Exact Sequences of Witt Groups

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Abstract

An exact octagon of Witt groups of central simple algebras with involution is constructed, extending an exact sequence of Parimala, Sridharan and Suresh and motivated by exact sequences obtained by Lewis. From this, we derive relations between the cardinality of certain Witt groups. An exact octagon of equivariant Witt groups is also obtained, thus generalizing a similar octagon constructed by Lewis for quaternion algebras.

1 Introduction

Base change is an important tool in the algebraic theory of quadratic forms and of hermitian forms over division algebras. For a field extension L/K (of characteristic different from 2), we can consider base change from K to L; if moreover the extension has finite degree, then we also have the Scharlau transfer. The situation is especially well understood when L/K is of odd degree or a quadratic extension: see the book of Scharlau [13] for these basic notions and results. Another classical result, due to Kneser and Springer, concerns hermitian forms over quaternion algebras (see the appendix written by Springer in [6]).

The Witt group (and Witt ring for quadratic forms) gives a very useful way to study quadratic and hermitian forms. The above results can be expressed very efficiently in this framework. One of the basic result in the theory of quadratic forms is a theorem of Pfister which determines the kernel of the restriction map $r^*_{L/K} : W(K) \to W(L)$ for a quadratic extension L/K. More precisely, this kernel is the ideal generated by the form $< 1, -\delta >$ where $L = K(\sqrt{\delta})$. One can express this result by the exactness of the sequence:

$$W(K) \xrightarrow{t} W(K) \xrightarrow{r^*_{L/K}} W(L) \tag{1}$$

[1] *2000 Mathematics Subject Classification.* Primary 11E39; Secondary 11E81.

[2] *Key words and phrases.* Witt group, hermitian form, central simple algebra with involution, exact sequence of Witt groups, isotropy, equivariant Witt group, Morita equivalence.

[3] The authors gratefully acknowledge support from the RT Network "K-theory, Linear Algebraic Groups and Related Structures" (contract HPRN-CT-2002-00287)

where t is multiplication by the 2-dimensional form $< 1, -\delta >$. By a result of Elman-Lam, the Scharlau transfer map $s_* : W(L) \longrightarrow W(K)$, can be used to embed (1) in the following exact triangle (cf. [13, Ch. 2, 5.10]):

$$\begin{tikzcd} W(K) \arrow[rr, "r^*"] & & W(L) \arrow[dl, "s_*"] \\ & W(K) \arrow[ul, "t"] & \end{tikzcd} \tag{2}$$

For a quadratic extension L/K (resp. a quaternion division algebra $(a, b)_K = D$) with nontrivial automorphism $-$ (resp. with canonical involution $-$), one can consider the trace map $W(L, -) \to W(K)$ (resp. $W(D, -) \to W(K)$). By a result of Jacobson, these maps are injective (cf. [13, Ch. 10, 1.1, 1.2, 1.7] and [5]).

In [12, Appendix 2], Milnor and Husemoller construct the following exact sequence of $W(K)$-modules:

$$0 \longrightarrow W(L, -) \longrightarrow W(K) \longrightarrow W(L) \tag{3}$$

where $-$ is the nontrivial automorphism of the quadratic extension L/K. The results concerning hermitian forms over quaternion division algebras are given in the papers of Lewis [8], [9]. He found the following exact sequence:

$$0 \longrightarrow W(D, -) \longrightarrow W(L, -) \longrightarrow W^{-1}(D, -) \longrightarrow W(L) \tag{4}$$

where $L = K(\sqrt{a}) \subset D$ is stable by $-$. In fact, in this sequence, D can be also split (cf. [13, Ch. 10, 3.2]).

Remark 1.1. Note that Lewis uses $W(D, \wedge)$ instead of $W^{-1}(D, -)$, where $\wedge$ is the orthogonal involution of D defined by $\widehat{i} = -i$ and $\widehat{j} = j$ where $\{1,\ i,\ j,\ ij\}$ are usual generators of the quaternion algebra D. In fact $W(D, \wedge) \simeq W^{-1}(D, -)$: this isomorphism is induced by the map which associates to a hermitian form h over $(D, \wedge)$, the skew-hermitian form ih over $(D, -)$.

In [2, Appendix 2], Parimala, Sridharan and Suresh obtain this crucial exact sequence of Witt groups:

$$W^{\varepsilon}(A, \sigma) \xrightarrow{\pi_1^{\varepsilon}} W^{\varepsilon}(\widetilde{A}, \sigma_1) \xrightarrow{\rho_1^{\varepsilon}} W^{-\varepsilon}(A, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(\widetilde{A}, \sigma_2) \tag{5}$$

where A is a central simple algebra with an involution σ of any kind and $\varepsilon = \pm 1$ (for the notations, see the beginning of section 2). The exact sequence (4) is a particular case of (5). In general the map π_1^{ε} is not injective: in fact, 5.3 characterizes the cases where π_1^{ε} is injective. In [2], (5) is used to prove Serre's conjecture II for classical groups. In [9], Lewis has found a longer exact sequence:

$$\begin{gathered} 0 \to W(D, -) \to W(L, -) \to W^{-1}(D, -) \to W(L) \to W^{-1}(D, -) \to \\ \to W(L, -) \to W(D, -) \to 0 \end{gathered} \tag{6}$$

This sequence was one of our main motivations in order to embed the exact sequence (5) in an even longer one. In fact we obtain an exact octagon:

Theorem 1.2. *There is an exact sequence of Witt groups (in fact of $W(K, \sigma|_K)$-modules):*

$$\begin{tikzcd}
 & W^{\varepsilon}(A,\sigma) \arrow[r, "\pi_1^{\varepsilon}"] & W^{\varepsilon}(\widetilde{A},\sigma_1) \arrow[dr, "\rho_1^{\varepsilon}"] & \\
W^{-\varepsilon}(\widetilde{A},\sigma_2) \arrow[ur, "\rho_2^{-\varepsilon}"] & & & W^{-\varepsilon}(A,\sigma) \arrow[d, "\pi_2^{-\varepsilon}"] \\
W^{\varepsilon}(A,\sigma) \arrow[u, "\pi_2^{\varepsilon}"] & & & W^{\varepsilon}(\widetilde{A},\sigma_2) \arrow[dl, "\rho_2^{\varepsilon}"] \\
 & W^{\varepsilon}(\widetilde{A},\sigma_1) \arrow[ul, "\rho_3^{\varepsilon}"] & W^{-\varepsilon}(A,\sigma) \arrow[l, "\pi_3^{-\varepsilon}"] &
\end{tikzcd}$$

(for the definition of the maps, see section 2).

Remark 1.3. Note that the exact sequence (6) is a particular case of 1.2.

Remark 1.4. One can find a similar exact octagon of Witt groups of Clifford algebras of quadratic forms with their canonical involution in [11].

The proof of 1.2 can be found in section 5. Because of the nature of Witt groups, the above octagon does not give, a priori, much information about the behavior of these maps with respect to isotropy in the semigroups of isometry classes of nondegenerate hermitian forms: in section 3, theorem 3.4 describes this behavior in the case of division algebras. We also deduce an alternative proof of 1.2 in the case of division algebras.

Section 4 contains the relation between the notion of isotropy over a central simple algebra A and isotropy over the division algebra D Brauer equivalent to A via Morita equivalence as well as the Witt decomposition. In fact we point out that there exists a Witt decomposition over a central simple algebra with involution. In particular, this implies the existence of an anisotropic form in every class of the Witt group.

In [10], Lewis constructs an exact octagon of Witt groups of forms invariant under the action of a finite group G (Witt groups of equivariant forms) for quaternion division algebras. in section 6, we show likewise that the octagon of 1.2 is exact if we replace Witt groups by equivariant Witt groups. More precisely:

Theorem 1.5. *We suppose that A satisfies the same hypotheses as at the beginning of section 2. If we replace $W^{\pm\varepsilon}(A,\sigma)$ by $W^{\pm\varepsilon}(G,A,\sigma)$ and $W^{\pm\varepsilon}(\widetilde{A},\sigma_i)$ by $W^{\pm\varepsilon}(G,\widetilde{A},\sigma_i)$ (for $i=1,\ 2$) in theorem 1.2, we obtain an exact octagon of $W(K,\sigma|_K)$-modules.*

In section 7, by using the exact octagon, we obtain the following result:

Corollary 1.6. *Let A be a K-central simple algebra with an involution σ of the first kind. Then we have $|W^\varepsilon(A,\sigma)||W^{-\varepsilon}(A,\sigma)| = |W(K)|$. In particular $W(K)$ is finite if and only if $W^\varepsilon(A,\sigma)$ and $W^{-\varepsilon}(A,\sigma)$ are finite.*

This result was well known for quaternion algebras: see [9].

We thank Eva Bayer-Fluckiger, Detlev Hoffmann and Emmanuel Lequeu for useful conversations about this work and for their comments on earlier versions of this paper.

2 Notation and Definition of the Maps

Let K be a field of characteristic different from 2. All the modules in this paper are supposed to be right modules which are finitely generated and all the ε-hermitian forms are supposed to be nondegenerate. Let A be a central simple algebra over K with an involution σ (of any kind). For $\varepsilon=\pm1$, let $S^\varepsilon(A,\sigma)$ denote the semigroup of isometry classes of ε-hermitian forms over (A,σ) and let $W^\varepsilon(A,\sigma)$ be the Witt group of (A,σ) (i.e., the quotient of the Grothendieck group corresponding to $S^\varepsilon(A,\sigma)$ by the subgroup generated by metabolic forms, an ε-hermitian form (V,h) being metabolic if there exists an A-submodule W of V such that $W=W^\perp$ for h).

Remark 2.1. As A is simple, there is of course no difference between the notions of metabolic and hyperbolic hermitian forms. We use any of these two notions subsequently (except in section 6).

First, we define the different maps involved in 1.2.As in [2], we suppose that there exist $\lambda,\ \mu\in A^*$ such that $\sigma(\lambda)=-\lambda$, $\sigma(\mu)=-\mu$, $\mu\lambda=-\lambda\mu$ and such that $L=K(\lambda)$ is a quadratic extension of K. We write $\widetilde{A}$ for the commutant of L in A: this is a central simple algebra over L . One can easily verify that $\mu\widetilde{A}=\widetilde{A}\mu$, $\mu^2\in\widetilde{A}$, $\sigma(\widetilde{A})=\widetilde{A}$ and $A=\widetilde{A}\oplus\mu\widetilde{A}$. We define two involutions on $\widetilde{A}$ in the following way: let $\sigma_1=\sigma|_{\widetilde{A}}$ and let $\sigma_2=\mathrm{Int}(\mu^{-1})\circ\sigma_1$ (where $\mathrm{Int}(\mu^{-1})(x)=\mu^{-1}x\mu$).

