WITT MOTIVES, TRANSFERS AND REDUCTIVE GROUPS

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Introduction

In this paper, we compute Grothendieck-Witt and Witt groups of representation categories of split reductive algebraic groups. We also define transfer maps between Witt groups with respect to proper morphisms and establish the base change and projection formulae for those. Then we use this to define the cateory of *Witt motives*. In forthcoming work, these results will hopefully be improved and combined to lead to the computation of Witt groups of twisted flag varietes.

Let us fix a base field F of characteristic different from 2. In his paper [28], Panin computes the K-groups of twisted flag varietes, generalizing results of Quillen on Brauer-Severi varieties and of Swan on quadrics. To this purpose, he constructs a category of K_0-motives with nice properties. This allows him to reduce the computations to K-groups of finite-dimensional separable F-algebras and to K_0 of representation categories of split reductive algebraic groups.

This paper is an attempt to apply the techniques of Panin to Grothendieck-Witt groups GW and Witt groups W instead of the Grothendieck group and higher K-groups. As usual, the Grothendieck-Witt group $\mathrm{GW}(\mathcal{A})$ of an abelian category with duality $(\mathcal{A}, *)$ is defined as the Grothendieck group of isomorphism classes of symmetric spaces (and identifying metabolic spaces with the associated hyperbolic spaces if $\mathcal{A}$ is not semi-simple, see Definition 1.6). Identifying the hyperbolic spaces with zero yields the Witt group $\mathrm{W}(\mathcal{A})$. Two examples we are interested in are vector bundles over a smooth F-schemes $(Vect(X), Hom_{O_X}(\ \ , O_X))$ and finite-dimensional representations $(Rep(G), Hom_F(\ \ , F))$ of a reductive algebraic group G. If $X = Spec(F) = G$, these to examples coincide and yield the classical Witt group of the field F.

The situation for Witt groups is much more complicated than for K-groups. When Panin wrote his paper, the Grothendieck group and even its ring structure for representation categories was already known. It is a free polynomial ring over $\mathbf{Z}$ with explicit generators, and moreover K_0 of the representation category of a parabolic group was known to be a finite free module over K_0 of the corresponding reductive group. Similar results had not yet been established for GW and W, and at least the result for parabolic groups seems to be wrong for GW and W. For an F-split reductive group G, we prove (see 2.16):

Theorem 0.1. *The Witt group* $\mathrm{W}(\mathrm{Rep}(G)) \oplus \mathrm{W}_-(\mathrm{Rep}(G))$ *can be mapped injectively by a morphism of* $\mathrm{W}(F)$*-algebras to* $\mathrm{W}(F) \otimes_{\mathbf{Z}} \mathbf{Z}[(X^+)^0]$*, where* $(X^+)^0$ *is the set of dominant characters fixed under the duality. This morphism becomes an isomorphism when* G *is semisimple and simply connected.*

The proof combines a careful study of the action of the duality on the given representation category (involving root systems, the Weil group etc.) and general structure theorems for Witt rings of abelian categories with duality.

The size of $\mathrm{W}(\mathrm{Rep}(G))$ can be very small. For instance if $G = T$ is a torus, one obtains $\mathrm{W}(Rep(T)) \cong \mathrm{W}(F)$.

The first section discusses the general theory of (Grothendieck-)Witt groups of an abelian category $\mathcal{A}$. In particular, we establish theorems that allow to describe $W(\mathcal{A})$ as module and even as an algebra over $W(F)$ quite explicitely. Most of the tools of the proofs were already in the literature, in particular [30] was a useful source to us.

Section 2 applies the result of section 1 to representation categories. This yields the above theorem and reduces the computation to Witt groups of division algebras.

Section 3 contains some results that will be necessary for computations in the equivariant category of *Witt motives* (or *W-motives*) discussed below.

The last section contains a construction of W-motives reminiscent to Panin's K_0-motives. As for K_0, the construction of this category relies heavily on the existence on transfers having good properties such as base change. We prove (see Lemma 4.14, Corollary 4.21 and Proposition 4.23)

Theorem 0.2. *Let $f : X \to Y$ be a proper map of relative dimension d between smooth noetherian schemes of finite Krull dimension and L a line bundle on Y. Then we can construct a transfer map of degree $-d$*

$$f_* : W^{*+d}(X, f^*L \otimes_{O_X} \omega_X) \to W^*(Y, L \otimes_{O_Y} \omega_Y)$$

which satisfies the base change and projection formula with respect to flat morphisms.

This is a consequence of a more general result (see Definition 4.11 and Theorem 4.20). Observe the twists and shifts that show up. The construction of the transfer maps is more tricky than one might expect as one has to keep track carefully of the dualites and isomorphisms between objects and their biduals involved. In the appendix we show that the natural isomorphism from the identity to the bidual and various other isomorphisms can be constructed from an internal Hom adjoint to the derived tensor product. We also show that this and other constructions such as the adjointness of $\mathbf{L}f^*$, $\mathbf{R}f_*$ and $f^!$ can be carried out in a way compatible with each other and with the triangulated structure, and moreover compatible with the various sign conditions of Balmer and Gille. These verifications - though not very surprising except maybe that there is a nice choice of sign conventions - are rather long, but there is no way to avoid them.

Theorem 0.2 allows us to construct the category of W-motives (see section 4.3) which are more complicated but similar in spirit to Panin's K_0-motives (and Manin's classical motives). We then construct a graph functor and explain the usual structures (pseudoabelian completion, tensor product) as well as an involution on this category of W-motives.

This paper is work in progress. We believe that the results mentioned above are sufficiently interesting to make them available now. Nevertheless, there is obviously still quite some work to do in order to compute the Witt group of twisted flag varieties. Panin's strategy has to be considerably modified as some results for K_0 (as the one on parabolic groups mentioned above) do not hold for GW and W. Also some details remain to be checked for seeing that the category of Witt motives generalizes well to the H-equivariant setting for H an algebraic group, and that one can enlarge this category with respect to semisimple algebras as Panin does for K_0 (compare Remark 4.30). We hope to settle these issues in forthcoming work.

Ch. Walter has computed the (Grothendieck-)Witt groups of projective bundles in [37] by different methods, and there is work in progress by him on quadrics. Pumplün [29] has some partial results about the classical Witt group of Brauer-Severi varieties, and very recently Nenashev [27] obtained some partial results about the Witt group of the standard hyperbolic quadric. It seems that the methods of

Pumplün and Nenashev do not generalize to obtain results about Witt groups of twisted flag varieties in general.

It would have been possible to rephrase the first section of this article in the language of Balmer's triangular Witt groups [3] (generalized to Grothendieck-Witt groups by Walter) and to replace many of the proofs in the first chapter on abelian categories by triangulated ones. But as all our categories are abelian, we only have non-zero triangular Witt groups in even degrees, which in light of the 4-periodicity of Balmer Witt groups amounts to study the Witt groups for symmetric and skew-symmetric forms simultaneously as we do. On the other hand, the description of transfer maps and almost everything else in the last section would have been impossible without using Balmer's derived approach of Witt groups.

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The first three sections of this preprint are almost identical to those of the one posted in July whereas the appendix is new. Some proofs of the former section 4 now appear in this appendix.

1. Abelian categories with duality

We fix a field F of characteristic not 2 throughout the whole article. In the first part of this section we recall some of the results of [30] in a convenient way to suite our further purposes.

1.1. Dévissage.

Let F be a field. Let $(\mathcal{A}, *, \eta)$ be an F-linear abelian category with duality (see e.g. [30]). We assume that the duality functor $* : \mathcal{A} \to \mathcal{A}^{op}$ is exact and F-linear. Unless specified, an object is always an object of $\mathcal{A}$.

Definition 1.1. Let A be an object of $\mathcal{A}$. A subobject B of A is an object B of $\mathcal{A}$ with a monomorphism $i_{B,A} : B \to A$. A quotient of A is an object C of $\mathcal{A}$ with an epimorphism $p_{A,C} : A \to C$.

Recall the definition of simple objects and their basic properties.

Definition 1.2. A nonzero object S of $\mathcal{A}$ is called simple when any of its subobjects is zero or isomorphic to S by its defining monomorphism to S.

Lemma 1.3. *Simple objects have the following obvious properties.*

(1) *Every quotient of a simple object is zero or isomorphic to it by its defining surjection.*
(2) *Every morphism to a simple object is a surjection or zero.*
(3) *Every morphism from a simple object is an injection or zero.*

By a finite filtration of an object M, we mean as usual a sequence of subobjects $0 = M_0 \subset M_1 \ldots M_n = M$. The length of this filtration is by definition the integer n. The quotients M_i/M_{i-1} of the filtration are the cokernels of the inclusion morphisms.

Definition 1.4. We say that an object has a finite length when it admits a finite filtration with simple quotients.

Remark 1.5. It is obvious that any object of finite length admits a simple subobject or quotient.

Recall (see e.g. [30]) the definition of the Grothendieck-Witt group $\mathrm{GW}(\mathcal{A})$ for an abelian category with duality $(\mathcal{A}, *, \eta)$. For this, we first introduce a new category $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{A}, *, \eta)$ whose objects are symmetric bilinear non-degenerate objects (a, ϕ) ("symmetric spaces" for short) of $\mathcal{A}$ (namely $\phi = \phi^* \circ \eta : A \overset{\cong}{\to} A^*$). A morphism from (A, ϕ) to (B, ψ) is an isomorphism i from A to B such that $i^* \circ \psi \circ i = \phi$.

Definition 1.6. For an abelian category with duality $(\mathcal{A}, *, \eta)$, the Grothendieck-Witt group $\mathrm{GW}(A)$ is defined as the quotient of the classical Grothendieck group $K_0(\mathcal{H}(\mathcal{A}))$ obtained by identifying metabolic spaces with the standard hyperbolic space of the associated lagrangian. For $\epsilon \in \{-, +\}$ we denote by $\mathrm{GW}_\epsilon(\mathcal{A})$ the Grothendieck-Witt group of the category with duality $(\mathcal{A}, *, \epsilon\eta)$ (with this notation, $\mathrm{GW}(\mathcal{A}) = \mathrm{GW}_+(\mathcal{A})$). The Witt group $\mathrm{W}_\epsilon(\mathcal{A})$ is obtained from $\mathrm{GW}_\epsilon(\mathcal{A})$ by identifying all hyperbolic (and thus all metabolic) spaces with zero.

We thus have an exact sequence of abelian groups $K(\mathcal{A}) \to \mathrm{GW}(\mathcal{A}) \to \mathrm{W}(\mathcal{A}) \to 0$, where the morphisms are given by the hyperbolic functor and the quotient map, respectively. Recall also that if multiplying with 2 is a global automorphism (*i.e.* all *Hom*-sets are uniquely 2-divisible, this is in particular the case if we have a forgetful functor to F-Vect and $char(F) \neq 2$) then any split metabolic space is isomorphic to a standard hyperbolic one, and the categories of quadratic and of symmetric bilinear forms are equivalent. We assume from now on that this condition on the invertibility of 2 is always satisfied.

Recall that a semisimple object is a direct sum of simple objects and let $\mathcal{A}_{ss}$ be the full category of $\mathcal{A}$ whose objects are semisimple. The duality obviously preserves $\mathcal{A}_{ss}$, it is therefore a full subcategory with duality of $\mathcal{A}$.

Theorem 1.7 (Dévissage)**.** *(See* [6, Theorem VIII.3.4] *and* [30, Theorem 6.10]*) Assume that all objects of $\mathcal{A}$ have finite length. Then the inclusion of $\mathcal{A}_{ss}$ in $\mathcal{A}$ induces isomorphisms*

$$K_0(\mathcal{A}_{ss}) \xrightarrow{\simeq} K_0(\mathcal{A})$$
$$\mathrm{GW}(\mathcal{A}_{ss}) \xrightarrow{\simeq} \mathrm{GW}(\mathcal{A})$$

and

$$\mathrm{W}(\mathcal{A}_{ss}) \xrightarrow{\simeq} \mathrm{W}(\mathcal{A})$$

1.2. Group structure of $\mathrm{GW}(\mathcal{A})$ and $\mathrm{W}(\mathcal{A})$.

Definition 1.8. Let M be an object. We denote $\mathcal{A}_M$ the full subcategory of $\mathcal{A}$ whose objects are the ones isomorphic to a finite direct sum of copies of M. This subcategory therefore only depends on the isomorphism class of M.

