is the function field of an anisotropic bilinear Pfister form and to relate this computation to some linking property of subfields of F. The computation of $H^{n+1}(L/F) = \ker(H^{n+1}(F) \to H^{n+1}(L))$ is much more involved and has been done in [A-Ba 2]. For any bilinear form φ over F we will denote by $F(\varphi)$ the function field of the quadric $\{\varphi(x, x) = 0\}$.

In Section 1 and 2 we compute $\nu_{F(\varphi)/F}(n)$ and $\overline{I}^n_{F(\varphi)/F}$ where $\varphi = \ll a_1, \ldots, a_n \gg$. In section 3 we extend these computations to $\nu_{F(\varphi)/F}(m)$, $\overline{I}^m_{F(\varphi)/F}$, for arbitrary $m \geq 1$. We will use the following notions and notations taken from [Ca]. A 2-basis of F is a subset $\{a_1, a_2, \ldots\} \subset F$ such that the elements $\{a^\varepsilon = \prod_i a_i^{\varepsilon_i} \mid \varepsilon = (\varepsilon_i)\}$ (here $\varepsilon = (\varepsilon_i)$, runs over all sequences with $\varepsilon_i = 0$ or 1, and with 0 almost everywhere), form a basis of F over F^2 as a vector space. This is equivalent with the fact that the forms $\frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_n}}{a_{i_n}}$, $i_1 < \cdots < i_n$ are a F-basis of Ω^n_F for all $n \geq 0$. Fixing such a 2-basis, let $[\Omega^n_F]^{[2]} = \bigoplus_{i_1 < \cdots < i_n} F^2 \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_n}}{a_{i_n}}$. Then the space $Z^n_F = \ker(d : \Omega^n_F \to \Omega^{n+1}_F)$ has the direct sum decomposition $Z^n_F = [\Omega^n_F]^{[2]} \oplus d\Omega^{n-1}_F$ and we get a homomorphism $C : Z^n_F \to \Omega^n_F$ given by

$$C\left(\sum_{i_1 < \cdots < i_n} c^2_{i_1 \cdots i_n} \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_n}}{a_{i_n}} + d\eta\right) = \sum_{i_1 < \cdots < i_n} c_{i_1 \cdots i_n} \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_n}}{a_{i_n}}$$

which obviously induces an isomorphism $C : Z^n_F / d\Omega^{n-1}_F \simeq \Omega^n_F$. Although the decomposition of Z^n_F depends on the choice of the 2-basis, the map C does not. We will call C the Cartier operator. It is easy to see that $\nu_F(n)$ is characterized by: $w \in \nu_F(n)$ if and only if $dw = 0$ and $C(w) = w$.

1. $\nu_{L/F}(n)$ FOR $L = F(\ll a_1, \ldots, a_n \gg)$

Let $\varphi = \ll a_1, \ldots, a_n \gg$ be an anisotropic n-fold bilinear Pfister form over F. This means that $\{a_1, a_2, \ldots, a_n\}$ are part of a 2-basis $\{a_1, a_2, \ldots\}$ of F. Let $F^2(a_1, \ldots, a_n)$ be the subfield of F generated by $a_1, \ldots, a_n$ over F^2 and let $F^2(a_1, \ldots, a_n)'$ the additive subgroup of pure elements $\bigoplus_{\varepsilon \neq 0} F^2 a^\varepsilon$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$. Thus $F^2(a_1, \ldots, a_n) = D_F(\varphi)$ is the set of elements of F represented by

φ (including 0) and $F^2(a_1, \dots, a_n)' = D_F(\varphi)'$ are the elements represented by the pure part φ' of φ. Let $L = F(\varphi)$ and set $\Omega^n_{L/F} = \ker(\Omega^n_F \to \Omega^n_L)$. Then we have

$$\nu_{L/F}(n) = \Omega^n_{L/F} \cap \nu_F(n).$$

In [A-Ba 2] we have computed $\Omega^m_{L/F}$ for any $m \geq 0$. The result is

1.1. Proposition.

$$\Omega^m_{L/F} = \begin{cases} 0 & , \quad \textit{if } m < n \\ \Omega^{m-n}_F \wedge \dfrac{da_1}{a_1} \wedge \dots \wedge \dfrac{da_n}{a_n} & , \quad \textit{if } m \geq n. \end{cases}$$