◇ **Definition of π_1^ε and π_2^ε**

We have two L-linear projections: $\pi_1 : A \to \widetilde{A} : a_1 + \mu a_2 \mapsto a_1$ and $\pi_2 : A \to \widetilde{A} : a_1 + \mu a_2 \mapsto a_2$. If $h : V \times V \to A$ is an ε-hermitian space over (A, σ), we define (for $i = 1, 2$) $\pi_i^\varepsilon(h) : V \times V \to \widetilde{A}$ by $\pi_i^\varepsilon(h)(x, y) = \pi_i(h(x, y))$. One readily verifies that $\pi_1^\varepsilon(h)$ is an ε-hermitian space over $(\widetilde{A}, \sigma_1)$ and that $\pi_2^\varepsilon(h)$ is a $-\varepsilon$-hermitian space over $(\widetilde{A}, \sigma_2)$. In order to see that π_1^ε and π_2^ε induce homomorphisms of Witt groups we have to prove that these maps respect regularity, isometry classes, orthogonality and hyperbolicity. All of these properties come from the fact that:

$$h(x, y) = 0 \quad \forall y \iff \pi_1^\varepsilon(h)(x, y) = 0 \quad \forall y \iff \pi_2^\varepsilon(h)(x, y) = 0 \quad \forall y$$

Hence, π_1^ε and π_2^ε induce homomorphisms of semigroups of isometry classes of nondegenerate hermitian forms and homomorphisms of Witt groups (again denoted by π_1^ε and π_2^ε)

$$\begin{array}{ll} \pi_1^\varepsilon : S^\varepsilon(A, \sigma) \to S^\varepsilon(\widetilde{A}, \sigma_1); & \pi_1^\varepsilon : W^\varepsilon(A, \sigma) \to W^\varepsilon(\widetilde{A}, \sigma_1) \\ \pi_2^\varepsilon : S^\varepsilon(A, \sigma) \to S^{-\varepsilon}(\widetilde{A}, \sigma_2); & \pi_2^\varepsilon : W^\varepsilon(A, \sigma) \to W^{-\varepsilon}(\widetilde{A}, \sigma_2) \end{array}$$

◇ **Definition of ρ_1^ε**

Let (V, f) be an ε-hermitian space over $(\widetilde{A}, \sigma_1)$. We associate to it $(V \otimes_{\widetilde{A}} A, \rho_1^\varepsilon(f))$ where $\rho_1^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha)\lambda f(x, y)\beta$ for $x,\ y \in V$ and $\alpha,\ \beta \in A$. We can easily verify that ρ_1^ε is well defined and that $(V \otimes_{\widetilde{A}} A, \rho_1^\varepsilon(f))$ is a $-\varepsilon$-hermitian space over (A, σ). Moreover ρ_1^ε induces homomorphisms:

$$\rho_1^\varepsilon : S^\varepsilon(\widetilde{A}, \sigma_1) \to S^{-\varepsilon}(A, \sigma); \quad \rho_1^\varepsilon : W^\varepsilon(\widetilde{A}, \sigma_1) \to W^{-\varepsilon}(A, \sigma)$$

◇ **Definition of ρ_2^ε**

Let (V, f) be an ε-hermitian space over $(\widetilde{A}, \sigma_2)$. We associate to it $(V \otimes_{\widetilde{A}} A, \rho_2^\varepsilon(f))$ where $\rho_2^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha)\lambda\mu f(x, y)\beta$ for $x,\ y \in V$ and $\alpha,\ \beta \in A$. One can verify that ρ_2^ε is well defined, that $(V \otimes_{\widetilde{A}} A, \rho_2^\varepsilon(f))$ is a $-\varepsilon$-hermitian space over (A, σ) and that ρ_2^ε induces homomorphism:

$$\rho_2^\varepsilon : S^\varepsilon(\widetilde{A}, \sigma_2) \to S^{-\varepsilon}(A, \sigma); \quad \rho_2^\varepsilon : W^\varepsilon(\widetilde{A}, \sigma_2) \to W^{-\varepsilon}(A, \sigma)$$

◇ **Definition of π_3^ε**

We define π_3^ε to be $\lambda\pi_1^\varepsilon$, so we obtain homomorphisms:

$$\pi_3^\varepsilon : S^\varepsilon(A, \sigma) \to S^{-\varepsilon}(\widetilde{A}, \sigma_1); \quad \pi_3^\varepsilon : W^\varepsilon(A, \sigma) \to W^{-\varepsilon}(\widetilde{A}, \sigma_1)$$

◇ **Definition of ρ_3^ε**

We define: $\rho_3^\varepsilon(f) = \rho_1^{-\varepsilon}(\lambda^{-1} f)$, i.e., $\rho_3^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha) f(x, y)\beta$ for $x, y \in V$ and $\alpha, \beta \in A$. By a straightforward verification we obtain homomorphisms:

$$\rho_3^\varepsilon : S^\varepsilon(\widetilde{A}, \sigma_1) \to S^\varepsilon(A, \sigma); \quad \rho_3^\varepsilon : W^\varepsilon(\widetilde{A}, \sigma_1) \to W^\varepsilon(A, \sigma)$$

Remark 2.2. Note that, in these definitions, ε is arbitrary so for example $\pi_1^{-\varepsilon}$ will be a homomorphism of Witt groups from $W^{-\varepsilon}(A, \sigma)$ to $W^{-\varepsilon}(\widetilde{A}, \sigma_1)$ and so on for the other maps.

A summary of these definitions can be found in table 8.1.

3 The Behavior of the Maps for Division Algebras

In this section (D, τ) is a division algebra with an involution τ of any kind. We assume that $(D, \widetilde{D}, \tau, \tau_1, \tau_2)$ satisfies the same hypotheses as the ones mentioned in section 2 for $(A, \widetilde{A}, \sigma, \sigma_1, \sigma_2)$.

Proposition 3.1. (i) *If $h =< \delta >$ is a one dimensional ε-hermitian form over (D, τ) (with $\delta = d_1 + \mu d_2$, $d_1,\ d_2 \in \widetilde{D}$) then the matrix of $\pi_1^\varepsilon(h)$ over $(\widetilde{D}, \tau_1)$ with respect to the basis $\{1, \mu\}$ is*

$$\begin{pmatrix} d_1 & \mu d_2 \mu \\ -\mu^2 d_2 & -\mu d_1 \mu \end{pmatrix}$$

(ii) *If P is the matrix of an ε-hermitian form f over $(\widetilde{D}, \tau_1)$ with respect to a basis B, then λP is the matrix of $\rho_1^\varepsilon(f)$ over (D, τ) with respect to the basis $B \otimes 1$.*
(iii) *If $h =< \delta >$ is a one dimensional $-\varepsilon$-hermitian form over (D, τ) (with $\delta = d_1 + \mu d_2$, $d_1,\ d_2 \in \widetilde{D}$) then the matrix of $\pi_2^{-\varepsilon}(h)$ over $(\widetilde{D}, \tau_2)$ with respect to the basis $\{1, \mu\}$ is*

$$\begin{pmatrix} d_2 & \mu^{-1} d_1 \mu \\ -d_1 & -\mu d_2 \mu \end{pmatrix}$$

(iv) *If P is the matrix of an ε-hermitian form f over $(\widetilde{D}, \tau_2)$ with respect to a basis B, then $\lambda \mu P$ is the matrix of $\rho_2^\varepsilon(f)$ over (D, τ) with respect to the basis $B \otimes 1$.*
(v) *For a $-\varepsilon$-hermitian form h over (D, τ), the matrix of $\pi_3^{-\varepsilon}(h)$ with respect to a basis B is λ times the matrix of $\pi_1^\varepsilon(h)$ with respect to the basis $B \cup \mu B$.*
(vi) *If P is the matrix of an ε-hermitian form f over $(\widetilde{D}, \tau_1)$ with respect to a basis B, then P is the matrix of $\rho_3^\varepsilon(f)$ over (D, τ) with respect to the basis $B \otimes 1$.*

Proof. (i) We have :
$h(1, 1) = d_1 + \mu d_2$, $h(1, \mu) = d_1 \mu + \mu d_2 \mu$, $h(\mu, 1) = -\mu d_1 - \mu^2 d_2$ and $h(\mu, \mu) = -\mu d_1 \mu - \mu^2 d_2 \mu$.
As $d_1,\ d_2 \in \widetilde{D}$, we can find the given matrix by applying π_1^ε to these relations.
(ii) If (V, f) is an ε-hermitian space over $(\widetilde{D}, \tau_1)$ and if $\{e_1, \ldots, e_n\}$ is a basis of V over $\widetilde{D}$ then $\{e_1 \otimes 1, \ldots, e_n \otimes 1\}$ is a basis of $V \otimes_{\widetilde{D}} D$ over D. With the definition of ρ_1^ε, we have: $\rho_1^\varepsilon(f)(e_i \otimes 1, e_j \otimes 1) = \lambda f(e_i, e_j)\ \forall\ i, j = 1, \ldots, n$.
(iii) By applying $\pi_2^{-\varepsilon}$ to the four relations found in (i), we deduce the given matrix for $\pi_2^{-\varepsilon}(h)$.
(iv) If (V, f) is an ε-hermitian space over $(\widetilde{D}, \tau_2)$ and if $\{e_1, \ldots, e_n\}$ is a basis of V over $\widetilde{D}$ then $\{e_1 \otimes 1, \ldots, e_n \otimes 1\}$ is a basis of $V \otimes_{\widetilde{D}} D$ over D. With the definition of ρ_2^ε, we have: $\rho_2^\varepsilon(f)(e_i \otimes 1, e_j \otimes 1) = \lambda \mu f(e_i, e_j)\ \forall\ i, j = 1, \ldots, n$.
(v) This is obvious by the definition of $\pi_3^{-\varepsilon}$.
(vi) If (V, f) is an ε-hermitian space over $(\widetilde{D}, \tau_1)$ and if $\{e_1, \ldots, e_n\}$ is a basis of V over $\widetilde{D}$ then $\{e_1 \otimes 1, \ldots, e_n \otimes 1\}$ is a basis of $V \otimes_{\widetilde{D}} D$ over D. By the definition of ρ_3^ε, we have: $\rho_3^\varepsilon(f)(e_i \otimes 1, e_j \otimes 1) = f(e_i, e_j)\ \forall\ i, j = 1, \ldots, n$. □

The summary of the previous proposition can be found in table 8.2.

Lemma 3.2. *If $\widetilde{D}$ is commutative then D is a quaternion division algebra, say $D = (a, b)_K$, $\widetilde{D} = L = K(\lambda)$ with $\lambda^2 = a \in K$ and $\mu^2 = b \in K$.*

Proof. $\widetilde{D}$ is the commutant of L in D, so it is a central simple algebra of center L. As $\widetilde{D}$ is commutative, we have $\widetilde{D} = L$ and the lemma readily follows. □

Lemma 3.3. *If E is a noncommutative central simple algebra and $\star$ an involution over E, then for $\varepsilon = \pm 1$, the set of nonzero ε-hermitian elements with respect to $\star$ is nonempty.*

Proof. This is a consequence of [13, Ch. 8, 7.5]. □

The following theorem determines completely the behavior of the maps π_1, ρ_1, π_2, ρ_2, π_3 and ρ_3 with respect to isotropy.

