Division Algebras Over Rational Function Fields in One Variable

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Abstract. Let A be a central simple algebra over the field of rational functions in one variable over an arbitrary field of characteristic different from 2. If the Schur index of A is not divisible by the characteristic and its ramification locus has degree at most 3, then A is Brauer-equivalent to the tensor product of a quaternion algebra and a constant central division algebra D. The index of A is computed in terms of D and the ramification of A. The result is used to construct various examples of division algebras over rational function fields.

Introduction

The Brauer group of the field of rational fractions in one variable over an arbitrary field F of characteristic zero has been described by D.K. Faddeev in [**5**]. To recall his result, we use the following notation: for any closed point p of the projective line $\mathbb{P}^1_F$, let F_p denote the residue field at p. Let $X(F_p)$ be the character group of the absolute Galois group

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Γ_p of F_p, i.e. the group of continuous homomorphisms

$$X(F_p) = \mathrm{Hom}(\Gamma_p, \mathbb{Q}/\mathbb{Z}).$$

For each closed point p on $\mathbb{P}^1_F$, there is a map $\partial_p\colon \operatorname{Br} F(t) \to X(F_p)$ known as the *ramification* (or *residue*) map at p (see [**6**, p. 18]). Faddeev proved that the following sequence is exact:

$$0 \to \operatorname{Br} F \to \operatorname{Br} F(t) \xrightarrow{\oplus\partial_p} \bigoplus_{p\in\mathbb{P}^{1(1)}_F} X(F_p) \xrightarrow{\sum \mathrm{cor}} X(F) \to 0, \tag{0.1}$$

where $\sum$cor is the sum of the corestriction (or norm) maps $X(F_p) \to X(F)$ and $\mathbb{P}^{1(1)}_F$ is the set of closed points on $\mathbb{P}^1_F$. See also [**6**, Example 9.21, p. 26], where a version in nonzero characteristic is given. If the characteristic of F is p, Faddeev's exact sequence still holds if the Brauer groups and the character groups are replaced by their subgroups of prime-to-p torsion.

Our aim is to obtain information on the index of Brauer classes in $\operatorname{Br} F(t)$ from their image under the ramification map $\partial = \oplus\partial_p$. We achieve this goal in some very specific cases. To describe them, we use the following terminology:

Definition. Let

$$\mathfrak{R} = \operatorname{Ker}\Big(\sum \mathrm{cor}\colon \bigoplus_{p\in\mathbb{P}^{1(1)}_F} X(F_p) \to X(F)\Big).$$

The elements in $\mathfrak{R}$ are called *ramification sequences*. For $\rho = (\chi_p) \in \mathfrak{R}$, the *support* of ρ is

$$\mathrm{supp}(\rho) = \{p \in \mathbb{P}^{1(1)}_F \mid \chi_p \neq 0\},$$

and χ_p is the *component* of ρ at p. Viewing $\mathrm{supp}(\rho)$ as a divisor on $\mathbb{P}^1_F$, we may consider the *degree* of the support of ρ:

$$\deg \mathrm{supp}(\rho) = \sum_{p\in\mathrm{supp}(\rho)} \deg p.$$

If p is a rational point (i.e., $F_p = F$), then $\mathrm{cor}\colon X(F_p) \to X(F)$ is the identity map. Therefore, the support of a ramification sequence cannot consist of a single rational point,

$$\deg \mathrm{supp}(\rho) \geq 2 \qquad \text{for all } \rho \in \mathfrak{R}.$$

In this work, we consider only 2-torsion ramification sequences (and may therefore assume only $\operatorname{char} F \neq 2$). Their set is denoted by ${}_2\mathfrak{R}$. For

$\rho \in {}_2\mathfrak{R}$ with $\deg \operatorname{supp}(\rho) = 2$, we determine in Corollaries 4.2 and 4.3 the quaternion $F(t)$-algebras Q such that $\partial(Q) = \rho$. If $\alpha \in \operatorname{Br} F(t)$ has torsion prime to $\operatorname{char} F$ and the same ramification sequence ρ, then by Faddev's exact sequence (0.1) there is a central simple F-algebra A and a quaternion $F(t)$-algebra Q such that α is the Brauer class of $A \otimes_F Q$,

$$\alpha = [A \otimes_F Q] \in \operatorname{Br} F(t).$$

The index of α is determined in terms of A and ρ in Theorem 4.1.

In Section 5, we obtain similar results for $\rho \in {}_2\mathfrak{R}$ with $\deg \operatorname{supp}(\rho) = 3$. In this case, we show in Corollary 5.2 that there is up to isomorphism a unique quaternion $F(t)$-algebra Q_ρ such that

$$\partial(Q_\rho) = \rho.$$

For any central simple F-algebra A, the index of $A \otimes_F Q_\rho$ is computed in Theorem 5.1 in terms of A and a quartic field extension E_ρ/F canonically associated with ρ.

The existence of quaternion $F(t)$-algebras with given 2-torsion ramification sequence ρ with support of degree at most 3 is also shown in a recent paper of Kunyavskiĭ, Rowen, Tikhonov and Yanchevskiĭ [**9**, Corollary 2.10], except in the case where the support consists of a single point of degree 3. Ramification sequences of quaternion algebras are further discussed in [**9**], which also gives (in Corollary 2.14) an example due to Faddeev of a ramification sequence $\rho \in {}_2\mathfrak{R}$ with $\deg \operatorname{supp}(\rho) = 4$ which is not the ramification sequence of a quaternion $F(t)$-algebra.

The main ingredients in the proofs of our main results are a very general observation on cochains with values in a left principal ideal domain (Section 1) and a reduction of 2-torsion ramification sequences ρ with $\deg \operatorname{supp}(\rho) \leq 3$ to a normal form, which is achieved in Section 3. Although the restriction to 2-torsion elements in $\mathfrak{R}$ is not necessary for some of the reduction steps in Section 3, an example given in an appendix to this paper shows that even for 3-torsion ramification sequences the general principles of Section 1 lead to conditions which are much more difficult to handle.

1. Cochains in the ring of fractions of a left PID

This section presents the basic tool that will be used in subsequent sections to prove that certain tensor products are division algebras. As it is very general, we cast it in the setting of left Principal Ideal Domains (PID) although we apply it only in the case of polynomial rings in one indeterminate over division rings.

Let R be a left PID, with Ore ring of fractions Q,

$$Q = \{b^{-1}a \mid a, b \in R, b \neq 0\}.$$

Let G be a finite group of automorphisms of R, that we extend to Q.

