RESOLVING G-TORSORS BY ABELIAN BASE EXTENSIONS

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Abstract. Let G be a linear algebraic group defined over a field k. We prove that, under mild assumptions on k and G, there exists a finite k-subgroup S of G such that the natural map $H^1(K, S) \longrightarrow H^1(K, G)$ is surjective for every field extension K/k.

We give two applications of this result in the case where k an algebraically closed field of characteristic zero and K/k is finitely generated. First we show that for every $\alpha \in H^1(K, G)$ there exists an abelian field extension L/K such that $\alpha_L \in H^1(L, G)$ is represented by a G-torsor over a projective variety. In particular, we prove that α_L has trivial point obstruction.

As a second application of our main theorem, we show that a (strong) variant of the algebraic form of Hilbert's 13th problem implies that the maximal abelian extension of K has cohomological dimension ≤ 1. The last assertion, if true, would prove conjectures of Bogomolov and Königsmann, answer a question of Tits and establish an important case of Serre's Conjecture II for the group E_8.

Contents

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1. Introduction

The starting point for this paper is the following theorem, which will be proved in Sections 2 and 3.

Theorem 1.1. *Let G be a linear algebraic group defined over a field k. Assume that one of the following conditions holds:*

(a) $\mathrm{char}(k) = 0$ *and k is algebraically closed, or*

(b) $\mathrm{char}(k) = 0$ *and G is connected,*

(c) G is connected and reductive.

Then there exists a finite k-subgroup S of G, such that the natural map $H^1(K, S) \longrightarrow H^1(K, G)$ is surjective for every field extension K/k.

Here, as usual $H^1(K, G)$ is the Galois cohomology set $H^1(\mathrm{Gal}(\overline{K}/K), G)$; cf. [$\mathrm{Se}_2$]. Recall that this set does not, in general, have a group structure, but has a marked element, corresponding to the trivial (or split) class, which is usually denoted by 1. Given a field extension L/K we will, as usual, denote the image of α under the natural map $H^1(K, G) \longrightarrow H^1(L, G)$ by α_L.

In the course of the proof of Theorem 1.1 we will construct the finite group S explicitly (see the beginning of Section 2); it is an extension of the Weyl group W of G by a finite abelian group. Moreover, if G is split and contains certain roots of unity then S can be chosen to be a constant subgroup of G; see Remark 3.1. We also note that Theorem 1.1(a) has a natural interpretation in the context of invariant theory, extending a result of Galitskii [Ga]; see Section 4. We also note that Theorem 1.1(a) can be deduced from the results of Bogomolov (see [CS, Lemma 7.3]); we are grateful to J-L. Colliot-Thélène for pointing this out to us. For the sake of completeness, we will include a self-contained proof of Theorem 1.1(a) in Sections 2.

Our applications of Theorem 1.1 are motivated by the following question, implicit in the work of Tits [T_2].

Problem 1.2. Let G be a connected algebraic group defined over an algebraically closed field of characteristic zero, K/k be a field extension and $\alpha \in H^1(K, G)$. Is it true that α can always be split by (i) a finite abelian field extension L/K or (ii) by a finite solvable field extension L/K?

Tits [T_2, Théorème 2] showed that Problem 1.2(ii) has an affirmative answer for every almost simple group of any type, other than E_8. (He also showed that for every such G, the solvable field extension L/K can be chosen so that each prime factor of $[L : K]$ is a torsion prime of G.) Note that if Problem 1.2(ii) has an affirmative answer for fields K of cohomological dimension ≤ 2, then we would be able to conclude, using an argument originally due to Chernousov, that $H^1(K, E_8) = \{1\}$, thus proving an important (and currently open) case of Serre's Conjecture II; for details, see [PR, Chapter 6] or [Gi, Théorème 11].

Recall that $\alpha \in H^1(K, G)$ is called *unramified* if for every rank one discrete valuation ring $k \subset R \subset K$, α lies in the image of the natural map $H^1(R, G) \longrightarrow H^1(K, G)$. We will say that $\alpha \in H^1(K, G)$ is *strongly unramified* if it is represented by a torsor over an irreducible complete variety X/k. In other words, $k(X) = K$, and α lies in the image of the natural map $H^1(X, G) \longrightarrow H^1(K, G)$, restricting a torsor over X to the generic point of X. (Note that after birationally modifying X, we may assume it is smooth and projective). A strongly unramified torsor is unramified; for a finite group G the converse is true as well (see Lemma 5.1 and Remark 5.2).

Split torsors are clearly strongly unramified, and it is natural to think of strongly unramified torsors as "close" to being split. Our first application of Theorem 1.1 below may thus be viewed as a "first approximation" to the assertion of Problem 1.2.

Theorem 1.3. *Let k be an algebraically closed field of characteristic zero, G/k be a linear algebraic group, K/k be a finitely generated field extension, and $\alpha \in H^1(K, G)$. Then there exists a finite abelian extension L/K, such that α_L is strongly unramified.*

Note that the group G in Theorem 1.3 is not assumed to be connected; in particular, the case where G is finite (Proposition 6.1) is key to our proof. On the other hand, in the case where G is connected, Theorem 1.3 does not imply an affirmative answer to Problem 1.2. Indeed, while it is natural to think of α_L as "close to split", it may be not be literally split, even in the case where G is connected and simply connected. To illustrate this point, we will use a theorem of Gabber [CG] to construct a smooth projective 3-fold X/k and a non-trivial class $\alpha \in H^1(k(X), G_2)$ such that α is strongly unramified; see Proposition 7.1. (Here G_2 denotes the (split) exceptional group of type G_2 defined over k.)

Another natural obstruction to $\alpha \in H^1(K, G)$ being split, is the so-called *fixed point obstruction*; see Section 8. We will show that if α is strongly unramified then it has trivial fixed point obstruction; see Proposition 8.1. Combining this result with Proposition 7.1 yields the following:

Corollary 1.4. *Let k be an algebraically closed field of characteristic zero, G/k be a linear algebraic group, K/k be a finitely generated field extension, and $\alpha \in H^1(K, G)$. Then there exists a finite abelian extension L/K, such that α_L has trivial fixed point obstruction.*

Our second application of Theorem 1.1 relies on the following (strong) variant of the algebraic form of Hilbert's 13th problem; see [Di] or [BR_2, Section 3].

Problem 1.5. Let k be an algebraically closed field of characteristic zero, S be a finite group and K/k be a field extension. Is it true that for every $\alpha \in H^1(K, S)$ there exists (i) an abelian extension L/K such that $\mathrm{ed}(\alpha_L) \leq 1$? or (ii) a solvable extension L/K such that $\mathrm{ed}(\alpha_L) \leq 1$?