1.2. Corollary.

$$\nu_{L/F}(m) = \begin{cases} 0 & , \quad \textit{if } m < n \\ \Omega^{m-n}_F \wedge \dfrac{da_1}{a_1} \wedge \dots \wedge \dfrac{da_n}{a_n} \bigcap \nu_F(m) & , \quad \textit{if } m \geq n. \end{cases}$$

For the sake of completeness we will give a sketch of the proof of 1.1 at the end of this section. The rest of this section is devoted to compute $\nu_{L/F}(n) = F\frac{da_1}{a_1} \wedge \dots \wedge \frac{da_n}{a_n} \cap \nu_F(n)$.

1.3. Lemma. *For any $a \in F$ the following assertions are equivalent*

1. $a\frac{da_1}{a_1} \wedge \dots \wedge \frac{da_n}{a_n} \in \nu_F(n)$.
2. $\wp(a) = a^2 - a \in F^2(a_1, \dots, a_n)'$.

Proof. Assume (1). Choose a 2-basis of F containing $a_1, \dots, a_n$, say $\{a_1, \dots, a_n, \dots, a_N\}$ (we can assume without restriction that this basis is finite). Then $a = \sum_\varepsilon c^2_\varepsilon a^\varepsilon$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ runs over $\{0,1\}^N$ and $a^\varepsilon = a_1^{\varepsilon_1} \cdots a_N^{\varepsilon_N}$. It follows

$$da = \sum_{i=1}^N D_i(a) da_i$$

with $D_i(a) = \sum_\varepsilon c^2_\varepsilon a_1^{\varepsilon_1} \cdots \widehat{a_i} \cdots a_N^{\varepsilon_N}$ where ε runs over all ε's with $\varepsilon_i = 1$. Since $a\frac{da_1}{a_1} \wedge \dots \wedge \frac{da_n}{a_n} \in \nu_F(n) \subset \ker(d)$, we obtain $d(a) \wedge da_1 \wedge \dots \wedge da_n = 0$, and this implies $d(a) = c_1 da_1 + \dots + c_n da_n$ with some $c_i \in F$. Hence $D_i(a) = 0$ for all $i > n$ and we conclude $c_\varepsilon = 0$ whenever $\varepsilon_i = 1$ for some $i > n$. Thus we have

$$a = \sum_{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)} c^2_\varepsilon a^\varepsilon \in F^2(a_1, \dots, a_n)$$

and

$$a\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n}=\sum_{\varepsilon}c_{\varepsilon}^2\frac{da_1}{a_1^{1-\varepsilon_1}}\wedge\cdots\wedge\frac{da_n}{a_n^{1-\varepsilon_n}}.$$

From our assumption (1) and the definition of the Cartier operator mentioned in the introduction, we obtain

$$\begin{aligned} a\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n} &= C\left(a\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n}\right)\\ &= \sum_{\varepsilon}C\left(c_{\varepsilon}^2\frac{da_1}{a_1^{1-\varepsilon_1}}\wedge\cdots\wedge\frac{da_n}{a_n^{1-\varepsilon_n}}\right). \end{aligned}$$

For any $\varepsilon\neq 0$, the form $c_{\varepsilon}^2\frac{da_1}{a_1^{1-\varepsilon_1}}\wedge\cdots\wedge\frac{da_n}{a_n^{1-\varepsilon_n}}$ is exact and hence $C(c_{\varepsilon}^2\frac{da_1}{a_1^{1-\varepsilon_1}}\wedge\cdots\wedge\frac{da_n}{a_n^{1-\varepsilon_n}})=0$. Thus we obtain

$$\begin{aligned} a\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n} &= C\left(c_0^2\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n}\right)\\ &= c_0\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n} \end{aligned}$$

i.e. $a=c_0$. Therefore $a=a^2+\sum_{\varepsilon\neq 0}c_{\varepsilon}^2a^{\varepsilon}$, and this is precisely (2).