Lemma 1.9. *Let S and S' be simple objects. They are isomorphic if and only if there is a nonzero morphism from an object of $\mathcal{A}_S$ to an object of $\mathcal{A}_{S'}$.*

Proof: The non-trivial implication follows as Lemma 1.3 implies that any morphism $S \to S'$ is either an isomorphism or zero. □

Definition 1.10. We denote by $\coprod_S \mathcal{A}_S$ the coproduct of the abelian categories $\mathcal{A}_S$ indexed over a set of isomorpism classes of simple objects (we assume that these do indeed form a set) of $\mathcal{A}$. Explicitely, objects in this category are objects isomorphic to finite direct sums of the objets S defining our index set, and morphisms can be seen as matrices with entries wich are either an isomorphism or 0.

Observe that the inclusion $\iota_S : \mathcal{A}_S \to \mathcal{A}$ is a full inclusion of abelian categories, although $\mathcal{A}_S$ might not be closed under extensions in $\mathcal{A}$. In fact, this inclusion factors through $\mathcal{A}_{ss}$ because any object of $\mathcal{A}_S$ is obviously semisimple. Furthermore, lemma 1.9 ensures that this factorisation makes $\coprod_S \mathcal{A}_S$ is a full subcategory of $\mathcal{A}_{ss}$.

The subcategories $\mathcal{A}_S$ won't be duality preserving in general. Consider again the set of isomorphism classes of simple objects. The duality of $\mathcal{A}$ induces an involution on this set. We denote $\mathcal{I}$ the set of orbits of this involution. Note that these orbits contain one or two elements. For $i \in \mathcal{I}$, we say that a simple object is of type i if it is in one of the isomorphism classes in the orbit i.

Definition 1.11. Let $i \in \mathcal{I}$. Let $\mathcal{A}_i$ denote the full subcategory of $\mathcal{A}$ whose objects are isomorphic to a direct sum of simple objects of type i and let ι_i be the inclusion functor of $\mathcal{A}_i$ in $\mathcal{A}$. As previously, it is clear that ι_i factors through $\mathcal{A}_{ss}$, and the functor $\mathcal{A}_i \to \mathcal{A}_{ss}$ is also denoted ι_i.

Remark 1.12. Let S be a simple object of type i. Note that if i contains only one element, $\mathcal{A}_i = \mathcal{A}_S$, and if i contains two elements, $\mathcal{A}_i$ is the coproduct of $\mathcal{A}_S$ and $\mathcal{A}_{S^*}$.

The subcategory $\mathcal{A}_i$ is stable by the duality of $\mathcal{A}$, so it comes equipped with an induced duality.

Proposition 1.13. *The functor $\coprod_i \iota_i : \coprod_i \mathcal{A}_i \to \mathcal{A}_{ss}$ is an equivalence of categories with dualities.*

Proof: It is essentialy surjective since semisimple object are direct sums of simple objects. It is faithfull since by lemma 1.9 and remark 1.12 there are no morphisms between an object of $\mathcal{A}_i$ and an object of $\mathcal{A}_j$ if $i \neq j$. It is full for $\iota_i : \mathcal{A}_i \to \mathcal{A}$ is already full. □

Corollary 1.14. *When all objects of $\mathcal{A}$ are of finite length, Proposition 1.13 and the dévissage theorem 1.7 yield isomorphisms*

$$\bigoplus_i K_0(\mathcal{A}_i) \xrightarrow{\oplus \iota_i} K_0(\mathcal{A})$$

$$\bigoplus_i \mathrm{GW}(\mathcal{A}_i) \xrightarrow{\oplus \iota_i} \mathrm{GW}(\mathcal{A})$$

$$\bigoplus_i \mathrm{W}(\mathcal{A}_i) \xrightarrow{\oplus \iota_i} \mathrm{W}(\mathcal{A}).$$

Remark 1.15. The first two isomporphisms are compatible in an obvious way with the forgetful morphism from GW to K_0. Moreover, looking at the proof of the dévissage theorem in [30, Theorem 6.7], one can see that a symmetric object decomposes as a sum (and not a difference) of symmetric objects in GW ("Jordan-Hölder"), so the types and the underlying objects appearing in the decomposition of the class of a symmetric object can be checked in K_0 by applying the forgetful functor. Of course, this is only true for classes of symmetric objets and not for differences of those.

Theorem 1.16. *Assume that all objects of $\mathcal{A}$ are of finite length. Let $\mathcal{A}'$ be a full subcategory of $\mathcal{A}$, stable by duality, subobjets and quotients. Then the inclusion functor from $\mathcal{A}'$ (with the induced duality) to $\mathcal{A}$ makes $\mathrm{GW}(\mathcal{A}')$ (resp. $\mathrm{W}(\mathcal{A}')$) a direct summand of $\mathrm{GW}(\mathcal{A})$ (resp. $\mathrm{W}(\mathcal{A})$). In the decomposition of 1.14, $\mathrm{GW}(\mathcal{A}')$ (resp. $\mathrm{W}(\mathcal{A}')$) identifies with $\bigoplus_{i \in I'} \mathrm{GW}(\mathcal{A}_i)$ (resp. $\mathrm{W}(\mathcal{A}_i)$) where I' contains only the indices corresponding to simple objets that are in $\mathcal{A}'$.*

Proof: The stability by duality ensures that one can induce a duality on $\mathcal{A}'$ such that the inclusion functor preserves the duality. The stability by subobjets and quotients ensures that $\mathcal{A}'_{ss}$ is a subcategory of $\mathcal{A}_{ss}$ and that all objects of $\mathcal{A}'$ are of finite length. The theorem is then obvious when restricting to $\mathcal{A}_{ss}$. □

Remark 1.17. In the setting of Theorem 1.16, a simple object of $\mathcal{A}$ that is in $\mathcal{A}'$ is also simple as an object of $\mathcal{A}'$, because $\mathcal{A}'$ is stable by subobjets and full.

1.3. **Identification of the summands.** In the following, we interpret each left-hand summand of the latter isomorphisms as Grothendieck-Witt groups of rings with involutions.

Lemma 1.18. *Let S be a simple object. Then* $\mathrm{Hom}(S, S)$ *is a division algebra whose center contains F and* $\mathrm{Hom}(S^*, S^*)$ *is isomorphic to its opposite algebra.*

Proof: Obvious. □

An antisymmetric isomorphism in $(\mathcal{A}, *, \eta)$ is a symmetric morphism in $(\mathcal{A}, *, -\eta)$. When we want to cover both symmetric and antisymmetric morphisms, we therefore use the locution ϵ-symmetric where $\epsilon \in \{-, +\}$ as before.

Definition 1.19. We say that a simple object S is of sign ϵ (resp. of sign 0) if there is an isomorphism $S \to S^*$ which is ϵ-symmetric (resp. if S and S^* are not isomorphic). As S determines i, we also say that S is of type (i, ϵ) or $(i, 0)$. We say that i is of sign ϵ if there is a simple object in $\mathcal{A}_i$ of sign ϵ, and we say that i is of sign 0 if a simple object in $\mathcal{A}_i$ is not isomorphic to its dual.

The following proposition explains that a simple objet that is not of sign 0 is of sign $+$ or $-$ (or maybe both). Thus, any i has a sign, and if it is not of sign 0, it may have both signs $+$ and $-$ simultaneously.

Proposition 1.20. [30, Proposition 2.5] *Let S be a simple object. If there is an isomorphism from S to S^*, then there is an ϵ (maybe both values work) such that an ϵ-symmetric isomorphism from S to S^* exists.*

However, the next lemma shows that if $\mathrm{Hom}(S, S) \simeq F$, then S cannot be of sign $+$ and $-$ at the same time.

Lemma 1.21. *Let S be a simple object such that* $\mathrm{Hom}(S, S) \simeq F$. *If $\varphi : S \to S^*$ is an isomorphism, then all isomorphisms from S to S^* are scalar multiples of φ, and are therefore ϵ-symmetric if φ is ϵ-symmetric. Moreover, any isomorphism from S to S^* is either symmetric or antisymmetric. Thus S is of sign either $+$, or $-$, or* 0 *(exclusively).*

Proof: Let ψ be another isomorphism. Then $\psi^{-1} \circ \phi = f \in \mathrm{Hom}(S, S) \simeq F$. This proves the first part of the claim. Applying this first result to $(\varphi^*)^{-1}$ yields $\varphi^* = f\varphi$ so $\varphi^* * = f^2\varphi$. Thus $f^2 = 1$ since $\varphi^* * = \varphi$ (for simplicity, we omit the isomorphism from S to S^{**} given with the duality, but the proof holds with it). □

Definition 1.22. [30, Lemma 1.2] Let M be an object isomorphic to its dual and m be an ϵ-symmetric isomorphism from M to M^*. Denote the F-algebra $\mathrm{Hom}(M, M)$ by E and define $\sigma_m : E \to E$ by $\sigma_m(e) = m^{-1} \circ e^* \circ m$. It is staight forward to check that σ_m is an F-linear involution on E (again, we have omitted the bidual isomorphism for simplicity).

The following results (1.23, 1.24, 1.25, 1.26 and 1.27) are easy consequences of [30, Proposition 2.4].

Proposition 1.23. *Let M be an object isomorphic to its dual by an ϵ-symmetric isomorphism m. Let* $E = \mathrm{Hom}(M, M)$ *and $\mathcal{P}(E)$ be the category of right E-modules with duality induced by σ_m. The functor*

$$\begin{array}{rccc} F_M : & \mathcal{A}_M & \longrightarrow & \mathcal{P}(E) \\ & N & \mapsto & \mathrm{Hom}(M, N) \end{array}$$

is a functor of categories with duality (as in [19, p. 80]*).*

Theorem 1.24. *Let S be a simple object of type (i, ϵ) (isomorphic to its dual) and s be an ϵ-symmetric isomorphism from S to S^*. Let D denote* $\mathrm{Hom}(S, S)$

and $(\mathcal{P}(D), d_s, \theta)$ *the category with duality induced by* σ_s *in the usual way (see e.g.* [30, Example 1.1]. *Then the functor* F_S *induces an equivalence of categories between* $\mathcal{H}(\mathcal{A}_S, *, \eta)$ *and* $\mathcal{H}(\mathcal{P}(D), d_s, \epsilon\theta)$ *by* $F_S(M, \lambda) = (Hom_{\mathcal{A}_S}(S, M), \tilde{\lambda})$ *where* $\tilde{\lambda}(g) = Hom_{\mathcal{A}_S}(S, s^{-1}(\lambda \circ g)^*\theta)$.

Corollary 1.25. *This equivalence of categories induces the isomorphisms*

$$\mathrm{GW}_{\epsilon_1}(\mathcal{A}_i) \xrightarrow{\simeq} \mathrm{GW}_{\epsilon_1\epsilon}(D, \sigma_s)$$

and

$$\mathrm{W}_{\epsilon_1}(\mathcal{A}_i) \xrightarrow{\simeq} \mathrm{W}_{\epsilon_1\epsilon}(D, \sigma_s).$$

Let S be a simple object of type $(i, 0)$ (nonisomorphic to its dual). It is easy to show that any symmetric object in $\mathcal{A}_i$ is a direct sum of the same number of copies of S and S^*, thus $\mathcal{H}(\mathcal{A}_i) = \mathcal{H}(\mathcal{A}_M)$, where $M = S \oplus S^*$. If $D = \mathrm{Hom}(S, S)$, then $\mathrm{Hom}(M, M) \simeq D \times D^{op}$. Let h be the hyperbolic form on M.

Theorem 1.26. *The functor* F_M *induces an equivalence of categories between* $\mathcal{H}(\mathcal{A}_i, *, \eta)$ *and* $\mathcal{H}(\mathcal{P}(D \times D^{op}), d_h, \theta)$.

Corollary 1.27. *This equivalence of categories induces the isomorphisms*

$$\mathrm{GW}_{\epsilon_1}(\mathcal{A}_i) \xrightarrow{\simeq} \mathrm{GW}_{\epsilon_1}(D \times D^{op}, \sigma_h)$$

and

$$\mathrm{W}_{\epsilon_1}(\mathcal{A}_i) \xrightarrow{\simeq} \mathrm{W}_{\epsilon_1}(D \times D^{op}, \sigma_h).$$

Theorems 1.24 and 1.26 may be sharpened, in fact we do have an equivalence of categories with duality.