Here $\mathrm{ed}(\alpha_L)$ denotes the essential dimension of α_L, i.e., the minimal value of $d \geq 0$ such that α_L lies in the image of the natural map $H^1(L_0, S) \longrightarrow H^1(L, S)$ for some intermediate field $k \subset L_0 \subset L$ with $\mathrm{trdeg}_k(L_0) = d$; see [BR$_1$], [BR$_2$] or [Re]. In particular, $\mathrm{ed}(\alpha_L) = 0$ if and only if α_L is split.

We do not know whether or not the assertions of Problem 1.5 are true (cf. Remark 9.4). However, using Theorem 1.1 we will show that, if true, they have some remarkable consequences.

Theorem 1.6. *Let k be an algebraically closed field of characteristic zero. and let K/k be a field extension. Denote the maximal abelian and the maximal solvable extensions of K by K_{ab} and K_{sol} respectively.*

(i) If Problem 1.5(i) has an affirmative answer then $\mathrm{cd}(K_{ab}) \leq 1$.

(ii) If Problem 1.5(ii) has an affirmative answer then $\mathrm{cd}(K_{sol}) \leq 1$.

Note that the inequality $\mathrm{cd}(K_{ab}) \leq 1$ is only known in a few cases (e.g., for K = a number field, or K = a p-adic field by class field theory, or for $K = \mathbb{C}((X))((Y))$ by a theorem of Colliot-Thélène, Parimala and Ojanguren [COP, Theorem 2.2]). If it were established, it would immediately imply an affirmative answer to Problem 1.2. Another important consequence would be a conjecture of Bogomolov [Bog, Conjecture 2], which asserts that $\mathrm{cd}\big(K_{ab}^{(p)}\big) \leq 1$, where $K^{(p)}$ is a maximal prime-to-p extension of K. On the other hand, an affirmative answer to Problem 1.5(ii) would imply that $\mathrm{cd}_p\big(K(p)\big) \leq 1$, where p is a prime number and $K(p)$ is the p-closure (i.e the maximal p–solvable extension) of K, thus giving an affirmative answer to a question of J. Königsmann; cf. [Koe, Question 5.3].

2. Proof of Theorem 1.1(a)

We begin with the following observation. Let k be a field of characteristic zero, G/k be a linear algebraic group, and $R_u(G)$ be the unipotent radical of G. Recall that G has a Levi decomposition, $G = R_u(G) \rtimes G_{\mathrm{red}}$, where G_{red} is a reductive subgroup of G, uniquely determined up to conjugacy. As usual, we shall refer to G_{red} as a *Levi subgroup* of G.

Lemma 2.1. *Let $i\colon G_{\mathrm{red}} \hookrightarrow G$ be a Levi subgroup of G. Then for any field extension K/k, the natural map*

$$i_*\colon H^1(K, G_{\mathrm{red}}) \longrightarrow H^1(K, G)$$

is a bijection.

Proof. Let $\pi\colon G \longrightarrow G/R_u(G)$ be the natural projection. By the Levi decomposition, $G_{\mathrm{red}} \overset{i}{\hookrightarrow} G \xrightarrow{\pi} G/R_u(G)$ is an isomorphism between G_{red} and $G/R_u(G)$. Thus

$$H^1(K, G_{\mathrm{red}}) \xrightarrow{i_*} H^1(K, G) \xrightarrow{\pi_*} H^1(K, G/R_u(G))$$

is a bijection between $H^1(K, G_{\mathrm{red}})$ and $H^1(K, G/R_u(G))$. By [Sa, Lemma 1.13], π_* is also a bijection. Hence, so is i_*. $\square$

Remark 2.2. Lemma 2.1 tells us that if the natural map

$$H^1(K, S) \longrightarrow H^1(K, G_{\text{red}})$$

is surjective then so is the natural map

$$H^1(K, S) \longrightarrow H^1(K, G) .$$

In particular, in the course of proving Theorem 1.1(a) and (b) we may replace G by G_{red} and thus assume that G is reductive.

We now proceed with the proof of Theorem 1.1(a). Let k be an algebraically closed field of characteristic zero and G be a linear algebraic group defined over k. As usual, we will identify G with its group of k-points $G(k)$. In view of Remark 2.2, we will assume that G (or equivalently, the connected component G^0 of G) is reductive.

Let T be a maximal torus of G and set $N = N_G(T)$ and $W = N_G(T)/T$. Then W is a finite group and N is an extension of W by T. Let $\mu = {}_nT$ be the group of n-torsion points of T, where $n = |W|$. Consider the exact sequences

$$1 \to T \to N \to W \to 1 \qquad \text{and} \qquad 1 \to \mu \to T \overset{\times n}{\to} T \to 1 .$$

The first sequence yields a class in $H^2(W, T)$. Since $n \cdot H^2(W, T) = 0$, the second sequence tells us that this class comes from $H^2(W, \mu)$. In terms of group extensions, it means that there exists an extension S of W by μ such that N is the push-out of S by the morphism $\mu \hookrightarrow T$. In particular, S is a finite subgroup of N of order $|W|^{\text{rank}(G)+1}$. We will now prove the following variant of Theorem 1.1.

Proposition 2.3. *Assume G is reductive and S is the finite subgroup of G constructed above. Then the map $H^1(K, S) \to H^1(K, G)$ is surjective for any field extension K/k.*

Proof. We claim that the natural map $H^1(K, N) \to H^1(K, G)$ is surjective for every field extension K/k. Indeed, let $\overline{K}$ be an algebraic closure of K. For any $[z] \in H^1(K, G)$ the twisted group ${}_zG^0$ is reductive and has a maximal torus Q. Viewing Q and T as maximal tori in $G^0(\overline{K})$, we see that they are $\overline{K}$-conjugate; the claim now follows from [Se$_2$, Lemma III.2.2.1].

It remains to prove that the map $H^1(K, S) \to H^1(K, N)$ is surjective. We will do this fiberwise, with respect to the map $p_* : H^1(K, N) \to H^1(K, W)$. Let $[a] = p_*([b]) \in H^1(K, W)$. A twisting argument [Se$_2$, I.5.5], shows that the map

$$H^1(K, {}_bT) \to p_*^{-1}([a])$$

is surjective; here ${}_bT$ denotes the torus T, twisted by the cocycle b. On the other hand, we have $q_* : H^1(K, S) \to H^1(K, W)$. Since $H^2(K, \mu) \to H^2(K, T)$ is injective and $\partial([a]) = 0$, where $\partial : H^1(K, W) \to H^2(K, T)$ is the connecting morphism, we conclude that $[a] \in \operatorname{Im} q_*$. It now suffices to prove that the map

$$H^1(K, {}_b\mu) \to H^1(K, {}_bT)$$

is surjective. The cokernel of this map is given by the exact sequence

$$H^1(K, {}_b\mu) \to H^1(K, {}_bT) \overset{\times n}{\to} H^1(K, {}_bT).$$

The torus ${}_bT$ is split by the Galois extension L/K given by $[a] \in H^1(K, W) = \mathrm{Hom}_{ct}(\mathrm{Gal}(\overline{K}/K), W)/\mathrm{Int}(W)$, the degree of this extension divides n. The restriction-corestriction formula $\times n = Cor_k^L \circ Res_k^L$ and the fact that $H^1(L, T) = 0$ (Hilbert's Theorem 90) imply that the map $\times n : H^1(K, {}_bT) \to H^1(K, {}_bT)$ is trivial. We conclude that the map $H^1(K, {}_b\mu) \to H^1(K, {}_bT)$ is surjective. Hence, the map $H^1(K, S) \to H^1(K, N)$ is surjective as well. □

3. Proof of Theorem 1.1(b) and (c)

In view of Remark 2.2 part (b) follows from part (c). The rest of this section will be devoted to proving part (c). We will consider three cases.