Theorem 3.4. (i) *Let $h \in S^{\varepsilon}(D, \tau)$. If $\widetilde{D}$ is commutative and $\varepsilon = 1$, then $\pi_1^{\varepsilon}(h) \in S^{\varepsilon}(\widetilde{D}, \tau_1)$ is isotropic if and only if h is isotropic. Otherwise, $\pi_1^{\varepsilon}(h)$ is isotropic if and only if there exists $c \in \widetilde{D}^*$ with $\tau_2(c) = -\varepsilon c$ such that h contains a subform isometric to $< \mu c >$.*
(ii) *Suppose that $f \in S^{\varepsilon}(\widetilde{D}, \tau_1)$. If $\widetilde{D}$ is commutative, $\varepsilon = 1$ and $\dim(f) = 2$, then $\rho_1^{\varepsilon}(f) \in S^{-\varepsilon}(D, \tau)$ is isotropic if and only if f is isotropic or f is isometric to the two dimensional anisotropic form $< c, -bc >$ where $c \in K^*$ and $b = \mu^2$ (see 3.2). Otherwise $\rho_1^{\varepsilon}(f)$ is isotropic if and only if there exist $d_1, d_2 \in \widetilde{D}$ such that $\tau_1(d_1) = \varepsilon d_1$, $\tau_2(d_2) = -\varepsilon d_2$ and $f = f_1 \perp f_2$ for some nondegenerate*

$$f_1 \simeq \begin{pmatrix} d_1 & \mu d_2 \mu \\ -\mu^2 d_2 & -\mu d_1 \mu \end{pmatrix}.$$

(iii) *For $h \in S^{-\varepsilon}(D, \tau)$, $\pi_2^{-\varepsilon}(h) \in S^{\varepsilon}(\widetilde{D}, \tau_2)$ is isotropic if and only if there exists $c \in \widetilde{D}^*$ with $\tau_1(c) = -\varepsilon c$ such that h contains a subform isometric to $< c >$.*
(iv) *For $f \in S^{\varepsilon}(\widetilde{D}, \tau_2)$, $\rho_2^{\varepsilon}(f) \in S^{-\varepsilon}(D, \tau)$ is isotropic if and only if there exist $d_1, d_2 \in \widetilde{D}$ such that $\tau_1(d_1) = -\varepsilon d_1$, $\tau_2(d_2) = \varepsilon d_2$ and $f = f_1 \perp f_2$ for some nondegenerate*

$$f_1 \simeq \begin{pmatrix} d_2 & \mu^{-1} d_1 \mu \\ -d_1 & -\mu d_2 \mu \end{pmatrix}.$$

(v) *Let $h \in S^{-\varepsilon}(D, \tau)$. If $\widetilde{D}$ is commutative and $\varepsilon = -1$, then $\pi_3^{-\varepsilon}(h) \in S^{\varepsilon}(\widetilde{D}, \tau_1)$ is isotropic if and only if h is isotropic. Otherwise, $\pi_3^{-\varepsilon}(h)$ is isotropic if and only if there exists $c \in \widetilde{D}^*$ with $\tau_2(c) = \varepsilon c$ such that h contains a subform isometric to $< \mu c >$.*
(vi) *Suppose that $f \in S^{\varepsilon}(\widetilde{D}, \tau_1)$. If $\widetilde{D}$ is commutative, $\varepsilon = -1$ and $\dim(f) = 2$, then $\rho_3^{\varepsilon}(f) \in S^{\varepsilon}(D, \tau)$ is isotropic if and only if f is isotropic or f is isometric to the two dimensional anisotropic form $< \lambda c, -\lambda bc >$ where $c \in K^*$ and $b = \mu^2$ (see 3.2). Otherwise $\rho_3^{\varepsilon}(f)$ is isotropic*

if and only if there exist $d_1, d_2 \in \widetilde{D}$ *such that* $\tau_1(d_1) = -\varepsilon d_1$, $\tau_2(d_2) = \varepsilon d_2$ *and* $f = f_1 \perp f_2$ *for some nondegenerate*

$$f_1 \simeq \begin{pmatrix} \lambda d_1 & \lambda\mu d_2 \mu \\ -\lambda\mu^2 d_2 & -\lambda\mu d_1 \mu \end{pmatrix}.$$

Proof. Let $\mathbb{H}_\varepsilon$ denote an ε-hyperbolic plane.
(i) First, suppose that $\widetilde{D}$ is commutative and $\varepsilon = 1$. By 3.2, we know that D is a quaternion division algebra. The equivalence comes from the fact that, in this case, the trace form of h is isotropic if and only if h is isotropic.

Next, suppose that the previous case is excluded. If h is anisotropic and $\pi_1^\varepsilon(h)$ is isotropic then we can find $x \in V - \{0\}$ such that $\pi_1^\varepsilon(h(x,x)) = 0$, i.e., $h(x,x) = \mu c$ with $c \in \widetilde{D}^*$. As $\tau(h(x,x)) = \varepsilon h(x,x)$, we conclude that $\tau_2(c) = -\varepsilon c$. It readily follows that h contains $< \mu c >$. Now consider the case where h is isotropic. We have $\mathbb{H}_\varepsilon \simeq < \mu c, -\mu c >$ for all $c \in \widetilde{D}^*$ with $\tau_2(c) = -\varepsilon c$, provided such a c exists. If $\varepsilon = -1$ then we can take $c = 1$. If $\varepsilon = 1$ then we only have to show that there exists $c \in \widetilde{D}$ such that $\tau_2(c) = -c$. By assumption $\widetilde{D}$ is noncommutative, and one can apply 3.3 to conclude. Conversely, suppose that h contains a subform isometric to $< \mu c >$ as in the assertion. By applying 3.1 (i) to the form $< \mu c >$, we easily deduce that $\pi_1^\varepsilon(h)$ is isotropic.
(ii) First suppose that $\widetilde{D}$ is commutative, $\varepsilon = 1$ and $\dim(f) = 2$. We are in the situation of 3.2. If f is isotropic or $f \simeq < c, -bc >$, it is obvious that $\rho_1^\varepsilon(f)$ is isotropic. Conversely suppose that for a two dimensional form $f = < c, d >$, $\rho_1^\varepsilon(f) = < \lambda c, \lambda d >$ is isotropic. So there exists $q \in D^*$ such that

$$\tau(q)\lambda c q + \lambda d = 0. \tag{7}$$

Write $q = z_1 + \mu z_2$ with $z_1,\ z_2 \in \widetilde{D}$. By replacing q with $z_1 + \mu z_2$ in (7), by using the fact that $\{1, \mu\}$ is an L-basis of D and by remarking that $\tau_2 = \mathrm{id}_L$, we obtain the following system:

$$\begin{cases} \tau_1(z_1) c z_1 + \tau_1(z_2) b c z_2 + d = 0 \\ z_1 z_2 = 0 \end{cases}$$

If $z_2 = 0$, then $\tau_1(z_1) c z_1 + d = 0$; this means that f is isotropic. If $z_1 = 0$ then $d = -\tau_1(z_2) b c z_2$, so $f \simeq < c, -\tau_1(z_2) b c z_2 > \simeq < c, -bc >$.

Next, suppose that the previous case is excluded. If f is anisotropic and $\rho_1^\varepsilon(f)$ is isotropic, let $z = x_1 \otimes 1 + y_1 \otimes \mu$ be a nonzero isotropic vector for $\rho_1^\varepsilon(f)$. By the definition of ρ_1^ε, we obtain $(\lambda f(x_1, x_1) - \mu\lambda f(y_1, y_1)\mu) + (\lambda f(x_1, y_1)\mu - \mu\lambda f(y_1, x_1)) = 0$ and so we obtain the following system:

$$\begin{cases} f(x_1, x_1) + \mu f(y_1, y_1)\mu = 0 \\ \mu f(y_1, x_1) + f(x_1, y_1)\mu = 0 \end{cases} \tag{8}$$

As f is anisotropic, thanks to this system, we can suppose that both x_1 and y_1 are nonzero. Moreover, x_1 and y_1 are linearly independent over $\widetilde{D}$. In fact, if x_1 and y_1 are linearly dependent then $x_1 = y_1 d$ with $d \in \widetilde{D}^*$ and by replacing x_1 with $y_1 d$ in (8), we obtain the following system:

$$\begin{cases} \tau(d) f(y_1, y_1) d + \mu f(y_1, y_1)\mu = 0 \\ \mu f(y_1, y_1) d + \tau(d) f(y_1, y_1)\mu = 0 \end{cases} \tag{9}$$

From the second equation of (9), we obtain that $\tau(d)f(y_1,y_1) = -\mu f(y_1,y_1)d\mu^{-1}$. By replacing $\tau(d)f(y_1,y_1)$ by $-\mu f(y_1,y_1)d\mu^{-1}$ in the first equation of (9), we obtain $\mu f(y_1,y_1)(-d\mu^{-1}d+\mu) = 0$. As the second factor is non zero for all $d \in \widetilde{D}$, $f(y_1,y_1) = 0$ which is a contradiction with the anisotropy of f. Now, y_1 and x_1 span a two dimensional subspace W over $\widetilde{D}$ and if we denote $d_1 = f(y_1,y_1)$, $d_2 = \mu^{-1}f(y_1,x_1)\mu^{-1}$, the matrix M of $f|_W$ with respect to the basis $\{y_1,x_1\}$ is exactly the one given in the proposition. As $f_1 = f|_W$ is nondegenerate (since f is anisotropic), we can write $f = f_1 \perp f_2$ so f contains the given form. Now consider the case where f is isotropic. If $\widetilde{D}$ is noncommutative, we take $d_1 = 0$ and we can find $d_2 \in \widetilde{D}$ such that $\tau_2(d_2) = -\varepsilon d_2$ and it is obvious that

$$\mathbb{H}_\varepsilon \simeq \begin{pmatrix} 0 & \mu d_2 \mu \\ -\mu^2 d_2 & 0 \end{pmatrix},$$

so f contains the given form. If $\widetilde{D}$ is commutative and $\varepsilon = -1$, we take $d_1 = 0$ and $d_2 = 1$ and $\mathbb{H}_{-1}$ is isometric to the matrix given in the proposition. If $\widetilde{D}$ is commutative, $\varepsilon = 1$ and $\dim(f) \geq 3$ then $f \simeq \mathbb{H}_1 \perp f_1$ with $\dim(f_1) \geq 1$. If $f_1 \simeq < a, ... >$ then $f \simeq < \mu a \mu, -\mu a \mu, a, ... >$ so, for $d_1 = a$, $d_2 = 0$, f contains the given form. Conversely, with the same notations as (ii), 3.1 (ii) and a straightforward calculation show that $(\mu, 1)$ is an isotropic vector for $\rho_1^\varepsilon(f_1)$.
(iii) If h is anisotropic and $\pi_2^{-\varepsilon}(h)$ is isotropic then we can find $x \in V - \{0\}$ such that $\pi_2^{-\varepsilon}(h(x,x)) = 0$, that is $h(x,x) = c \in \widetilde{D}^*$. We conclude as in (i). If h is isotropic and $\varepsilon = 1$ then $h \simeq < \lambda, -\lambda > \perp h_1$ and we can take $c = \lambda$. If h is isotropic and $\varepsilon = -1$, $h \simeq \mathbb{H}_1 \perp h_1$ and all we have to do is to find $c \in \widetilde{D}^*$ such that $\tau_1(c) = c$; we can take $c = 1$.