Conversely, assuming $a=a^2+\sum_{\varepsilon\neq 0}c_{\varepsilon}^2a^{\varepsilon}$, $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)\in\{0,1\}^n$, we obtain

$$a\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n}=a^2\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n}+\sum_{\varepsilon\neq 0}c_{\varepsilon}^2\frac{da_1}{a_1^{1-\varepsilon_1}}\wedge\cdots\wedge\frac{da_n}{a_n^{1-\varepsilon_n}}$$

and hence $C(a\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n})=a\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n}$ and this implies (1). ∎

Putting 1.2 and 1.3 together we obtain

1.4. Theorem. *Let $L=F(\ll a_1,\ldots,a_n\gg)$. Then*

$$\nu_{L/F}(n)=\left\{a\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_n}{a_n}|a\in F\ \textit{with}\ \wp(a)\in F^2(a_1,\ldots,a_n)'\right\}.$$

Proof of Proposition (1.1). The function field $L=F(\varphi)$, where $\varphi=\ll a_1,\ldots,a_n\gg$, can be described as follows: for any $\varepsilon\in\{0,1\}^n$, different from $(0,\ldots,0)$, let X_{ε} be a variable and set $K=F(X_{\varepsilon})$ for the field generated by all this variables over F. Then $L=K(\sqrt{T})$, where $T=\sum_{\varepsilon\neq 0}a^{\varepsilon}X_{\varepsilon}^2$ (T is the pure part of $\ll a_1,\ldots,a_n\gg$). We proceed now in three steps:

1. $\Omega^m_{K/F}=0$. This is clear choosing a 2-basis $\mathcal{B}$ of F and enlarging it to a 2-basis $\mathcal{B}\cup\{X_{\varepsilon},\varepsilon\neq 0\}$ of K.

2. If $E = F(\sqrt{a})$, $a \in F\backslash F^2$, then $\Omega^m_{E/F} = \Omega^{m-1}_F \wedge da$. Since $a \notin F^2$, we can choose a 2-basis of F containing a, say $\mathcal{B} = \{a, c_i \ \ i \in I\}$. Then $\{\sqrt{a}, c_i \ \ i \in I\}$ is a 2-basis of E. If $w \in \Omega^m_{E/F}$, set $w = (\sum_{i_1<\cdots<i_{m-1}} a_{i_1\cdots i_{m-1}} dc_{i_1}\wedge\cdots\wedge dc_{i_{m-1}})\wedge da + \sum_{i_1<\cdots<i_m} a_{i_1\cdots i_m} dc_{i_1}\wedge\cdots\wedge dc_{i_m}$. Then in Ω^m_E we get $\sum_{i_1<\cdots<i_m} a_{i_1\cdots i_m} dc_{i_1}\wedge\cdots\wedge dc_{i_m} = 0$ and hence $a_{i_1\cdots i_m} = 0$ for all $i_1 < \cdots < i_m$. This proves the claim.
3. $\Omega^m_{L/F} = \Omega^{m-n}_F \wedge \dfrac{da_1}{a_1} \wedge \cdots \wedge \dfrac{da_n}{a_n}$, and if $m < n$, $\Omega^m_{L/F} = 0$. We just consider the case $m \geq n$. We choose a 2-basis $\mathcal{B} = \{a_1, \ldots, a_n, \ldots\}$ of F and take $w \in \Omega^m_{L/F}$. From (2) we get $w = u\wedge dT$, $u \in \Omega^{m-1}_K$, where $dT = k_1da_1+\cdots+k_nda_n$. $k_1, \ldots, k_n$ being certain quadratic polynomials over F. Inserting $da_1 = k_1^{-1}(k_2da_2+\cdots+k_nda_n)+k_1^{-1}dT$ into u, we see that we may assume u free from terms containing da_1 in its basis expansion with respect to the 2-basis $\mathcal{B}\cup\{X_\varepsilon, \ \ \varepsilon \neq 0\}$ of K. Write $w = w_0+w_1\wedge da_1$ with forms w_0, w_1 free from forms containing da_1. Then in Ω^m_K we have $w_0 + w_1 \wedge da_1 = u \wedge k_1da_1 + u \wedge (k_2da_2 + \cdots + k_nda_n)$, i.e. $(w_1 + k_1u) \wedge da_1 = w_0 + u \wedge (k_2da_2 + \cdots + k_nda_n)$. Since the form on the right of this equation does not contain da_1 in the 2-basis expansion, we obtain $w_1 = k_1u$ and therefore $k_1w = w_1 \wedge dT$. Everything in this equation is defined over $F[X_\varepsilon, \ \ \varepsilon \neq 0]$, so that a simple specialization of the variables proves that da_1 divides w in Ω^m_F. Thus w is divisible by $da_1, \ldots, da_n$ and hence by $da_1 \wedge \ldots \wedge da_n$. This proves the claim.