Case 1. Let G be a quasi-split adjoint group. We denote by T a maximal quasi-split torus in G, $N = N_G(T)$ and $W = N_G(T)/T$. For every root $\alpha \in \Sigma = \Sigma(G, T)$, where Σ is the root system of G with respect to T, the corresponding subgroup $G_\alpha \leq G$ is isomorphic (over a separable closure of k) to either SL_2 or PSL_2.

Let $T_\alpha = T \cap G_\alpha$ and let $w_\alpha \in N_{G_\alpha}(T_\alpha)$ be a representative of the Weyl group of G_α with respect to the maximal torus T_α given by a matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By Galois' criteria for rationality, the group L generated by all w_α is k-defined. One easily checks that the intersection $L \cap T$ belongs to the 2-torsion subgroup of T; in particular, L is finite.

Let $\mu = {}_nT$ be the n-torsion subgroup of T where n is the cardinality of the Weyl group W. Consider the subgroup S_1 of N generated by L and μ. Now, arguing as in the proof of Proposition 2.3, and using the fact that T and T/μ are permutation tori (and hence both have trivial Galois cohomology in dimension 1), one checks that the canonical map $H^1(K, S_1) \to H^1(K, N)$ is surjective for every extension K/k. In the course of the proof of Proposition 2.3 we showed that $H^1(K, N) \to H^1(K, G)$ is surjective. Then the composite $H^1(K, S_1) \to H^1(K, N) \to H^1(K, G)$ is surjective as well.

Case 2. Let G be an adjoint k-group. Denote by G_0 the quasi-split adjoint group of the same inner type as G. One knows (see [T$_1$]) that $G = {}_a(G_0)$ is the twisted form of G_0 for an appropriate cocycle $a \in Z^1(k, G_0)$. If S_1 is the subgroup of G_0 constructed in Case 1, we may assume without loss of generality that a takes values in S_1. Let $S_2 = {}_aS_1$ and consider the diagram

$$\begin{array}{ccc} H^1(K, S_1) & \xrightarrow{\pi_1} & H^1(K, G_0) \\ \uparrow f_S & & \uparrow f_G \\ H^1(K, S_2) & \xrightarrow{\pi_2} & H^1(K, G)\,. \end{array}$$

Here f_S and f_G are natural bijections. Since π_1 is surjective, so is π_2.

Case 3. Let G be a connected reductive k-group. It is an almost direct product of the semisimple k-group $H = [G, G]$ and the central k-torus C of G. Let Z be the center of H. Clearly, we have $C \cap H \leq Z$. Consider the group $G_3 = G/Z$ and a natural morphism $f : G \to G_3$. By our construction, G_3 is the direct product of the torus $C/C \cap H$ and the adjoint group $H_3 = H/Z$.

Let S_3 be the subgroup constructed in Case 2 for H_3 and let $\mu = {}_n(C/C \cap H)$ be the n-torsion subgroup of the torus $C/C \cap H$, where n is the index of the minimal extension of k splitting C. Then for any extension K/k a natural morphism $H^1(K, \mu \times S_3) \to H^1(K, G_3)$ is surjective. We claim that $S = f^{-1}(\mu \times S_3)$ is as required, i.e. $H^1(K, S) \to H^1(K, G)$ is surjective.

Indeed, the exact sequences $1 \to Z \to G \to G_3 \to 1$ and $1 \to Z \to S \to S_3 \to 1$ give rise to a commutative diagram

$$\begin{array}{ccccccc}
H^1(K,Z) & \to & H^1(K,G) & \xrightarrow{g_1} & H^1(K,G_3) & \xrightarrow{g_2} & H^2(K,Z) \\
\uparrow & & \uparrow f_1 & & \uparrow f_2 & & \uparrow id \\
H^1(K,Z) & \to & H^1(K,S) & \xrightarrow{h_1} & H^1(K,\mu \times S_3) & \xrightarrow{h_2} & H^2(K,Z)
\end{array}$$

Here g_2, h_2 are the corresponding connected homomorphisms. Let $[a] \in H^1(K, G)$ and $[b] = g_1([a])$. Since f_2 is surjective, there is a class $[c] \in H^1(\mu \times S_3)$ such that $f_2([c]) = [b]$. Since $h_2([c]) = g_2 f_2([c]) = 0$, there is $[d] \in H^1(K, S)$ such that $h_1([d]) = [c]$. Thus two classes $[a]$ and $f_1([d])$ have the same image in $H^1(K, G_3)$. By a twisting argument, one gets a surjective map $H^1(K, {}_dZ) \to g^{-1}(g_1([a]))$. Since $Z \subset S$ and hence ${}_dZ \subset {}_dS$, we have $[a] \in \operatorname{Im} f_1$. This completes the proof of Theorem 1.1. $\square$

Remark 3.1. Our argument shows that if G is split and k contains certain roots of unity, then the subgroup S in parts (b) and (c) can be taken to be a constant group.

More precisely, in part (c), k needs to have a primitive root of unity of degree $n = |W(G_{\rm ss})| \cdot |Z(G_{\rm ss})|$, where $W(G_{\rm ss})$ and $Z(G_{\rm ss})$ denote, respectively, the Weyl group and the center of the semisimple part $G_{\rm ss}$ of G.

The same is true in part (b), except that G needs to be replaced by $G_{red} = G/R_u(G)$ in the above definition of n.

4. An invariant-theoretic interpretation

For the rest of this paper k will be an algebraically closed field of characteristic zero, K will be a finitely generated extension of k and G will be a linear algebraic group defined over k. In this section we will introduce some terminology in this context and discuss an invariant-theoretic interpretation of Theorem 1.1(a).

Recall that every element of $H^1(K, G)$ is uniquely represented by a primitive generically free *G-variety* V, up to birational isomorphism. That is, $k(V)^G = K$, the rational quotient map $\pi \colon V \dashrightarrow V/G$ is a torsor over the generic point of V/G, and this torsor is α; see [Po, 1.3]. (Here "V is primitive" means that G transitively permutes the irreducible components of V. In particular, if G is connected then V is irreducible.)

