COHOMOLOGICAL INVARIANTS AND R-TRIVIALITY OF ADJOINT CLASSICAL GROUPS

GRÉGORY BERHUY, MARINA MONSURRÒ, AND JEAN-PIERRE TIGNOL

Abstract. Using a cohomological obstruction, we construct examples of absolutely simple adjoint classical groups of type ${}^2A_n$ with $n \equiv 3 \bmod 4$, C_n or ${}^1D_n$ with $n \equiv 0 \bmod 4$, which are not R-trivial hence not stably rational.

Introduction

For an algebraic group G defined over a field F, let $G(F)/R$ be the group of R-equivalence classes introduced by Manin in [6]. The algebraic group G is called *R-trivial* if $G(L)/R = 1$ for every field extension L/F. It was established by Colliot-Thélène and Sansuc in [2] (see also [7, Proposition 1]) that the group G is R-trivial if the variety of G is stably rational.

In this paper, we focus on the case where G is an absolutely simple classical group of adjoint type. Adjoint groups of type ${}^1A_n$ or B_n are easily seen to be rational (see [7, pp. 199, 200]). Voskresenskiĭ and Klyachko [11, Cor. of Th. 8] proved that adjoint groups of type ${}^2A_n$ are rational if n is even, and Merkurjev [7, Prop. 4] showed that adjoint groups of type C_n are stably rational for n odd. On the other hand, Merkurjev also produced in [7] examples of adjoint groups of type ${}^2A_3$ ($= {}^2D_3$) and of type ${}^2D_n$ for any $n \geq 4$ which are not R-trivial, hence not stably rational. Examples of adjoint groups of type ${}^1D_4$ which are not R-trivial were constructed by Gille in [3].

The goal of the present paper is to construct examples of adjoint groups of type ${}^2A_n$ with $n \equiv 3 \bmod 4$ and of adjoint groups of type C_n or ${}^1D_n$ with $n \equiv 0 \bmod 4$ which are not R-trivial. Our constructions are based on Merkurjev's computation in [7] of the group of R-equivalence classes of adjoint classical groups, which we now recall briefly. According to Weil (see [4, §26]), every absolutely simple classical group of adjoint type over a field F of characteristic different from 2 can be obtained as the connected component of the identity in the automorphism group of a central[1] simple algebra with involution (A, σ) over F. Let $\mathbf{Sim}(A, \sigma)$ be the algebraic group of similitudes of (A, σ), defined (as a group scheme) by

$$\mathbf{Sim}(A, \sigma)(E) = \{u \in A \otimes_F E \mid (\sigma \otimes \mathrm{Id})(u)u \in E^\times\}$$

2000 *Mathematics Subject Classification.* 11E57, 20G15.

The authors gratefully acknowledge support from the TMR network "K-theory and linear algebraic groups" (ERB FMRX CT97-0107). The third author is also grateful to the National Fund for Scientific Research (Belgium) for partial support.

[1] We use the same terminology as in [4]. In particular, the center of A is F if σ is of the first kind; it is a quadratic étale extension of F if σ is of the second kind.

for every commutative F-algebra E, and let $\mathbf{PSim}(A,\sigma)$ be the group of projective similitudes,

$$\mathbf{PSim}(A,\sigma) = \mathbf{Sim}(A,\sigma)/R_{K/F}(\mathbf{G}_{\mathrm{m},K})$$

where K is the center of A. The connected component of the identity in these groups is denoted by $\mathbf{Sim}^+(A,\sigma)$ and $\mathbf{PSim}^+(A,\sigma)$ respectively. We let $\mathrm{Sim}(A,\sigma)$, $\mathrm{PSim}(A,\sigma)$, $\mathrm{Sim}^+(A,\sigma)$ and $\mathrm{PSim}^+(A,\sigma)$ denote the corresponding groups of F-rational points:

$$\mathrm{Sim}(A,\sigma) = \mathbf{Sim}(A,\sigma)(F), \qquad \mathrm{PSim}(A,\sigma) = \mathbf{PSim}(A,\sigma)(F), \quad \text{etc.}$$

The group $\mathbf{PSim}^+(A,\sigma)$ is canonically isomorphic (under the map which carries every similitude g to the induced inner automorphism $\mathrm{Int}(g)$) to the connected component of the identity in the automorphism group of (A,σ). To describe the group of R-equivalence classes of $\mathrm{PSim}^+(A,\sigma)$, consider the homomorphism

$$\mu\colon\ \mathbf{Sim}(A,\sigma) \to \mathbf{G}_{\mathrm{m}}$$

which carries every similitude to its multiplier

$$\mu(g) = \sigma(g)g.$$

Let $G^+(A,\sigma) = \mu\big(\mathrm{Sim}^+(A,\sigma)\big) \subset F^\times$ and $NK^\times = \mu(K^\times) \subset F^\times$ (so $NK^\times = F^{\times 2}$ if $K = F$). Let also $\mathrm{Hyp}(A,\sigma)$ be the subgroup of $F^\times$ generated by the norms of the finite extensions L of F such that (A,σ) becomes hyperbolic after scalar extension to L. In [7, Theorem 1], Merkurjev shows that the multiplier map μ induces a canonical isomorphism

$$\mathrm{PSim}^+(A,\sigma)/R \simeq G^+(A,\sigma)/\big(NK^\times \cdot \mathrm{Hyp}(A,\sigma)\big). \tag{1}$$