■

2. $I^n_{L/F}$ FOR $L = F(\ll a_1, \ldots, a_n \gg)$

Using the isomorphism $\alpha : \nu_{L/F}(n) \cong \overline{I}^n_{L/F}$ and 1.4 we can now describe $\overline{I}^n_{L/F}$. But in order to apply α to $a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \in \nu_{L/F}(n)$ we must express this form in terms of pure logarithmic differentials.

2.1. Examples. 1. *If $n = 1$, i.e. $a = a^2 + c^2a_1$ (see (1.4)), then*

$$a\frac{da_1}{a_1} = \frac{a^2a_1}{aa_1}\frac{da_1}{a_1} = \frac{a^2a_1}{c^2a_1^2 + a^2a_1}\frac{da_1}{a_1} = \frac{d(c^2a_1^2 + a^2a_1)}{c^2a_1^2 + a^2a_1}$$

and hence $\alpha(a\frac{da_1}{a_1}) = \ll c^2a_1^2 + a^2a_1 \gg \mod I_F$.

2. *If $n = 2$, i.e. $a = a^2 + c_1^2a_1 + c_2^2a_2 + c_3^2a_1a_2 = a' + a_1a''$, with $a' = a^2 + c_2^2a_2$, $a'' = c_1^2 + c_3^2a_2 \in F^2(a_2)$, a simple computation*

shows that $d(aa_1) \wedge da_2 = a' da_1 \wedge da_2$, *and hence*

$$\begin{aligned} a\frac{da_1}{a_1} \wedge \frac{da_2}{a_2} &= \frac{a^2}{a'}\frac{d(aa_1)}{aa_1} \wedge \frac{da_2}{a_2} = \frac{a^2}{a'}\frac{da_2}{a_2} \wedge \frac{d(aa_1)}{aa_1} \\ &= \frac{d(a_2 a')}{a_2 a'} \wedge \frac{d(aa_1)}{aa_1}. \end{aligned}$$

Thus $\alpha(a\frac{da_1}{a_1} \wedge \frac{da_2}{a_2}) = \ll a_2 a', aa_1 \gg \mod I_F^2$.

Let us now consider the general situation. According to (1.3) we have $a = a^2 + \sum_{\varepsilon \neq 0} c_\varepsilon^2 a^\epsilon = a' + a_1 a''$ with $a', a'' \in F^2(a_2, \ldots, a_n)$. We claim that there exist elements $x_i \in F^2(a_i, \ldots, a_n)^*$ with $a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$. The cases $n = 1, 2$ are clear because of (2.1). We have

$$a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} = \frac{a^2}{a'}\frac{da_2}{a_2} \wedge \cdots \wedge \frac{da_n}{a_n} \wedge \frac{d(aa_1)}{aa_1}.$$

Notice that the form $\frac{a^2}{a'}\frac{da_2}{a_2} \wedge \cdots \wedge \frac{da_n}{a_n}$ is defined over $F^2(a_2, \ldots, a_n)$. Set $\overline{a} = a^2/a'$. We show that $\wp(\overline{a}) \in F^2(a_2, \ldots, a_n)'$, so the claim follows by induction. In fact, we have $\wp(\overline{a}) = \overline{a}^2 + \overline{a} = a^2(a^2 + a')/a^2$, and since

$$a' = a^2 + \sum_{\eta \neq 0} c_\eta^2 a_2^{\eta_2} \cdots a_n^{\eta_n},$$

$\eta \in \{0, 1\}^{n-1}$, it follows $\wp(\overline{a}) = \sum_{\eta \neq 0} \left(\frac{ac_\eta}{a'}\right)^2 a_2^{\eta_2} \cdots a_n^{\eta_n} \in F^2(a_2, \ldots, a_n)'$. From $a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ we obtain $\alpha(a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}) = \ll x_1, \ldots, x_n \gg \mod I_F^{n+1}$, and this shows $\overline{I}_{L/F}^n \subset \{\ll x_1, \ldots, x_n \gg \mod I_F^{n+1} \mid x_1, \ldots, x_n \in F^2(a_i, \ldots, a_n)^*\}$. It is easy to check that the other inclusion also holds. Thus we have shown.