If S is a closed subgroup of G and $\alpha \in H^1(K,S)$ is represented by a generically free S-variety V_0, then the image of α in $H^1(K,G)$ is represented by the G-variety $G *_S V_0$, which is, by definition, the rational quotient of $G \times V_0$ for the S-action given by $s\colon (g,v_0) \mapsto (gs^{-1}, s{\cdot}v_0)$. We shall denote the image of (g,v_0) in this quotient by $[g,v_0]$. Note that a rational quotient is, a priori, only defined up to birational isomorphism; however, a regular model for $G *_S V_0$ can be chosen so that the G-action on $G \times V_0$ (by translations on the first factor) descends to a regular G-action on $G *_S V_0$, making the rational quotient map $G \times V_0 \dashrightarrow G *_S V_0$ G-equivariant (via $g' \cdot [g,v_0] \mapsto [g'g,v_0]$); see [Re, 2.12]. If S is a finite group and V_0 is a quasi-projective S-variety (which will be the case in the sequel) then we may take $G *_S V_0$ to be the geometric quotient for the S-action on $G \times V_0$, as in [PV, Section 4.8].

Now let V be a G-variety. An S-invariant subvariety $V_0 \subset V$ is called a (G,S)-section if

(a) $G \cdot V_0$ is dense in V and

(b) V_0 has a dense open S-invariant subvariety U such that for $g \cdot u \in V_0$ for some $u \in U$ implies $g \in S$.

The above definition is due to Katsylo [Ka]; sometimes a (G,S)-section is also called a *standard relative section* (see [Po, 1.7.6]) or *a relative section* with normalizer S (see [PV, Section 2.8]). A G-variety V is birationally isomorphic to $G *_S V_0$ for some S-variety V_0 if and only if V has a (G,S)-section; see [PV, Section 2.8]. In this context Theorem 1.1(a) can be rephrased as follows:

Theorem 1.1$'$: *Every generically free G-variety has a (G,S)-section, where S is a finite subgroup of G.*

Recall that a subvariety V_0 of a generically free G-variety V is called a *Galois quasisection* if the rational quotient map $\pi\colon V_0 \dashrightarrow V/G$ restricts to a dominant map $V_0 \dashrightarrow V/G$, and the induced field extension $k(V_0)/k(V)^G$ is Galois. If V_0 is a Galois quasisection then the finite group $\Gamma(V_0) := \mathrm{Gal}(k(V_0)/k(V)^G)$ is called the Galois group of V_0; see [Ga] or [Po, (1.1.1)]. (Note $\Gamma(V_0)$ is not required to be related to G in any way.) The following theorem is due to Galitskii [Ga]; cf. also [Po, (1.6.2) and (1.17.6)].

Theorem 4.1. *If G is connected then every generically free G-variety has a Galois quasisection.*

A (G,S)-section is clearly a Galois quasisection with Galois group S. Hence, Theorem 1.1$'$ (or equivalently, Theorem 1.1(a)) may be viewed as an extension of Theorem 4.1. Note that the Galois group $\Gamma(V_0)$ of the Galois quasisection V_0 constructed in the proof of Theorem 4.1 is isomorphic to a subgroup of the Weyl group $W(G)$; cf. [Po, Remark 1.6.3]. On the other hand, the group S in our proof of Theorem 1.1(a), is an extension of $W(G)$ by a finite abelian group. Enlarging the finite group S may thus be viewed as “the price to be paid” for a section with better properties.

5. Preliminaries on G-covers

Let G be a finite group. We shall call a finite morphism $\pi\colon X' \longrightarrow X$ of algebraic varieties a *G-cover*, if X is irreducible, G acts on X', so that π maps every G-orbit in X' to a single point in X, and π is a G-torsor over a dense open subset U of X. We will express the last condition by saying that π is *unramified* over U. Restricting π to the generic point of X, we obtain a torsor $\alpha \in H^1(k(X), G)$ over Spec $k(X)$. In this situation we shall say that π represents α. If a cover $\pi\colon X' \longrightarrow X$ is unramified over all of X, then we will simply say that π is *unramified*.

Recall from the Introduction, that we call $\alpha \in H^1(K, G)$ *unramified* if it lies in the image of $H^1(R, G) \longrightarrow H^1(K, G)$ for every discrete valuation ring $k \subset R \subset K$ and *strongly unramified*, if it is represented by an unramified G-cover $\pi\colon X' \longrightarrow X$ over a projective variety X.

Lemma 5.1. *Let G be a finite group, K be a finitely generated extension of an algebraically closed base field k of characteristic zero, and $\alpha \in H^1(K, G)$. Then the following assertions are equivalent:*

(a) α is represented by a projective G-variety V (in the sense of Section 4), such that every element $1 \neq g \in G$ acts on V without fixed points,

(b) α is strongly unramified, and

(c) α is unramified.

Note that condition (b) can be rephrased by saying that α has trivial fixed point obstruction; see Section 8.

Proof. *(a)* $\Rightarrow$ *(b)*: The G-action on V has a geometric quotient $\pi\colon V \longrightarrow X$, where X is a projective variety; cf., e.g., [PV, Section 4.6]. We claim that π is a torsor over X. Indeed, we can cover V by G-invariant affine open subsets V_i. The quotient variety X is then covered by affine open subsets $X_i = \pi(V_i)$, moreover, $\pi_i = \pi_{|U_i}\colon U_i \longrightarrow \pi(U_i)$ is the geometric quotient for the G-action on U_i; see [PV, Theorem 4.16]. It is thus enough to show that $\pi_i\colon V_i \longrightarrow X_i$ is a torsor for each i; this is an immediate corollary of the Luna Slice Theorem; see, e.g., [PV, Theorem 6.1].

(b) $\Rightarrow$ (c): Suppose α is represented by a G-torsor $V \longrightarrow X$, where X is a projective variety with $k(X) = K$. We want to prove that for any discrete valuation ring $R \subset K$ the class α belongs to the image $H^1(R, G) \to H^1(K, G)$.

Indeed, the ring R dominates a point in X; denote this point by D. Consider the canonical map Spec $R \to X$ sending the closed point in Spec R to D and the generic point of Spec R into the generic point of X. Take the fiber product $(\mathrm{Spec}\ R) \times_X V$. It follows immediately from the construction that the G-torsor

$$(\mathrm{Spec}\ R) \times_X V \to \mathrm{Spec}\ R$$

is as required, i.e. its image under the map $H^1(R,G) \to H^1(K,G)$ is α.