LEMMA. *Suppose $(c_\sigma)_{\sigma\in G}$ is a family of elements in $Q^\times$ such that*

$$c_{\sigma\tau}^{-1}\sigma(c_\tau)c_\sigma \in R \qquad \text{for all } \sigma,\ \tau \in G.$$

There exists $q \in Q^\times$ such that

$$\sigma(q)c_\sigma q^{-1} \in R \qquad \text{for all } \sigma \in G.$$

PROOF. From the definition of Q, it follows that for each $\sigma \in G$ the left ideal

$$I_\sigma = \{g \in R \mid \sigma(g)c_\sigma \in R\}$$

is nonzero. As R is left Ore (as it is a left PID) and G is finite, the intersection

$$I = \bigcap_{\sigma\in G} I_\sigma = \{g \in R \mid \sigma(g)c_\sigma \in R \text{ for all } \sigma \in G\}$$

is a nonzero left ideal. Let $q \in R$ be such that

$$I = Rq,$$

hence $q \neq 0$. For all $\tau \in G$ we have $\tau(q)c_\tau \in R$ since $q \in I$. Moreover, for $\sigma \in G$,

$$\sigma\big(\tau(q)c_\tau\big)c_\sigma = \sigma\tau(q)\sigma(c_\tau)c_\sigma = \big(\sigma\tau(q)c_{\sigma\tau}\big)\big(c_{\sigma\tau}^{-1}\sigma(c_\tau)c_\sigma\big).$$

The first factor on the right side lies in R because $q \in I$, and the second factor lies in R by hypothesis. Therefore,

$$\sigma\big(\tau(q)c_\tau\big)c_\sigma \in R \qquad \text{for all } \sigma,\ \tau \in G,$$

hence

$$\tau(q)c_\tau \in I \qquad \text{for all } \tau \in G,$$

and therefore $\tau(q)c_\tau \in Rq$ for all $\tau \in G$. $\square$

The only application of this lemma in the present paper is the following: Let E be a (finite-dimensional) central division algebra over an arbitrary field K, and let x be an indeterminate over K. The ring $E[x] = E \otimes_K K[x]$ is a left (and right) PID with Ore ring of fractions $E(x) = E \otimes_K K(x)$. Let α be an automorphism of finite order n of $E[x]$, and let $g \in K[x]$. Consider the algebra

$$\Delta(E(x), \alpha, g) = E(x) \oplus E(x)y \oplus \cdots \oplus E(x)y^{n-1} \tag{1.1}$$

where multiplication is defined by

$$y^n = g \qquad \text{and} \qquad yf = \alpha(f)y \text{ for } f \in E(x).$$

This algebra may be viewed as the factor ring of the skew-polynomial ring $E(x)[y;\alpha]$ by the ideal generated by $y^n - g$.

THEOREM 1.1. *Suppose n is a prime and α does not restrict to the identity on $K(x)$. The algebra $\Delta(E(x),\alpha,g)$ is a division algebra if and only if there is no $f \in E[x]$ such that*

$$\alpha^{n-1}(f)\ldots\alpha(f)f = g. \tag{1.2}$$

PROOF. By [**7**, Theorem 1.3.16, p. 17] (see also [**1**, Theorem 11.12, p. 184]), the algebra $\Delta(E(x),\alpha,g)$ is not division if and only if there exists $f \in E(x)$ for which (1.2) holds. Therefore, it suffices to show that if (1.2) holds for some $f \in E(x)$, then it also holds for some $f \in E[x]$.

Suppose $f \in E(x)^\times$ satisfies (1.2), and let

$$c_{\alpha^i} = \alpha^{i-1}(f)\ldots\alpha(f)f \qquad \text{for } i = 0, \ldots, n-1.$$

Then, for i, $j = 0, \ldots, n-1$,

$$c_{\alpha^{i+j}}^{-1}\alpha^i(c_{\alpha^j})c_{\alpha^i} = \begin{cases} 1 & \text{if } i+j < n, \\ g & \text{if } i+j \geq n. \end{cases}$$

We may therefore apply the lemma to find $q \in E(x)^\times$ such that

$$\alpha(q)c_\alpha q^{-1} \in E[x].$$

Let $f' = \alpha(q)c_\alpha q^{-1} = \alpha(q)fq^{-1}$. We have

$$\alpha^{n-1}(f')\ldots\alpha(f')f' = qgq^{-1} = g,$$

hence $f' \in E[x]$ satisfies (1.2). $\square$

We consider two types of examples, where $\Delta(E(x),\alpha,g)$ turns out to be the tensor product of a "constant" division algebra and a quaternion algebra. One more example is discussed in the Appendix.

1.1. Quaternion algebras with a constant slot. Let D be a central division algebra over a field F of characteristic different from 2, and let $a \in F^\times$ be such that D does not contain a square root of a. Then $E = D \otimes_F F(\sqrt{a})$ is a central division algebra over $K = F(\sqrt{a})$, and the nontrivial automorphism of K/F extends to an automorphism α of E which is the identity on D. Let t be an indeterminate over F and $g \in F[t]$. The algebra $\Delta(E(t),\alpha,g)$ of (1.1) is a tensor product:

$$\Delta(E(t),\alpha,g) = D \otimes_F (a,g)_{F(t)}.$$

Theorem 1.1 shows that this tensor product has zero divisors if and only if there exists $f \in E[t]$ such that

$$\alpha(f)f = g.$$

Since the degree of the left side is even, the following result is clear:

COROLLARY 1.2. *Let D be a central division F-algebra which does not contain a square root of a. For every $g \in F[t]$ of odd degree, the tensor product $D \otimes_F (a, g)_{F(t)}$ is a division algebra.*

We now consider a case where $\deg g = 2$.

PROPOSITION 1.3. *Let D be a central division F-algebra which does not contain a square root of a, and let $b \in F^\times$. The tensor product $D \otimes_F (a, t^2 - b)_{F(t)}$ is not a division algebra if and only if D contains an element s such that $s^2 = ab$.*

PROOF. As observed in the beginning of this subsection, the tensor product $D \otimes_F (a, t^2 - b)_{F(t)}$ is not a division algebra if and only if there exists $f \in E[t]$ such that

$$\alpha(f)f = t^2 - b. \tag{1.3}$$

Comparing degrees, we see that if f exists it must be of the form

$$f = ut + v \qquad \text{for some } u,\, v \in E.$$

Substituting in (1.3), we obtain

$$\alpha(u)u = 1, \qquad \alpha(u)v + \alpha(v)u = 0, \qquad \alpha(v)v = -b. \tag{1.4}$$

Letting $x = \sqrt{a}\alpha(u)v$, computation shows that the existence of u, $v \in E$ satisfying (1.4) is equivalent to the existence of u, $x \in E$ satisfying

$$\alpha(u)u = 1, \qquad \alpha(x) = uxu^{-1}, \qquad x^2 = ab. \tag{1.5}$$

If D contains s such that $s^2 = ab$, then (1.5) holds with $u = 1$ and $x = s$.

Conversely, suppose (1.5) holds for some u, $x \in E$. If $u = 1$, then $\alpha(x) = x$ hence $x \in D$ and we may set $s = x$. If $u \neq 1$, then letting $w = u - 1 \in E^\times$ we have

$$\alpha(w)u = 1 - u = -w.$$

For $s = wxw^{-1}$, it follows that

$$\alpha(s) = \alpha(w)\alpha(x)\alpha(w)^{-1} = \alpha(w)uxu^{-1}\alpha(w)^{-1} = wxw^{-1} = s,$$

so $s \in D$, and $s^2 = ab$. $\square$

1.2. Quaternion algebras with an indeterminate slot. Let D be a central division algebra over a field F of characteristic different from 2 and let t be an indeterminate over F. For $g \in F[t]$, we may consider the tensor product $D \otimes_F (t, g)_{F(t)}$ as a special case of the construction in (1.1):

$$D \otimes_F (t, g)_{F(t)} = \Delta\bigl(D(x), \alpha, g(x^2)\bigr)$$

where $x^2 = t$ and α is the automorphism of $D(x)$ defined by

$$\alpha(x) = -x \qquad \text{and} \qquad \alpha(d) = d \text{ for } d \in D.$$

Therefore, Theorem 1.1 shows that this tensor product is not a division algebra if and only if there exists $f \in D[x]$ such that

$$\alpha(f)f = g(x^2). \tag{1.6}$$

We just consider one example where $\deg g = 3$.