The converse is an easy consequence of 3.1 (iii).
(iv) If f is anisotropic and $\rho_2^\varepsilon(f)$ is isotropic, let $z = x_1 \otimes 1 + y_1 \otimes \mu$ be an isotropic vector for $\rho_2^\varepsilon(f)$. With a straightforward computation we find the same system as in the proof of (ii). Proceeding in the same way, we can suppose that x_1 and y_1 are nonzero and span a two dimensional subspace W over $\widetilde{D}$. If $d_1 = -f(x_1,y_1)$ and $d_2 = f(y_1,y_1)$ then the matrix of $f|_W$ with respect to the basis $\{y_1,x_1\}$ is exactly the one given in the proposition. Now consider the case where f is isotropic. If $\varepsilon = 1$, we take $d_2 = 0$ and $d_1 = \lambda$. If $\varepsilon = -1$, we take $d_2 = 0$ and $d_1 = 1$.

Conversely, $(\mu, 1)$ is an isotropic vector for $\rho_2^\varepsilon(f_1)$.
(v) If $\widetilde{D}$ is commutative and $\varepsilon = -1$ then the equivalence between $\pi_3^{-\varepsilon}(h)$ being isotropic and h being isotropic readily comes from (i).

Next, we suppose that the previous case is excluded. If h is anisotropic and $\pi_3^{-\varepsilon}(h)$ is isotropic we can conclude as in (i) and (iii). If h is isotropic then $h \simeq < \mu c, -\mu c > \perp h_1$ for all $c \in \widetilde{D}^*$ such that $\tau_2(c) = \varepsilon c$ and we only have to find such a c. If $\widetilde{D}$ is noncommutative this is clear. If $\widetilde{D}$ is commutative then $\tau_2 = \mathrm{id}_L$. As $\varepsilon = 1$ we can take $c = 1$. Conversely, we only have to apply 3.1 (v).
(vi) First, suppose that $\widetilde{D}$ is commutative, $\varepsilon = -1$ and $\dim(f) = 2$ then we are in the situation of 3.2. The proof goes as in (ii). We leave it to the reader.

Next suppose that the previous case is excluded. If f is anisotropic and $\rho_3^\varepsilon(f)$ is isotropic, let $z = x_1 \otimes 1 + y_1 \otimes \mu$ be an isotropic vector for $\rho_3^\varepsilon(f)$. We have the following system:

$$\begin{cases} f(x_1, x_1) - \mu f(y_1, y_1)\mu = 0 \\ f(x_1, y_1)\mu - \mu f(y_1, x_1) = 0 \end{cases}$$

Now, let $d_1 = \lambda^{-1} f(y_1, y_1)$ and $d_2 = \mu^{-1}\lambda^{-1} f(y_1, x_1)\mu^{-1}$. Let W be the two dimensional $\widetilde{D}$-subspace generated by x_1 and y_1 (the proof of the fact that x_1 and y_1 are linearly independent over $\widetilde{D}$ is similar to (ii)). The matrix of $f|_W$ with respect to the basis $\{y_1, x_1\}$ is exactly the one given in the proposition (the form $f|_W$ is nondegenerate because f is anisotropic). Now consider the case where h is isotropic. If $\widetilde{D}$ is noncommutative, we take $d_1 = 0$ and $d_2 \in \widetilde{D}$ such that $\tau_2(d_2) = \varepsilon d_2$. If $\widetilde{D}$ is commutative and $\varepsilon = 1$, $f \simeq \mathbb{H}_1 \perp f_1$ and we take $d_2 = 1$ and $d_1 = 0$. If $\widetilde{D}$ is commutative, $\varepsilon = -1$ and $\dim(f) \geq 3$, we conclude as in (ii). Conversely, $(\mu, 1)$ is an isotropic vector for $\rho_3^\varepsilon(f_1)$. $\square$

In particular, if f and h are anisotropic, we obtain table 8.3. From 3.1 and 3.4, we obtain the following result:

Corollary 3.5. *The octagon of 1.2 is exact when A is a division algebra.*

4 Isotropy of Hermitian Forms over Central Simple Algebras

The goal of this section is to prove that there exists a Witt decomposition for nondegenerate ε-hermitian forms over a central simple algebra with involution (A, σ), i.e., for any form h, $h \simeq h_1 \perp h_2$ with h_1 hyperbolic and h_2 anisotropic (a notion that we have to define) unique up to isometry.

The notion of isotropy that we are going to use is the following:

Definition 4.1. A nondegenerate ε-hermitian space (V, h) over (A, σ) is said to be isotropic if there exists $x \in V - \{0\}$ such that $h(x, x) = 0$.

Remark 4.2. In the case where A is a division algebra, this notion is exactly the one usually given in the literature.

Now, we recall a crucial notion: Morita-equivalence. The following definition is based upon [4].

Definition 4.3. Let (A, σ) and (B, τ) be two central simple algebras over K such that σ and τ are two K/k-involutions. Let $\delta = 1$ if σ and τ are of the second kind or of the first kind and of the same type and $\delta = -1$ if σ and τ are of the first kind and of different type.
A δ-Morita equivalence $((A, \sigma), (B, \tau), M, N, f, g, \nu)$ between the algebra with involutions (A, σ)

and (B,τ) is a tuple consisting of :
• an (A,B)-bimodule M (i.e., a left A-module and a right B-module with compatible structures);
• a (B,A)-bimodule N;
• two nonzero bimodule homomorphisms $f : M\otimes_B N \to A$ and $g : N\otimes_A M \to B$ which are associative, i.e., $f(m\otimes n).m' = m.g(n\otimes m')$ et $g(n\otimes m).n' = n.f(m\otimes n')$ for all $m,\ m'\in M$, $n,\ n'\in N$;
• a linear bijective map $\nu : M \to N$ which verifies $\nu(amb) = \tau(b)\nu(m)\sigma(a)$ for all $a\in A$, $m\in M$, $b\in B$.

Remark 4.4. Note that we do not suppose that σ and τ are of the same type as in [4]:that is why we call this notion δ-Morita equivalence.

Remark 4.5. In fact one can prove that f (resp. g) is a bimodule isomorphism between $M\otimes_B N$ and A (resp. $N\otimes_A M$ and B), see [4, §1.1].

Now, we suppose that $B = D$ denotes the division algebra Brauer equivalent to A. By Albert's theorem, we know that there exists an involution τ over D such that τ is of the same kind as σ. By [4, §1.4], one can find $M,\ N,\ f,\ g$ and ν as in 4.3 and $\delta\in\{\pm 1\}$ such that $((A,\sigma),(D,\tau),M,N,f,g,\nu)$ is a δ-Morita equivalence. We can define semigroup homomorphisms:

$$F : S^{\varepsilon}(A,\sigma)\to S^{\delta\varepsilon}(D,\tau);\quad (V,h)\mapsto (V\otimes_A M, b_0h)$$
$$G : S^{\delta\varepsilon}(D,\tau)\to S^{\varepsilon}(A,\sigma);\quad (W,\phi)\mapsto (W\otimes_D N, b_0'\phi)$$

where:

$$\begin{array}{rcll} (b_0h)(v\otimes m, v'\otimes m') &=& g(\nu(m)\otimes h(v,v')m') & \forall v,\ v'\in V, m,\ m'\in M \\ (b_0'\phi)(w\otimes n, w'\otimes n') &=& f(\nu^{-1}(n)\otimes \phi(w,w')n') & \forall w,\ w'\in W, n,\ n'\in N. \end{array}$$

In fact, we can prove that F is a semigroup isomorphism and G is its inverse and that they induce isomorphisms of Witt groups. The details of proofs can be found in [7, I.9, 3.5].

Remark 4.6. From now on, when we will use an argument involving Morita theory, we will implicitly refer to [7, I.9].

We have:

Lemma 4.7. (i) *M is a simple left A-module and N is a simple right A-module.* (ii) *The maps F and G respect the rank of hermitian spaces (recall that the rank of a hermitian space (V,h) over (A,σ), where V is a right (resp. left) A-module, is defined to be the positive integer n such that $V\simeq T^n$ where T is a simple right (resp. left) A-module).*

Proof. (i) We prove it for N, the proof for M being similar. As A is a simple algebra, N is a semisimple right A-module and we can write $N \simeq T^n$ where $n \in \mathbb{N} - \{0\}$ and T is a simple right A-module. But, we know from 4.5 that $N \otimes_A M \simeq D$ as (D, D)-bimodules so:

$$D \overset{g^{-1}}{\simeq} N \otimes_A M \simeq (T \otimes_A M)^n$$

as right D-vector spaces. A dimension argument shows that $n = 1$ and N is simple as a right A-module.
(ii) We prove it for F, the proof for G is similar. If (V, h) is a right A-module of rank n then we have $V \simeq T^n$ where T is a simple right A-module; by (i), we can take $T = N$. As D is a division algebra, D is a simple right D-module and we have:

$$V \otimes_A M \simeq (N \otimes_A M)^n \overset{g}{\simeq} D^n.$$

So we deduce that the rank of $V \otimes_A M$ is n. □

Now, we can prove (for fixed A, σ and D as before):

Proposition 4.8. *(V, h) is isotropic over (A, σ) if and only if for every δ-Morita equivalence $((A, \sigma), (D, \tau), M, N, f, g, \nu)$, $F(V, h)$ is isotropic over (D, τ).*