For any positive integer d, let $H^d(F,\mu_2)$ be the degree d cohomology group of the absolute Galois group of F with coefficients $\mu_2 = \{\pm 1\}$. In Section 3 we consider the case where σ is of the first kind. If it is orthogonal, we assume further that its discriminant is trivial. Assuming the index of A divides $\frac{1}{2}\deg A$, we construct a homomorphism

$$\Theta_1\colon\ \mathrm{PSim}^+(A,\sigma)/R \to H^4(F,\mu_2),$$

and give examples where this homomorphism is nonzero, hence $\mathrm{PSim}^+(A,\sigma)/R \neq 1$. Similarly, if σ is of the second kind and the exponent of A divides $\frac{1}{2}\deg A$, we construct in Section 4 a homomorphism

$$\Theta_2\colon\ \mathrm{PSim}^+(A,\sigma)/R \to H^3(F,\mu_2)$$

and show that this map is nonzero in certain cases. In all the examples where we show $\Theta_1 \neq 0$ or $\Theta_2 \neq 0$, the algebra with involution has the form $(A,\sigma) = (B,\rho) \otimes (C,\tau)$ where ρ is an orthogonal involution which admits improper similitudes.

Throughout the paper, the characteristic of the base field F is different from 2.

1. Improper similitudes

Let (A,σ) be a central simple F-algebra with orthogonal involution of degree $n = 2m$. The group of similitudes $\mathbf{Sim}(A,\sigma)$ is denoted $\mathbf{GO}(A,\sigma)$. This group is not connected. Its connected component of the identity $\mathbf{GO}^+(A,\sigma)$ is defined by the equation

$$\mathrm{Nrd}_A(g) = \mu(g)^m,$$

where Nrd_A is the reduced norm. We denote by $\mathrm{GO}(A,\sigma)$ and $\mathrm{GO}^+(A,\sigma)$ the group of F-rational points

$$\mathrm{GO}(A,\sigma) = \mathbf{GO}(A,\sigma)(F), \qquad \mathrm{GO}^+(A,\sigma) = \mathbf{GO}^+(A,\sigma)(F).$$

The elements in $\mathrm{GO}^+(A,\sigma)$ are called *proper similitudes*, and those in the nontrivial coset

$$\mathrm{GO}^-(A,\sigma) = \{g \in \mathrm{GO}(A,\sigma) \mid \mathrm{Nrd}_A(g) = -\mu(g)^m\}$$

are called *improper similitudes.*

For example, if $m = 1$ (i.e. A is a quaternion algebra), then every orthogonal involution has the form $\sigma = \mathrm{Int}(q) \circ \gamma$, where γ is the canonical involution, q is an invertible pure quaternion and $\mathrm{Int}(q)$ is the inner automorphism induced by q, mapping $x \in A$ to qxq^{-1}. It is easily checked that

$$\mathrm{GO}^+(A,\sigma) = F(q)^\times \qquad \text{and} \qquad \mathrm{GO}^-(A,\sigma) = q'F(q)^\times,$$

where q' is a unit which anticommutes with q. Therefore, $\mathrm{GO}^-(A,\sigma) \neq \varnothing$.

If $m > 1$, the existence of improper similitudes is an important restriction on A and σ, since it implies that A is split by the quadratic étale F-algebra $F[\sqrt{\operatorname{disc}\sigma}]$, where $\operatorname{disc}\sigma$ is the discriminant of σ, see [9, Theorem A] or [4, (13.38)]. In particular, the index of A satisfies $\operatorname{ind} A \leq 2$, i.e. A is Brauer-equivalent to a quaternion algebra. Moreover, if m is even, then $-1 \in \mathrm{Nrd}_A(A)$, see [9, Corollary 1.13]. There is no other restriction on A, as the following proposition shows.

1. **Proposition.** *Let H be an arbitrary quaternion F-algebra and let m be an arbitrary integer. If m is even, assume $-1 \in \mathrm{Nrd}_H(H^\times)$. Then the algebra $M_m(H)$ carries an orthogonal involution which admits improper similitudes.*

Proof. Suppose first m is odd. Let i, j be elements in a standard quaternion basis of H. We set

$$\sigma = t \otimes (\mathrm{Int}(i) \circ \gamma) \qquad \text{on } M_m(H) = M_m(F) \otimes_F H,$$

where γ is the canonical involution on H. It is readily verified that $1 \otimes j$ is an improper similitude of σ.