We now give some values of i for which the Witt group $\mathrm{W}(\mathcal{A}_i)$ is always zero.

Theorem 1.28. *Let* D *be a division algebra and equip* $D \times D^{op}$ *with the standard involution* σ_h *(exchanging* D *and* D^{op}*). Then the Witt group* $\mathrm{W}(D \times D^{op}, \sigma_h)$ *is zero.*

Proof: This follows as the hyperbolic functor $K_0(D) \to \mathrm{GW}(D \times D^{op}, \sigma_h)$ is an isomorphism because any symmetric object in $\mathcal{P}(D \times D^{op})$ is hyperbolic, see [18, p. 263]. □

The following result is classical, see e.g. [31, Theorem 7.8.1].

Theorem 1.29. *Endow* F *with the trivial involution. The Witt group* $\mathrm{W}_-(F)$ *is zero.*

Note that the two last theorems imply that the corresponding Grothendieck-Witt groups are isomorphic to $\mathbf{Z}$.

Remark 1.30. As we have seen in 1.20, in the general case, there are four mutualy exclusive possibilities for i. It can be of sign $+$, $-$, $+$ and $-$, or 0. In the last case, 1.28 says that $\mathrm{W}(\mathcal{A}_i)$ is zero, and 1.29 says that $\mathrm{W}(\mathcal{A}_i)$ is zero in the last three cases if the endomorphism ring of a simple object in $\mathcal{A}_i$ is just F.

Chosing

- a simple object S_i of $\mathcal{A}_i$ for each i and denoting $\mathrm{Hom}(S, S)$ by D_i
- an antisymmetric isomorphism a_i from S_i to S_i^* if i is of sign $-$
- a symmetric isomorphism s_i if i is of sign $+$ but not of sign $-$

and combining the results of 1.7, 1.14, 1.25, 1.27 and 1.28, one gets the following.

Theorem 1.31. *Assume that all objects in* $\mathcal{A}$ *are of finite length. Then we have an isomorphism of abelian groups*

$$\mathrm{GW}_\epsilon(\mathcal{A}) \simeq \bigoplus_{i \notin \{0,-\}} \mathrm{GW}_\epsilon(D_i, \sigma_{s_i}) \bigoplus_{i \in \{-\}} \mathrm{GW}_{-\epsilon}(D_i, \sigma_{a_i}) \bigoplus_{i \in \{0\}} \mathrm{GW}_\epsilon(D_i \times D_i^{op}, \sigma_h)$$

where $i \in \{\ldots\}$ is a slight abuse of notation to say that i has the signs in the set $\{\ldots\}$. This induces the isomorphism

$$\mathrm{W}_\epsilon(\mathcal{A}) \simeq \bigoplus_{i \notin \{0,-\}} \mathrm{W}_\epsilon(D_i, \sigma_{s_i}) \bigoplus_{i \in \{-\}} \mathrm{W}_{-\epsilon}(D_i, \sigma_{a_i})$$

Note that the indices i in the first summand are all of sign $+$ and that the Grothendieck-Witt groups appearing in the last summand are all isomorphic to $\mathbf{Z}$. Moreover, if $\epsilon = 1$ (resp. -1) all the Witt groups in the last (resp. first) summand are zero when $D_i \simeq F$.

1.4. **Monoidal structure and algebra structure of $GW(\mathcal{A})$ and $\mathrm{W}(\mathcal{A})$.** From now on, we assume that our category $\mathcal{A}$ is symmetric monoidal and has a unit object, denoted $\mathbf{1}$. Recall that this means that for any object M we have an isomorphism $\gamma_M : M \xrightarrow{\cong} \mathbf{1} \otimes M$ satisfying the standard coherence conditions (see [17]). Moreover, we assume that $F \xrightarrow{\cong} \mathrm{Hom}(\mathbf{1}, \mathbf{1})$, and $\mathbf{1}$ is isomorphic to its dual by a symmetric morphism $\phi_{\mathbf{1}}$. We also assume that we have natural isomorphisms $\rho : A^* \otimes B^* \xrightarrow{\cong} (A \otimes B)^*$. We will assume that ρ is compatible with respect to η in the sense that the square of [19, p. 80] commutes as well as the following

Condition. Let (M, ψ) be an ϵ-symmetric space. Then ψ equals $\gamma^* \rho(\phi_{\mathbf{1}} \otimes \psi)\gamma : M \to M^*$. (For our purposes it would be sufficient to require this for simple M.)

The monoidal structure induces a product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. This product is assumed to be bi-exact. Observe that this reduces to a product $\otimes_i : \mathcal{A}_{\mathbf{1}} \times \mathcal{A}_i \to \mathcal{A}_i$.

The product induces a product on the category of symmetric spaces by setting $(A, \phi) \otimes (B, \psi) := (A \otimes B, \rho \circ \phi \otimes \psi)$. Observe that $(\mathbf{1}, \phi_{\mathbf{1}})$ is a unit for the symmetric monoidal category of symmetric spaces in $\mathcal{A}$.

Remark 1.32. We could replace the above conditions by the stronger condition of the existence of a (monoidal) forgetful functor $\mathcal{A} \to Vect(F)$ such that we have a neutral Tannakian category. By the main theorem of Tannaka theory (see e.g. [9, Theorem 2.11]), the category $\mathcal{A}$ is then equivalent to the category of representations of an affine group scheme flat over F. This is indeed the example we are mainly interested in.

Definition 1.33. We denote by $\mathrm{GW}^{\mathrm{gr}}(\mathcal{A})$ (resp. $\mathrm{W}^{\mathrm{gr}}(\mathcal{A})$) the $\mathbf{Z}/2$-graded abelian group with $\mathrm{GW}_+(\mathcal{A})$ (resp. $\mathrm{W}_+(\mathcal{A})$) in degree 0 and $\mathrm{GW}_-(\mathcal{A})$ (resp. $\mathrm{W}_-(\mathcal{A})$) in degree -1.

Remark 1.34. By [5, Proposition 5.2], the graded triangular Witt groups are zero in odd degree for the bounded derived category of an abelian category. As all categories we consider are abelian, the group $\mathrm{GW}^{\mathrm{gr}}(\mathcal{A})$ is isomorphic to the $\mathbf{Z}/4$-graded Witt group of Balmer.

From the above discussion, we immediately get the following.

Lemma 1.35. *For a symmetric monoidal category with duality $(\mathcal{A}, *, \eta, \otimes)$, the product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ induces a ring structure on the group $\mathrm{GW}(\mathcal{A})$ (resp. $\mathrm{GW}^{\mathrm{gr}}(\mathcal{A})$) and equips this ring with a $\mathrm{GW}(\mathcal{A}_{\mathbf{1}})$-algebra structure. The same is true after replacing GW by W everywhere. The $\mathrm{GW}(\mathcal{A}_{\mathbf{1}})$ (resp. $\mathrm{W}(\mathcal{A}_{\mathbf{1}})$) algebra $\mathrm{GW}^{\mathrm{gr}}(\mathcal{A})$ (resp. $\mathrm{W}^{\mathrm{gr}}(\mathcal{A})$) is then a superalgebra ($\mathbf{Z}/2$-graduated).*

Proof: The statement about GW is clear. For W recall that the tensor product of a metabolic space with a symmetric space is again metabolic as $\otimes$ is exact. For $\mathrm{GW}^{\mathrm{gr}}$ and W^{gr}, just recall that the tensor product of an ϵ_1-symmetric space by an ϵ_2-symmetric space is an $\epsilon_1\epsilon_2$-symmetric space. □

Proposition 1.36. *The product of Lemma 1.35 restricts to a product*

$$\mu_i : \mathrm{GW}(\mathcal{A}_{\mathbf{1}}) \times \mathrm{GW}_\epsilon(\mathcal{A}_i) \to \mathrm{GW}_\epsilon(\mathcal{A}_i).$$

Proof: This follows directly from the isomorphism $\mathbf{1} \otimes M \simeq M$ for any object$M$. □

We also need the following for algebras with involutions. Let (D, σ) and (D', σ') be F-algebras with involutions. The tensor product (over F) of a D-module and a D'-module is a $D \otimes_F D'$-module. This defines a biexact functor $\otimes : D{-}\mathrm{mod} \times D'{-}\mathrm{mod} \to D \otimes_F D'{-}\mathrm{mod}$. This functor induces the products

$$\mathrm{GW}_\epsilon(D, \sigma) \times \mathrm{GW}_{\epsilon'}(D', \sigma') \longrightarrow \mathrm{GW}_{\epsilon\epsilon'}(D \otimes_F D', \sigma \otimes \sigma')$$

and

$$\mathrm{W}_\epsilon(D, \sigma) \times \mathrm{W}_{\epsilon'}(D', \sigma') \longrightarrow \mathrm{W}_{\epsilon\epsilon'}(D \otimes_F D', \sigma \otimes \sigma')$$

As a particular case, we get the products

$$\mu_D : \mathrm{GW}_\epsilon(F) \times \mathrm{GW}_{\epsilon'}(D, \sigma) \longrightarrow \mathrm{GW}_{\epsilon\epsilon'}(D, \sigma)$$

and

$$\mathrm{W}_\epsilon(F) \times \mathrm{W}_{\epsilon'}(D, \sigma) \longrightarrow \mathrm{W}_{\epsilon\epsilon'}(D, \sigma)$$

Proposition 1.37. *For any i of sign ϵ, the following diagramm is commutative.*

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{A}_{\mathbf{1}}) \times \mathrm{GW}_{\epsilon'}(\mathcal{A}_i) \arrow[r, "\mu_i"] \arrow[d, "F_{\mathbf{1}} \times F_i"] & \mathrm{GW}_{\epsilon'}(\mathcal{A}_i) \arrow[d, "F_i"] \\
\mathrm{GW}(F) \times \mathrm{GW}_{\epsilon\epsilon'}(D_i, \sigma_i) \arrow[r, "\mu_{D_i}"] & \mathrm{GW}_{\epsilon\epsilon'}(D_i, \sigma_i)
\end{tikzcd}$$

For any i of sign 0, the following diagramm is commutative.

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{A}_{\mathbf{1}}) \times \mathrm{GW}_\epsilon(\mathcal{A}_i) \arrow[r] \arrow[d] & \mathrm{GW}_\epsilon(\mathcal{A}_i) \arrow[d] \\
\mathrm{GW}(F) \times \mathrm{GW}_\epsilon(D_i \times D_i^{op}, \sigma_h) \arrow[r] & \mathrm{GW}_\epsilon(D_i \times D_i^{op}, \sigma_h)
\end{tikzcd}$$

Proof: This is a painful but straightforward computation, involving naturality of the ρ, the commutativity of the square of [19, p. 80] and Condition 1.4. □

Corollary 1.38. *Let S be a simple object in $\mathcal{A}_i$ such that $\mathrm{Hom}(S, S) \simeq F$. We have the two following possibilities.*

(1) *i is of sign ϵ. Equip S with an ϵ-symmetric isomorphism f from S to S^*. The morphism $.(S, f) : \mathrm{GW}(\mathcal{A}_{\mathbf{1}}) \to \mathrm{GW}_\epsilon(\mathcal{A}_i)$ is an isomorphism of $\mathrm{GW}(\mathcal{A}_{\mathbf{1}})$-modules.*
(2) *i is of sign 0. The morphism $.H(S) : \mathrm{GW}(\mathcal{A}_{\mathbf{1}}) \to \mathrm{GW}(\mathcal{A}_i) \simeq \mathbf{Z}$ is a surjective morphism of $\mathrm{GW}(\mathcal{A}_{\mathbf{1}})$-modules.*

Proof: Use the commutative diagram of Proposition 1.37, Corollaries 1.25 and 1.27 and observe that both properties are obvious on the lower line. □

Corollary 1.39. *In the case i of sign ϵ in the previous lemma, the isomorphim $.(S, f)$ induces an isomorphism $\mathrm{W}(\mathcal{A}_{\mathbf{1}}) \to \mathrm{W}(\mathcal{A}_i)$ of $\mathrm{W}(\mathcal{A}_{\mathbf{1}})$-modules. The case i of sign 0 just gives the trivial surjection $\mathrm{W}(\mathcal{A}_{\mathbf{1}})$ onto 0.*

As a consequence of Proposition 1.37, we have the following.