2.2. Theorem. *If* $L = F(\ll a_1, \ldots, a_n \gg)$, *then*

$$\overline{I}_{L/F}^n = \left\{\overline{\ll x_1, \ldots, x_n \gg} \mid x_1, \ldots, x_n \in F^2(a_1, \ldots, a_n)^*\right\}.$$

Using the fact that $\overline{I}_{L/F}^n$ is a group, we obtain

2.3. Corollary. *For any* $x_1, \ldots, x_n, y_1, \ldots, y_n \in F^2(a_1, \ldots, a_n)^*$, *there exist* $z_1, \ldots, z_n \in F^2(a_1, \ldots, a_n)^*$ *such that*

$$\ll x_1, \ldots, x_n \gg + \ll y_1, \ldots, y_n \gg = \ll z_1, \ldots, z_n \gg \mod I_F^{n+1}.$$

One can interpret this result as a sort of n-linkage property of the subfields $F^2(a_1, \dots, a_n)$ relative to the field F in the sense of the following definition. Let $E \subset F$ be a subfield. E is called n-linked relative to F if for all $x_1, \dots, x_n, y_1, \dots, y_n \in E^*$, there exist $z_1, \dots, z_n \in E^*$ with $\ll x_1, \dots, x_n \gg + \ll y_1, \dots, y_n \gg = \ll z_1, \dots, z_n \gg \mod I_F^{n+1}$ in $W(F)$.

3. $\nu_{L/F}(m)$ FOR $L = F(\ll a_1, \dots, a_n \gg)$

The aim of this section is to extend the results of section 2 to arbitrary m, i.e. we will compute $\nu_{L/F}(m)$ for $m \geq 1$. Since $\nu_{L/F}(m) = \Omega^m_{L/F} \cap \nu_F(m)$, we conclude $\nu_{L/F}(m) = 0$ if $m < n$ (see (1.1)). Thus we assume $m \geq n$, and hence $\nu_{L/F}(m) = \Omega_F^{m-n} \wedge \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \cap \nu_F(m)$. In order to characterize the forms $w \in \Omega_F^{m-n}$ with $w \wedge \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \in \nu_F(m)$ we generalize the operator $\wp(a) = a^2 - a$ as follows (see [Ka], [A-Ba 2]). Let $\{a_1, \dots, a_n, \dots, a_N\}$ be a 2-basis of F which we assume without restriction to be finite. Then any form $\eta \in \Omega^q_F$, $(q \geq 1)$ can be written in a unique way as

$$\eta = \sum_{i_1 < \cdots < i_q} c_{i_1 \cdots i_q} \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_q}}{a_{i_q}}.$$

We define $\wp : \Omega^q_F \to \Omega^q_F$ by

$$\begin{aligned} \wp(\eta) &= \sum_{i_1 < \cdots < i_q} \wp(c_{i_1 \cdots i_q}) \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_q}}{a_{i_q}} \\ &= \eta^{[2]} - \eta \end{aligned}$$

where $\eta^{[2]} = \sum_{i_1 < \cdots < i_q} c^2_{i_1 \cdots i_q} \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_q}}{a_{i_q}}$.

This definition obviously depends on the choice of the 2-basis $\{a_i\}$, but if one changes the 2-basis, the new $\wp$ operator values differs from the former values by exact forms, i.e. $\wp(\eta) \mod d\Omega_F^{q-1}$ is independent of the 2-basis. With this notation we have

3.1. Lemma. *Let $m \geq n$ and $w \in \Omega_F^{m-n}$. Fix a 2-basis $\{a_1, \dots, a_n, \dots, a_N\}$ of F. Then the following assertions are equivalent*

1. $w \wedge \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \in \nu_F(m)$.
2. $\wp(w) \in \sum_{\varepsilon \neq 0} a^\varepsilon \left[\Omega_F^{m-n}\right]^{[2]} + d\Omega_F^{m-n-1} + \sum_{i=1}^n \Omega_F^{m-n-1} \wedge da_i$ *(where $a^\varepsilon = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ and $[\Omega_F^q]^{[2]}$ denotes the group of all $\eta^{[2]}$, $\eta \in \Omega_F^q$).*

If $m = n$, we recover (1.3). Since the proof of (3.1) follows the same pattern as the proof of (1.3), we will skip some details.