Suppose next m is even, and $q \in H$ satisfies $\mathrm{Nrd}_H(q) = -1$. We pick a quaternion basis 1, i, j, $k = ij$ such that i commutes with q, and set

$$\sigma = \mathrm{Int}\,\mathrm{diag}(j, i, \ldots, i) \circ (t \otimes \gamma) \qquad \text{and} \qquad g = \mathrm{diag}(j, qj, \ldots, qj).$$

Again, computation shows that g is an improper similitude of σ. □

Necessary and sufficient conditions for the existence of improper similitudes for a given involution σ are not known if $m \geq 4$. For $m = 2$ (resp. $m = 3$), Corollary (15.9) (resp. (15.26)) in [4] shows that $\mathrm{GO}^-(A,\sigma) \neq \varnothing$ if and only if the Clifford algebra $C(A,\sigma)$ has outer automorphisms (resp. outer automorphisms which commute with its canonical involution). (For $m = 2$ another equivalent condition is that A is split by the center of $C(A,\sigma)$, see [4, (15.11)] or [9, Prop. 1.15].) We use this fact to prove the following result:

2. **Proposition.** *Let (A,σ) be a central simple F-algebra with orthogonal involution of degree 4. Assume that A is not split and* $\operatorname{disc}\sigma \neq 1$. *Then there exists a field extension L/F such that A_L is not split and* $\mathrm{GO}^-(A_L,\sigma_L) \neq \varnothing$.

Proof. By hypothesis, $F(\sqrt{\operatorname{disc}\sigma})$ is a quadratic field extension of F. We denote it by K for simplicity and let ι be its nontrivial F-automorphism. The Clifford algebra $C = C(A,\sigma)$ is a quaternion K-algebra. Let X be the Severi-Brauer variety of $C \otimes_K {}^{\iota}C$ and let L be the function field of its Weil transfer:

$$L = F\bigl(R_{K/F}(X)\bigr).$$

Then $(C \otimes_K {}^{\iota}C) \otimes_K KL$ splits, so C_{KL} is isomorphic to ${}^{\iota}C_{KL}$, which means that C_{KL} has outer automorphisms. By [4, (15.9)], it follows that $\mathrm{GO}^-(A_L,\sigma_L) \neq \varnothing$.

On the other hand, by [9, Corollary 2.12], the kernel of the scalar extension map $\mathrm{Br}(F) \to \mathrm{Br}(L)$ is generated by the corestriction of $C\otimes_K {}^{\iota}C$. Since this corestriction is trivial, A_L is not split. □

2. Trace forms

In this section, A is a central simple F-algebra of even degree with an involution σ of the first kind. We consider the quadratic forms T_A and T_σ on A defined by

$$T_A(x) = \operatorname{Trd}_A(x^2), \qquad T_\sigma(x) = \operatorname{Trd}_A\bigl(\sigma(x)x\bigr) \qquad \text{for } x \in A,$$

where Trd_A is the reduced trace on A. We denote by T_σ^+ (resp. T_σ^-) the restriction of T_σ to the space $\mathrm{Sym}(\sigma)$ of symmetric elements (resp. to the space $\mathrm{Skew}(\sigma)$ of skew-symmetric elements), so that

$$T_A = T_\sigma^+ \perp -T_\sigma^- \qquad \text{and} \qquad T_\sigma = T_\sigma^+ \perp T_\sigma^-. \tag{2}$$

Recall that if σ is orthogonal the (signed) discriminant $\operatorname{disc} T_\sigma^+$ is equal to the discriminant $\operatorname{disc}\sigma$ up to a factor which depends only on the degree of A, see for instance [4, (11.5)]. In the following, we denote by I^nF the n-th power of the fundamental ideal IF of the Witt ring WF.

3. **Lemma.** *Let σ, σ_0 be two involutions of the first kind on A.*

- *If σ and σ_0 are both symplectic, then $T_\sigma^+ - T_{\sigma_0}^+ \in I^3F$.*
- *If σ and σ_0 are both orthogonal, then $\operatorname{disc}(T_\sigma^+ - T_{\sigma_0}^+) = \operatorname{disc}\sigma \operatorname{disc}\sigma_0$. Moreover, if $\operatorname{disc}\sigma = \operatorname{disc}\sigma_0$, then $T_\sigma^+ - T_{\sigma_0}^+ \in I^3F$.*