(c) $\Rightarrow$ (a): Let V be a smooth projective G-variety representing α and let $\pi\colon V \longrightarrow X$ be the geometric quotient. Note that X is normal. We want to show that every $1 \neq g \in G$ acts on V without fixed points. Assume the contrary: $gv = v$ for some $v \in V$. By [RY$_2$, Theorem 9.3] (with $s = 1$ and $H_1 = <g>$), after performing a sequence of blowups with smooth G-invariant centers on V, we may assume that the fixed point locus V^g of g contains a divisor $D \subset V$. If $R = \mathcal{O}_{X,\pi(D)}$ is the local ring of the divisor $\pi(D)$ in X then α does not lie in the image of the natural morphism $H^1(R,G) \longrightarrow H^1(K,G)$, a contradiction. $\square$

Remark 5.2. Our proof of the implication (b) $\Rightarrow$ (c) does not use the fact that G is a finite group. This implication is valid for every linear algebraic group G.

6. Proof of Theorem 1.3

Let S be the finite subgroup of G given by Theorem 1.1(i). Then $\alpha \in H^1(K,G)$ is the image of some $\beta \in H^1(K,S)$. Examining the diagram

$$\begin{array}{ccc} H^1(X,S) & \longrightarrow & \beta_L \in H^1(L,S) \\ \downarrow & & \downarrow \qquad \downarrow \\ H^1(X,G) & \longrightarrow & \alpha_L \in H^1(L,G)\,, \end{array}$$

where X is a complete variety and $L = k(X)$, we see that if Theorem 1.3 holds for S then it holds for G.

From now on we may assume that G is a finite group. In this case Theorem 1.3 can be restated as follows.

Proposition 6.1. *Let G be a finite group, k be an algebraically closed base field of characteristic zero, K/k be a finitely generated extension, and $\alpha \in H^1(K,G)$. Then there exists an abelian field extension L/K such that α_L is represented by an unramified G-cover $\pi\colon Z' \longrightarrow Z$, where Z and Z' are projective varieties.*

The rest of this section will be devoted to proving Proposition 6.1. We begin with the following lemma.

Lemma 6.2. *Let G be a finite group. Then every $\alpha \in H^1(K,G)$ is represented by a G-cover $\pi\colon X' \longrightarrow X$ such that*

(a) X' is normal and projective,

(b) X is smooth and projective,

(c) there exists a normal crossing divisor D on X such that π is unramified over $X - D$.

Proof. Suppose α is represented by a G-Galois algebra K'/K. We may assume without loss of generality that K' is a field. Indeed, otherwise α is the image of some $\alpha_0 \in H^1(K, G_0)$, where G_0 is a proper subgroup of G, and we can replace G by G_0 and α by α_0.

Choose a smooth projective model Y/k for K/k and let $\phi\colon Y' \longrightarrow Y$ be the normalization of Y in K'. Then Y' is projective (see [Mu, Theorem III.8.4, p. 280]), and by uniqueness of normalization (see [Mu, Theorem III.8.3, pp. 277 - 278]), G acts on Y' by regular morphisms, so that $k(Y')$ is isomorphic to K' as a G-field (see [Mu, pp. 277 - 278]). We have thus shown that α can be represented by a cover $\phi\colon Y' \longrightarrow Y$ satisfying conditions (a) and (b). We will now birationally modify this cover to obtain another cover $\pi\colon X' \longrightarrow X$ which satisfies condition (c) as well.

The cover ϕ is unramified over a dense open subset of Y; denote this subset by U. Set $E = Y - U$, and resolve E to a normal crossing divisor D via a birational morphism $\gamma\colon X \longrightarrow Y$. Now consider the diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow \pi & & \downarrow \phi \\ X & \xrightarrow{\gamma} & Y\,, \end{array}$$

where X' is the normalization of X in K'. By our construction X is smooth and X' is normal. Moreover, since γ is an isomorphism over U, π is unramified over $X - D = \phi^{-1}(U)$, as desired. □

We are now ready to complete the proof of of Proposition 6.1. Our argument will be based on [GM, Theorem 2.3.2], otherwise known as "Abhyankar's Lemma", which describes the local structure of a covering, satisfying conditions (a) - (c) of Lemma 6.2, in the etale topology. We thank K. Karu for bringing this result to our attention.

Let $\pi\colon X' \longrightarrow X$ be a G-cover of projective varieties representing α and satisfying conditions (a) - (c) of Lemma 6.2. Denote the irreducible components of D by $D_1, \dots, D_s$.

Since X is smooth, each $x \in X$ has an affine open neighborhood U_x where each D_j is principal, i.e., is given by $\{a_{x,j} = 0\}$ for some $a_{x,j} \in \mathcal{O}_X(U_x)$ (possibly $a_{x,j} = 1$ for some x and j). By quasi-compactness, finitely many of these open subsets, say, $U_{x_1}, \dots, U_{x_n}$ cover X. To simplify our notation, we set $U_i = U_{x_i}$ and $a_{ij} = a_{x_i,j}$.

Now let b_{ij} be an $|G|$th root of a_{ij} in the algebraic closure of $K = k(X)$ and $L = K(b_{ij})$, where i ranges from 1 to n and j ranges from 1 to s. Suppose $\gamma\colon Z \longrightarrow X$ is the normalization of X in L and $Z' = X' \times_X Z$. Since we are assuming that k is algebraically closed of characteristic zero (and in particular, k contains a primitive $|G|$th root of unity), L/K is an abelian extension. It is also easy to see from our construction that Z and Z' are projective, Z is normal, and the natural projection $\pi'\colon Z' \longrightarrow Z$ is a G-cover, which represents $\alpha_L \in H^1(L.G)$. To sum up, we have constructed the following diagram of morphisms:

$$\begin{array}{ccc} Z' & \to & X' \\ \downarrow\ \psi & & \downarrow\ \pi \\ Z & \xrightarrow{\gamma} & X\,. \end{array}$$

It remains to show that the G-cover ψ is unramified. Suppose we want to show that ψ is unramified at $z_0 \in Z$. Since the open sets $U_1, \ldots, U_n$ cover X, $x_0 = \gamma(z_0)$ lies in U_i for some $i = 1, \ldots, n$. By Abhyankar's lemma [GM, Theorem 2.3.2], there exists an abelian subgroup $H \simeq \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_s\mathbb{Z}$ of G (possibly with $n_j = 1$ for some j) and a (Kummer) H-Galois cover

$$V_j = \{(x, t_1, \ldots, t_s) \mid t_1^{n_1} = a_{i,1}\,,\ \ldots,\ t_s^{n_s} = a_{i,s}\} \subset U_j \times \mathbb{A}^s\,,$$

such that the G-covers $\pi\colon X' \longrightarrow X$ and $\phi\colon V_j *_H G \longrightarrow U_j$ are isomorphic over an etale neighborhood of x_0 in X. (Here the natural projection $V_j \longrightarrow U_j$ is an H-cover, and $V_j *_H G \longrightarrow U_j$ is the G-cover induced from it; for a definition of $C_j *_H G$, see Section 4.)