Proof of (3.1) Assume (2), i.e. we have

$$w = w^{[2]} + \sum_{\varepsilon \neq 0} a^{\varepsilon} A_{\varepsilon}^{[2]} + dB + \sum_{i=1}^{n} E_i \wedge da_i$$

with $A_i \in \Omega_F^{m-n}$, $B, E_1, \ldots, E_n \in \Omega_F^{m-n-1}$. It follows $d(w \wedge \eta) = 0$, where $\eta = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$. Applying the Cartier Operator to $w \wedge \eta$ we get

$$C(w \wedge \eta) = C(w^{[2]} \wedge \eta) = w \wedge \eta,$$

because $a^{\varepsilon} A_{\varepsilon}^{[2]} \wedge \eta$, $dB \wedge \eta$ are exact forms. This implies $w \wedge \eta \in \nu_F(m)$. Let us now assume (1). Set $w = \sum_{i_1 < \cdots < i_{m-n}} c_{i_1 \ldots i_{m-n}} \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_{m-n}}}{a_{m-n}}$. In what follows all computations will be modulo $< da_1, \ldots, da_n >$. Let $k > n$ be the maximal index with $w = R_0 + a_k R_1$, R_0, R_1 differential forms generated by $\frac{da_{n+1}}{a_{n+1}}, \cdots, \frac{da_N}{a_N}$ over $F^2(a_1, \ldots, a_{k-1})$. Let us write $R_0 = M_0 + M_1 \wedge \frac{da_k}{a_k}$, $R_1 = M_2 + M_3 \wedge \frac{da_k}{a_k}$ with forms $M_0, \ldots, M_3$ not containing the differential da_k and with coefficients in $F^2(a_1, \ldots, a_{k-1})$. It follows

$$dw = dM_0 + a_k dM_2 + dM_1 \wedge \frac{da_k}{a_k} + a_k(dM_3 + M_2) \wedge \frac{da_k}{a_k}$$

and since $d(w \wedge \eta) = 0$, we get $d(M_0 \wedge \eta) = 0$, $d(M_2 \wedge \eta) = 0$, $d(M_1 \wedge \frac{da_k}{a_k} \wedge \eta) = 0$ and $(dM_3 + M_2) \wedge \frac{da_k}{a_k} \wedge \eta = 0$. Because of the choice made above, we obtain $d(M_0) = 0$, $dM_2 = 0$, $dM_1 = 0$, $dM_3 + M_2 = 0$. Thus $w = M_0 + M_1 \wedge \frac{da_k}{a_k} + d(a_k M_3)$. Let $w' = M_0 + M_1 \wedge \frac{da_k}{a_k}$. It follows $d(w' \wedge \eta) = dw' \wedge \eta = 0$. Moreover w' is generated by $\frac{da_{n+1}}{a_{n+1}}, \ldots, \frac{da_N}{a_N}$ over $F^2(a_1, \ldots, a_{k-1})$.

Repeating the above procedure with w' we finally arrive at a decomposition

$$w = w_0 + dM$$

with w_0 generated by $\frac{da_{n+1}}{a_{n+1}}, \ldots, \frac{da_N}{a_N}$ over $F^2(a_1, \ldots, a_n)$, i.e. $w = \sum_{\mu} c_{\mu} \frac{da_{\mu}}{a_{\mu}} + dM$, with $c_{\mu} \in F^2(a_1, \ldots, a_n)$, and μ running over all $m-n$-tuples $\mu = (i_1, \ldots, i_{m-n})$ of integers with $n+1 \leq i_1 < \cdots < i_{m-n} \leq N$ and where $a_{\mu} = a_{i_1} \cdots a_{i_{m-n}}$, $da_{\mu} = da_{i_1} \wedge \cdots \wedge da_{i_{m-n}}$.