Proof. The symplectic case has been considered in [1, Theorem 4]. For the rest of the proof, we assume that σ and σ_0 are both orthogonal. By [4, (11.5)], there is a factor $c \in F^\times$ such that

$$\operatorname{disc} T_\sigma^+ = c \operatorname{disc}\sigma \qquad \text{and} \qquad \operatorname{disc} T_{\sigma_0}^+ = c \operatorname{disc}\sigma_0,$$

hence

$$\operatorname{disc}(T_\sigma^+ - T_{\sigma_0}^+) = \operatorname{disc} T_\sigma^+ \operatorname{disc} T_{\sigma_0}^+ = \operatorname{disc}\sigma \operatorname{disc}\sigma_0.$$

To complete the proof, observe that the Witt-Clifford invariant $e_2(T_\sigma^+)$ (or, equivalently, the Hasse invariant $w_2(T_\sigma^+)$) depends only on $\operatorname{disc}\sigma$ and on the Brauer class of A, as was shown by Quéguiner [10, p. 307]. Therefore, if $\operatorname{disc}\sigma = \operatorname{disc}\sigma_0$, then $e_2(T_\sigma^+) = e_2(T_{\sigma_0}^+)$, hence $T_\sigma^+ - T_{\sigma_0}^+ \in I^3F$ by a theorem of Merkurjev. □

We next compute the Arason invariant $e_3(T_\sigma^+ - T_{\sigma_0}^+) \in H^3(F,\mu_2)$ in the special case where σ and σ_0 decompose. We use the following notation: $[A] \in H^2(F,\mu_2)$ is the cohomology class corresponding to the Brauer class of A under the canonical isomorphism $H^2(F,\mu_2) = {}_2\mathrm{Br}(F)$. For $a \in F^\times$ we denote by (a) the cohomology class corresponding to the square class of a under the canonical isomorphism $H^1(F,\mu_2) = F^\times/F^{\times 2}$.

4. **Lemma.** *Suppose $A = B \otimes_F C$ for some central simple F-algebras B, C of even degree. Let ρ and ρ_0 be orthogonal involutions on B and let τ be an involution of the first kind on C. Let also $\sigma = \rho \otimes \tau$ and $\sigma_0 = \rho_0 \otimes \tau$.*

If τ (hence also σ and σ_0) is symplectic, then

$$e_3(T_\sigma^+ - T_{\sigma_0}^+) = \begin{cases} 0 & \text{if } \deg C \equiv 0 \bmod 4, \\ (\operatorname{disc}\rho \operatorname{disc}\rho_0) \cup [C] & \text{if } \deg C \equiv 2 \bmod 4. \end{cases}$$

If τ (hence also σ and σ_0) is orthogonal, then

$$e_3(T_\sigma^+ - T_{\sigma_0}^+) = \begin{cases} (\operatorname{disc}\rho \operatorname{disc}\rho_0) \cup (\operatorname{disc}\tau) \cup (-1) & \text{if } \deg C \equiv 0 \bmod 4, \\ (\operatorname{disc}\rho \operatorname{disc}\rho_0) \cup \big((\operatorname{disc}\tau) \cup (-1) + [C]\big) & \text{if } \deg C \equiv 2 \bmod 4. \end{cases}$$

Proof. The decomposition

$$\operatorname{Sym}(\sigma) = \big(\operatorname{Sym}(\rho) \otimes \operatorname{Sym}(\tau)\big) \oplus \big(\operatorname{Skew}(\rho) \otimes \operatorname{Skew}(\tau)\big)$$

yields

$$T_\sigma^+ = T_\rho^+ T_\tau^+ + T_\rho^- T_\tau^- \qquad \text{in } WF.$$

Since $T_B = T_\rho^+ - T_\rho^-$ we may eliminate T_ρ^- in the equation above to obtain

$$T_\sigma^+ = T_\rho^+ T_\tau^+ + (T_\rho^+ - T_B) T_\tau^-.$$

Similarly,

$$T_{\sigma_0}^+ = T_{\rho_0}^+ T_\tau^+ + (T_{\rho_0}^+ - T_B) T_\tau^-$$

and subtracting the two equalities yields

$$T_\sigma^+ - T_{\sigma_0}^+ = (T_\rho^+ - T_{\rho_0}^+) T_\tau^+ + (T_\rho^+ - T_{\rho_0}^+) T_\tau^- = (T_\rho^+ - T_{\rho_0}^+) T_\tau.$$

Since $\deg C$ is even, we have $T_\tau \in I^2 F$ (see [4, (11.5)]), hence

$$e_3(T_\sigma^+ - T_{\sigma_0}^+) = \big(\operatorname{disc}(T_\rho^+ - T_{\rho_0}^+)\big) \cup e_2(T_\tau) \qquad \text{in } H^3(F, \mu_2).$$

By Lemma 3 we have

$$\operatorname{disc}(T_\rho^+ - T_{\rho_0}^+) = \operatorname{disc}\rho \operatorname{disc}\rho_0.$$

The computation of $e_2(T_\tau)$ in [10, Theorem 1] or [5] completes the proof. □

Remark. If σ and σ_0 are symplectic, the Arason invariant $e_3(T_\sigma^+ - T_{\sigma_0}^+)$ is the *discriminant* $\Delta_{\sigma_0}(\sigma)$ investigated in [1].