Now recall that by our construction the elements $b_{ij} \in L = k(Z)$ satisfy $b_{ij}^{|G|} = a_{ij} \in \mathcal{O}_X(U_{ij})$. In particular, they are integral over U_j and thus they are regular function on $\gamma^{-1}(U_j)$. Since n_j divides $|G|$ for every $j = 1, \ldots, s$, the pull-back of ϕ to Z splits over an etale neighborhood of z_0; hence, so does $\psi =$ pull-back of π. In other words, ψ is unramified at z_0, as claimed. This completes the proof of Proposition 6.1. $\square$

7. An example

Proposition 7.1. *Let k be an algebraically closed base field of characteristic zero such that* $\mathrm{trdeg}_{\mathbb{Q}}(k) \geq 3$. *(Note that the last condition is satisfied by every uncountable field.) Then there exist a smooth projective 3-fold X/k with function field $K = k(X)$ and a non-trivial class $\alpha \in H^1(K, G_2)$ such that α is strongly unramified.*

Note that no such examples can exist if X is a curve or a surface, since in this case $H^1(k(X), G_2) = \{1\}$; see [BP]. Our proof parallels a similar construction for $G = \mathrm{PGL}_n$ by Colliot-Thélène and Gabber [CG]; see Remark 7.2 below.

Proof. Let E_1, E_2, E_3 be elliptic curves. For $i = 1, 2, 3$ choose $p_i, q_i \in E_i$ so that $p_i \ominus q_i$ is a point of order 2. (Here $\ominus$ denotes denotes subtraction with respect to the group operation on E_i.) Then $2p_i - 2q_i$ is a principal divisor on E_i and $p_i - q_i$ is not; see, e.g., [Si, Corollary 3.5]. Thus $2p_i - 2q_i = \mathrm{div}(f_i)$, where $f_i \neq 0$ is a rational function on E_i, which is not a complete square. Adjoining $\sqrt{f_i}$ to $k(E_i)$, we obtain an irreducible unramified $\mathbb{Z}/2\mathbb{Z}$-cover $\pi_i\colon E_i' \longrightarrow E_i$. (Note that by the Hurwitz formula, E_i' is also an elliptic curve.)

Now set $X = E_1 \times E_2 \times E_3$ and $K = k(X)$, $S = (\mathbb{Z}/2\mathbb{Z})^3$, and consider the element $\beta \in H^1(k(X), S)$, represented by the S-cover

$$\pi = (\pi_1, \pi_2, \pi_3) \colon E_1' \times E_2' \times E_3' \longrightarrow E_1 \times E_2 \times E_3 = X\,.$$

Since π is an unramified cover, β is strongly unramified.

We now recall that the exceptional group G_2/k contains a unique (up to conjugacy) maximal elementary abelian 2-groups $i \colon S = (\mathbb{Z}/2\mathbb{Z})^3 \hookrightarrow G_2$. Set $\alpha = i_*(\beta) \in H^1(K, G_2)$. Since β is strongly unramified, so is α. It thus remains to show that $\alpha \neq 1$ in $H^1(K, G_2)$ (for a suitable choice of E_i and E_i').

The cohomology set $H^1(K, G_2)$ classifies octonion algebras or equivalently, 3-fold Pfister forms; cf. [Se$_3$, Theorem 9]. By [GMS, §22.10], the map

$$H^1(K, S) = \Big(K^\times/(K^\times)^2\Big)^3 \xrightarrow{i_*} H^1(K, G_2)$$

is non-trivial; hence, it sends $(a_1, a_2, a_3) \in \Big(K^\times/(K^\times)^2\Big)^3$ to the class of the 3-Pfister form $\langle\langle a_1, a_2, a_3\rangle\rangle$; see [GMS, Theorem 27.15]. By our construction, $\beta \in H^1(K, S)$ corresponds to $(f_1, f_2, f_3) \in \Big(K^\times/(K^\times)^2\Big)^3$. Thus $\alpha = i_*(\beta)$ is non-split in $H^1(K, G_2)$ if and only if the 3-fold Pfister form $\langle\langle f_1, f_2, f_3\rangle\rangle$ is nonsplit or, equivalently, if $(f_1) \cup (f_2) \cup (f_3) \neq 0$ in $H^3(k(X), \mathbb{Z}/2\mathbb{Z})$; see [EL, Corollary 3.3].

Since we are assuming that $\mathrm{trdeg}_{\mathbb{Q}}(k) \geq 3$, we can choose elliptic curves E_1, E_2 and E_3 so that their j-invariants are algebraically independent over $\mathbb{Q}$. We now appeal to a theorem of Gabber ([CG, p. 144]), which says that $(f_1) \cup (f_2) \cup (f_3) \neq 0$ in $H^3(k(X), \mathbb{Z}/2\mathbb{Z})$. Hence, $\alpha \neq 1$ in $H^1(K, G)$, as claimed. This completes the proof of Proposition 7.1. □

Remark 7.2. A similar construction yields the following (cf. [CG, Proposition 11]):

Let k be an algebraically closed base field of characteristic zero such that $\mathrm{trdeg}_{\mathbb{Q}}(k) \geq 2$ *and $n \geq 2$. Then there exist a smooth projective surface X/k with function field $K = k(X)$ and a non-trivial class $\alpha \in H^1(K, \mathrm{PGL}_n)$ such that α is strongly unramified.*

Here $X = E_1 \times E_2$ is a product of two elliptic curves whose j-invariants are algebraically independent over k, p_i and $q_i \in E_i$ are chosen so that $p_i \ominus q_i$ is a point of order n on E_i, $S = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, and $i \colon S \hookrightarrow \mathrm{PGL}_n$ is given by

$$i(1,0) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \zeta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \zeta^{n-1} \end{pmatrix} \quad \text{and} \quad i(0,1) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $\zeta \in k$ is a primitive nth root of unity. The rest of the argument is unchanged. □

8. The fixed point obstruction

We now recall the notion of *fixed point obstruction* from [RY_3, Introduction]. Suppose $\alpha \in H^1(K, G)$ is represented by a generically free primitive G-variety V (as in Section 4). We shall say that a subgroup of G is *toral* if it lies in a subtorus of G and *non-toral* otherwise. If V (or any G-variety birationally isomorphic to it) has a smooth point fixed by a non-toral diagonalizable subgroup $H \subset G$, then we shall say that V (or equivalently, α) has *non-trivial fixed point obstruction*; cf. [RY_3, Introduction]. Note that after birationally modifying V, we may assume that V is smooth and complete (or even projective, see, e.g., [RY_2, Proposition 2.2]), and that the fixed point obstruction can be detected on any such model. In other words, if V and V' are smooth complete birationally isomorphic G-varieties then $V^H = \emptyset$ if and only if $(V')^H = \emptyset$ for any diagonalizable subgroup $H \subset G$; see [RY_1, Proposition A2]. If $V^H = \emptyset$ for every diagonalizable non-toral subgroup $H \subset G$ (and V is smooth and complete), then we will say that V, or equivalently α, has *trivial fixed point obstruction*.