Proof. Let $x \in V - \{0\}$ be such that $h(x, x) = 0$ and $((A, \sigma), (D, \tau), M, N, f, g, \nu)$ be a δ-Morita equivalence. We can easily see that there exists $m \in M$ such that $x \otimes m \neq 0$; in fact if $x \otimes m = 0$ for all $m \in M$ then we have $V_1 \otimes_A M = 0$ where $V_1 = xA \neq 0$. Now N is a simple right A-module by 4.7, so we have $V_1 = \oplus_{i=1}^d N$, $d \geqslant 1$. Therefore we conclude that $0 = V_1 \otimes_A M \simeq \oplus_{i=1}^d (N \otimes_A M) \overset{g}{\simeq} D^d$ which is a contradiction. Now $x \otimes m$ is clearly an isotropic vector for $b_0 h$ so $F(V, h)$ is isotropic. If $F(V, h)$ is isotropic and if $((A, \sigma), (D, \tau), M, N, f, g, \nu)$ is a δ-Morita equivalence, let $y \neq 0$ be an isotropic vector for $b_0 h$. By the same argument as before one can find $n \in N$ such that $y \otimes n \neq 0$. Using the definition of $F^{-1} (= G)$ we see that $y \otimes n$ is an isotropic vector for $b'_0 b_0 h$ for all $n \in N$. But $(V \otimes_A M \otimes_D N, b'_0 b_0 h)$ is isometric to (V, h) so we can conclude the existence of an $x \in V - \{0\}$ such that $h(x, x) = 0$. □

Using this proposition and the fact that the Witt decomposition exists over (D, τ), we conclude the existence of a Witt decomposition over (A, σ). Namely, if (V, h) is an ε-hermitian space over (A, σ), then $F(V, h) \simeq \phi_1 \perp \phi_2$ where ϕ_1 is hyperbolic and ϕ_2 anisotropic over (D, τ). We have :

$$(V, h) \simeq (G \circ F)(V, h) \simeq G(\phi_1) \perp G(\phi_2)$$

By the previous proposition, $G(\phi_1)$ is hyperbolic and we can show that $G(\phi_2)$ is anisotropic. By the same type of argument, we can show that this decomposition is unique up to isometry because it is the case over (D, τ). So:

Corollary 4.9. (i) *There exists a Witt decomposition over (A, σ).*
(ii) *For all $[h] \in W^\varepsilon(A, \sigma)$, there exists an anisotropic form h_0 over (A, σ) such that $[h] = [h_0]$ in $W^\varepsilon(A, \sigma)$.*

Remark 4.10. This result should be well known, but we could not find an explicit statement of it in the literature.

5 Exactness of the Octagon for Central Simple Algebras

Let (A,σ) be a central simple algebra over K ($\operatorname{char} K \neq 2$) with involution. We make the same hypotheses as in 2: namely we suppose that there exist $\lambda,\ \mu \in A^*$ such that $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$, $\mu\lambda = -\lambda\mu$ and that $L = K(\lambda)$ is a quadratic extension of K. We write $\widetilde{A}$ for the commutant of L in A. We have $\mu\widetilde{A} = \widetilde{A}\mu$, $\mu^2 \in \widetilde{A}$, $\sigma(\widetilde{A}) = \widetilde{A}$ and $A = \widetilde{A} \oplus \mu\widetilde{A}$. We define two involutions on $\widetilde{A}$ in this way: let $\sigma_1 = \sigma|_{\widetilde{A}}$ and let $\sigma_2 = \operatorname{Int}(\mu^{-1}) \circ \sigma_1$.

Proof of theorem 1.2: thanks to [2, Appendix 2], we already know that $\operatorname{im} \pi_1^\varepsilon = \ker \rho_1^\varepsilon$ and $\operatorname{im} \rho_1^\varepsilon = \ker \pi_2^{-\varepsilon}$. First, we prove that this sequence is a complex from $\pi_2^{-\varepsilon}$ up to ρ_3^ε.

$\rho_2^\varepsilon \pi_2^{-\varepsilon} = 0$: let (V,h) be a $(-\varepsilon)$-hermitian space over (A,σ), so $\pi_2^{-\varepsilon}(h)$ is an ε-hermitian form over $(\widetilde{A},\sigma_2)$ and $\rho_2^\varepsilon\pi_2^{-\varepsilon}(h)$ will be a $(-\varepsilon)$-hermitian form over (A,σ). It is enough to find a self orthogonal right A-submodule of $V \otimes_{\widetilde{A}} A$ with respect to $\rho_2^\varepsilon\pi_2^{-\varepsilon}(h)$. Let

$$W = \{x.\mu \otimes 1 + x \otimes \mu |\ x \in V\}. \tag{10}$$

Now, W is readily seen to be a right A-submodule of $V_{\widetilde{A}} \otimes_{\widetilde{A}} A$ and an easy calculation shows that this space is a totally isotropic subspace of $\rho_2^\varepsilon\pi_2^{-\varepsilon}(h)$. By dimension count over K, we have $W = W^\perp$ (with respect to $\rho_2^\varepsilon\pi_2^{-\varepsilon}(h)$) and so $\rho_2^\varepsilon\pi_2^{-\varepsilon}(h)$ is hyperbolic.

$\pi_3^{-\varepsilon}\rho_2^\varepsilon = 0$: let (V,h) be an ε-hermitian space over $(\widetilde{A},\sigma_2)$. Let

$$W' = \{x \otimes 1 |\ x \in V\} \subset V \otimes_{\widetilde{A}} A. \tag{11}$$

Then W' is a $\widetilde{A}$-submodule of $V_A \otimes_{\widetilde{A}} A$ and it is a totally isotropic subspace for $\pi_3^{-\varepsilon}\rho_2^\varepsilon(h)$. By a dimension argument one has $W' = W'^\perp$ and so $\pi_3^{-\varepsilon}\rho_2^\varepsilon(h)$ is hyperbolic.

$\rho_3^\varepsilon\pi_3^{-\varepsilon} = 0$: it is obvious from the definition that $\rho_3^\varepsilon\pi_3^{-\varepsilon} = 0$.

Next we prove that the sequence is exact from $\pi_2^{-\varepsilon}$ up to π_1^ε.

$\ker(\rho_2^\varepsilon) \subset \operatorname{im}(\pi_2^{-\varepsilon})$: let (W,f) be an ε-hermitian form over $(\widetilde{A},\sigma_2)$ such that $\rho_2^\varepsilon(f)$ is hyperbolic. We may assume that f is anisotropic thanks to 4.9. There exists an A-submodule W_1 of $W \otimes_{\widetilde{A}} A$ such that $W_1^\perp = W_1$ (with respect to $\rho_2^\varepsilon(f)$). Let $W \otimes \mu = \{w \otimes \mu | w \in W\}$. Let $w_1 \in W_1 \cap (W \otimes \mu)$ ($w_1 = w \otimes \mu$ with $w_1 \in W_1$ and $w \in W$). As $\rho_2^\varepsilon(f)(w_1,w_1) = 0$, $f(w,w) = 0$ and so $w_1 = w \otimes \mu = 0$ since f is anisotropic. Moreover $\dim_K W_1 = \frac{1}{2}\dim_K W \otimes_{\widetilde{A}} A = \dim_K W \otimes \mu$ so $W \otimes_{\widetilde{A}} A = W_1 \oplus (W \otimes \mu)$ as $\widetilde{A}$-modules. This implies that for all $w \in W$, there exists $w' \in W$ such that $w \otimes 1 + w' \otimes \mu \in W_1$. Since A is a free $\widetilde{A}$-module, w' is uniquely determined by w and we write $J(w) := w'$. By definition of J, we have $J^2(w) = w\mu^{-2}$ and $J(wa) = J(w)\mu a \mu^{-1}$ for all $w \in W$ and $a \in \widetilde{A}$. As $W_1 = W_1^\perp$, $\rho_2^\varepsilon(f)(x \otimes 1 + J(x) \otimes \mu, y \otimes 1 + J(y) \otimes \mu) = 0$ for all

$x,\ y \in W$ and we obtain the following system:

$$\begin{cases} f(x,y) + \mu f(J(x), J(y))\mu = 0 \\ f(x, J(y))\mu + \mu f(J(x), y) = 0 \end{cases} \tag{12}$$

By means of J, we define an A-module structure over W by

$$w.\mu = J(w)\mu^2$$

for all $w \in W$. We denote by W_J the A-module W equipped with this new action. Let h be the map defined by

$$h(x,y) = \mu f(x, J(y))\mu + \mu f(x,y) \tag{13}$$

for all $x,\ y \in W_J$. By the definition of J and (12), we conclude that (W_J, h) is a $(-\varepsilon)$-hermitian space over (A, σ). Let us show that h is sesquilinear on the left with respect to σ: h is clearly biadditive, so it suffices to show this fact for μ and for elements of $\widetilde{A}$. We have:

$$\begin{aligned} h(x.\mu, y) &= h(J(x)\mu^2, y) \\ &= \mu f(J(x)\mu^2, J(y))\mu + \mu f(J(x)\mu^2, y) \\ &= \mu^3 f(J(x), J(y))\mu + \mu^3 f(J(x), y) \\ &= -\mu^2 f(x,y) - \mu^2 f(x, J(y))\mu \quad \text{(using (12))} \\ &= \sigma(\mu) h(x,y) \end{aligned}$$

Sesquilinearity on the left for elements of $\widetilde{A}$ and linearity on the right are done in the same way. Let us prove that h is $-\varepsilon$-hermitian with respect to σ:

$$\begin{aligned} h(y,x) &= \mu f(y, J(x))\mu + \mu f(y,x) \\ &= \varepsilon\sigma(f(J(x), y))\mu^2 + \varepsilon\sigma(f(x,y))\mu \\ &= -\varepsilon\sigma(\mu^{-1} f(x, J(y))\mu)\mu^2 - \varepsilon\sigma(\mu f(x,y)) \quad \text{(using (12))} \\ &= -\varepsilon\sigma(h(x,y)) \end{aligned}$$

If h is degenerate, by (13), there exists a non zero $x \in W$ such that $f(x,y) = 0$ for all $y \in W$. In particular, this implies that f is isotropic which is a contradiction. Now, (W_J, h) is the antecedent of (W, f) by $\pi_2^{-\varepsilon}$, i.e. $((W_J)_{\widetilde{A}}, \pi_2^{-\varepsilon}(h))$ is isometric to (W, f) (the isometry is given by the identity map). We conclude that $\ker(\rho_2^{\varepsilon}) \subset \mathrm{im}(\pi_2^{-\varepsilon})$.