3. Involutions of the first kind

In this section, A is a central simple F-algebra of even degree, and σ is an involution of the first kind on A. We assume $\operatorname{ind} A$ divides $\frac{1}{2}\deg A$, i.e. $A \simeq M_2(A_0)$ for some central simple F-algebra A_0, so that A carries a hyperbolic involution σ_0 of the same type as σ. If σ is orthogonal, we assume $\operatorname{disc}\sigma = 1$ ($= \operatorname{disc}\sigma_0$), so that in all cases $T_\sigma^+ - T_{\sigma_0}^+ \in I^3 F$, by Lemma 3.

5. **Proposition.** *The map $\theta_1\colon \operatorname{Sim}(A,\sigma) \to H^4(F,\mu_2)$ defined by*

$$\theta_1(g) = \big(\mu(g)\big) \cup e_3(T_\sigma^+ - T_{\sigma_0}^+)$$

induces a homomorphism

$$\Theta_1\colon \operatorname{PSim}^+(A,\sigma)/R \to H^4(F,\mu_2).$$

Moreover, for all $g \in \operatorname{Sim}(A,\sigma)$, we have

$$\theta_1(g) \cup (-1) = 0 \qquad \text{in } H^5(F,\mu_2).$$

Proof. In view of the isomorphism (1), it suffices to show that for every finite field extension L/F such that $(A,\sigma)\otimes_F L$ is hyperbolic and for every $x\in L^\times$,

$$\big(N_{L/F}(x)\big)\cup e_3(T^+_\sigma - T^+_{\sigma_0}) = 0 \qquad \text{in } H^4(F,\mu_2).$$

The projection formula yields

$$\big(N_{L/F}(x)\big)\cup e_3(T^+_\sigma - T^+_{\sigma_0}) = \operatorname{cor}_{L/F}\big((x)\cup e_3(T^+_\sigma - T^+_{\sigma_0})_L\big).$$

Since σ_L is hyperbolic, the involutions σ_L and $(\sigma_0)_L$ are conjugate, hence

$$e_3(T^+_\sigma - T^+_{\sigma_0})_L = 0.$$

For the last equality, observe that (2) yields the following equations in WF:

$$T_\sigma + T_A = \langle 1,1\rangle T^+_\sigma \qquad \text{and} \qquad T_{\sigma_0} + T_A = \langle 1,1\rangle T^+_{\sigma_0},$$

hence

$$T_\sigma - T_{\sigma_0} = \langle 1,1\rangle (T^+_\sigma - T^+_{\sigma_0}).$$

Since σ_0 is hyperbolic, we have $T_{\sigma_0}=0$. Moreover, for $g\in \operatorname{Sim}(A,\sigma)$ the map $x\mapsto gx$ is a similitude of T_σ with multiplier $\mu(g)$, hence

$$\langle 1,-\mu(g)\rangle T_\sigma = \langle 1,-\mu(g)\rangle\langle 1,1\rangle(T^+_\sigma - T^+_{\sigma_0}) = 0.$$

Since

$$e_5\big(\langle 1,-\mu(g)\rangle\langle 1,1\rangle(T^+_\sigma - T^+_{\sigma_0})\big) = \theta_1(g)\cup(-1),$$

the proposition follows. □

6. **Proposition.** *Let $(A,\sigma)=(B,\rho)\otimes(C,\tau)$, where B and C are central simple F-algebras of even degree and ρ, τ are involutions of the first kind. Suppose* $\operatorname{ind} B$ *divides* $\frac12 \deg B$ *and ρ is orthogonal. For $g\in \mathrm{GO}^-(B,\rho)$, we have $g\otimes 1\in \operatorname{Sim}^+(A,\sigma)$ and*

$$\theta_1(g\otimes 1) = \begin{cases} 0 & \textit{if } \deg C\equiv 0 \bmod 4,\\ [B]\cup[C] & \textit{if } \deg C\equiv 2 \bmod 4.\end{cases}$$

Proof. For $g\in \mathrm{GO}(B,\rho)$, we have

$$\sigma(g\otimes 1)g\otimes 1 = \rho(g)g = \mu(g)$$

and

$$\operatorname{Nrd}_A(g\otimes 1) = \operatorname{Nrd}_B(g)^{\deg C},$$

so $g\otimes 1\in \operatorname{Sim}^+(A,\sigma)$.

Since $\operatorname{ind} B$ divides $\frac12\deg B$, we may find a hyperbolic orthogonal involution ρ_0 on B, and set $\sigma_0=\rho_0\otimes\tau$, a hyperbolic involution on A of the same type as σ.