If α is split (i.e., $\alpha = 1$ in $H^1(K, G)$) then by [RY_2, Lemma 4.3] α has trivial fixed point obstruction. We will now extend this result as follows.

Proposition 8.1. *If $\alpha \in H^1(K, G)$ is strongly unramified then α has trivial fixed point obstruction.*

Proof. Let $\overline{G}$ be a smooth projective $G \times G$-variety, which contains G as a dense open orbit. (Here we are viewing G as a $G \times G$-variety with respect to left and right multiplication). To construct $\overline{G}$, we use a theorem of Kambayashi, which says that G can be $G \times G$-equivariantly embedded into $\mathbb{P}(V)$ for some linear representation $G \times G \longrightarrow \mathrm{GL}(V)$; see [PV, Theorem 1.7]. Taking the closure of G in $\mathbb{P}(V)$, and $G \times G$-equivariantly resolving its singularities, we obtain $\overline{G}$ with desired properties.

For $\overline{g} \in \overline{G}$, we will write $g_1 \cdot \overline{g} \cdot g_2^{-1}$ instead of $(g_1, g_2) \cdot \overline{g}$; the reason for this notation is that for $\overline{g} \in G$, $(g_1, g_2) \cdot \overline{g} = g_1 \overline{g} g_2^{-1} \in G$.

Since α is strongly unramified, it can be represented by a G-torsor $\pi \colon Z \longrightarrow X$ over a smooth projective irreducible variety X. (Here $K = k(X)$.) We will now construct a smooth complete G-variety $\overline{Z}$ representing α (i.e., birationally isomorphic to Z) by "enlarging" each fiber of π from G to $\overline{G}$.

Let $U_i \to X$, $i \in I$ be an etale covering which trivializes π. Then π is described by the transition maps $f_{ij} \colon U_{ij} \times G \longrightarrow U_{ij} \times G$ on the pairwise "overlaps" U_{ij}; here each f_{ij} is an automorphism of the trivial G-torsor $U_{ij} \times G$ on U_{ij}. (Here G acts trivially on U_{ij} and by left translations on itself.) These transition maps satisfy a cocycle condition (for Cech cohomology) which expresses the fact that they are compatible on triple "overlaps" U_{hij}. It is easy to see that f_{ij} is given by the formula

$$f_{ij}(u, g) = (u, g \cdot h_{ij}(u)), \tag{1}$$

for some morphism $h_{ij} \colon U_{ij} \longrightarrow G$. (In fact, $h_{ij}(u) = \mathrm{pr}_2 \circ f_{ij}(u, 1_G)$, where $\mathrm{pr}_2 \colon U_{ij} \times G \longrightarrow G$ is the projection to the second factor.) Formula (1) can now be used to extend f_{ij} to a G-equivariant automorphism

$$\overline{f_{ij}}\colon U_{ij} \times \overline{G} \longrightarrow U_{ij} \times \overline{G}\,,$$

where G acts on $\overline{G}$ on the left. Since f_{ij} satisfies the cocycle condition and G is dense in $\overline{G}$, we conclude that $\overline{f_{ij}}$ satisfy the cocycle condition as well. By descent theory, the transition maps $\overline{f_{ij}}$ patch together to yield a variety $\overline{Z}$ and a commutative diagram of morphisms

$$\begin{tikzcd} Z \arrow[rr, hook] \arrow[dr, "\pi"'] & & \overline{Z} \arrow[dl, "\overline{\pi}"] \\ & X & \end{tikzcd}$$

which locally (in the etale topology) looks like

$$\begin{tikzcd} U_i \times G \arrow[rr, hook] \arrow[dr, "\pi"'] & & U_i \times \overline{G} \arrow[dl, "\overline{\pi}"] \\ & U_i & \end{tikzcd}$$

(The maps π and $\overline{\pi}$ in the second diagram are projections to the first component.) It is now easy to see that $\overline{Z}$ is smooth and proper over X and $Z \hookrightarrow \overline{Z}$ is a G-equivariant open embedding. Indeed, these properties can be checked locally (in the etale topology) on X, where they are immediate from the second diagram. Note also that since $\overline{Z}$ is proper over X, and X is projective over k, $\overline{Z}$ is complete as a k-variety.

Having constructed a smooth complete model $\overline{Z}$ for α, we are now ready to show that α has trivial fixed point obstruction. Indeed, suppose a diagonalizable subgroup H of G has a fixed point in $z \in \overline{Z}$. We want to show that H is toral in G. Indeed, let F be the fiber of $\overline{\pi}$ containing z. By our construction $F \simeq \overline{G}$ as G-varieties (here $\overline{G}$ is viewed as a G-variety with respect to the left G-action). We conclude that H has a fixed point in $\overline{G}$. Since $\overline{G}$ has G as a G-invariant dense open subset, it is split as a G-variety (i.e., it represents the trivial class in $H^1(k, G)$), [RY$_2$, Lemma 4.3] now tells us that H is toral. This shows that α has trivial fixed point obstruction, thus completing the proof of Proposition 8.1. □

Remark 8.2. The fact that G acts on $\overline{G}$ both on the right and on the left was crucial in the construction of $\overline{Z}$ in the above proof. The action on the right was used to glue the transition maps $\overline{f_{i,j}}$ together, and the action on the left to define a G-action on $\overline{Z}$. If G could only act on $\overline{G}$ on one side, we would still be able to construct $\overline{Z}$ as a variety; however, we would no longer be able to define a G-action on it, extending the G-action on Z.

Remark 8.3. Non-split $\alpha \in H^1(K, \mathrm{PGL}_p)$ with trivial fixed point obstruction were constructed in [RY$_3$] for all odd primes p. It was not previously known whether or not such α could exists in $H^1(K, G)$, if G is connected and simply connected; Propositions 7.1 and 8.1 provide such an example with $G = G_2$. (Remark 7.2 provides further examples with $G = \mathrm{PGL}_n$.) Note however, that the elements of $H^1(K, \mathrm{PGL}_p)$ constructed in [RY$_3$] are

are not strictly unramified (or even unramified), because there K is a purely transcendental extension of k; see [RY$_3$, Theorem 4].

9. Proof of Theorem 1.6

Recall that a field F has cohomological dimension ≤ 1 if and only if the Brauer group $\mathrm{Br}(F')$ is trivial for any separable finite field extension F'/F; see [Se$_2$, Proposition II.3.5]. It will be convenient for us to work with étale K-algebras, rather than just separable field extension of K. Recall that a K-étale algebra is a finite product $E = K_1 \times K_2 \times \cdots \times K_n$ of finite separable extensions K_i/K. The Brauer group of E is $\mathrm{Br}(E) = \oplus_i \mathrm{Br}(K_i)$; an element of this group is represented by an n-tuple $\mathcal{A} = (\mathcal{A}_i/K_i)_{i=1,..,n}$ of central simple algebras. Note that $\mathcal{A}$ is an Azumaya algebra over E. Given a field F, we have

$$\text{(2)} \qquad \mathrm{cd}(F) \leq 1 \iff \mathrm{Br}(E) = 0 \text{ for any étale algebra } E/F;$$

see [Se$_2$, Proof of Theorem III.2.2.1] or [FJ, Lemma 10.11].