Theorem 1.40. *The first isomorphism in 1.31 is a $\mathrm{GW}(\mathcal{A}_{\mathbf{1}})$-module isomorphism, and the second is a $\mathrm{W}(\mathcal{A}_{\mathbf{1}})$-module isomorphism.*

2. Reductive groups

In this section, we apply the results of the preceding section to the abelian category of F-linear representations of an F-split reductive group G.

2.1. **Representations of linear algebraic groups.**

2.1.1. *General properties.* Let H be a linear algebraic group defined over F. Then $\mathrm{Rep}(H)$, the category of finite dimensional F-linear representations of H is an abelian category with the standard duality $* = Hom_F(\ \ , F)$ on the underlying F-vector space. The monoidal structure on $\mathrm{Rep}(H)$ is given by the tensor product of the underlying vector spaces, and $\mathbf{1}$ is given by the one-dimensional vector space F on which H acts by the identity. All objects in $\mathrm{Rep}(H)$ are of finite length, this is an evident consequence of they being of finite dimension as vector spaces, so the requirements of Theorem 1.31 - and therefore 1.40 - are satisfied. A simple object of $\mathrm{Rep}(H)$ is an irreducible representation of H. To be coherent with Bourbaki ([8]) or Serre ([32]), we usually denote an irreducible representation by the letter E.

Lemma 2.1. *Let H and H' be algebraic groups over F and let $\alpha : H \to H'$ be a morphism defined over F. Then*

(1) *the restriction morphism induces a faithfull functor of categories with duality from $\mathrm{Rep}(H')$ to $\mathrm{Rep}(H)$. The image of this functor is stable by subobjets and quotients.*
(2) *If α is surjective (in the category of algebraic groups), $\mathrm{Rep}(H')$ is a full subcategory of $\mathrm{Rep}(H)$. In the decomposition of 1.14, $\mathrm{GW}^{\mathrm{gr}}(\mathrm{Rep}(H'))$ (resp. $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(H'))$) is isomorphic to a direct summand of $\mathrm{GW}^{\mathrm{gr}}(\mathrm{Rep}(H))$ (resp. $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(H))$), namely the one containing the summands corresponding to all the indices i such that a simple module of type i is in $\mathrm{Rep}(H')$ (resp. only the indices of nonzero sign).*

Proof: Point 1 is clear. Point 2 is a straightforward consequence of 1.16 if we prove that the subcategory is full. This follows from the fact that the restriction of a representation to H is the same underlying vector space, with an algebraic action of H through H' by α. Since α is a surjection in the caterory of algebraic groups, then any morphism on the underlying vector space, commuting with the action of H through α will also commute with the action of H'. □

2.1.2. *Properties of* $\mathrm{Rep}(G)$. Let G denote an F-split reductive group.

Proposition 2.2. *Let E be an irreducible representation. Then the canonical inclusion $F \to \mathrm{Hom}(E, E)$ is an isomorphism.*

Proof: Any division ring over F_{sep} (a separable closure of F) is just F_{sep}, so the property is true if $F = F_{sep}$. According to [35, Theorem 2.5], an irreducible representation E stays irreducible after any scalar extension. The inclusion $\mathrm{Hom}_F(E, E) \otimes F_{sep} \subset \mathrm{Hom}_{F_{sep}}(E_{F_{sep}}, E_{F_{sep}}) \simeq F_{sep}$ therefore yields the result, since it implies that $\mathrm{Hom}_F(E, E)$ is one dimensional as an F vector space. □

Corollary 2.3. *The hypothesis of Lemma 1.21 are fulfilled for any irreducible representation, so the sign of an index i is unique.*

The following properties for reductive groups are classical, see [7]. Our exposition and notations follow closely [33, chapters 7 and 8]. For the properties of abstract root systems, the reference is [8].

Let T denote a split maximal torus of G, $X = X(T)$ its character group (morphisms from T to $\mathbf{G}_m$) . The roots of G with respect to T are the non trivial elements α of X such that the adjoint representation of G in its Lie algebra restricted to T acts through α on a non trivial subspace. The set of roots of G with respect to T is denoted by $\phi = \phi(G,T)$. It is a finite subset of X, and generates a subgroup of X denoted by Q. Let V' be the subspace generated by Q (or ϕ) of the $\mathbf{R}$-vector space $V = X \otimes \mathbf{R}$. The set ϕ forms a reduced root system (in the sense of [8, chapter VI]) of V'. To each root α corresponds an element $\alpha^\vee$ in $(V')^*$ (see [8, ch. VI, §1, def. 1]). The elements p of V' such that $\alpha^\vee(p) \in \mathbf{Z}$ for all $\alpha \in \phi$ are called the weights of ϕ. They form a subgroup of V' called P wich contains $X \cap V'$. The Weyl group $W = N_G(T)/C_G(T)$ of (G,T) acts naturally on V and preserves V'. It is naturally identified with the Weyl group of ϕ through its action on V'. There is a subgroup N of X (namely the characters wich are zero on the connected component of the identity of the intersection of T and the derived subgroup of G) such that W acts trivially on N and such that any scalar product on V invariant by W makes V' and $V'' = N \otimes \mathbf{R}$ orthogonal. Any character $x \in X$ can therefore be written uniquely $x = x' + x''$ with $x \in X \cap V'$ and $x'' \in N$. A Weyl chamber C of V is just a $C' \times V''$, where C' is a Weyl chambers of V'. The Weyl group acts simply tranisitively on their set.

Proposition 2.4. *Let R denote the radical of G. It is, by definition, the maximal closed, connected, normal solvable sugroup of G.*

(1) *It is a torus defined over F and the quotient G/R is a split semisimple group G_{ss}.*
(2) *The split maximal torus T of G contains R, and the quotient T/R is a split maximal torus T' of G_{ss}.*
(3) *The restriction of the caracters of T' to T is the subgroup $X \cap V'$.*

Proof: The radical of a reductive group is a subtorus of the identity component of G. It is contained in any maximal torus, therefore in T. Since T is split, R is defined over F and split (see [33, 13.2.5, (1)]), and the quotient T/R is a torus defined over F. It is maximal in G/R, otherwise T would not be maximal. □

When G is semisimple, we therefore have $V = V'$ and Q is a subgroup of finite index in X. The group G is then said to be simply connected if $P = X$ and adjoint if $Q = X$.

Theorem 2.5. *Let H be an semisimple algebraic group.*

(1) *There are unique (up to isomorphism) groups H_{sc} and H_{adj} such that H_{sc} is simply connected, H_{adj} is adjoint and there are central isogenies (surjective morphisms with finite kernel in the center of the group) $H_{sc} \to H$ and $H \to H_{adj}$.*
(2) *Let G be a split semisimple algebraic group. Then G_{sc} and G_{adj} are also split. The above isogenies induce isogenies $\tilde{T} \to T$ and $T \to \bar{T}$ with the same kernels, where $\tilde{T}$ (resp. $\bar{T}$) is a split maximal torus of G_{sc} (resp. G_{adj}).*
(3) *Taking the morphisms to $\mathbf{G}_m$ ("dualizing"), one obtains inclusions $X(\bar{T}) \subset X \subset X(\tilde{T})$ where $X(\tilde{T})$ can be identified with P in $V = X \otimes \mathbf{R}$ and $X(\bar{T})$ can be identified with Q.*

Proof: For point 1, see [20, Theorem 26.7], for point 2 and 3, see [20, Central isogenies, p. 357]. □

2.2. **The Witt and Grothendieck-Witt groups of** $\mathrm{Rep}(G)$**.** The irreducible representations of a torus T are the one dimensional representations on which T acts through a character $x \in X$. The category $\mathrm{Rep}(T)$ of finite dimensional representations of T is semisimple and therefore every representation L of T decomposes as a direct sum of one dimensional ones. The characters through which T acts on these are called the weights of T in L. This gives a natural isomorphism from $K_0(\mathrm{Rep}(T))$ to $\mathbf{Z}[X]$. By restricting a representation L of G to T, one therefore obtains an element in $\mathbf{Z}[X]$ (denoted by $\mathrm{ch}_G(L)$ in [32, 3.6], we shall do alike). The multiplication in $\mathbf{Z}[X]$ is compatible with the tensor product of representations.

Choosing a B a Borel subgroup of G containing T, one gets a set of positive roots in ϕ, and a corresponding basis $(\alpha_1, \dots, \alpha_n)$ of ϕ. Let $(\omega_1, \dots, \omega_n)$ be the dual basis of $(\alpha_1^\vee, \dots, \alpha_n^\vee)$. It is a basis of V' (as vector space) and P (as free group). A weight $p \in P$ (resp. a character $x \in X$) is said to be dominant if $\alpha_i^\vee(p) \geq 0$ (resp. $\alpha_i^\vee(x) \geq 0$) for all $i \in \{1, \dots, n\}$. Denote P^+ (resp. X^+) the set of dominant weights (resp. characters). They are exactly the set of weights (resp. characters) that are sums with positive integral coefficients of fundamental weights.

Proposition 2.6. *(see [32, 3.6, Lemma 5]) Let x be an element of X^+. There exists a unique (up to isomorphism) irreducible representation E_x of G such that $\mathrm{ch}_G(E_x) = x + \sum_i x_i$, where $x_i < x$ (using the order defined above). Moreover, every irreducible representation of G appears that way.*

Since the duality permutes the irreducible representations, it acts (by a permutation of order at most 2) on X^+ through the bijection of Proposition 2.6. We say that an element $x \in X^+$ is selfdual when it is fixed by this action of the duality. In fact this permutation extends to a linear automorphism of $V = V'$ which preserves X. This follows from the following.

Lemma 2.7. *Let W be the Weyl group of ϕ (and therefore of G). There is an element $w_C \in W$ such that $(E_x)^* = E_{-w_C(x)}$. Thus, $E_x \simeq (E_x)^*$ if and only if $x = -w_C(x)$. We denote $-w_C(x)$ by x^*. We denote this element w_0 (as usually done) when a Weyl chamber C has been chosen once for all to fix an order on X.*

Proof: Since the positive elements with respect to C are the negative elements with respect to $-C$, the highest weight of $(E_x)^*$ with respect to $-C$ is $-x$. There is a unique element $w_C \in W$ that sends $-C$ to C (W acts simply transitively on the set of Weyl chambers), and $w(-x)$ is the highest weight of $(E_x)^*$ with respect to C. □

We then denote by $s(x)$ the sign of E_x. We then also say that an element in $\mathrm{GW}^{\mathrm{gr}}(\mathrm{Rep}(G))$ (resp. $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G))$) is of type x when it is in $\mathrm{GW}(F).(E_x, \phi_x)$, where ϕ_x is an isomorphism from E_x to $(E_x)^*$. This is coherent with our previous notation (1.11), identifying x with the isomorphism class of irreducible representations corresponding to x through the previous proposition.

2.2.1. *Restriction to the semisimple case.*

Lemma 2.8. *Let x be a character. If $E_x \simeq (E_x)^*$, then $x \in V'$.*

Proof: Write $x = x' + n$, $x' \in X \cap V'$ and $n \in N$. Since W acts trivially on N, $-w_0(x) = x$ implies $-w_0(x') = x'$ and $-n = n$. □

Corollary 2.9. *The restriction morphism from $\mathrm{Rep}(G_{ss})$ to $\mathrm{Rep}(G)$ induces an isomorphism from $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G_{ss}))$ to $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G))$.*

Proof: The semisimple quotient exists and is defined over F thanks to Proposition 2.4. We can therefore apply Lemma 2.1 (2). For any $x \in (X \cap V')^+$, we can consider the irreducible representation E_x of G_{ss} because $X \cap V'$ is identified with the dominant characters of G_{ss} by Proposition 2.4 (3). The restriction of E_x to G is still irreducible by Remark 1.17, and naturally coincides with the E_x of G, according to its highest weight. The Lemma 2.8 then says that the simple representations isomorphic to their dual all comme from G_{ss}. □

2.2.2. *The semisimple case.*

Theorem 2.10. *For each selfdual dominant character x, choose an isomorphism $\phi_x : E_x \to (E_x)^*$. It is automatically $s(x)$-symmetric, thanks to Lemma 1.21. The classes of the elements (E_x, ϕ_x) form a basis of the* $\mathrm{W}(F)$*-module* $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G))$*. If we only take the classes of elements of sign $+$, we get a basis of the* $\mathrm{W}(F)$*-module* $\mathrm{W}(\mathrm{Rep}(G))$.