$\ker(\pi_3^{-\varepsilon}) \subset \mathrm{im}(\rho_2^{\varepsilon})$: let (V, h) be a $(-\varepsilon)$-hermitian space over (A, σ) such that $\pi_3^{-\varepsilon}(h)$ is hyperbolic. We can assume that h is anisotropic thanks to 4.9. There exists an $\widetilde{A}$-submodule W of $V_{\widetilde{A}}$ such that $W^{\perp} = W$ (with respect to $\pi_3^{-\varepsilon}(h)$). This implies that $h(x,y) \in \mu\widetilde{A}$, for all $x,\ y \in W$. We define a map $f : W \times W \to \widetilde{A}$ where:

$$f(x,y) = \mu^{-1}\lambda^{-1} h(x,y) \tag{14}$$

for $x,\ y \in W$. Since h is anisotropic, f is nondegenerate, we can easily see that (W, f) is an ε-hermitian form over $(\widetilde{A}, \sigma_2)$ and $(W \otimes_{\widetilde{A}} A, \rho_2^{\varepsilon}(f))$ is isometric to (V, h) via $\Phi(w \otimes a) = wa$ for

all $w \in W$, $a \in A$. We conclude that $\ker \pi_3^{-\varepsilon} \subset \operatorname{im} \rho_2^{\varepsilon}$.
The exactness of

$$W^{-\varepsilon}(A,\sigma) \xrightarrow{\pi_3^{-\varepsilon}} W^{\varepsilon}(\widetilde{A},\sigma_1) \xrightarrow{\rho_3^{\varepsilon}} W^{\varepsilon}(A,\sigma)$$

follows from theexactness of $W^{\varepsilon}(A,\sigma) \xrightarrow{\pi_1^{\varepsilon}} W^{\varepsilon}(\widetilde{A},\sigma_1) \xrightarrow{\rho_1^{\varepsilon}} W^{-\varepsilon}(A,\sigma)$; in fact we have:

$$\ker(\rho_3^{\varepsilon}) = \lambda \ker(\rho_1^{-\varepsilon}) = \lambda \operatorname{im}(\pi_1^{-\varepsilon}) = \operatorname{im}(\pi_3^{-\varepsilon}).$$

The exactness of

$$W^{\varepsilon}(\widetilde{A},\sigma_1) \xrightarrow{\rho_3^{\varepsilon}} W^{\varepsilon}(A,\sigma) \xrightarrow{\pi_2^{\varepsilon}} W^{-\varepsilon}(\widetilde{A},\sigma_2)$$

follows from the exactness of

$$W^{\varepsilon}(\widetilde{A},\sigma_1) \xrightarrow{\rho_1^{\varepsilon}} W^{-\varepsilon}(A,\sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(\widetilde{A},\sigma_2).$$

The exactness at $W^{-\varepsilon}(\widetilde{A},\sigma_2)$ follows from the exactness at $W^{\varepsilon}(\widetilde{A},\sigma_2)$.
Finally, we have $\ker \pi_1^{\varepsilon} = \lambda^{-1} \ker \pi_3^{\varepsilon} = \lambda^{-1} \operatorname{im} \rho_2^{-\varepsilon} = \operatorname{im} \rho_2^{-\varepsilon}$ which shows the exactness of

$$W^{-\varepsilon}(\widetilde{A},\sigma_2) \xrightarrow{\rho_2^{-\varepsilon}} W^{\varepsilon}(A,\sigma) \xrightarrow{\pi_1^{\varepsilon}} W^{\varepsilon}(\widetilde{A},\sigma_1)$$

thus completing the proof. □

Remark 5.1. The proof of 1.2 is based on the one given in [10] for quaternion algebras. Nevertheless, one can also give a proof of it by an induction argument as in [2, Appendix 2].

We need the following trivial but useful lemma:

Lemma 5.2. *Suppose that B is a central simple algebra over K with an involution τ. Then*
(i) $W(B,\tau) = 0$ *if and only if B is split and τ is of the first kind and of symplectic type.*
(ii) $W^{-1}(B,\tau) = 0$ *if and only if B is split and τ is of the first kind and of orthogonal type.*

Proof. For (i), if B is split and τ is of the first kind and of symplectic type it is clear that $W(B,\tau) = 0$. Conversely let suppose that $W(B,\tau) = 0$. First we show that B is split. In fact let B_0 be the Brauer-equivalent division algebra of B. By Albert's theorem there exists an involution τ_0 on B_0 of the same kind than τ. By Morita theory, $W(B,\tau) \simeq W^{\varepsilon}(B_0,\tau_0)$, where $\varepsilon = 1$ if τ and τ_0 are both of the second kind or of the first kind and of the same type, otherwise $\varepsilon = -1$. So we conclude that $W^{\varepsilon}(B_0,\tau_0) = 0$. If B is not split then the same holds for B_0. But in this case, 3.3 implies that the set of the nondegenerate ε-hermitian forms of dimension 1 is nonempty, in particular $W^{\varepsilon}(B_0,\tau_0) \neq 0$ which is a contradiction. So B is split. Now we prove that τ is of the first kind. In fact if τ is of the second kind then by Morita theory we have $W(B,\tau) \simeq W(K,\tau|_K)$. So we conclude that $W(K,\tau|_K) = 0$ which is impossible. Thus τ is of the first kind. Finally τ is of symplectic type, otherwise by Morita theory we obtain $0 = W(B,\tau) \simeq W(K)$ which is impossible. The proof of (ii) is similar. □

Corollary 5.3. *The map π_1^ε is injective if and only if L is a splitting field of A, σ is of the first kind and either $\varepsilon = 1$ and σ is symplectic, or $\varepsilon = -1$ and σ is orthogonal.*

Proof. First, one can easily see that σ and σ_2 are of the same kind and of different type when they are of the first kind. If L is a splitting field of A, then by [13, Ch. 8, 5.4], $\widetilde{A}$ is split. If $\varepsilon = 1$ and σ is of the first kind and of symplectic type or $\varepsilon = -1$ and σ is of the first kind and of orthogonal type then, by Morita theory, we conclude that $W^{-\varepsilon}(\widetilde{A}, \sigma_2) = 0$. By the exactness of the octagon at $W^\varepsilon(A, \sigma)$, π_1^ε becomes injective.

Conversely if π_1^ε is injective then again by the exactness of the octagon at $W^\varepsilon(A, \sigma)$, $\mathrm{im}\,\rho_2^{-\varepsilon} = 0$. We show that $W^{-\varepsilon}(\widetilde{A}, \sigma_2) = 0$. If $W^{-\varepsilon}(\widetilde{A}, \sigma_2) \neq 0$, and as δ-Morita equivalence preserves the rank by 4.7, we can find a nondegenerate $-\varepsilon$-hermitian form of rank one over $(\widetilde{A}, \sigma_2)$ by considering the division algebra Brauer-equivalent to $\widetilde{A}$. As $\rho_2^{-\varepsilon}$ preserves the rank we conclude that there exists a nondegenerate ε-hermitian form of rank one in the image of $\rho_2^{-\varepsilon}$, which is a contradiction, since the image of $\rho_2^{-\varepsilon}$ in $W^\varepsilon(A, \sigma)$ is zero. So $W^{-\varepsilon}(\widetilde{A}, \sigma_2) = 0$. Now the result follows from 5.2. $\square$

Remark 5.4. If A is a split quaternion algebra and σ is symplectic, then the exact octagon of 1.2 becomes:

$$0 \to W(L, -) \to W(K) \to W(L) \to W(K) \to W(L, -) \to 0$$

where L/K is a quadratic extension with a nontrivial automorphism $- = \sigma|_L$. If A is a quaternion division algebra and σ is symplectic then 1.2 becomes:

$$0 \to W(A, \sigma) \to W(L, \sigma|_L) \to W^{-1}(A, \sigma) \to W(L) \to W^{-1}(A, \sigma) \to$$

$$\to W(L, \sigma|_L) \to W(A, \sigma) \to 0$$

where L is a maximal subfield of A which is stable by the involution. These sequences can be found in [9].

6 Exact octagon of Witt groups of equivariant forms

Let K be a field of characteristic different from 2 and G be a finite group. Let A be a central simple algebra over K with an involution σ. We denote by $A[G]$ the group algebra of G over A. Let $\varepsilon = \pm 1$.

Definition 6.1. We say that an ε-hermitian space (M, h) over (A, σ) is a G-space if:
• G acts on M on the right and for all $g \in G$, the map $M \to M : m \mapsto m.g$ is A-linear on the right,
• we have $h(m.g, n.g) = h(m, n)$ for all $m,\ n \in M$ and for all $g \in G$.
In this case, we will say that h is a G-form. It is obvious that M is a right $A[G]$-module.

We say that two ε-hermitian G-spaces (M,h) and (M',h') are isometric if there exists an isomorphism $\Phi : M \to M'$ of right $A[G]$-modules such that

$$h'(\Phi(m),\Phi(n)) = h(m,n),$$

for all $m,\ n \in M$. If M is a right $A[G]$-module, $M^* = \mathrm{Hom}_A(M,A)$ has a natural structure of right $A[G]$-module: $(f.g)(m) = f(m.g^{-1})$ if $f \in M^*$, $g \in G$ and $m \in M$. A G-space (M,h) over (A,σ) is said to be nondegenerate if $M \to M^* :\ x \mapsto h(x,-)$ is an isomorphism of A-modules. One can define the hyperbolic ε-hermitian G-space associated to such an M by $(M \oplus M^*, \mathbb{h}_M)$, where

$$\mathbb{h}_M(m \oplus f, m' \oplus f') = f(m') + \varepsilon\sigma(f'(m)),$$

for all $m,\ m' \in M$ and $f,\ f' \in M^*$.

Remark 6.2. If char $K \nmid |G|$ then, by Maschke's theorem, the group algebra $A[G]$ is semisimple. Thanks to that, we can show that an ε-hermitian G-space (M,h) is hyperbolic if and only if it is metabolic (i.e. if there exists a right $A[G]$-submodule N of M such that $N = N^{\perp}$ for h): the proof can be adapted from [1, Corollary 1.4]. This fact will not be used here.

Now, one can construct a group (as for ε-hermitian forms) called the Witt group of ε-hermitian G-forms (also called the Witt group of equivariant forms) which will be denoted by $W^{\varepsilon}(G,A,\sigma)$ (i.e., the quotient of the Grothendieck group corresponding to isometry classes of nondegenerate ε-hermitian G-forms by the subgroup generated by metabolic forms). An element of this group is denoted by $[(M,h)]$ where (M,h) is an ε-hermitian G-space over (A,σ).

We can easily see that the maps involved in the octagon 1.2 induce group homomorphisms between the corresponding Witt groups of hermitian G-forms: if W is an $\widetilde{A}[G]$-module then $W \otimes_{\widetilde{A}} A$ is an $A[G]$-module, where G acts on $W \otimes_{\widetilde{A}} A$ by $(w \otimes a).g = w.g \otimes a$ for $w \in W,\ a \in A,\ g \in G$.

The notion of anisotropy for G-forms that we will use is the following (as in [3, p. 29]):

Definition 6.3. An ε-hermitian G-space (M,h) over (A,σ) is said to be anisotropic if for all $A[G]$-submodules N of M, we have $N \cap N^{\perp} = 0$ (for h).

Remark 6.4. Note that this notion of anisotropy coincides with the usual notion of anisotropy (see section 4) when (M,h) is an ε-hermitian space over a simple algebra with involution. But in the case of ε-hermitian G-forms, this notion of anisotropy is weaker than the usual one. For example, let $q : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be the quadratic form defined by $q(x,y) = x^2 + y^2$ and $G = \{1, \theta\}$ where θ is the reflection in the hyperplane orthogonal to $(1,0)$. Then, q is a quadratic G-form which is isotropic as a quadratic form but anisotropic as a G-form in the sense of 6.3.