Proposition 1.4. *Let D be a central division F-algebra and a, b, $c \in F$. The tensor product*

$$D \otimes_F (t, a^2 - bt + ct^2 - t^3)_{F(t)}$$

is not a division algebra if and only if D contains an element s such that

$$s^4 - 2cs^2 - 8as + (c^2 - 4b) = 0. \tag{1.7}$$

Proof. Let $g = a^2 - bt + ct^2 - t^3 \in F[t]$. If $s \in D$ satisfies (1.7), then the polynomial

$$f = a + \tfrac{1}{2}(s^2 - c)x + sx^2 + x^3$$

satisfies $\alpha(f)f = g(x^2)$, hence, by the observations at the beginning of this subsection, the tensor product $D \otimes (t, g)_{F(t)}$ is not a division algebra.

Conversely, if $D \otimes (t, g)_{F(t)}$ is not a division algebra, then (1.6) holds for some $f \in D[x]$, which is necessarily of the form

$$f = rx^3 + sx^2 + ux + v \qquad \text{for some } r,\ s,\ u,\ v \in D.$$

Comparing the leading coefficients and the constant terms in (1.6) yields

$$r^2 = 1 \qquad \text{and} \qquad v^2 = a^2.$$

Changing the sign of f and/or changing f into $\alpha(f)$ if necessary, we may assume $r = 1$ and $v = a$. Expanding (1.6) then yields the following relations between s and u:

$$s^2 - 2u = c, \qquad u^2 - 2as = b.$$

Eliminating u, we obtain (1.7). □

REMARK. The cubic resolvent of the quartic equation (1.7) is $g(t) = 0$. The conceptual relation between g and the quartic polynomial in (1.7) will be made explicit in Section 5.

2. Transformations of the projective line

A choice of projective coordinates in the projective line $\mathbb{P}^1_F$ over an arbitrary field F is an identification

$$\mathbb{P}^1_F = \mathrm{Proj}(F[u, v])$$

where u, v are indeterminates of degree 1 over F. When projective coordinates are chosen, the rational points of $\mathbb{P}^1_F$ are identified with $F \cup \{\infty\}$ in such a way that $a \in F$ corresponds to the homogeneous ideal $(u - av)F[u, v]$, and ∞ to $vF[u, v]$. We then identify the field of rational functions on $\mathbb{P}^1_F$ with $F(t)$, where $t = uv^{-1}$, and $\mathbb{P}^1_F \setminus \{\infty\} = \mathbb{A}^1_F = \mathrm{Spec}(F[t])$.

It is well-known that the group of transformations of $\mathbb{P}^1_F$ is simply transitive on the triples of rational points, see for instance [**2**, § III.3]. In this section, we prove an analogous result for divisors consisting of a single point of degree 3, and for divisors consisting of a rational point and a point of degree 2.

For clarity, we treat separately the various cases, although a uniform proof should be possible.

PROPOSITION 2.1. *Let p, p' be rational points on $\mathbb{P}^1_F$ and q, q' be closed points of degree 2. The projective transformations of $\mathbb{P}^1_F$ which map p to p' and q to q' are in one-to-one correspondence with the F-isomorphisms $F_q \xrightarrow{\sim} F_{q'}$.*

PROOF. After a projective transformation, we may assume $p = p'$ and choose projective coordinates such that p is the point at infinity. The projective transformations which leave p invariant then are the transformations of the affine line $\mathbb{A}^1_F$.

Viewing q and q' as closed points on $\mathbb{A}^1_F = \mathrm{Spec}(F[t])$, consider monic irreducible polynomials π, $\pi' \in F[t]$ of degree 2 such that

$$q = \pi F[t] \qquad \text{and} \qquad q' = \pi' F[t].$$

The affine transformation $t \mapsto at + b$ (with a, $b \in F$, $a \neq 0$) maps q to q' if and only if

$$\pi(at + b)F[t] = \pi'(t)F[t]. \tag{2.1}$$

Let $\tau' \in F_{q'}$ be the image of t under the canonical epimorphism $F[t] \to F[t]/(\pi') = F_{q'}$. Equation (2.1) holds if and only if π is the minimal polynomial of $a\tau' + b$ over F. Since every element in $F_{q'}$ has the form $a\tau' + b$ for some a, $b \in F$, affine transformations which map q to q' are in one-to-one correspondence with the elements in $F_{q'}$ whose minimal polynomial is π, hence also with the F-isomorphisms $F_q \xrightarrow{\sim} F_{q'}$. $\square$

PROPOSITION 2.2. *Let r, r' be closed points of degree 3 on $\mathbb{P}^1_F$. The projective transformations of $\mathbb{P}^1_F$ which map r to r' are in one-to-one correspondence with the F-isomorphisms $F_r \xrightarrow{\sim} F_{r'}$.*

PROOF. Choose coordinates to represent $\mathbb{P}^1_F$ as $\mathrm{Proj}(F[u,v])$, where u, v are indeterminates of degree 1. We may then find homogeneous irreducible polynomials π, $\pi' \in F[u,v]$ of degree 3 such that

$$r = \pi F[u,v] \qquad \text{and} \qquad r' = \pi' F[u,v].$$

After scaling, we may assume the coefficients of u^3 in π and π' are 1. The projective transformation $u \mapsto au + bv$, $v \mapsto cu + dv$ (where a, b, c, $d \in F$ and $ad - bc \neq 0$) maps r to r' if and only if

$$\pi(au + bv, cu + dv)F[u,v] = \pi'(u,v)F[u,v]$$

or, equivalently (after de-homogeneizing),

$$\pi(at + b, ct + d)F[t] = \pi'(t,1)F[t]. \tag{2.2}$$

Let $\tau' \in F_{r'}$ be the image of t under the canonical epimorphism $F[t] \to F[t]/\big(\pi'(t,1)\big) = F_{r'}$. Equation (2.2) holds if and only if $\pi(a\tau' + b, c\tau' + d) = 0$, which means that $\pi(t,1)$ is the minimal polynomial of $\frac{a\tau'+b}{c\tau'+d}$. As in the proof of Proposition 2.1, it now suffices to establish the following result:

Claim: For every $x \in F_{r'}$, $x \notin F$, there exist a, b, c, $d \in F$, uniquely determined up to a scalar factor, such that $x = \frac{a\tau'+b}{c\tau'+d}$ and $ad - bc \neq 0$.

Since $\tau' \notin F$ and $x \neq 0$, we have

$$\dim_F(F\tau' + F) = \dim_F x(F\tau' + F) = 2,$$

hence, since $\dim_F F_{r'} = 3$,

$$\dim_F\big((F\tau' + F) \cap x(F\tau' + F)\big) \geq 1.$$

If the inequality is strict, then

$$x(F\tau' + F) = F\tau' + F, \tag{2.3}$$

hence $x \in F\tau' + F$. Since $x \notin F$, it follows that $Fx + F = F\tau' + F$, hence $\tau' \in Fx + F$ and (2.3) implies that $F\tau' + F$ is a subalgebra of $F_{r'}$. This is a contradiction since τ' has degree 3. Therefore,

$$\dim_F\bigl((F\tau' + F) \cap x(F\tau' + F)\bigr) = 1.$$

We may thus find a, b, c, $d \in F$, uniquely determined up to a scalar, such that $a\tau' + b = x(c\tau' + d) \neq 0$, hence

$$x = \frac{a\tau' + b}{c\tau' + d}.$$

If $ad - bc = 0$, then $x \in F$, a contradiction. $\square$

Viewing projective transformations as changes of projective coordinates, we readily deduce from Propositions 2.1 and 2.2 (and from the "second fundamental theorem of projective geometry" [**2**, § III.3]):