Proof: This is just a rephrasing of 1.31, 1.38 and 1.40 for $\mathrm{Rep}(G)$. □

Remark 2.11. Such a theorem cannot be stated for $\mathrm{GW}(\mathrm{Rep}(G))$, not because we lack generators, but because the map $\mathrm{GW}(F) \to \mathrm{GW}(\mathrm{Rep}(G)_{E_x})$ given by the product by an element in $\mathrm{GW}(\mathrm{Rep}(G)_{E_x})$ is never injective, when x is of sign 0 (see 1.38). For this reason, it is not a free module.

Lemma 2.12. *The duality, acting on P by $-w_0$ permutes the fundamental weights $\omega_1, \dots, \omega_n$.*

Proof: The fundamental weights form a basis of the monoid P^+, meaning that every element of P^+ can be written uniquely as a sum with positive integral coefficients of fundamental weights. Such a basis is unique, because P^+ is isomorphic to $\mathbf{N}^n$ as a monoid. The element $-w_0$ acts linearly and bijectively on P, and preserves P^+, therefore it has to send a basis on a basis. □

Let σ denote the permutation such that $\omega_{\sigma(i)} = (\omega_i)^*$. It is of order at most 2, therefore its orbits σ_l have at most two elements. Let o_l denote $\sum_{i\in\sigma_l} \omega_{\sigma(i)}$.

Lemma 2.13. *Let x be a dominant character. We have $x = x^*$ ($E_x \simeq (E_x)^*$) if and only if $x = \sum_l n_l o_l$ with $n_l \in \mathbf{N}$, and l running through the orbits of σ.*

Proof: It is a consequence of Lemmas 2.7 and 2.12. □

2.3. **Algebra structure of the Witt group of a reductive group.** We now give a description of $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G))$ as a $\mathrm{W}(F)$-algebra. It will turn out (see Theorem 2.16) to be a subring of a polynomial ring in a finite number of variables on $\mathrm{W}(F)$ (and the whole polynomial ring when G is semisimple simply connected).

Lemma 2.14. *Let $x, y \in X^+$ be selfdual. Let $\phi_x : E_x \to (E_x)^*$ and $\phi_y : E_y \to (E_y)^*$ be isomorphisms. Then $x + y = (x + y)^*$ and there exists an isomorphism $\phi_{x+y} : E_{x+y} \to (E_{x+y})^*$ and elements T_i of type $i < x + y$ such that*

$$(E_x, \phi_x).(E_y, \phi_y) = (E_{x+y}, \phi_{x+y}) + \sum_i T_i$$

in $\mathrm{GW}(\mathrm{Rep}(G))$*, and so also in* $\mathrm{W}(\mathrm{Rep}(G))$.

Proof: The character $x+y$ is selfdual since $(x+y)^* = -w_0(x+y) = -w_0(x) - w_0(y) = x + y$. The required equality can be checked with the forgetful functor to K_0, by remark 1.15. Since $K_0(\mathrm{Rep}(G))$ injects in $K_0(\mathrm{Rep}(T))$ by [32, 3.6, Theorem 4], the equality can be checked in $K_0(\mathrm{Rep}(T))$, where it is clear, for all the weights of $E_x \otimes E_y$ but $x + y$ are strictly smaller than $x + y$, therefore E_{x+y} must appear once and the other simple modules must have smaller highest weights. □

Lemma 2.15. *If $x = x^*$ and $y = y^*$, we have $s(x).s(y) = s(x + y)$.*

Proof: This can be checked by restricting the representations E_x and E_y to $\mathrm{Rep}(T)$. The result is then implied by the fact that E_{x+y} is then a direct summand of $E_x \otimes E_y$. □

Let $(X^+)^0$ (resp. $(P^+)^0$) be the subset of X^+ (resp. P^+) fixed by duality. It is a submonoid of X^+ (resp. P^+) by 2.14. For any monoid M, let $\mathbf{Z}[M]$ denote the monoid ring over M. An inclusion $M \subset M'$ of monoids induces an inclusion $\mathbf{Z}[M] \subset \mathbf{Z}[M']$. We denote by $e^m \in \mathbf{Z}[M]$ the element corresponding to $m \in M$.

Theorem 2.16. *When G is simply connected, there is an isomorphism of* $\mathrm{W}(F)$*-algebras from* $\mathrm{W}(F)\otimes_{\mathbf{Z}}\mathbf{Z}[(X^+)^0]$ *to* $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G))$*. When G is not necessarily simply connected, then* $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G))$ *is just a subalgebra of* $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G_{sc}))$*.*

Proof: Let us first prove the theorem in the simply connected case. We then have $X = P$, and $(X^+)^0$ is therefore a free $\mathbf{Z}$-module with the o_i as a base (see 2.13). The ring $\mathrm{W}(F)\otimes\mathbf{Z}[(X^+)^0]$ is therefore a polynomial ring over $\mathrm{W}(F)$ with a finite number of variables (e^{o_i}). For each o_i, we choose an automorphism ϕ_{o_i} of E_{o_i}. We then consider the morphism f sending e^{o_i} to the class of (E_{o_i},ϕ_{o_i}). By Lemma 2.14, for any $x\in(X^+)^0$, we have $f(e^x) = (E_x,\phi_x)+\sum T_x$, for some isomorphism ϕ_x and for some T_x of type smaller than x. By 2.10, the (E_x,ϕ_x) form a base of the $\mathrm{W}(F)$-module $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G))$, so do the $f(e^x)$ by [8, Ch. VI, §3, n°4, Lemma 4], hence the result in the simply connected case.

For the general case, just use 2.1 point 2 and 2.5 point 1. □

2.4. **Parabolic subgroups.**

Theorem 2.17. *Let G be an algebraic group over F. Suppose its unipotent radical U is defined over F. Then the reductive group $G_{red} = G/U$ is defined over F, and the morphism $G\to G_{red}$ induces an isomorphism* $\mathrm{GW}^{\mathrm{gr}}(\mathrm{Rep}(G_{red}))\simeq\mathrm{GW}^{\mathrm{gr}}(\mathrm{Rep}(G))$ *(resp.* $\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G_{red}))\simeq\mathrm{W}^{\mathrm{gr}}(\mathrm{Rep}(G))$*).*

Proof: If we show that U acts trivially on any simple representation of G, it will show that any irreducible representation of G comes from G/U. The result is then a consequence of Lemma 2.1, point 2, with $\alpha : G\to G/U$. Let E be a simple representation of G. By Borel's theorem (see [33, Theorem 6.2.6]) applied to the unipotent - therefore solvable - group U, the restriction of E to U has a nontrivial subrepresentation E_1 on which U acts trivially. But since U is normal in G, this subrepresentation is stable by G, and since E is irreducible, we must have $E_1 = E$. □

Corollary 2.18. *Let G be a split reductive group, P a parabolic subgroup of G containing a split maximal torus T of G. Then* $\mathrm{GW}^{\mathrm{gr}}(\mathrm{Rep}(P_{red}))\simeq\mathrm{GW}^{\mathrm{gr}}(\mathrm{Rep}(P))$*.*

Proof: By [33, Proposition 16.1.1], the unipotent radical of P is defined over F, then apply Theorem 2.17. □

3. A few functors

3.1. **The Picard group of a monoidal category.**

Let $(\mathcal{A},\otimes)$ be an abelian monoidal category as in section 1.4.

Definition 3.1. Let $\mathrm{Pic}(\mathcal{A})$ be the set of the isomorphism classes of objets A of $\mathcal{A}$ such that there exists another object A' such that $A\otimes A'\simeq\mathbf{1}$. The tensor product induces a group structure on $\mathrm{Pic}(\mathcal{A})$. We call this group the Picard group of $\mathcal{A}$.

Proposition 3.2. *Any (lax) monoidal functor of monoidal categories induces a morphism on the Picard groups. In particular, an equivalence of monoidal categories from $\mathcal{A}$ to $\mathcal{A}'$ induces an isomorphism* $\mathrm{Pic}(\mathcal{A})\simeq\mathrm{Pic}(\mathcal{A}')$*.*

Example 3.3.
1. Let $(\mathrm{Vect}(X),\otimes_{\mathcal{O}_X})$ be the category of vector bundles (locally free sheaves of finite dimension) over a scheme X. (All schemes in this article are assumed to be seperated.) Then $\mathrm{Pic}(\mathrm{Vect}(X))$ is the usual Picard group $\mathrm{Pic}(X)$.
2. Let H be an algebraic group and X an H-variety. By this we mean a morphism $H\times X\to X$ satisfying the standard properties. Let $\mathrm{Vect}^H(X)$ be the category of H-equivariant vector bundles over X. This is a full subcategory of the category of H-equivariant-O_X-modules. We say that

an H-equivariant-O_X-modules is coherent if the underlying O_X-module is, and we denote their category by $Coh^H(X)$. See [34] and [23, section 3] for the precise definitions and basic properties. We define the equivariant Picard group of X by $\mathrm{Pic}^H(X) = \mathrm{Pic}(\mathrm{Vect}^H(X))$. Of course, for $H = \{1\}$ we obtain example (1).

(3) Let H be an algebraic group. An element of $\mathrm{Pic}(\mathrm{Rep}(H))$ is the isomorphy class of a 1-dimensional representation. Its inverse is given by the dual representation. This defines a natural isomorphism between $\mathrm{Pic}(\mathrm{Rep}(H))$ and the group $\mathrm{X}(H) = \mathrm{Hom}(H, \mathbf{G}_m)$ of characters of H. We identify these two groups by this isomorphism.

Lemma 3.4. *There is a forgetful functor from $\mathrm{Pic}^G(X)$ to $\mathrm{Pic}(X)$. This functor is surjective if $X = G/P$, for P a parabolic subgroup of an semisimple simply connected group G.*

Proof: See [25, Corollary 1.6]. □

3.2. The Ind and Res functors.

Definition 3.5. Let G be a linear algebraic group over the field F and P a closed subgroup of G. There are well known equivalences of categories $\mathrm{Res} : \mathrm{Vect}^G(G/P) \to \mathrm{Rep}(P)$ and $\mathrm{Ind} : \mathrm{Rep}(P) \to \mathrm{Vect}^G(G/P)$ inverse to eachother (see [28, §1]).

When E and E' are representations of G, the representations $\Lambda^k(E)$, $S^k(E)$, $E^{\otimes k}$ (exterior, symmetric and tensor products) and $\mathrm{Hom}_F(E, E')$ are well defined and well-known. Similarly, the same constructions exist for vector bundles and equivariant vector bundles over X.

Lemma 3.6. (1) *The* Ind *and* Res *functors commute (in an obvious way) with these constructions.*
(2) *They therefore induce isomorphisms inverse to each other between $\mathrm{X}(P)$ and $\mathrm{Pic}^G(G/P)$.*

The categories $\mathrm{Rep}(P)$ and $\mathrm{Vect}^G(X)$ can be endowed with different dualities. Let U (resp. L) be a 1-dimensional representation of P (resp. a G-equivariant line bundle over X). The functor $\mathrm{Hom}_F(-, U)$ (resp. $\mathrm{Hom}(-, L)$) is a duality functor on $\mathrm{Rep}(P)$ (resp. $\mathrm{Vect}^G(X)$). The morphism of functor ϖ_U (resp. ϖ_L) identifying an object with its bidual is just the usual evaluation. We use the notations $(.)^*$ for the usual dual ($L = \mathcal{O}_X$), $(.)^{\vee L}$ when we use the line bundle L, and sometimes just $(.)^\vee$ when the line bundle L is clear.

Proposition 3.7. *The functor* Ind *induces an equivalence of categories with duality from $(\mathrm{Rep}(P), \mathrm{Hom}(-, L), \varpi_L)$ to $(\mathrm{Vect}^G(G/P), \mathrm{Hom}(-, \mathrm{Ind}(L)), \varpi_{\mathrm{Ind}(L)})$.*

Proof: This is a simple consequence of Lemma 3.6. □
Of course, we can state the analogous theorem with Res.