If τ is symplectic, Lemma 4 yields

$$e_3(T^+_\sigma - T^+_{\sigma_0}) = \begin{cases} 0 & \text{if } \deg C\equiv 0 \bmod 4,\\ (\operatorname{disc}\rho)\cup[C] & \text{if } \deg C\equiv 2 \bmod 4.\end{cases}$$

The proposition follows by taking the cup-product with $\big(\mu(g)\big)$, since $\big(\mu(g)\big)\cup(\operatorname{disc}\rho)=[B]$ by [9, Theorem A] (see also [4, (13.38)]).

Suppose next τ is orthogonal. By Lemma 4,

$$e_3(T^+_\sigma - T^+_{\sigma_0}) = \begin{cases} (\operatorname{disc}\rho)\cup(\operatorname{disc}\tau)\cup(-1) & \text{if } \deg C\equiv 0 \bmod 4,\\ (\operatorname{disc}\rho)\cup\big((\operatorname{disc}\tau)\cup(-1)+[C]\big) & \text{if } \deg C\equiv 2 \bmod 4.\end{cases}$$

Using again the equation $\big(\mu(g)\big) \cup (\operatorname{disc}\rho) = [B]$ and taking into account the equation $(-1) \cup [B] = 0$, which follows from [9, Corollary 1.13], we obtain the formula for $\theta_1(g \otimes 1)$. □

Using Proposition 6, it is easy to construct examples where $\theta_1 \neq 0$. For these examples, the map Θ_1 of Proposition 5 is not trivial, hence $\mathbf{PSim}^+(A,\sigma)$ is not R-trivial.

7. **Corollary.** *Let Q, H be quaternion F-algebras satisfying*

$$(-1) \cup [H] = 0 \text{ in } H^3(F,\mu_2) \qquad \text{and} \qquad [H] \cup [Q] \neq 0 \text{ in } H^4(F,\mu_2).$$

Let $A = M_{2r}(H) \otimes M_s(Q)$, where r is arbitrary and s is odd. Let ρ be an orthogonal involution on $M_{2r}(H)$ which admits improper similitudes (see Lemma 1), and let τ be any involution of the first kind on $M_s(Q)$. Then $\mathbf{PSim}^+(A, \rho \otimes \tau)$ is not R-trivial.

To obtain explicit examples, we may take for F the field of rational fractions in four indeterminates $F = \mathbb{C}(x_1, y_1, y_2, y_2)$ and set $H = (x_1, y_1)_F$, $Q = (x_2, y_2)_F$. Note that the degree of A can be any multiple of 8 and that the conditions on Q and H in Corollary 7 imply $\operatorname{ind} A = 4$. Indeed, if there is a quadratic extension of F which splits Q and H, then $[H] \cup [Q]$ is a multiple of $(-1) \cup [H]$.

Other examples can be obtained from Proposition 2.

8. **Corollary.** *Let (B,ρ) be a central simple algebra of degree 4 and index 2 with orthogonal involution of nontrivial discriminant over a field F_0. Let $F = F_0(x,y)$ be the field of rational fractions in two indeterminates x, y over F_0, and let (C,τ) be a central simple F-algebra with involution of the first kind such that*

$$\deg C \equiv 2 \bmod 4 \qquad \text{and} \qquad [C] = (x) \cup (y) \in H^2(F,\mu_2).$$

Then $\mathbf{PSim}^+(B \otimes C, \rho \otimes \tau)$ is not R-trivial.

Proof. Proposition 2 yields an extension L_0/F_0 such that ρ_{L_0} admits an improper similitude g and B_{L_0} is not split. Set $L = L_0(x,y)$. By Proposition 6,

$$g \otimes 1 \in \mathbf{Sim}^+(B \otimes C, \rho \otimes \tau)(L) \qquad \text{and} \qquad \theta_1(g \otimes 1) = [B_L] \cup (x) \cup (y).$$

Since $[B_{L_0}] \neq 0$, taking successive residues for the x-adic and the y-adic valuations shows that $\theta_1(g \otimes 1) \neq 0$. Therefore, $\mathbf{PSim}^+(B \otimes C, \rho \otimes \tau)(L)/R \neq 1$, hence $\mathbf{PSim}^+(B \otimes C, \rho \otimes \tau)$ is not R-trivial. □

4. Involutions of the second kind

We assume in this section that (A,σ) is a central simple algebra with unitary involution over F. In this case, the group of similitudes is connected,

$$\mathbf{Sim}^+(A,\sigma) = \mathbf{Sim}(A,\sigma) \qquad \text{and} \qquad \mathbf{PSim}^+(A,\sigma) = \mathbf{PSim}(A,\sigma).$$

We denote by K the center of A and write $K = F[X]/(X^2 - \alpha)$. We assume the degree of A is even, $\deg A = n = 2m$, and denote by $D(A,\sigma)$ the discriminant algebra of (A,σ) (see [4, §10] for a definition).