Now, one can prove a proposition analogous to [3, proposition 2]:

Proposition 6.5. *If $[(M,h)] \in W^{\varepsilon}(G,A,\sigma)$, $[(M,h)] \neq 0$ then we can find an anisotropic ε-hermitian G-space (M',h') over (A,σ) such that $[(M,h)] = [(M',h')]$.*

Proof. The proof goes as in [3, proposition 2]. □

Now, we prove theorem 1.5:

Theorem *We suppose that A satisfies the same hypotheses as at the beginning of section 2. If we replace $W^{\pm\varepsilon}(A,\sigma)$ by $W^{\pm\varepsilon}(G,A,\sigma)$ and $W^{\pm\varepsilon}(\widetilde{A},\sigma_i)$ by $W^{\pm\varepsilon}(G,\widetilde{A},\sigma_i)$ (for $i=1,\ 2$) in theorem 1.2, we obtain an exact octagon of $W(K,\sigma|_K)$-modules.*

Proof. We only have to adapt the proof of 1.2. Let us keep the same notations. The fact that the considered octagon is a complex readily follows from the proof of 1.2 as we can easily verify that there exist right G-modules structures over W and W' (see (10) and (11) for the definition of these spaces).

To show that this sequence is exact, we use proposition 6.5 to exhibit an anisotropic representative (in the sense of definition 6.3) of each form in the considered kernel.

Let us show that $\ker(\rho_2^\varepsilon)\subset \mathrm{im}(\pi_2^{-\varepsilon})$. We only point at the changes to the original proof. So let (W,f) be an anisotropic ε-hermitian G-space (in the sense of 6.3) lying in the kernel of ρ_2^ε. Thanks to 6.2, we know that there exists an $A[G]$-submodule W_1 of $W\otimes_{\widetilde{A}}A$ such that $W_1=W_1^\perp$. To show that $W_1\cap(W\otimes\mu)=0$, we consider the following subspace: $V=\{w\in W|w\otimes\mu\in W_1\}$. It is an $\widetilde{A}$-submodule of W and if $w\in W$, $g\in G$ we have: $w.g\otimes\mu=(w\otimes\mu).g\in W_1$ as W_1 is a right G-module. So, V is an $\widetilde{A}[G]$-submodule of W. Now, if $v,\ v'\in V$ then $\rho_2^\varepsilon(f)(v\otimes\mu,v'\otimes\mu)=0$ and we have $f(v,v')=0$. So $V\subset V^\perp$ and as (W,f) is anisotropic, we deduce that $V=0$ and that $W_1\cap(W\otimes\mu)=0$. Now, $W\otimes_{\widetilde{A}}A=W_1\oplus(W\otimes\mu)$ as $\widetilde{A}[G]$-modules and this implies that for all $w\in W$, there exists $w'\in W$ such that $w\otimes 1+w'\otimes\mu\in W_1$. One can define the map $J:W\to W$ as in the proof of 1.2 by $J(w)=w'$(the map J is well-defined because f is anisotropic). Thanks to the previous uniqueness, we have $J(w.g)=J(w).g$ for all $w\in W$, $g\in G$. Thanks to that, we easily show that W_J is an $A[G]$-module. We define h as in (13). If h is degenerate then there exists $x\neq 0$ in W_J such that $h(x,y)=0$ for all $y\in W_J$. We deduce that $f(x,y)=0$ for all $y\in W$ and this shows that $x\in W\cap W^\perp$ which is a contradiction to the anisotropy of (W,f). We can conclude as in 1.2.

Now let us show that $\ker(\pi_3^{-\varepsilon})\subset\mathrm{im}(\rho_2^\varepsilon)$. Let (V,h) be an anisotropic ε-hermitian G-space lying in the kernel of $\pi_3^{-\varepsilon}$. Then there exists an $\widetilde{A}[G]$-submodule W of $V_{\widetilde{A}}$ such that $W=W^\perp$. As in the proof of 1.2, we define a map f as in (14) and all we have to do is to show that the ε-hermitian form f is nondegenerate. If f is degenerate, let U be the right $A[G]$-module generated by W. Then U is an $A[G]$-submodule of V. Now there exists $x\in W$ such that $h(x,y)=0$ for all $y\in W$ and this implies that $h(x,y)=0$ for all $y\in U$ and we have $x\in U\cap U^\perp$ which is a contradiction to the anisotropy of (V,h) thus completing the proof. □

Remark 6.6. If the group G is trivial, then 1.2 is a special case of 1.5.

7 Order of Witt group

Let L/K be a finite extension. One can ask for the relation between the orders of $W(K)$ and $W(L)$. If L/K is an extension of odd order then by the weak version of Springer's theorem there is a canonical injection $W(K) \hookrightarrow W(L)$ so the finiteness of $W(L)$ implies that of $W(K)$. If L/K is an extension of even degree, then this property fails. However, for a quadratic extension L/K the finiteness of $W(K)$ implies that of $W(L)$; what is easy to see is that by the exact triangle of Elman-Lam (2) one has: $|W(L)| \leqslant |W(K)|^2$. In [9] the defect of this inequality is calculated; in fact, in [9] the following relation has been proved:

$$|W(L)||W(L,-)|^2 = |W(K)|^2 \tag{15}$$

where $-$ is the nontrivial automorphism of L/K. We have the same situation for a quaternion algebra Q with its symplectic involution $-$. In this case the finiteness of $W(K)$ implies that of $W(Q,-)$ by the exact sequence of Jacobson. In fact by [9] we have

$$|W^{\varepsilon}(Q,-)||W^{-\varepsilon}(Q,-)| = |W(K)| \tag{16}$$

More generally, as stated in 1.6, we have:
Corollary *Let A be a K-central simple algebra with an involution σ of the first kind. Then we have $|W^{\varepsilon}(A,\sigma)||W^{-\varepsilon}(A,\sigma)| = |W(K)|$. In particular $W(K)$ is finite if and only if $W^{\varepsilon}(A,\sigma)$ and $W^{-\varepsilon}(A,\sigma)$ are finite.*

Proof. By Merkurjev's theorem A is similar to a multi-quaternion algebra, say $A \sim Q_1 \otimes \ldots \otimes Q_n$. By Morita theory we have $W^{\varepsilon}(A,\sigma) \simeq W^{\varepsilon'}(Q_1 \otimes \ldots \otimes Q_n, \sigma_1 \otimes \ldots \otimes \sigma_n)$ where $\varepsilon' = \varepsilon$ or $\varepsilon' = -\varepsilon$ and σ_i is the canonical involution of Q_i for $1 \leqslant i \leqslant n$. So in order to prove the statement, we can suppose that:

$$(A,\sigma) = (Q_1 \otimes \ldots \otimes Q_n, \sigma_1 \otimes \ldots \otimes \sigma_n).$$

We proceed by induction on n. If $n = 0$, i.e., A is split, the statement becomes $|W^{\varepsilon}(K)||W^{-\varepsilon}(K)| = |W(K)|$ which is true because $\{W^{\varepsilon}(K),\ W^{-\varepsilon}(K)\} = \{W(K), 0\}$. If $n = 1$ the statement is a consequence of (16). Suppose that $n > 1$. Suppose that $Q_n = (a,b)_K$ where $(a,b)_K$ is the quaternion algebra generated by i and j with $i^2 = a$, $j^2 = b$ and $ij = -ji$ where $a,\ b \in K$. Take the split quaternion algebra $Q'_n = (a,1)_K$ generated by i' and j' with $i'^2 = a$, $j'^2 = 1$ and $i'j' = -j'i'$. If we compare the exact octagon of 1.2 for $(A,\sigma) = (Q_1 \otimes \ldots \otimes Q_{n-1} \otimes Q_n, \sigma_1 \otimes \ldots \otimes \sigma_{n-1} \otimes \sigma_n)$ with $\lambda = 1 \otimes \ldots \otimes 1 \otimes i$ and $\mu = 1 \otimes \ldots \otimes 1 \otimes j$ and also for $(A',\sigma') = (Q_1 \otimes \ldots \otimes Q_{n-1} \otimes Q'_n, \sigma_1 \otimes \ldots \otimes \sigma_{n-1} \otimes {\sigma'}_n)$ (${\sigma'}_n$ is the canonical involution of Q'_n) with $\lambda' = 1 \otimes \ldots \otimes 1 \otimes i'$ and $\mu' = 1 \otimes \ldots \otimes 1 \otimes j'$ we deduce that $|W^{\varepsilon}(A,\sigma)||W^{-\varepsilon}(A,\sigma)| = |W^{\varepsilon}(A',\sigma')||W^{-\varepsilon}(A',\sigma')|$. By Morita theory we have $W^{\varepsilon}(A',\sigma') \simeq W^{-\varepsilon}(A'',\sigma'')$ and $W^{-\varepsilon}(A',\sigma') \simeq W^{\varepsilon}(A'',\sigma'')$ where

$$(A'',\sigma'') = (Q_1 \otimes \ldots \otimes Q_{n-1}, \sigma_1 \otimes \ldots \otimes \sigma_{n-1})$$

So by induction hypothesis we obtain $|W^{\varepsilon}(A,\sigma)||W^{-\varepsilon}(A,\sigma)| = |W(K)|$. □

Remark 7.1. If A is a quaternion algebra over K, and σ is the canonical involution of A, then for $K = \mathbb{Q}_p$, both groups $W^{\pm 1}(A, \sigma)$ are finite. For $K = \mathbb{R}$, the group $W^{-1}(A, \sigma)$ is finite and $W^1(A, \sigma)$ is infinite.

Corollary 7.2. *Let A be a quaternion algebra over K with an involution involution σ of the second kind. Then $|W(A, \sigma)| = |W(K, \sigma|_K)|$.*

Proof. By a theorem of Albert [13, Ch. 8, 11.2] $(A, \sigma) = (A_0 \otimes_k K, \sigma_0 \otimes \sigma|_K)$ where k is the fixed field of σ in K, A_0 is quaternion algebra over k and σ_0 is its canonical involution. Suppose that $A_0 = (a, b)_k$ where $(a, b)_k$ is the quaternion algebra generated by i and j with $i^2 = a$, $j^2 = b$ and $ij = -ji$ where $a,\ b \in k$. Take the split quaternion algebra $A_0' = (a, 1)_k$ generated by i' and j' with $i'^2 = a$, $j'^2 = 1$ and $i'j' = -j'i'$. Let σ_0' be the canonical involution of A_0'. If we compare the exact octagon of 1.2 for $(A_0 \otimes_k K, \sigma_0 \otimes \sigma|_K)$ with $\lambda = i \otimes 1$ and $\mu = j \otimes 1$ and for $(A', \sigma') := (A_0' \otimes_k K, \sigma_0' \otimes \sigma|_K)$ with $\lambda' = i' \otimes 1$ and $\mu' = j' \otimes 1$, we deduce that $|W^\varepsilon(A, \sigma)||W^{-\varepsilon}(A, \sigma)| = |W^\varepsilon(A', \sigma')||W^{-\varepsilon}(A', \sigma')|$. As $W^\varepsilon(A, \sigma) \simeq W^{-\varepsilon}(A, \sigma) \simeq W(A, \sigma)$ and $W^\varepsilon(A', \sigma') \simeq W^{-\varepsilon}(A', \sigma') \simeq W(A', \sigma')$ we deduce that $|W(A, \sigma)| = |W(A', \sigma')|$. By Morita theory $W(A', \sigma') \simeq W(K, \sigma|_K)$ because A' is split. This implies the result. $\square$

Corollary 7.3. *Let $A = Q_1 \otimes_K \ldots \otimes_K Q_n$ be a a multi-quaternion algebra over K with the involution $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_n$ where σ_i is an involution of Q_i of the second kind for $i = 1, \ldots, n$. Then $|W(A, \sigma)| = |W(K, \sigma|_K)|$.*

Proof. The argument is similar to 1.6: we use an induction on n and the case $n = 1$ has already been proved in 7.2. $\square$

The previous corollary gives the motivation to ask the following question for which we do not know the answer.