3.3. **A few well-know sheafs.** The Ind functor gives us an isomorphism from $\mathrm{X}(P)$ to $\mathrm{Pic}^G(G/P)$ for a closed subgroup of an algebraic group G. By Remark 3.4, it is clear that the canonical sheaf $\omega_{(G/P)}$ (i.e. the maximal exterior power of the cotangent sheaf) can be endowed with a natural G action. Nevertheless, we would like to understand from which character of P it comes from. This is the result of the following proposition.

Proposition 3.8. *Let Ad_G and Ad_P denote the adjoint representations of G and P. Let $\Lambda^{max} V = \Lambda^{\dim V} V$ denote the maximal exterior power of a representation. We have the isomorphism*

$$\omega_{G/P} \simeq \mathrm{Ind}(\Lambda^{max}(\mathrm{res}_P^G(Ad_G)^*) \otimes \Lambda^{max}(Ad_P))$$

Proof: The algebraic group G acts on the variety G by conjugation. The point 1 is fixed by this action, so it induces a linear action of G on the tangent space T_1G to the variety G at 1. This is, by definition the adjoint representation Ad_G of G. The subgroup P acts similarly on T_1G/P and we have an exact sequence of P representations

$$0 \to T_1P \to T_1G \to T_1G/P \to 0.$$

Dualizing, and taking the maximal exterior powers, we get the result. □

We also give the analogous result for the usual $\mathcal{O}_X(i)$, when $X = \mathbf{P}^{n-1} = SL_n/P$, where P is the subgroup of SL_n fixing the line generated by the first coordinate.

Theorem 3.9. *Let T be the diagonal torus of SL_n. Let $t_1 \in \mathrm{X}(T)$ be the character corresponding to the first coordinate.*

(1) *The restriction morphism* $\mathrm{res} : \mathrm{X}(P) \to \mathrm{X}(T)$ *identifies $X(P)$ with $\mathbf{Z}$ enerated by t_1 in $X(T)$.*
(2) *Let* $\mathrm{res}(p_1) = t_1$*. Then* $\mathcal{O}_{\mathbf{P}^{n-1}}(-i) = \mathrm{Ind}((p_1)^i)$*.*

Proof: We just have to prove this for $\mathcal{O}(-1)$, in which case it is a straightforward consequence of its definition and the identification of G/P with $\mathbf{P}^{n-1}$. □

4. Witt motives

In this section, we will construct transfer maps between (Grothendieck-)Witt groups with respect to proper morphisms and establish some properties such as the base change and the projection formula. In contrast to K_0, the transfer maps for (Grothendieck-)Witt groups will shift the degree and twist the duality. Using section 3, it seems straigthforward to generalize the construction of transfer maps to the H-equivariant setting for an algebraic group H, but we have not checked this in full detail, so the careful reader should assume $H = \{1\}$. These transfer maps and their properties are then used for the construction of the categories $\mathbf{GW}^H$ and $\mathbf{W}^H$ of Grothendieck-Witt motives and Witt motives with respect to an algebraic group H. This category is the analogue of the category $\mathbf{K}^H$ of [28, section 6] which is the crucial construction for Panin's computations.

Everything in the sequel is true both for GW and W, so we just state everything for W.

We observe that in some very special cases there are already constructions that deserve the name transfer map. In particular, for any projection map $\pi : \mathbf{P}^n \times X \to X$, Walter establishes maps $W^i(\mathbf{P}^n \times X, \pi^*L(-n-1)) \to W^{i-n}(X, L)$ [37, p. 24] which using in particular Theorem 5.6, Proposition 5.11 and p.23/24 of *loc. cit.* can be seen to be natural with respect to X. Also, there seems to be work in progress by C. Walter on the construction of transfer maps in a very general setting (which should presumably yield the same transfer maps as those we constructed). There are also transfer constructions for Witt groups with respect to certain finite maps and closed embeddings in the affine space [13], [38], but not for other projective morphisms which is what we need.

4.1. Some derived categories.

In order to define transfers, it will be necessary to consider larger categories than just $D^b(Vect(X))$. We denote $D^b(X)$ the bounded derived category of sheaves of O_X-modules and by $D^b_c(X)$ and $D^b_{qc}(X)$ the full subcategories of complexes with coherent resp. quasi-coherent cohomology. For X noetherian regular of finite Krull dimension, the inclusion $D^b(Vect(X)) \to D^b_c(X)$ is an equivalence of triangulated categories.

The inclusion $D^b_c(X) \to D^b_{qc}(X)$ is fully faithful by definition, but never an equivalence. Recall that for X locally noetherian, the functor $D^b(QCoh(X)) \to D^b_{qc}(X)$

is an equivalence, see e.g. [16, Corollary II.7.19]. Under suitable assumptions (see appendix A, and in particular A.8.2), the category $D^b(X)$ has an internal Hom denoted by $\mathbf{R}\underline{Hom}_X$ or just $\mathbf{R}\underline{Hom}$ which is right adjoint to $\otimes^{\mathbf{L}}_{O_X}$ and restricts to internal Homs in $D^b_c(X)$ and $D^b_{qc}(X)$.

Remark 4.1. *Sign conventions.* First of all, we use chain complexes, as in Balmer's work (i.e. the differential in degree n is $d_n : A_n \to A_{n-1}$). The sign conventions that we then use are discussed in A.8.1.

4.2. **Transfers.** If $f : X \to Y$ is proper and Y locally noetherian, then there is a functor $\mathbf{R}f_* : D^+_{qc}(X) \to D^+_{qc}(Y)$ and similar for D^+_c (see [16, p. 88-89]). The construction of transfers for Witt groups along $\mathbf{R}f_*$ will rely on the following duality theorem due to Grothendieck-Verdier (-Hartshorne-Deligne) (see [36, Proposition 3, p. 404]):

Theorem 4.2. *Let $f : X \to Y$ be a proper morphism of noetherian schemes of finite Krull dimension. Then there is a functor $f^! : D^+_{qc}(Y) \to D^+_{qc}(X)$ and a natural transformation Tr_f such that for all $F \in D^-_{qc}(X)$, $G \in D^+_{qc}(Y)$ the composition*

$$\tilde{\alpha} : \mathbf{R}f_*\mathbf{R}\underline{Hom}(F, f^!G) \stackrel{a\mathbf{R}f_*}{\to} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, \mathbf{R}f_*f^!G) \stackrel{Tr_f}{\to} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, G)$$

is an isomorphism in $D^+_{qc}(Y)$.

Applying the global section functor $\mathbf{R}\Gamma(Y, \quad)$ and using the isomorphism of functors $\mathbf{R}\Gamma(X, \quad) \stackrel{\simeq}{\to} \mathbf{R}\Gamma(Y, \mathbf{R}f_*(\quad))$ (see [16, II.Proposition 5.2]), the isomorphism of the theorem becomes an isomorphism $\mathbf{R}\underline{Hom}(F, f^!G) \stackrel{\simeq}{\to} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, G)$ in $D^b(\mathbf{Ab})$. Observe also that since f is proper, the above statement remains true after replacing qc by c everywhere by [16, Proposition II.2.2 and p. 383].

Applying H^0, we obtain (see [36, Theorem 1]):

Corollary 4.3. *In the above situation, the functors $Rf_* : D^+_{qc}(X) \to D^+_{qc}(Y)$ and $f^! : D^+_{qc}(Y) \to D^+_{qc}(X)$ form an adjoint pair.*

Proof: Apply H^0 to the isomorphism $\mathbf{R}\underline{Hom}(F, f^!G) \stackrel{\simeq}{\to} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, G)$ in $D^b(\mathbf{Ab})$. $\square$

Remark 4.4. In fact, Verdier proceeds in the other direction. That is, he first states Corollary 4.3 and then deduces Theorem 4.2 using the projection formula. We will also use Corollary 4.3 and the projection formula to construct a natural isomorphism $\alpha : \mathbf{R}f_*\mathbf{R}\underline{Hom}(F, f^!G) \to \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, G)$ (see Theorem A.24), but in a different way than Verdier.

Being part of an adjoint pair, the functor $f^! : D^+_{qc}(X) \to D^+_{qc}(Y)$ and the natural transformation Tr_f are unique up to unique isomorphism, see [36, p. 394]. There are at least two different ways to construct them and to prove the isomorphism of the theorem (see also [26] for still another approach). One is to use residual complexes as Hartshorne [16] does. The other is to use apply the techniques of [36] as done by Deligne in the appendix of [16]. We will use this second construction. Although $f^!$ and Tr_f are unique up to unique isomorphism, this does not automatically mean that we can say explicitely how the isomorphism between the constructions of Hartshorne and Deligne looks like.

We now explain why this theorem is useful to define transfers. Given a line bundle L on a Gorenstein scheme X (for instance $L = O_Y$, or $L = \omega_Y := \omega_{Y/F}$ the canonical sheaf if X is smooth over F), the functor $* := *_L := \underline{Hom}_{O_X}(\quad, L)$ is a duality functor on $Vect(X)$ with the natural isomorphism $\varpi : Id \stackrel{\simeq}{\to} **$ defined below. This induces a duality on the triangulated category $\mathbf{R}\underline{Hom}(\quad, L)$ on $D^b(Vect(X))$

where L is considered as a complex concentrated in degree 0. We will work with the larger category $D^b_c(X)$ instead on which $\mathbf{R}\underline{Hom}(\ \ , L)$ is still a duality (see [11, 2.5.3 p. 115] and [16, Theorem V.3.1]) and gives rise to the so-called *coherent Witt groups* (compare [11, Definition 2.16]). For the precise definition of the ϖ with respect to $\mathbf{R}\underline{Hom}(\ \ , L)$ on $D^b_c(X)$ we use and for a comparison with the signs choosen in [11, p. 112] see Remark 4.5 and sections A.4 and A.8 of the appendix. As we always work with coherent Witt groups (instead of derived categories of vector bundles), we denote these simply by $W^*(X)$:

Definition 4.5. Let L be a line bundle on a scheme X which is Gorenstein noetherian of finite Krull dimension. Then we define

$$W^i(X, L) := W(D^b_c(X), \mathbf{R}\underline{Hom}(\ \ , L), \varpi_L).$$

Beware that following Hartshorne, in the notation $\mathbf{R}\underline{Hom}(\ \ , L)$ the $\mathbf{R}$ means $\mathbf{R}_I\mathbf{R}_{II}$, so we replace the line bundle L by an injective resolution. We also have locally free resolutions if X is quasiprojective. Moreover, the derived functor of Hom using projective resolutions if those exist (denoted by $\mathbf{R}_{II}\mathbf{R}_I$ in Hartshorne) is canonically isomorphic to the one defined via injective resolutions (see [16, p. 65/66, 91]), but we will not use this in the sequel. *From now on, we assume that all schemes are noetherian of finite Krull dimension and Gorenstein.*

Remark 4.6. First, we have a natural transformation

$$\varpi_{Y,K} : Id \to \mathbf{R}\underline{Hom}(\mathbf{R}\underline{Hom}(\ \ , K), K))$$

for any bounded complex K (see A.3.2). We say that K is a dualizing complex if $\varpi_{Y,K}$ is an isomorphism. The fact that X is Gorenstein ensures that $\mathcal{O}_X$ is a dualizing complex. Moreover, [16, Theorem V.3.1] implies that any dualizing complex of finite injective dimension is isomorphic to a shifted line bundle.

Recall that as X is noetherian regular of finite Krull dimension, the inclusion $(D^b(Vect(X)), *_L) \to (D^b_c(X), *_L)$ is an equivalence of triangulated categories with duality, inducing a non-canonical isomorphism between the associated Witt groups (the proof of [11, Corollary 2.17.2] for O_X carries over to arbitrary line bundles L). We also obtain a map $f^* : W^*(Y, O_Y) \to W^*(X, O_X)$ between coherent Witt groups for $f : X \to Y$ a flat morphism [12, p. 221].

The techniques of the appendix yield such maps f^* for other dualities provided X and Y are regular.