9. **Lemma.** *$D(A,\sigma)$ is split if (A,σ) is hyperbolic.*

Proof. The lemma is clear if A is split, for then σ is adjoint to a hyperbolic hermitian form h and $[D(A,\sigma)] = (\alpha) \cup (\operatorname{disc} h)$ by [4, (10.35)]. The general case is reduced to the case where A is split by scalar extension to the field of functions $L = F\big(R_{K/F}(\mathrm{SB}(A))\big)$ of the Weil transfer of the Severi-Brauer variety of A. Indeed, $A \otimes_F L$ is split and the scalar extension map $\mathrm{Br}(F) \to \mathrm{Br}(L)$ is injective by [9, Corollary 2.12]. □

10. **Proposition.** *Suppose $A^{\otimes m}$ is split. The map $\theta_2\colon \operatorname{Sim}(A,\sigma) \to H^3(F,\mu_2)$ defined by*

$$\theta_2(g) = \big(\mu(g)\big) \cup [D(A,\sigma)]$$

induces a homomorphism

$$\Theta_2\colon \operatorname{PSim}(A,\sigma)/R \to H^3(F,\mu_2).$$

Moreover, for any $g \in \operatorname{Sim}(A,\sigma)$,

$$\theta_2(g) \cup (\alpha) = 0 \quad \text{in } H^4(F,\mu_2).$$

Proof. In view of the isomorphism (1), it suffices to show that for every finite field extension L/F such that $(A,\sigma) \otimes_F L$ is hyperbolic and for every $x \in L^\times$,

$$\big(N_{L/F}(x)\big) \cup [D(A,\sigma)] = 0 \qquad \text{in } H^3(F,\mu_2),$$

and that for every $\lambda \in K^\times$,

$$\big(N_{K/F}(\lambda)\big) \cup [D(A,\sigma)] = 0 \qquad \text{in } H^3(F,\mu_2).$$

As in the proof of Proposition 5, we are reduced by the projection formula to proving that $D(A,\sigma)$ is split by K and by every extension L/F such that $(A,\sigma)\otimes L$ is hyperbolic. The latter assertion follows from the lemma. On the other hand, by [4, (10.30)] and by the hypothesis on B we have

$$[D(A,\sigma)_K] = [\lambda^m A] = m[A] = 0.$$

To prove the last part, we use the trace form T_σ defined as in Section 2,

$$T_\sigma(x) = \operatorname{Trd}_A(\sigma(x)x) \qquad \text{for } x \in A,$$

and its restrictions T_σ^+, T_σ^- to $\operatorname{Sym}(A,\sigma)$ and $\operatorname{Skew}(A,\sigma)$ respectively. In the case of involutions of unitary type we have

$$T_\sigma = T_\sigma^+ \perp T_\sigma^- = \langle 1, -\alpha\rangle T_\sigma^+.$$

The computation of the Clifford algebra of T_σ^+ in [4, (11.17)] shows that $T_\sigma \in I^3F$ and

$$e_3(T_\sigma) = (\alpha) \cup [D(A,\sigma)].$$

Now, for $g \in \operatorname{Sim}(A,\sigma)$ the map $x \mapsto gx$ is a similitude of T_σ with multiplier $\mu(g)$, hence $\langle 1, -\mu(g)\rangle T_\sigma = 0$ in WF. Taking the image under e_4 yields

$$0 = \big(\mu(g)\big) \cup e_3(T_\sigma) = \theta_2(g) \cup (\alpha).$$

□

11. **Remarks.** (1) If $\operatorname{ind} A$ divides $\frac{1}{2}\deg A$, so that A carries a hyperbolic unitary involution σ_0, then [4, (11.17)] and Lemma 9 yield

$$[D(A,\sigma)] = e_2(T_\sigma^+ - T_{\sigma_0}^+).$$

This observation underlines the analogy between θ_2 and the map θ_1 of Proposition 5. Note however that no hypothesis on the index of A is required in Proposition 10.

(2) For $g \in \operatorname{Sim}(A,\sigma)$, the equation $\theta_2(g) \cup (\alpha) = 0$ implies that $\theta_2(g)$ lies in the image of the corestriction map $\operatorname{cor}_{K/F}\colon H^3(K,\mu_2) \to H^3(F,\mu_2)$, by [4, (30.12)]. On the other hand, if the characteristic does not divide m, Corollary 1.18 of [8] yields an explicit element $\xi \in H^3(K,\mu_m^{\otimes 2})$ such that $\operatorname{cor}_{K/F}(\xi) = \theta_2(g)$. In particular, if m is odd it follows that $\theta_2 = 0$.