Let $c_{\mu} = \sum_{\varepsilon} c_{\mu,\varepsilon}^2 a^{\varepsilon}$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$, $a^{\varepsilon} = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$. Then

$$w = \sum_{\varepsilon} a^{\varepsilon} \left(\sum_{\mu} c_{\mu,\varepsilon}^2 \frac{da_{\mu}}{a_{\mu}} \right) + dM.$$

Set $A_\varepsilon = \sum_\mu c_{\mu,\varepsilon} \frac{da_\mu}{a_\mu}$. Thus we have

$$w = A_0^{[2]} + \sum_{\varepsilon \neq 0} a^\varepsilon A_\varepsilon^{[2]} + dM.$$

Using the fact that $a^\varepsilon A_\varepsilon^{[2]} \wedge \eta,\ dM \wedge \eta \in d\Omega_F^{m-n-1}$, we conclude

$$C(w \wedge \eta) = C\left(A_0^{[2]} \wedge \eta\right) = A_0 \wedge \eta,$$

i.e. $w \wedge \eta = A_0 \wedge \eta$. It follows $w = A_0 + H$ with $H \in < \frac{da_1}{a_1}, \dots, \frac{da_n}{a_n} >$.

Therefore

$$w = w^{[2]} + \sum_{\varepsilon \neq 0} a^\varepsilon A_\varepsilon^{[2]} + dM + H'$$

with $H' \in \sum_{i=1}^n \Omega_F^{m-n-1} \wedge \frac{da_i}{a_i}$. This proves the lemma. ∎

3.2. Corollary. *Let $L = F(\ll a_1, \dots, a_n \gg)$. Then*

1. *If $m < n$, $\nu_{L/F}(m) = 0$.*
2. *If $m \geq n$, then*

$$\nu_{L/F}(m) = \{w \wedge \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \quad | \quad w \in \Omega_F^{m-n},$$

$$\textit{with } \wp(w) \in \sum_{\varepsilon \neq 0} a^\varepsilon \left[\Omega_F^{m-n}\right]^{[2]} \quad + \quad d\Omega_F^{m-n-1} + \sum_{i=1}^{n} \Omega_F^{m-n-1} \wedge da_i\}.$$

Using this decomposition of $\nu_{L/F}(m)$ and lemma (2.5) in [Ka], we can show

3.3. Corollary. *Let $L = F(\ll a_1, \dots, a_n \gg)$. Then*

1. *If $m < n$, $\overline{I}_{L/F}^m = 0$*
2. *If $m \geq n$,*

$$\overline{I}_{L/F}^m = \left\{\overline{\psi \ll x_1, \dots, x_n \gg} \mid \psi \in I_F^{m-n},\ x_1, \dots, x_n \in F^2(a_1, \dots, a_n)^*\right\}$$

We will omit the details of the proof.

References

[A-Ba 1] Aravire, R., Baeza, R.: *Milnor's K-theory and quadratic forms over fields of characteristic two.* Comm. Algebra, 20 (4), 1087-1107 (1992).

[A-Ba 2] Aravire, R., Baeza, R.: *The behavior of quadratic and differential forms under function field extensions in characteristic two.* Preprint 2000.

[Ca] Cartier, P.: *Questions de rationalité des diviseurs en géometrie algebrique.* Bull. Soc. Math. France, 86, 177-251 (1958).

[Ka] Kato, K.: *Symmetric bilinear forms, quadratic forms and Milnor K-theory in characteristic two.* Inv. Math. 66, 493-510 (1982).

[Mi] Milne, J.S.: *Duality in flat cohomology of a surface* . Ann. Sc. Ec. N. Sup. 9, 171-202 (1976).

[Sa] Sah, C.H.: *Symmetric bilinear forms and quadratic forms.* J. of Algebra 20, 144-160 (1972).

Roberto Aravire
Departamento de Ciencias Físicas y Matemáticas
Universidad Arturo Prat, Casilla 121, Iquique, Chile
e-mail: raravire@cec.unap.cl.

Ricardo Baeza
Instituto de Matemática y Física
Universidad de Talca, Casilla 747, Talca, Chile.
Departamento de Matemáticas, Facultad de Ciencias
Universidad de Chile, Casilla 653, Santiago, Chile.
e-mail: rbaeza@inst-mat.utalca.cl.