Corollary 2.3. (1) *Let p_1, p_2, $p_3 \in \mathbb{P}_F^{1(1)}$ be distinct rational points and let λ_1, λ_2, $\lambda_3 \in F[u,v]$ be homogeneous polynomials of degree 1 which are pairwise distinct up to scalars. There is a choice of projective coordinates $\mathbb{P}_F^1 = \mathrm{Proj}(F[u,v])$ such that*

$$p_1 = \lambda_1 F[u,v], \quad p_2 = \lambda_2 F[u,v], \quad p_3 = \lambda_3 F[u,v].$$

(2) *Let $p \in \mathbb{P}_F^{1(1)}$ be a rational point and $q \in \mathbb{P}_F^{1(1)}$ be a point of degree 2. Let $\lambda \in F[u,v]$ be a homogeneous polynomial of degree 1 and $\pi \in F[u,v]$ be a homogeneous polynomial of degree 2 such that $F_q \simeq F[t]/\bigl(\pi(t,1)\bigr)$. There is a choice of projective coordinates $\mathbb{P}_F^1 = \mathrm{Proj}(F[u,v])$ such that*

$$p = \lambda F[u,v], \qquad q = \pi F[u,v].$$

(3) *Let $r \in \mathbb{P}_F^{1(1)}$ be a point of degree 3 and $\pi \in F[u,v]$ be a homogeneous polynomial of degree 3 such that $F_r \simeq F[t]/\bigl(\pi(t,1)\bigr)$. There is a choice of projective coordinates $\mathbb{P}_F^1 = \mathrm{Proj}(F[u,v])$ such that*

$$r = \pi F[u,v].$$

3. 2-torsion ramification sequences

In this section, we assume that the characteristic of the base field F is different from 2. We use projective transformations to set 2-torsion ramification sequences with support of degree at most 3 into a standard form.

Let Γ be the absolute Galois group of F and $\mu_2 = \{\pm 1\} \subset F^\times$. Since $\operatorname{char} F \neq 2$, we may identify the 2-torsion characters of Γ with square classes of F, via the isomorphisms

$$ {}_2X(F) = \operatorname{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}) = H^1(\Gamma, \mu_2) = F^\times/F^{\times 2}. $$

A ramification sequence $\rho \in {}_2\mathfrak{R}$ can therefore be viewed as an element in the kernel of the norm map

$$ \bigoplus_{p \in \mathbb{P}_F^{1(1)}} F_p^\times/F_p^{\times 2} \to F^\times/F^{\times 2}. \tag{3.1} $$

Suppose first $\deg \operatorname{supp}(\rho) = 2$.

PROPOSITION 3.1. *Let $\rho \in {}_2\mathfrak{R}$ be a 2-torsion ramification sequence with* $\deg \operatorname{supp}(\rho) = 2$*.*

(1) *If* $\operatorname{supp}(\rho) = \{p_1, p_2\}$ *(two distinct rational points), there is a nonsquare $a \in F^\times$ and a choice of projective coordinates $\mathbb{P}_F^1 =$* $\operatorname{Proj}(F[u, v])$ *such that*

$$ p_1 = (u - av)F[u, v], \qquad p_2 = (u + av)F[u, v], $$

and the nontrivial components of ρ are

$$ aF^{\times 2} \text{ at } p_1, \qquad aF^{\times 2} \text{ at } p_2. $$

(2) *If* $\operatorname{supp}(\rho) = \{q\}$ *with* $\deg q = 2$*, there are nonsquares a, $b \in F^\times$ in different square classes and a choice of projective coordinates* $\mathbb{P}_F^1 = \operatorname{Proj}(F[u, v])$ *such that*

$$ q = (u^2 - bv^2)F[u, v] $$

and the component of ρ at q is $aF_q^{\times 2}$.

PROOF. (1) The nontrivial components of ρ are square classes in $F_{p_1} = F$ and $F_{p_2} = F$. These square classes coincide, since ρ is in the kernel of the norm map (3.1). We may therefore find $a \in F^\times$ such that the nontrivial components of ρ are $aF^{\times 2}$ at p_1 and p_2. By Corollary 2.3, we may choose projective coordinates such that $p_1 = (u - av)F[u, v]$ and $p_2 = (u + av)F[u, v]$. This completes the proof of (1).

(2) We may find a nonsquare $b \in F^\times$ such that $F_q \simeq F(\sqrt{b})$, hence, by Corollary 2.3, there are projective coordinates such that $q = (u^2 - bv^2)F[u, v]$. The component of ρ at q is a square class in the kernel of the norm-induced map

$$ F_q^\times/F_q^{\times 2} \to F^\times/F^{\times 2}. $$

As an application of Hilbert's Theorem 90, it is easily seen that the kernel of this map is the image of the map $F^\times/F^{\times 2} \to F_q^\times/F_q^{\times 2}$ induced by the inclusion $F \subset F_q$. (See [**10**, p. 202].) Therefore, there is a nonsquare $a \in F^\times$ such that the component of ρ at q is $aF_q^{\times 2}$. □

We now turn to the case where the support of ρ has degree 3. Three cases arise:

(I) $\operatorname{supp}(\rho) = \{p_1, p_2, p_3\}$ where p_1, p_2, p_3 are three distinct rational points;

(II) $\operatorname{supp}(\rho) = \{p, q\}$ where p is a rational point and q is a point of degree 2;

(III) $\operatorname{supp}(\rho) = \{r\}$, a single point of degree 3.

In case (I), let $a_iF^{\times 2}$ be the component of ρ at p_i, for $i = 1$, 2, 3. Since these components are nontrivial, $a_i \notin F^{\times 2}$ for all i. However, $a_1a_2a_3 \in F^{\times 2}$ since ρ is in the kernel of the norm map (3.1), hence we may substitute a_1a_2 for a_3. Therefore, discarding the trivial components of ρ, we may write

$$\rho = (a_1F_{p_1}^{\times 2}, a_2F_{p_2}^{\times 2}, a_1a_2F_{p_3}^{\times 2}).$$

Applying Corollary 2.3, we may choose projective coordinates so that $\infty \neq p_1$, p_2, p_3, and in fact we may assume

$$p_1 = (t - a_1)F[t], \quad p_2 = (t - a_2)F[t], \quad p_3 = (t - a_1a_2)F[t].$$

In case (II), let $y \in F_q^\times$ be such that the component of ρ at q is $yF_q^{\times 2}$. Since ρ is in the kernel of the norm map (3.1), the component of ρ at p is

$$N_{F_q/F}(y)F^{\times 2} \in F^\times/F^{\times 2}.$$

Since this component is not trivial, we have $y \notin F$. Let $X^2 - mX + a$ be the minimal polynomial of y over F, so $a = N_{F_q/F}(y)$ and

$$\rho = (aF_p^{\times 2}, yF_q^{\times 2}).$$

We may apply Corollary 2.3 to choose coordinates such that

$$p = (t - a)F[t], \qquad q = (t^2 - mt + a)F[t].$$

In case (III), let $x \in F_r^\times$ be a nonsquare such that the component of ρ at r is $xF_r^{\times 2}$,

$$\rho = (xF_r^{\times 2}).$$

Since ρ is in the kernel of the norm map (3.1), we have

$$N_{F_r/F}(x) = a^2 \qquad \text{for some } a \in F^\times.$$

If $x \in F$ or if F_r/F is not separable, then $a^2 = x^3$, and we get a contradiction since $x \notin F_r^{\times 2}$. Let $X^3 - cX^2 + bX - a^2$ be the minimal polynomial of x over F. This is an irreducible separable polynomial in $F[X]$. By Corollary 2.3, we may assume

$$r = (t^3 - ct^2 + bt - a^2)F[t].$$

We summarize the special choice of coordinates as follows:

Proposition 3.2. *In all the cases (I), (II), (III), we may choose projective coordinates in* $\mathbb{P}^1_F$ *so that* $\infty \notin \operatorname{supp}(\rho)$ *and for each* $p \in \operatorname{supp}(\rho)$ *the nontrivial component of* ρ *at* p *is* $\tau F_p^{\times 2}$*, where* τ *is the image of* t *under the canonical map* $F[t] \to F[t]/p$.