The following explicit computation yields examples where $\theta_2 \neq 0$.

12. **Proposition.** *Let ι be the nontrivial automorphism of K/F, and assume*

$$(A,\sigma) = (B,\rho) \otimes_F (K,\iota)$$

for some central simple F-algebra with orthogonal involution (B,ρ) of degree $n = 2m$. Assume m is even. For $g \in \mathrm{GO}^-(B,\rho)$ we have $g \otimes 1 \in \operatorname{Sim}(A,\sigma)$ and

$$\theta_2(g \otimes 1) = (\alpha) \cup [B].$$

Proof. For $g \in \mathrm{GO}^-(B,\rho)$,

$$\sigma(g \otimes 1) g \otimes 1 = \rho(g) g = \mu(g),$$

so $g \otimes 1 \in \operatorname{Sim}(A,\sigma)$. By [4, (10.33)], we have

$$[D(A,\sigma)] = m[B] + (\alpha) \cup (\operatorname{disc}\rho).$$

Since m is even, the first term on the right side vanishes. The proposition follows by taking the cup-product with $\big(\mu(g)\big)$, since $[B] = \big(\mu(g)\big) \cup (\operatorname{disc}\rho)$ by [9, Theorem A] (see also [4, (13.38)]). $\square$

Remark. If m is odd in Proposition 12, then the definition of θ_2 requires the extra hypothesis that B is split by K. Computation then shows that $\theta_2(g \otimes 1) = 0$ for all $g \in \mathrm{GO}^-(B,\rho)$, as follows also from Remark 11.2 above.

13. **Corollary.** *Let r be an arbitrary integer. Let H be a quaternion F-algebra, $\alpha \in F^\times$, $K = F[X]/(X^2 - \alpha)$, and let ι be the nontrivial automorphism of K/F. Assume*

$$(-1) \cup [H] = 0 \text{ in } H^3(F,\mu_2) \qquad \text{and} \qquad (\alpha) \cup [H] \neq 0 \text{ in } H^3(F,\mu_2).$$

Let ρ be an orthogonal involution on $M_{2r}(H)$ which admits improper similitudes (see Lemma 1). Then $\mathbf{PSim}(M_{2r}(H) \otimes_F K, \rho \otimes \iota)$ is not R-trivial.

As in the previous section (see Corollary 8), alternative examples can be constructed from Proposition 2:

14. **Corollary.** *Let (B,ρ) be a central simple algebra of degree 4 with orthogonal involution over a field F_0. Assume B is not split and $\operatorname{disc}\rho \neq 1$. Let $F = F_0(x)$ be the field of rational fractions in one indeterminate over F_0, let $K = F(\sqrt{x})$ and let ι be the nontrivial automorphism of K/F. The group $\mathbf{PSim}(B \otimes_{F_0} K, \rho \otimes \iota)$ is not R-trivial.*

Note that this corollary also follows from [7, Theorem 3].

References

[1] G. Berhuy, M. Monsurrò, J.-P. Tignol, *The discriminant of a symplectic involution.* To appear in Pacific J. of Math.

[2] J.-L. Colliot-Thélène, J.-J. Sansuc, *La R-équivalence sur les tores.* Ann. scient. Éc. Norm. Sup., 4^e série, **10**, 175–230 (1997).

[3] P. Gille, *Examples of non-rational varieties of adjoint groups.* J. Algebra **193**, 728–747 (1997).

[4] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions.* Amer. Math. Soc. Coll. Pub. **44**, AMS, Providence, RI, 1998.

[5] D.W. Lewis, *A note on trace forms and involutions of the first kind.* Exposition. Math. **15**, 265–272 (1997).

[6] Y. Manin, *Cubic forms.* North-Holland, Amsterdam, 1974.

[7] A. Merkurjev, *R-equivalence and rationality problem for semisimple adjoint classical algebraic groups.* Pub. Math. IHES **46**, 189–213 (1996).

[8] A. Merkurjev, R. Parimala, J.-P. Tignol, *Invariants of quasi-trivial tori and the Rost invariant.* Preprint.

[9] A. Merkurjev, J.-P. Tignol, *The multipliers of similitudes and the Brauer group of homogeneous varieties.* J. reine angew. Math. **461**, 13–47 (1995).

[10] A. Quéguiner, *Cohomological invariants of algebras with involution.* J. Algebra **194**, no 1, 299–330 (1997).

[11] V.E. Voskresenkiĭ, A.A. Klyachko, *Toroidal Fano varieties and root system.* Math. USSR Izvestija **24**, 221–244 (1985).

Département de mathématiques, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland

E-mail address: gregory.berhuy@epfl.ch, marina.monsurro@epfl.ch

Institut de Mathématique Pure et Appliquée, Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

E-mail address: tignol@math.ucl.ac.be