Question 7.4. *Let A be a K-central simple algebra with an involution σ of the second kind. Is it true that $|W(A, \sigma)| = |W(K, \sigma|_K)|$?*

Remark 7.5. Using 1.5 and applying the same type of arguments, we can show that, if A is a central simple algebra with an involution σ of the first kind, then $|W^\varepsilon(G, A, \sigma)||W^{-\varepsilon}(G, A, \sigma)| = |W(G, K)||W^{-1}(G, K)|$. In 7.3 one can replace the Witt groups by equivariant Witt groups.

8 APPENDIX

Table 8.1.

Map		*Definition*
$S^\varepsilon(A,\sigma) \overset{\pi_1^\varepsilon}{\to} S^\varepsilon(\widetilde{A},\sigma_1)$	$(V_A,h)\mapsto(V_{\widetilde{A}},\pi_1^\varepsilon(h))$	$\pi_1^\varepsilon(h)(x,y)=\pi_1(h(x,y))$
$S^\varepsilon(A,\sigma) \overset{\pi_2^\varepsilon}{\to} S^{-\varepsilon}(\widetilde{A},\sigma_2)$	$(V_A,h)\mapsto(V_{\widetilde{A}},\pi_2^\varepsilon(h))$	$\pi_2^\varepsilon(h)(x,y)=\pi_2(h(x,y))$
$S^\varepsilon(A,\sigma) \overset{\pi_3^\varepsilon}{\to} S^{-\varepsilon}(\widetilde{A},\sigma_1)$	$(V_A,h)\mapsto(V_{\widetilde{A}},\pi_3^\varepsilon(h))$	$\pi_3^\varepsilon(h)(x,y)=\lambda\pi_1^\varepsilon(h(x,y))$
$S^\varepsilon(\widetilde{A},\sigma_1) \overset{\rho_1^\varepsilon}{\to} S^{-\varepsilon}(A,\sigma)$	$(W,f)\mapsto(W\otimes_{\widetilde{A}}A,\rho_1^\varepsilon(f))$	$\rho_1^\varepsilon(f)(x\otimes\alpha,y\otimes\beta)=\sigma(\alpha)\lambda f(x,y)\beta$
$S^\varepsilon(\widetilde{A},\sigma_2) \overset{\rho_2^\varepsilon}{\to} S^{-\varepsilon}(A,\sigma)$	$(W,f)\mapsto(W\otimes_{\widetilde{A}}A,\rho_2^\varepsilon(f))$	$\rho_2^\varepsilon(f)(x\otimes\alpha,y\otimes\beta)=\sigma(\alpha)\lambda\mu f(x,y)\beta$
$S^\varepsilon(\widetilde{A},\sigma_1) \overset{\rho_3^\varepsilon}{\to} S^{\varepsilon}(A,\sigma)$	$(W,f)\mapsto(W\otimes_{\widetilde{A}}A,\rho_3^\varepsilon(f))$	$\rho_3^\varepsilon(f)(x\otimes\alpha,y\otimes\beta)=\sigma(\alpha)f(x,y)\beta$

Table 8.2.

Map	*Form*	*Conditions*	*Image*
$S^\varepsilon(D,\tau)\overset{\pi_1^\varepsilon}{\to}S^\varepsilon(\widetilde{D},\tau_1)$	$<\delta_1,\ldots,\delta_n>$; $\delta_i=d_{2i-1}+\mu d_{2i}$, $d_{2i-1},d_{2i}\in\widetilde{D}$	$\tau_1(d_{2i-1})=\varepsilon d_{2i-1}$ $\tau_2(d_{2i})=-\varepsilon d_{2i}$	$\bigoplus_{i=1}^n\begin{pmatrix} d_{2i-1} & \mu d_{2i}\mu \\ -\mu^2 d_{2i} & -\mu d_{2i-1}\mu\end{pmatrix}$
$S^\varepsilon(\widetilde{D},\tau_1)\overset{\rho_1^\varepsilon}{\to}S^{-\varepsilon}(D,\tau)$	$<\gamma_1,\ldots,\gamma_n>$; $\gamma_i\in\widetilde{D}$	$\tau_1(\gamma_i)=\varepsilon\gamma_i$	$<\lambda\gamma_1,\ldots,\lambda\gamma_n>$
$S^{-\varepsilon}(D,\tau)\overset{\pi_2^{-\varepsilon}}{\to}S^\varepsilon(\widetilde{D},\tau_2)$	$<\delta_1,\ldots,\delta_n>$; $\delta_i=d_{2i-1}+\mu d_{2i}$, $d_{2i-1},d_{2i}\in\widetilde{D}$	$\tau_1(d_{2i-1})=-\varepsilon d_{2i-1}$ $\tau_2(d_{2i})=\varepsilon d_{2i}$	$\bigoplus_{i=1}^n\begin{pmatrix} d_{2i} & \mu^{-1}d_{2i-1}\mu \\ -d_{2i-1} & -\mu d_{2i}\mu\end{pmatrix}$
$S^\varepsilon(\widetilde{D},\tau_2)\overset{\rho_2^\varepsilon}{\to}S^{-\varepsilon}(D,\tau)$	$<\gamma_1,\ldots,\gamma_n>$; $\gamma_i\in\widetilde{D}$	$\tau_2(\gamma_i)=\varepsilon\gamma_i$	$<\lambda\mu\gamma_1,\ldots,\lambda\mu\gamma_n>$
$S^{-\varepsilon}(D,\tau)\overset{\pi_3^{-\varepsilon}}{\to}S^\varepsilon(\widetilde{D},\tau_1)$	$<\delta_1,\ldots,\delta_n>$; $\delta_i=d_{2i-1}+\mu d_{2i}$, $d_{2i-1},d_{2i}\in\widetilde{D}$	$\tau_1(d_{2i-1})=-\varepsilon d_{2i-1}$ $\tau_2(d_{2i})=\varepsilon d_{2i}$	$\bigoplus_{i=1}^n\begin{pmatrix} \lambda d_{2i-1} & \lambda\mu d_{2i}\mu \\ -\lambda\mu^2 d_{2i} & -\lambda\mu d_{2i-1}\mu\end{pmatrix}$
$S^\varepsilon(\widetilde{D},\tau_1)\overset{\rho_3^\varepsilon}{\to}S^{\varepsilon}(D,\tau)$	$<\gamma_1,\ldots,\gamma_n>$; $\gamma_i\in\widetilde{D}$	$\tau_1(\gamma_i)=\varepsilon\gamma_i$	$<\gamma_1,\ldots,\gamma_n>$

Table 8.3. [1]

$\pi_1^\varepsilon(h)$ *is hyperbolic* $\Rightarrow h\simeq<\mu\gamma_1,\ldots,\mu\gamma_n>$	$\tau_2(\gamma_i)=-\varepsilon\gamma_i$	$S^\varepsilon(D,\tau)\overset{\pi_1^\varepsilon}{\to}S^\varepsilon(\widetilde{D},\tau_1)$
$\rho_1^\varepsilon(f)$ *is hyperbolic* $\Rightarrow f\simeq\bigoplus_{i=1}^n\begin{pmatrix} d_{2i-1} & \mu d_{2i}\mu \\ -\mu^2 d_{2i} & -\mu d_{2i-1}\mu\end{pmatrix}$	$\tau_1(d_{2i-1})=\varepsilon d_{2i-1}$ $\tau_2(d_{2i})=-\varepsilon d_{2i}$	$S^\varepsilon(\widetilde{D},\tau_1)\overset{\rho_1^\varepsilon}{\to}S^{-\varepsilon}(D,\tau)$
$\pi_2^{-\varepsilon}(h)$ *is hyperbolic* $\Rightarrow h\simeq<\gamma_1,\ldots,\gamma_n>$	$\tau_1(\gamma_i)=-\varepsilon\gamma_i$	$S^{-\varepsilon}(D,\tau)\overset{\pi_2^{-\varepsilon}}{\to}S^\varepsilon(\widetilde{D},\tau_2)$
$\rho_2^\varepsilon(f)$ *is hyperbolic* $\Rightarrow f\simeq\bigoplus_{i=1}^n\begin{pmatrix} d_{2i} & \mu^{-1}d_{2i-1}\mu \\ -d_{2i-1} & -\mu d_{2i}\mu\end{pmatrix}$	$\tau_1(d_{2i-1})=-\varepsilon d_{2i-1}$ $\tau_2(d_{2i})=\varepsilon d_{2i}$	$S^\varepsilon(\widetilde{D},\tau_2)\overset{\rho_2^\varepsilon}{\to}S^{-\varepsilon}(D,\tau)$
$\pi_3^{-\varepsilon}(h)$ *is hyperbolic* $\Rightarrow h\simeq<\mu\gamma_1,\ldots,\mu\gamma_n>$	$\tau_2(\gamma_i)=\varepsilon\gamma_i$	$S^{-\varepsilon}(D,\tau)\overset{\pi_3^{-\varepsilon}}{\to}S^\varepsilon(\widetilde{D},\tau_1)$
$\rho_3^\varepsilon(f)$ *is hyperbolic* $\Rightarrow f\simeq\bigoplus_{i=1}^n\begin{pmatrix} \lambda d_{2i-1} & \lambda\mu d_{2i}\mu \\ -\lambda\mu^2 d_{2i} & -\lambda\mu d_{2i-1}\mu\end{pmatrix}$	$\tau_1(d_{2i-1})=-\varepsilon d_{2i-1}$ $\tau_2(d_{2i})=\varepsilon d_{2i}$	$S^\varepsilon(\widetilde{D},\tau_1)\overset{\rho_3^\varepsilon}{\to}S^{\varepsilon}(D,\tau)$

[1] provided that f and h are anisotropic.

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