Lemma 9.1. *The following are equivalent:*

(a) $cd(K_{ab}) \leq 1$,

(b) For any étale algebra E/K, the restriction map $\mathrm{Br}(E) \longrightarrow \mathrm{Br}(E \otimes_K K_{ab})$ is trivial.

Moreover, the lemma remains true if K_{ab} is replaced by K_{sol}.

Proof. (a) $\Rightarrow$ (b): immediate from (2).

(b) $\Rightarrow$ (a): Let B/K_{ab} be an étale algebra. There exists a finite abelian subextension K'/K of K_{ab}/K and an étale algebra B'/K' such that $B' \otimes_{K'} K_{ab} = B$. We have

$$B = \varinjlim_{K' \subset L \subset K_{ab}} B' \otimes_{K'} L,$$

where the limit is taken on subfields L of K_{ab} finite over K'. Consequently,

$$\mathrm{Br}(B) = \varinjlim_{K' \subset L \subset K_{ab}} \mathrm{Br}(B' \otimes_{K'} L),$$

and (b) implies that $\mathrm{Br}(B) = 0$. (a) now follows from (2).

The proof remains unchanged if K_{ab} is replaced by K_{sol}. □

We now turn to the proof of Theorem 1.6(i). We start with the group $G = (\mathrm{PGL}_n)^m \rtimes \mathrm{S}_m$. By Theorem 1.1(a), there exists a finite subgroup S of G such that the natural homomorphism $H^1(K, S) \longrightarrow H^1(K, G)$ is surjective The group S_m is the automorphism group of the trivial étale algebra, so by Galois descent the set $H^1(K, \mathrm{S}_m)$ classifies m-dimensional étale algebras. By [Se$_2$, Corollary I.5.4.2], the fiber of the map $H^1(K, G) \longrightarrow H^1(K, \mathrm{S}_m)$ at $[E] \in H^1(k, \mathrm{S}_m)$ is

$$H^1\Big(K, {}_E\left(\mathrm{PGL}_n{}^m\right)\Big)/{}_E(\mathrm{S}_m)\,,$$

with ${}_E(\mathrm{PGL}_n{}^m)$ and ${}_E(\mathrm{S}_m)$ are the twisted groups by the étale algebra E/K. Since $G \to S_m$ has a section, the map ${}_EG(K) \to {}_E(S_m)(K)$ is surjective. Then ${}_E(S_m)$ acts trivially on $H^1(K, {}_E(\mathrm{PGL}_n{}^m))$ and hence the fiber at $[E]$ is $H^1(K, {}_E(\mathrm{PGL}_n{}^m))$. By definition of the Weil restriction, we have ${}_E(\mathrm{PGL}_n^m) = R_{E/k}(\mathrm{PGL}_n)$. We identify $H^1\Big(K,_E(\mathrm{PGL}_n^m)\Big) = H^1(E, \mathrm{PGL}_n)$ by the Shapiro isomorphism. Thus

$$H^1(K, G) = \bigsqcup_{[E] \in H^1(K, \mathrm{S}_m)} H^1(E, PGL_n)\,.$$

An element of $H^1(K, G)$ is then given by an Azumaya algebra $\mathcal{A}/E$ of degree n defined over a K-étale algebra E of rank m. By Theorem 1.1(a), every class $[\mathcal{A}/E]$ comes from a class $\alpha \in H^1(K, S)$.

We now apply the assertion of Problem 1.5(i) to the group S and the class α. There exists an abelian extension L/K, a k-curve C and a map $k(C) \subset L$ such that the restriction of the class α in $H^1(L, S)$ belongs to the image of $H^1(k(C), S) \longrightarrow H^1(L, S)$. The commutative diagram of restriction maps

$$\begin{array}{ccccc} H^1(K,S) & \longrightarrow & H^1(L,S) & \longleftarrow & H^1(k(C),S) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(K,G) & \longrightarrow & H^1(L,G) & \longleftarrow & H^1(k(C),G) \end{array}$$

shows that there exists an étale algebra $E'/k(C)$ and an Azumaya algebra $\mathcal{A}'/E'$ such that

$$E \otimes_K L \xrightarrow{\sim} E' \otimes_{k(C)} L, \quad \text{and} \quad \mathcal{A}' \otimes_{E'} (E' \otimes_{k(C)} L) \xrightarrow{\sim} \big(\mathcal{A} \otimes_E (E \otimes_K L)\big).$$

Since $\mathrm{cd}(k(C)) \leq 1$ (see [Se$_2$, §II.3]), $\mathcal{A}'/A$ is the split Azumaya algebra of rank n. We conclude that $\mathcal{A} \otimes_E (E \otimes_K L)/(E \otimes_K L)$ is the split Azumaya algebra of rank n. This shows that the map $\mathrm{Br}(E) \to \mathrm{Br}(E \otimes K_{ab})$ is trivial for any étale algebra E/K. Lemma 9.1 now tells us that $\mathrm{cd}(K_{ab}) \leq 1$. This concludes the proof of Theorem 1.6(i).

The proof of part (ii) is exactly the same, except that the field extension L/K, constructed at the beginning of previous paragraph, is now solvable, rather than abelian. □

Remark 9.2. A similar argument shows that the conjecture of Bogomolov stated at the end of the Introduction (see also [Bog, Conjecture 2]), is a consequence of the following weaker form of Problem 1.5(i) (which is also open):

Problem 1.5′: *Let k be an algebraically closed field of characteristic zero, S be a finite group, K/k be a field extension and p be a prime integer. Is it true that for every $\alpha \in H^1(K, S)$ there exists a finite extension $[K' : K]$ of degree prime to p and an abelian extension L/K' such that* $\mathrm{ed}(\alpha_L) \leq 1$*?*

Remark 9.3. The group G in Problem 1.5 can be replaced by the symmetric group S_n. In other words, if the assertion of Problem 1.5(i) or (ii) holds for $G = \mathrm{S}_n$, for every $n \geq 1$ then the same assertion holds for every finite group G.

Remark 9.4. The third author would like to take this opportunity to correct a misstatement he made in [BR_1, Introduction]. The identity $d'(6) = 2$, which is attributed to Abhyankar [A] at the bottom of p. 161 in [BR_1], would, if true, give a negative answer to Problem 1.5(ii) for the symmetric group $G = \mathrm{S}_6$. In fact, the version of Hilbert's 13th problem considered in [A] is quite different from ours; the base extensions that are allowed there are integral ring extensions, rather than field extensions. For this reason the identity $d'(6) = 2$ does not follow from the results of [A] and, to the best of our knowledge, Problem 1.5 is still open.

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