Proof. Corollary 2.3 yields projective coordinates for which the ramification points have the required form. In case (II), the element $y \in F_q^\times$ is thus identified with one of the roots of $X^2 - mX + a$ in $F[t]/q$. If it is not identified with τ, Proposition 2.1 still allows a change of coordinates which leaves p and q invariant and maps y to τ. Similarly, in case (III) we may assume $x \in F_r^\times$ is identified with τ by Proposition 2.2. □

4. Algebras with 2-torsion ramification sequence of degree 2

In this section, the base field F is assumed to be of characteristic different from 2. Let $\rho \in {}_2\mathfrak{R}$ be a 2-torsion ramification sequence whose support has degree 2. Proposition 3.1 shows that, after a change of projective coordinates, we may assume that either

- (I) there exists a nonsquare $a \in F^\times$ such that $\operatorname{supp}(\rho) = \{a, -a\}$ and the nontrivial components of ρ are $aF^{\times 2}$, or
- (II) there exist nonsquares a, $b \in F^\times$ in different square classes such that $\operatorname{supp}(\rho) = \{q = (u^2 - bv^2)F[u, v]\}$ and the nontrivial component of ρ is $aF_q^{\times 2}$.

In each case, it is not difficult to find a quaternion $F(t)$-algebra Q with ramification $\partial(Q) = \rho$, as we proceed to show.

Recall that for any closed point $p \in \mathbb{P}_F^{1(1)}$, the image of a quaternion algebra $(f, g)_{F(t)}$ under the ramification map ∂_p is given as follows:

$$\partial_p(f, g)_{F(t)} = (-1)^{v(f)v(g)} \overline{f^{v(g)} g^{-v(f)}} F_p^{\times 2} \in F_p^\times / F_p^{\times 2}, \tag{4.1}$$

where v is the discrete valuation of $F(t)$ corresponding to p and $^-$ denotes the residue map from the valuation ring of v to its residue field F_p. This follows from the description of the residue map (tame map) in Milnor's K-theory and the functoriality of the norm residue homomorphism, see

[**3**, Theorem 2.3]. Using this description, it is easily verified that $\partial(a, t^2 - a^2)_{F(t)}$ is as in (I) above, and that $\partial(a, t^2 - b)_{F(t)}$ is as in (II). Therefore, if $\alpha \in \operatorname{Br} F(t)$ has torsion prime to $\operatorname{char} F$ and a ramification sequence $\partial(\alpha)$ as in (I) (resp. (II)), then by Faddeev's exact sequence (0.1), there exists a central simple F-algebra A such that

$$\alpha = [A \otimes_F (a, t^2 - a^2)_{F(t)}] \qquad \big(\text{resp. } \alpha = [A \otimes_F (a, t^2 - b)_{F(t)}]\big).$$

We first compute the index $\operatorname{ind} \alpha$ in both cases simultaneously by considering quaternion algebras of the type $(a, t^2 - b)_{F(t)}$ where a, $b \in F^\times$ satisfy $a \notin F^{\times 2}$ and $ab \notin F^{\times 2}$ (thus allowing $b = a^2$). The specific features of each case will be discussed next.

Theorem 4.1. *For every central simple F-algebra A and a, $b \in F^\times$ such that $a \notin F^{\times 2}$ and $ab \notin F^{\times 2}$,*

$$\operatorname{ind}(A \otimes_F (a, t^2 - b)_{F(t)}) = 2 \gcd\big\{\operatorname{ind}\big(A \otimes F(\sqrt{a})\big), \operatorname{ind}\big(A \otimes F(\sqrt{ab})\big)\big\}.$$

Note that if $a \in F^{\times 2}$ or $ab \in F^{\times 2}$, then $(a, t^2 - b)_{F(t)} = (ab, t^2 - b)_{F(t)}$ is split, hence obviously

$$\operatorname{ind}(A \otimes_F (a, t^2 - b)_{F(t)}) = \operatorname{ind} A.$$

Proof. By general principles (see for instance [**4**, p. 68]), for any quaternion $F(t)$-algebra Q we have

$$\operatorname{ind}(A \otimes_F Q) \text{ divides } 2 \operatorname{ind}(A)$$

and, similarly, substituting $A \otimes_F Q$ for A,

$$\operatorname{ind}(A) = \operatorname{ind}(A \otimes_F Q \otimes_{F(t)} Q) \text{ divides } 2 \operatorname{ind}(A \otimes_F Q).$$

Therefore,

$$\operatorname{ind}(A \otimes_F Q) \in \big\{\tfrac{1}{2} \operatorname{ind} A,\ \operatorname{ind} A,\ 2 \operatorname{ind} A\big\}. \tag{4.2}$$

Moreover, if $\operatorname{ind}(A \otimes_F Q) = \frac{1}{2} \operatorname{ind} A$, then letting D (resp. D') be a division algebra Brauer-equivalent to A (resp. $A \otimes_F Q$), we have $\deg D(t) = \operatorname{ind} A$ and $\deg D' = \frac{1}{2} \operatorname{ind} A$, hence

$$\deg D(t) = \deg(D' \otimes_{F(t)} Q).$$

Since $D(t)$ and $D' \otimes_{F(t)} Q$ are Brauer-equivalent, it follows that

$$D(t) \simeq D' \otimes_{F(t)} Q,$$

hence $D(t)$ contains a subalgebra isomorphic to Q.

In the case where $Q = (a, t^2 - b)_{F(t)}$, the last condition implies that $D(t)$ contains elements f, g such that $fg = -gf$ and $g^2 = t^2 - b$. Since

$D(t) = D \otimes_F F(t)$, we may find f_0, $g_0 \in F[t]$ and f_1, $g_1 \in D[t]$ such that $f = f_1 f_0^{-1}$ and $g = g_1 g_0^{-1}$. Then, clearing denominators,

$$f_1 g_1 = -g_1 f_1 \qquad \text{and} \qquad g_1^2 = (t^2 - b) g_0^2.$$

From these equations, it follows that the leading coefficient of g_1 is in F and that it anticommutes with the leading coefficient of f_1, a contradiction. Therefore, the choices in (4.2) restrict to

$$\text{(4.3)} \qquad \operatorname{ind}(A \otimes_F (a, t^2 - b)_{F(t)}) \in \{\operatorname{ind} A,\ 2 \operatorname{ind} A\}.$$

Clearly, $\operatorname{ind}(A \otimes_F (a, t^2 - b)_{F(t)}) = 2 \operatorname{ind} A$ holds if and only if $D \otimes_F (a, t^2 - b)_{F(t)}$ is a division algebra, which occurs if and only if D does not contain a copy of $F(\sqrt{a})$ nor of $F(\sqrt{ab})$, by Proposition 1.3. Now, by [**4**, p. 67], we have

$$\operatorname{ind}\bigl(A \otimes F(\sqrt{a})\bigr) = \begin{cases} \operatorname{ind} A & \text{if } F(\sqrt{a}) \not\hookrightarrow D, \\ \frac{1}{2} \operatorname{ind} A & \text{if } F(\sqrt{a}) \hookrightarrow D. \end{cases}$$