Proposition 4.7. *Let $f : X \to Y$ be a flat morphism of regular schemes and M a dualizing complex on Y. Then there is a natural morphism $f^* : W^*(Y, M) \to W^*(X, f^*M)$ induced by an exact functor of categories with dualities.*

Proof: This follows from Theorem A.31 which hypotheses are satisfied by the end of the proof of Theorem 4.10 and [16, Proposition II.5.8]. □

Now if we have a proper morphism $f : X \to Y$, we want to find dualizing complexes M on X (i.e., $M \in D^b_c(X)$ such that $D_M := \mathbf{R}\underline{Hom}(\ \ , M)$ is a duality on $D^b_c(X)$) such that $\mathbf{R}f_*$ can be extended to a functor of triangulated categories with duality $(\mathbf{R}f_*, \alpha) : (D^b_c(X), D_M, \varpi) \to (D^b_c(Y), *_L, \varpi)$. By definition, α must be a natural isomorphism $\alpha : \mathbf{R}f_*\mathbf{R}\underline{Hom}(F, M) \xrightarrow{\simeq} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, L)$. The duality theorem above tells us that this might be possible if we choose M to be isomorphic to $f^!L$. We also have the following:

Lemma 4.8. *If $f : X \to Y$ is a smooth proper morphism of relative dimension d and L a line bundle on Y, then there is a natural isomorphism $\beta : f^!L \xrightarrow{\simeq} f^*L \otimes \omega_{X/Y}[d]$. If moreover $g : Y \to Z$ is also smooth, we have an isomorphism $\omega_{X/Z} \simeq$*

$f^*(\omega_{Y/Z}) \otimes \omega_{X/Y}$. If f as above and $h : V \to Y$ arbitrary, then $\tilde{h}^*(\omega_{X/Y}) \cong \omega_{X\times_Y V/V}$.

Proof: See [16, p. 143, p. 419-421] or [36, Theorem 3] for the first claim and [16, p. 142, p. 141] for the second and third one. □

If f is a closed embedding of codimension d which is locally complete intersection (e. g. the graph of a morphism), then it is still possible to define $\omega_{X/Y}$ (see [16, p. 141]). and one may establish the second and third isomorphisms again using [16, p. 142, p. 141]. The description of $f^!L$ can be generalized easily to non-smooth morphisms when using absolute rather that relative canonical sheafs, which turns out to be more natural for our later purposes anyway.

Lemma 4.9. *(B. Kahn) Let $f : X \to Y$ be a proper morphism of relative dimension d between smooth noetherian schemes of finite Krull dimension and L a line bundle on Y. Then there is a natural isomorphism $\beta : f^!L \overset{\cong}{\to} f^*L \otimes f^*\omega_Y^{-1} \otimes \omega_X[d]$.*

Proof: Recall that $f^!L \cong Lf^*L \otimes^{\mathbf{L}} f^!\mathcal{O}_Y$ [16, p. 419-420]. Let p_X and p_Y be the projections from X and Y to Spec(F). We have $p_X = p_Y \circ f$, so, by adjunction $p_X^! = f^! \circ p_Y^!$, hence, by Lemma 4.8 and the above isomorphism

$$\omega_X[\dim X] \cong p_X^!\mathcal{O}_F \cong f^!p_Y^!\mathcal{O}_F \cong f^!\omega_Y[\dim Y] \cong Lf^*\omega_Y[\dim Y] \otimes^L f^!O_Y$$

which yields the formula for $L = O_Y$. Now apply the above isomorphism again for the general case. □

If $f : X \to Y$ is a proper map of schemes as above (separated, noetherian of finite Krull dimension), then $\mathbf{R}f_*$ restricts to a functor $\mathbf{R}f_* : D^b_c(X) \to D^b_c(Y)$.

Of course, we now have to show that this α indeed defines a functor of triangulated categories with duality. See Proposition A.26 for the comparison of this construction of α with the $\tilde{\alpha}$ of Verdier.

Theorem 4.10. *If in addition to the hypothesis of Theorem 4.2 X and Y are regular, then the functor*

$$(\mathbf{R}f_*, \alpha) : (D^b_c(X), *_{f^!L}, \varpi_X) \to (D^b_c(Y), *_L, \varpi_Y)$$

is a functor of triangulated categories with duality.

Proof: We want to apply Theorem A.32 with $\mathcal{D} = D^b_c(X)$, $\mathcal{C} = D^b_c(Y)$ (see also subsection A.8.2, beware that we often reduce to either D^+ or D^-), $E = \mathbf{L}f^*$ (as defined in [16, Proposition II.4.4]), $F = \mathbf{R}f_*$ and $G = f^!$. We know that $\varpi_{f^!L}$, ϖ_L are isomorphisms because L is a line bundle and therefore the complex $f^!L$ is quasiisomorphic to a shifted line bundle. The projection formula morphisms

$$(\mathbf{R}f_*D_{f^!L}A)\otimes\mathbf{R}f_*A \to \mathbf{R}f_*((f^*\mathbf{R}f_*D_{f^!L}A)\otimes A)$$

and

$$(D_L\mathbf{R}f_*A)\otimes\mathbf{R}f_*A \to \mathbf{R}f_*((f^*D_L\mathbf{R}f_*A)\otimes A)$$

are isomorphisms by [16, Proposition II.5.6]. For the adjunctions aef and afg, we use [16, Corollary II.5.11] and Corollary 4.3. The isomorphism ep is provided by [16, Proposition II.5.9], and the isomorphisms c, tp_1 and tp_2 are the obvious ones given by subsections A.8.1 and A.8.2. It remains to define te and to check that **TE**,**TEP1**, **TEP2** and **EPC** hold. Replacing all complexes by flat ones, we are reduced to study chain complexes. Thus we see that we may choose $te = Id : Tf^* \to f^*T$ as f^* is defined degreewise, and the commutativity of the three squares is immediatly checked degreewise using that in each diagram the only sign that appears (namely ϵ^{tp2}, resp. ϵ^{tp1}, resp. ϵ^c) does appear twice. □

Having done all this, we can finally define the desired transfer maps between Witt groups.

Definition 4.11. Let $f : X \to Y$ be a proper map between noetherian schemes of finite Krull dimension and L a line bundle on Y. Then we define the transfer map

$$f_* : W^*(X, f^! L) \to W^*(Y, L)$$

to be the map induced by the triangulated duality preserving functor $(\mathbf{R}f_*, \alpha)$ above.

The transfer respects compositions.

Lemma 4.12. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two proper maps and N a line bundle on Z. Then we have $(g \circ f)_* = g_* \circ f_* : \mathrm{W}^*(X, (g \circ f)^! N) \to \mathrm{W}^*(Z, N)$.*

Proof: One has to check $(\mathbf{R}(g \circ f)_*, \alpha_{g \circ f}) = (\mathbf{R}g_*, \alpha_g) \circ (\mathbf{R}f_*, \alpha_f)$ where the right hand side is defined by 4.19 below. This immediately follows using among others that $Tr_g \circ (\mathbf{R}g_* Tr_f g^!) = Tr_{g \circ f}$. □

Remark 4.13. If f is a smooth finite morphism, then $\omega_{X/Y} = O_X$, so by Lemma 4.8 the above transfer map becomes $f_* : W^*(X, f^* L) \to W^*(Y, L)$. In particular, we get some version of the classical Scharlau transfer [31] $f_* : W^*(X) \to W^*(Y)$ if $L = O_Y$.

For $L = \omega_Y$, the transfer map becomes

$$f_* : W^*(X, f^! \omega_Y) \to W^*(Y, \omega_Y).$$

Using the isomorphism of Lemma 4.8, Lemma 4.9 and the fact that an isomorphism of dualizing complexes induces an isomorphism of categories with dualities and thus of Witt groups, we deduce from Theorem 4.10 a transfer map $f_* : W^*(X, \omega_X[d]) \to W^*(Y, \omega_Y)$ (or $f_* : W^*(X, f^* L \otimes_{O_X} \omega_X[d]) \to W^*(Y, L \otimes_{O_Y} \omega_Y)$ for some line bundle L on Y).

Applying A.17, we therefore have:

Lemma 4.14. *Under the assumptions of Lemma 4.9, the above transfer map induces a transfer map of degree $-d$*

$$f_* : W^{*+d}(X, f^* L \otimes_{O_X} \omega_X) \to W^*(Y, L \otimes_{O_Y} \omega_Y).$$

Lemma 4.15. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two proper maps and N a line bundle on Z. Then we have $(g \circ f)_* = g_* \circ f_* : \mathrm{W}^*(X, (g \circ f)^* N \otimes \omega_X) \to \mathrm{W}^*(Z, N \otimes \omega_Z)$.*

Proof: We want to reduce this to Lemma 4.12. Recall that from Lemma 4.9, we have isomorphisms

$$\beta_f : f^! M \xrightarrow{\simeq} f^* M \otimes f^* \omega_Y^{-1} \otimes \omega_X[d]$$

and

$$\beta_g : g^! N \xrightarrow{\simeq} g^* N \otimes g^* \omega_Z^{-1} \otimes \omega_Y[d']$$

Fix an isomorphism $\lambda : N \otimes \omega_Z^{-1} \simeq N'$. Then the proof of Lemma 4.14 shows that starting with the isomorphism

$$\lambda_* : \mathrm{W}^*(Z, N) \to \mathrm{W}^*(Z, N' \otimes \omega_Z)$$

we obtain two isomorphisms

$$\mathrm{W}^*(X, f^! g^! N) \to \mathrm{W}^*(X, f^* g^* N' \otimes \omega_X)$$

applying either first g^* and then f^* or directly $(g \circ f)^*$. The lemma follows as one can show that these two coincide. To check this, one uses among others that

$$\beta_{g \circ f} : (g \circ f)^! N \otimes \omega_X^{-1} \to f^* g^* N \otimes f^* g^* \omega_Z^{-1} \otimes f^* \omega_Y \otimes f^* \omega_Y^{-1} \otimes \omega_X \otimes \omega_X^{-1}$$

and $f^*(\beta_g \otimes Id) \circ \beta_f$ are equal. □

4.3. **Another category.** Before we prove the related properties of transfers and pull-backs for Witt groups, we introduce a new category in which those properties can be expressed nicely.

Let L, L', M, M' and N, N' be vector bundles over X, Y and Z, respectively, and assume we have morphisms $p_{X/Z} : X \to Z$ and $p_{Y/Z} : Y \to Z$. Let $V = X \times_Z Y$ be the cartesian product of X by Y over Z. We denote $L \boxtimes_N M$ the vector bundle $p^*_{V/X}(L) \otimes (p_{V/X} \circ p_{X/Z})^*(N) \otimes p^*_{V/Y}(M)$ over $X \times_Z Y$. When we write $L \boxtimes M$, we mean that Z is the point and that N is trivial. We therefore get a vector bundle over $X \times Y$. We identify

- $(L \boxtimes_N M) \otimes (L' \boxtimes_{N'} M') = (L \otimes L') \boxtimes_{N \otimes N'} (M \otimes M')$
- $\omega_{X \times_Z Y} = \omega_X \boxtimes_{\omega_Z^{-1}} \omega_Y$

where the last equality follows from Lemma 4.8 provided everything is smooth. When $f : X' \to X$ and $g : Y' \to Y$, we also identify

- $(f \times g)^*(L \boxtimes M) = f^*L \boxtimes g^*M.$

Now, let $f : X \to Y$ and g be two composable morphism and P and P' be line bundles over the target of g. We identify

- $f^*(\mathcal{O}_Y) = \mathcal{O}_X$
- $f^* \circ g^*(P) = (g \circ f)^*(P)$
- $f^*(P) \otimes f^*(P') = f^*(P \otimes P')$

Finally, we denote L^{-1} the dual line bundle of L and we identify

- $L \otimes L^{-1} = \mathcal{O}_X$

Of course, we could avoid all those identifications by working with the canonical isomorphism involved, but the proofs would become completely unreadable.

Definition 4.16. Let $\mathcal{L}$ denote the category whose objects are pairs (X, L) where X is a smooth scheme and L is a line bundle over X. A morphism from (X, L) to (Y, M) is a pair (f, ϕ) where f is a morphism from X to Y and $\phi : f^*(M) \simeq L$ is an isomorphism of vector bundles. The composition is defined by $(g, \psi) \circ (f, \phi) = (g \circ f, \phi \circ f^*(\psi))$.

Associativity in $\mathcal{L}$ is clear. There is an obvious faithful functor from the category of smooth schemes to this category sending X to $(X, \mathcal{O}_X)$ and $f : X \to Y$ to $(f, Id_{\mathcal{O}_X})$. To keep notations concise, we denote by X and f the images of X and f by this functor. We denote pt the object $(SpecF, \mathcal{O}_{SpecF})$. The reader discouraged by all these notations might want to restrict his attention to the case where all the L and M are just the structure sheafs.