Similarly,

$$\operatorname{ind}\bigl(A \otimes F(\sqrt{ab})\bigr) = \begin{cases} \operatorname{ind} A & \text{if } F(\sqrt{ab}) \not\hookrightarrow D, \\ \frac{1}{2} \operatorname{ind} A & \text{if } F(\sqrt{ab}) \hookrightarrow D. \end{cases}$$

Summing up, we have $\operatorname{ind}(A \otimes (a, t^2 - b)_{F(t)}) = 2 \operatorname{ind} A$ if and only if

$$\operatorname{ind}\bigl(A \otimes F(\sqrt{a})\bigr) = \operatorname{ind} A = \operatorname{ind}\bigl(A \otimes F(\sqrt{ab})\bigr).$$

Therefore, by (4.3), $\operatorname{ind}(A \otimes (a, t^2 - b)_{F(t)}) = \operatorname{ind} A$ if and only if

$$\operatorname{ind}\bigl(A \otimes F(\sqrt{a})\bigr) = \tfrac{1}{2} \operatorname{ind} A \quad \text{or} \quad \operatorname{ind}\bigl(A \otimes F(\sqrt{ab})\bigr) = \tfrac{1}{2} \operatorname{ind} A.$$

$\square$

We now consider separately the two cases for the support of the ramification sequence.

4.1. Two rational points. Suppose $\rho \in {}_2\mathfrak{R}$ is a ramification sequence whose support consists of two rational points. Let $aF^{\times 2}$ be the component of ρ at these two points.

Corollary 4.2. *The quaternion algebras Q over $F(t)$ with $\partial(Q) = \rho$ form a 1-parameter family. For any quaternion algebra Q in this family and any central simple F-algebra A,*

$$\operatorname{ind}(A \otimes_F Q) = 2 \operatorname{ind}\bigl(A \otimes_F F(\sqrt{a})\bigr).$$

Proof. By Proposition 3.1, we may assume the two rational points in the support of ρ are a and $-a$, so that

$$\partial(a, t^2 - a^2)_{F(t)} = \rho.$$

Now, assume Q is a quaternion $F(t)$-algebra such that $\partial(Q) = \rho$. By Faddeev's exact sequence (0.1), there is a central simple F-algebra B such that

$$[Q] = [B \otimes_F (a, t^2 - a^2)_{F(t)}] \qquad \text{in } \operatorname{Br} F(t).$$

Then $\operatorname{ind}(B \otimes_F (a, t^2 - a^2)_{F(t)}) = 2$, and Theorem 4.1 implies that B is split by $F(\sqrt{a})$. Therefore, B is Brauer-equivalent to a quaternion algebra $(a, \lambda)_F$ for some $\lambda \in F^\times$, and

$$Q \simeq \big(a, \lambda(t^2 - a^2)\big)_{F(t)}.$$

For any central simple F-algebra A, the index of $A \otimes_F Q$ is computed by Theorem 4.1:

$$\operatorname{ind}(A \otimes_F Q) = \operatorname{ind}(A \otimes_F (a, \lambda)_F \otimes_F (a, t^2 - a^2)_{F(t)}) = 2 \operatorname{ind}\big(A \otimes_F (a, \lambda)_F \otimes F(\sqrt{a})\big).$$

□

Remark. If the two rational points in the support of ρ are chosen to be 0 and ∞, the family of quaternion algebras with $\partial(Q) = \rho$ is

$$\{(a, \lambda t)_{F(t)} \mid \lambda \in F^\times\}.$$

The formula $\operatorname{ind}(A \otimes_F (a, \lambda t)_{F(t)}) = 2 \operatorname{ind}\big(A \otimes F(\sqrt{a})\big)$ was obtained by a different proof in [**11**, Proposition 2.4].

4.2. Single point of degree 2. Suppose $\rho \in {}_2\mathfrak{R}$ is a ramification sequence whose support consists of a single point q of degree 2.

Corollary 4.3. *The quaternion $F(t)$-algebras Q with $\partial(Q) = \rho$ form two 1-parameter families.*

Proof. As observed in Proposition 3.1, we may find nonsquares a, $b \in F^\times$ in different square classes such that $q = (u^2 - bv^2)F[u, v]$, hence $F_q = F(\sqrt{b})$, and the nontrivial component of ρ is $aF_q^{\times 2}$. Then,

$$\partial(a, t^2 - b)_{F(t)} = \rho.$$

If Q is a quaternion $F(t)$-algebra with $\partial(Q) = \rho$, then by Faddeev's exact sequence (0.1) there exists a central simple F-algebra B such that

$$[Q] = [B \otimes_F (a, t^2 - b)_{F(t)}] \qquad \text{in } \operatorname{Br} F(t).$$

By Theorem 4.1, the equation $\operatorname{ind}(B\otimes_F(a,t^2-b)_{F(t)})=2$ implies that B is split by $F(\sqrt{a})$ or by $F(\sqrt{ab})$, hence B is Brauer-equivalent to $(a,\lambda)_F$ or to $(ab,\lambda)_F$ for some $\lambda\in F^\times$. Therefore, we have either

$$Q\simeq \big(a,\lambda(t^2-b)\big)_{F(t)} \qquad\text{or}\qquad Q\simeq \big(ab,\lambda(t^2-b)\big)_{F(t)}.$$

□

Remark. Of course, the two families in the corollary above are not disjoint. For $\lambda\in F^\times$ there exists $\mu\in F^\times$ such that

$$\big(ab,\lambda(t^2-b)\big)_{F(t)}=\big(a,\mu(t^2-b)\big)_{F(t)}$$

if and only if $(b,\lambda)_F$ is split by $F(\sqrt{a})$. The easy proof is omitted.

Remark. As in Corollary 4.2, we may use Theorem 4.1 to determine the index of $A\otimes_F\big(a,\lambda(t^2-b)\big)_{F(t)}$ for any $\lambda\in F^\times$, but the result depends on the parameter λ:

$$\operatorname{ind}\big(A\otimes_F\big(a,\lambda(t^2-b)\big)_{F(t)}\big)= 2\gcd\big\{\operatorname{ind}\big(A\otimes F(\sqrt{a})\big),\ \operatorname{ind}\big(A\otimes(a,\lambda)_F\otimes F(\sqrt{ab})\big)\big\}.$$

Similarly,

$$\operatorname{ind}\big(A\otimes_F\big(ab,\lambda(t^2-b)\big)_{F(t)}\big)= 2\gcd\big\{\operatorname{ind}\big(A\otimes(ab,\lambda)_F\otimes F(\sqrt{a})\big),\ \operatorname{ind}\big(A\otimes F(\sqrt{ab})\big)\big\}.$$

5. Algebras with 2-torsion ramification sequence of degree 3

As in the preceding section, the base field F is assumed to be of characteristic different from 2. Let $\rho\in{}_2\mathfrak{R}$ be a 2-torsion ramification sequence with support of degree 3. We use the same notation as in Section 3, and assume projective coordinates have been chosen as in Proposition 3.2. For each $p\in\operatorname{supp}(\rho)$, let $P_\rho\in F[t]$ be the monic irreducible polynomial such that $p=P_pF[t]$. Let also

$$P_\rho=\prod_{p\in\operatorname{supp}(\rho)}P_p,\qquad F_\rho=F[t]/P_\rho F[t]=\prod_{p\in\operatorname{supp}(\rho)}F_p,$$

and let $x_\rho\in F_\rho$ be the image of t in F_ρ. By Proposition 3.2, we have $\rho=x_\rho F_\rho^{\times2}$. Moreover, as observed in Section 3, F_ρ is an étale F-algebra.

With the notation and choice of coordinates above, it is readily verified that the quaternion algebra

$$Q=(t,-P_\rho)_{F(t)} \tag{5.1}$$

satisfies $\partial Q = \rho$ in each case (I), (II), and (III). Therefore, if $\alpha \in \operatorname{Br} F(t)$ has torsion prime to $\operatorname{char} F$ and satisfies $\partial\alpha = \rho$, then Faddeev's exact sequence (0.1) yields a central simple F-algebra A such that

$$\alpha = [A \otimes_F (t, -P_\rho)_{F(t)}].$$

In order to determine the index $\operatorname{ind}\alpha$, we use a quartic étale F-algebra E_ρ canonically associated with ρ, viewed as the quadratic extension $F_\rho(\sqrt{x_\rho})/F_\rho$. The construction of a quartic étale F-algebra from a quadratic étale extension of a cubic étale F-algebra is discussed at length in [**8**, §5]. We briefly recall the cohomological version of this construction.

For any integer n, let $\mathfrak{S}_n$ be the symmetric group on n elements. The conjugation action of the wreath product $\mathfrak{S}_3 \wr \mathfrak{S}_2$ on its four Sylow 3-subgroups yields a map

$$s\colon\ \mathfrak{S}_3 \wr \mathfrak{S}_2 \to \mathfrak{S}_4.$$

Let Γ be the absolute Galois group of F. The Galois cohomology set $H^1(\Gamma, \mathfrak{S}_3 \wr \mathfrak{S}_2)$ (for the trivial Γ-action) classifies the isomorphism classes of quadratic extensions of cubic étale F-algebras, so ρ defines an element in this set (corresponding to the extension $F_\rho(\sqrt{x_\rho})/F_\rho$). Consider the map induced by s,

$$s_*\colon\ H^1(\Gamma, \mathfrak{S}_3 \wr \mathfrak{S}_2) \to H^1(\Gamma, \mathfrak{S}_4).$$

Since $H^1(\Gamma, \mathfrak{S}_4)$ classifies quartic étale F-algebras up to isomorphism, the image of ρ under this map defines a quartic étale F-algebra E_ρ.

An explicit description of E_ρ is given in [**8**, §5.4]. Note that in each case (I, II or III), x_ρ generates F_ρ as an F-algebra, with minimal polynomial P_ρ of the form

$$P_\rho = t^3 - ct^2 + bt - a^2 \in F[t].$$

(The norm $N_{F_\rho/F}(x_\rho)$ is a square because ρ lies in the kernel of the sum of corestriction maps $\bigoplus_p X(F_p) \to X(F)$.) Applying [**8**, Corollary 5.22], we may describe E_ρ as the quartic F-algebra generated by an element with minimal polynomial

$$X^4 - 2cX^2 - 8aX + (c^2 - 4b). \tag{5.2}$$

Remarkably, the classification in [**8**, §6.3] shows that the F-algebra E_ρ is a field in each case (I, II, or III). In case (I), we have

$$P_\rho = (t - a_1)(t - a_2)(t - a_1a_2), \qquad E_\rho \simeq F(\sqrt{a_1}, \sqrt{a_2}),$$

and (5.2) is the minimal polynomial of $\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_1a_2}$. In case (II), E_ρ/F is a quartic 2-extension, with intermediate quadratic extension $F(\sqrt{a})$. In case (III), E_ρ/F is a quartic extension whose Galois closure has Galois group $\mathfrak{S}_4$ or the alternating group $\mathfrak{A}_4$.

Remark. The table in [**8**, §6.3] is set up from the perspective of the quartic algebra. The correspondence with the notation used here is the following: $Q = E_\rho$, $\mathcal{R}(Q) = F_\rho$, $\Lambda_2(Q) = F_\rho(\sqrt{x_\rho})$.

Theorem 5.1. *For every central simple F-algebra A and $P_\rho \in F[t]$ as above,*

$$\operatorname{ind}(A \otimes_F (t, -P_\rho)_{F(t)}) = \begin{cases} \operatorname{ind} A & \textit{if } \operatorname{ind}(A \otimes_F E_\rho) = \frac{1}{4}\operatorname{ind} A, \\ 2\operatorname{ind} A & \textit{otherwise.} \end{cases}$$

Proof. Let D be a central division F-algebra Brauer-equivalent to A. As observed in the beginning of the proof of Theorem 4.1 (see (4.2)),

$$\operatorname{ind}(A \otimes_F (t, -P_\rho)_{F(t)}) = \tfrac{1}{2}\operatorname{ind} A,\ \operatorname{ind} A \text{ or } 2\operatorname{ind} A.$$

Moreover, the first case occurs only if $D(t)$ contains a subalgebra isomorphic to $(t, -P_\rho)_{F(t)}$. In particular, it must contain a square root of t, hence $D(t)$ does not remain a division algebra over $F(t)(\sqrt{t})$. This is impossible, since $F(t)(\sqrt{t})$ is a purely transcendental extension of F. Therefore, only two possibilities remain:

$$\operatorname{ind}(A \otimes_F (t, -P_\rho)_{F(t)}) = \operatorname{ind} A \text{ or } 2\operatorname{ind} A.$$

The first case occurs if and only if $D \otimes (t, -P_\rho)_{F(t)}$ is not a division algebra. By Proposition 1.4, this condition is equivalent to the existence of a subalgebra isomorphic to E_ρ in D, hence also, by [**4**, Theorem 12, p. 67], to

$$\operatorname{ind}(D \otimes_F E_\rho) = \tfrac{1}{4}\operatorname{ind} D.$$

□

Corollary 5.2. *For any ramification sequence $\rho \in {}_2\mathfrak{R}$ whose support has degree* 3*, there is (up to isomorphism) a unique quaternion $F(t)$-algebra Q_ρ such that*

$$\partial Q_\rho = \rho.$$

Proof. By a suitable choice of projective coordinates, we may assume ρ has the special form of Proposition 3.2. As observed above (see (5.1)), the quaternion algebra

$$Q = (t, -P_\rho)_{F(t)}$$

satisfies $\partial Q = \rho$. If Q' is a quaternion $F(t)$-algebra such that $\partial Q' = \rho$, then by Faddeev's exact sequence (0.1), there is a central division F-algebra D such that Q' is Brauer-equivalent to $D \otimes_F Q$. By Theorem 5.1, the equality $\operatorname{ind}(D \otimes_F Q) = 2$ leads to $\operatorname{ind} D = 1$ (hence $Q' \simeq Q$), or $\operatorname{ind} D = 2$. The latter case occurs only if

$$\operatorname{ind}(D \otimes E_\rho) = \tfrac{1}{4} \operatorname{ind} D = \tfrac{1}{2},$$

which is impossible. □

Thus, if ρ is in the special form of Proposition 3.2,

$$Q_\rho = (t, -P_\rho)_{F(t)}.$$

The index of any Brauer class α such that $\partial\alpha = \rho$ is determined by Theorem 5.1.

Corollary 5.3. *If a, $b \in F^\times$ are nonsquares in different square classes and D is a quaternion division F-algebra, the tensor product*

$$D \otimes_F \big(t, a(b-t)\big)_{F(t)}$$

is a division algebra.

Proof. Inspection shows that $\big(t, a(b-t)\big)_{F(t)}$ ramifies only at 0, b, and ∞, with respective ramification $abF^{\times 2}$, $bF^{\times 2}$, and $aF^{\times 2}$. Therefore, for a suitable change of variables,

$$\big(t, a(b-t)\big)_{F(t)} = \big(t', (a-t')(b-t')(ab-t')\big)_{F(t')}.$$

Theorem 5.1 shows that $D \otimes_F \big(t, a(b-t)\big)_{F(t)}$ is not a division algebra if and only if

$$\operatorname{ind}\big(D \otimes_F F(\sqrt{a}, \sqrt{b})\big) = \tfrac{1}{4} \operatorname{ind} D = \tfrac{1}{2},$$

which is absurd. □

As a particular case, one may take $F = \mathbb{Q}_p$ (the field of p-adic numbers) with $p \neq 2$, $a \in \mathbb{Z}_p$ a nonsquare unit, $b = p$ and $D = (a, p)_{\mathbb{Q}_p}$. Corollary 5.3 shows that the tensor product

$$(a, p)_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \big(t, a(p-t)\big)_{\mathbb{Q}_p(t)}$$

is a division algebra over $\mathbb{Q}_p(t)$. Note that by a theorem of Saltman [**14**, Theorem 3.4], division algebras of exponent 2 over $\mathbb{Q}_p(t)$ have index 2 or 4. The Jacob–Tignol example of a biquaternion division algebra over $\mathbb{Q}_p(t)$ given in the appendix of [**14**] is not essentially different from the example above, but the arguments used to prove it to be a division algebra are quite different.

Appendix: Tensor product with a symbol of degree 3

Theorem 1.1 has been used so far to investigate tensor products with quaternion algebras only. In this appendix, we give another application, where the prime n in the statement of Theorem 1.1 is 3 instead of 2. This will illustrate the fact that Theorem 1.1 leads to more complex conditions for $n > 2$.

Let F be a field of characteristic different from 3, containing a primitive cube root of unity ζ. For a, $b \in F^\times$, we denote by $(a,b)_{\zeta,F}$ the *symbol algebra* of degree 3 generated by two elements i, j subject to

$$i^3 = a, \qquad ji = \zeta ij, \qquad j^3 = b.$$

(See [**4**, §11], where this type of algebra is called *power norm residue algebra.*)

PROPOSITION 5.4. *Let D be a central division F-algebra, let a, $b \in F^\times$ and let t be an indeterminate over F. The tensor product*

$$D \otimes_F (t, at+b)_{\zeta,F(t)} \tag{5.3}$$

is not a division algebra if and only if D contains elements u, v satisfying

$$\begin{aligned} u^3 &= a, & u(uv - vu) &= \zeta(uv - vu)u, \\ v^3 &= b, & v(uv - vu) &= \zeta^2(uv - vu)v. \end{aligned} \tag{5.4}$$

PROOF. The tensor product (5.3) can be viewed as a special case of the Δ-construction in (1.1),

$$D \otimes_F (t, at+b)_{\zeta,F(t)} = \Delta(D(x), \alpha, ax^3 + b)$$

where $x^3 = t$ and α is the automorphism of $D(x)$ defined by

$$\alpha(x) = \zeta x, \qquad \alpha(d) = d \quad \text{for } d \in D.$$

(Compare Section 1.2.) By Theorem 1.1, this algebra is not a division algebra if and only if there exists $f \in D[x]$ such that

$$\alpha^2(f)\alpha(f)f = ax^3 + b. \tag{5.5}$$

Comparing degrees, it is clear that f must be of the form $f = ux + v$ for some u, $v \in D$ if it exists. Comparing coefficients of like powers of x in (5.5) (and using $\zeta^2 = -1 - \zeta$) yields (5.4). □

For instance, the tensor product (5.3) is not a division algebra if $D \supset F(\sqrt[3]{a}, \sqrt[3]{b})$, since then (5.4) has a solution where u and v commute. This is not the only case, however:

COROLLARY 5.5. *The tensor product*

$$(b,a)_{\zeta,F} \otimes_F (t, at+b)_{\zeta,F(t)}$$

is not a division algebra.

PROOF. The elements u, $v \in (b,a)_{\zeta,F}$ such that

$$v^3 = b, \qquad uv = \zeta vu, \qquad u^3 = a$$

satisfy (5.4). □

A sharp contrast is given by the following positive result:

COROLLARY 5.6. *Suppose a, b are indeterminates over a field k containing a primitive cube root of unity, and let $F = k(a,b)$. The tensor product*

$$(a,b)_{\zeta,F} \otimes_F (t, at+b)_{\zeta,F(t)}$$

is a division algebra.

PROOF. Embedding F into the field of iterated Laurent series $\hat{F} = k((a))((b))$, we show the stronger result that

$$(a,b)_{\zeta,\hat{F}} \otimes_{\hat{F}} (t, at+b)_{\zeta,\hat{F}(t)}$$

is a division algebra.

Observe that the canonical (a,b)-adic valuation on $\hat{F}$ (with values in $\mathbb{Z}\times\mathbb{Z}$) extends to a totally ramified valuation ν on $(a,b)_{\zeta,\hat{F}}$. The standard generators i, j of $(a,b)_{\zeta,\hat{F}}$ satisfy

$$\nu(i) = \big(\tfrac{1}{3}, 0\big), \qquad \nu(j) = \big(0, \tfrac{1}{3}\big).$$

Suppose u, $v \in (a,b)_{\zeta,\hat{F}}$ satisfy (5.4). Then

$$\nu(u) = \big(\tfrac{1}{3}, 0\big) = \nu(i), \qquad \nu(v) = \big(0, \tfrac{1}{3}\big) = \nu(j),$$

hence, by [**12**, §3], letting $^{-}$ denote the residue map,

$$\overline{v^{-1}u^{-1}vu} = \overline{j^{-1}i^{-1}ji} = \zeta$$

and

$$\overline{v^{-1}u^{-2}vu^2} = \overline{j^{-1}i^{-2}ji^2} = \zeta^2.$$

On the other hand, from the equation

$$u(uv-vu) = \zeta(uv-vu)u$$

it follows that

$$u^2v + \zeta^2 uvu + \zeta vu^2 = 0$$

hence, dividing by u^2v,

$$1 + \zeta^2 v^{-1}u^{-1}vu + \zeta v^{-1}u^{-2}vu^2 = 0.$$

Taking the residue of each side yields $3 = 0$. □

Remark. Letting $K = k(a, b, t) = F(t)$ and

$$D_1 = (a, b)_{\zeta,K}, \qquad D_2 = (t, at + b)_{\zeta,K},$$

Corollaries 5.5 and 5.6 show that $D_1 \otimes_K D_2$ is a division algebra while $D_1^{\mathrm{op}} \otimes_K D_2$ is not a division algebra. The first examples of this type are in [**12**, §5]. The example above is essentially the same as the Tignol–Wadsworth example in degree 3.

Remark. Suppose D is a central division F-algebra of degree 3. If $u, v \in D$ satisfy (5.4) and $uv - vu \neq 0$, then

$$(v, uv - vu, u)$$

is a chain of length 2 of Kummer elements, in the sense of Rost [**13**]. The example in [**13**, Appendix] of a division algebra containing two Kummer elements which cannot be related by a chain of length 2 yields, by Proposition 5.4, a division algebra of exponent 3 and degree 9 over the field of rational fractions in two indeterminates $\mathbb{F}_7(t_1, t_2)$.

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