There is a well defined contravariant functor W^i from the subcategory of flat morphisms of $\mathcal{L}$ to the category of abelian groups that send an object (X, L) on the corresponding Witt group $W^i(X, L)$. The morphism (f, ϕ) is sent to the composition $W^i(Y, M) \to W^i(X, f^*(M)) \to W^i(X, L)$, where the second map is induced by ϕ and the first is the classical pull-back on Witt groups. For obvious reasons, we denote this morphism $(f, \phi)^*$.

We can also define the push-forwards (or transfer) for a morphism (f, ϕ) if f is as in Definition 4.11. This is a morphism $(f, \phi)_*$ from $W^{i+d}(X, M \otimes \omega_X)$ to $W^i(Y, L \otimes \omega_Y)$ when f is of dimension d. It is given by the isomorphism $W^{i+d}(X, f^*(L) \otimes \omega_X) \simeq W^{i+d}(X, M \otimes \omega_X)$ induced by ϕ composed with the morphism of 4.14. One can check that $((g, \psi) \circ (f, \phi))_* = (g, \psi)_* \circ (f, \phi)_*$.

For reasons that will be clear later, we define the twist of an object $c(X, L) = (X, L \otimes \omega_X)$. We can therefore interpret the transfer for a morphism (f, ϕ) as a morphism from $W^{i+d}(c(X, L))$ to $W^i(c(Y, M))$.

Next, we prove the base change formula for Witt groups. We need to study the following technical condition.

Definition 4.17. Let (F, α_F) and (G, α_G) be two exact fuctors $(A, D_A, \varpi_A) \to (B, D_B, \varpi_B)$ between exact categories with duality. We say that $\sigma : F \Rightarrow G$ is a natural transformation (resp. isomorphism) between duality preserving functors if $\sigma : F \Rightarrow G$ is a natural transformation (resp. isomorphism) between functors and the square

$$\begin{tikzcd} FD_A \arrow[r, "\alpha_F"] \arrow[d, "\sigma_{D_A}"'] & D_B F \\ GD_A \arrow[r, "\alpha_G"'] & D_B G \arrow[u, "D_B \sigma"'] \end{tikzcd}$$

is commutative. If F and G are exact functors between triangulated categories, we say that $\sigma : F \Rightarrow G$ is a triangulated natural transformation (resp. isomorphism) between duality preserving functors if moreover $\sigma T = T\sigma$.

Lemma 4.18. *If σ is a natural isomorphism between duality preserving functors as above, then the two maps $W(A) \to W(B)$ induced by F and G coincide. If moreover σ is a triangulated isomorphism, then the two induced maps between graded Witt groups $W^*(A) \to W^*(B)$ coincide.*

Proof: It is straightforward to check that the images with respect to F and G of some symmetric space in A are isomorphic as symmetric spaces in B, and similar for the shifted dualities in the triangulated setting. □

It is possible to compose duality preserving functors between triangulated categories with dualities.

Definition 4.19. Let

$$(F, \eta) : (A, D_A, \varpi_A) \to (B, D_B, \varpi_B)$$

and

$$(G, \rho) : (B, D_B, \varpi_B) \to (C, D_C, \varpi_C)$$

be two duality preserving functors of triangulated categories with duality. Then we define their composition $(GF, \rho\eta)$ by

$$(GF, (\rho F) \circ (G\eta)) : (A, D_A, \varpi_A) \to (C, D_C, \varpi_C).$$

It is straightforward to check that $(GF, \rho\eta)$ is a duality preserving functor.

Theorem 4.20. *Assume that we have a cartesian square of regular schemes*

$$\begin{tikzcd} V \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

where g and g' are flat and f and f' are proper and satisfy the hypotheses of Definition 4.11. Let N be a line bundle on X. Then we have a commutative square of Witt groups

$$\begin{tikzcd} W^*(V, g'^* f^! N) \arrow[d, "f'_*"'] & W^*(X, f^! N) \arrow[l, "g'^*"'] \arrow[d, "f_*"] \\ W^*(Y, g^* N) & W^*(Z, N) \arrow[l, "g^*"] \end{tikzcd}$$

Proof: The square of Witt groups is induced by the following diagram of cateories with duality

$$\begin{tikzcd}
(D^b_c(V), \mathbf{R}\underline{Hom}(\ \ , g'^* f^! N)) \arrow[d, "(Id,c)"'] & (D^b_c(X), \mathbf{R}\underline{Hom}(\ \ , f^! N)) \arrow[l, "(g'^*,id)"'] \arrow[dd, "(\mathbf{R}f_*,\alpha)"] \\
(D^b_c(V), \mathbf{R}\underline{Hom}(\ \ , f'^! g^* N)) \arrow[d, "(\mathbf{R}f'_*,\alpha')"'] & \\
(D^b_c(Y), \mathbf{R}\underline{Hom}(\ \ , g^* N)) & (D^b_c(Z), \mathbf{R}\underline{Hom}(\ \ , N)) \arrow[l, "(g^*,id)"]
\end{tikzcd}$$

where c is the canonical isomorphism of [36, Theorem 2]. We may now apply Lemma 4.18 to the functors $F = g^* \circ \mathbf{R}f_*$ and $G = \mathbf{R}f'_* \circ Id \circ g'^*$. The required natural isomorphism σ is given by [2, p. 84, p. 285] and [1, p. 290]. The hypothesis in Definition 4.18 is then precisely the commutativity of the square of functors and natural isomorphisms

$$\begin{tikzcd}
g^* \mathbf{R}f_* \mathbf{R}\underline{Hom}(\ \ , f^! N) \arrow[r, "id \circ \alpha"] \arrow[d, "\sigma \circ \mathbf{R}\underline{Hom}(\ \ , f^! N)"'] & \mathbf{R}\underline{Hom}(\ \ , g^* N) g^* \mathbf{R}f_* \\
\mathbf{R}f'_* g'^* \mathbf{R}\underline{Hom}(\ \ , f^! N) \arrow[r, "\alpha' \circ c \circ id"'] & \mathbf{R}\underline{Hom}(\ \ , g^* N) \mathbf{R}f'_* g'^* \arrow[u, "\mathbf{R}\underline{Hom}(\ \ , g^* N) \circ \sigma"']
\end{tikzcd}$$

This can be shown using adjunctions and their standard properties, in particular the fact that the two different definitions of c in [36, p. 401] coincide. □

Corollary 4.21. *Let (X, L), (Y, M), (Z, N) and (V, P) in $\mathcal{L}$ be such that X, Y, Z, V and the morphisms f, f', g and g' between them are as in Theorem 4.20 and f, f' satisfy the hypotheses of Lemma 4.9. Let (f, ϕ), (g, ψ), (f', ϕ') and (g', ψ') have sources and targets as follows:*

$$\begin{tikzcd}
 & (V,P) \arrow[r, "(f',\phi')"] & (Y,M) & \\
c(V,P) \arrow[d, "(g',\psi')"'] & & & c(Y,M) \arrow[d, "(g,\psi)"] \\
c(X,L) & & & c(Z,N) \\
 & (X,L) \arrow[r, "(f,\phi)"'] & (Z,N) &
\end{tikzcd}$$

Assume that $(\phi' \otimes Id) \circ (f')^*(\psi) = (\psi' \otimes Id) \circ (g')^*(\phi)$ (the source of these morphisms is $(g \circ f')^*(N \otimes \omega_Z) = (f \circ g')^*(N \otimes \omega_Z)$ and their target is $P \otimes (\mathcal{O}_X \boxtimes_{\mathcal{O}_Z} \omega_Y)$. Then the two morphism $(g, \psi)^* \circ (f, \phi)_* = (f', \phi')_* \circ (g', \psi')^* : W^*(X, L \otimes \omega_X) \to W^{*-d}(Y, M \otimes \omega_Y)$ coincide.

Proof: This follows from Theorem 4.20 and Lemma 4.14. □

Products. Observe that there is a product μ on $\oplus_L W^*(\ \ , L)$ induced by the (left) product of [14, Theorem 3.1] (see also [37, p.7/8]). Using the fact that for any vector bundles V and W over X one has $\Delta_X^*(V \times W) = V \times_X W$, one sees that the product μ factors through the exterior product (with $X = Y$)

$$\lambda_{(X,L),(Y,M)} : W^*(X,L) \times W^*(Y,M) \xrightarrow{\lambda} W^*(X \times Y, L \boxtimes M)$$

namely as $W^*(X,L) \times W^*(X,M) \xrightarrow{\lambda} W^*(X \times X, L \boxtimes M) \xrightarrow{\Delta_X^*} W^*(X, L \otimes M)$. The factorization follows as the pairing of exact categories with duality

$$(Vect(X), \mathrm{Hom}(\ \ , L)) \times (Vect(X), \mathrm{Hom}(\ \ , M)) \to (Vect(X), \mathrm{Hom}(\ \ , L \otimes M))$$

is dualizing in the sense of [14, Definition 1.11] and factors as a dualizing pairing through $(Vect(X \times X), \mathrm{Hom}(\ \ , L \boxtimes M))$ which induces a factorization of the dualizing pairing for the corresponding $D^b(Vect(\ \ ))$. The very same construction applies to D^b_{qc} and D^b_c.

Lemma 4.22. *The exterior product λ commutes with pull-backs and push-forwards. This means that if $f : X \to X'$ and $g : Y \to Y'$ are morphisms of schemes which are flat (resp. proper of pure dimension), we have the equalities*

$$\lambda(f^*(x'), g^*(y')) = (f \times g)^* \circ \lambda(x', y')$$

and

$$\lambda(f_*(x), g_*(y')) = (f \times g)_* \circ \lambda(x, y).$$

Proof: The first equality is a slight improvement of [14, Theorem 3.2] but is not more difficult to show. For the second equality, pick two complexes F and G on X and Y with forms. The claim then reduces essentially to the "Künneth" isomorphism $(f \times g)_*(F \boxtimes G) \simeq f_*(F) \boxtimes g_*(G)$ which can be checked by an explicit computation. □

Proposition 4.23. *For $(f, \phi) : (X, L) \to (Y, M)$ and $(f, \phi') : (X, L') \to (Y, M')$ with f as in Definition 4.11, the projection formula $(f, \phi \otimes \phi')_*(a.(f, \phi')^*(b)) = (f, \phi)_*(a).b$ holds for any $a \in W^i(X, L \otimes \omega_X)$ and $b \in W^j(Y, M')$.*

Proof: The result follows from the previous lemma and Corollary 4.21 applied to the following cartesian diagram.

$$\begin{tikzcd}
(X, L \otimes L') \arrow[r, "((Id\times f)\circ\Delta_X, Id\otimes\phi')"] \arrow[d, "(f,\phi\otimes\phi')"'] & (X \times Y, L \boxtimes M') \arrow[d, "(f\times Id, \phi\boxtimes Id)"] \\
(Y, M \otimes M') \arrow[r, "(\Delta_Y, Id)"'] & (Y \times Y, M \boxtimes M').
\end{tikzcd}$$

□

Recall that in the previous section (see Example 1) we introduced the category $Coh^H(X)$ of H-equivariant coherent O_X-modules and its full subcategory $Vect(X)^H$ of H-vector bundles on an H-scheme X for a given algebraic group H. The functor $Coh^H(\ \ )$ is contravariant for flat H-maps and covariant for H-projective morphisms (see [34, p. 543]). It is a non-trivial task to extend everything we did in this section so far to the equivariant setting. For instance, the existence of injective resolutions is not clear (that one has enough projectives follows from [34, Corollary 5.3]). *From now on we make the assumption that all the previous definitions and results of section 4 carry over to the H-equivariant setting.* We will hopefully discuss the details of this in forthcoming work. Therefore, the following is true unconditionally only for the non-equivariant setting ($H = 1$), otherwise the assumption has to be used.

Given an H-scheme X and and H-line bundle L on it, we write $W^{*,H}(X, L)$ for the Witt group of the derived category $D^{b,H}_c(X)$ of H-equivariant O_X-modules with coherent cohomology with respect to the duality induced by $*_L$.

4.4. **Categories of motives.** Now we are ready to define the category $\mathbf{W}^H$ of H-Witt motives.

Definition 4.24. Let $\mathcal{PL}$ be the full subcategory of $\mathcal{L}$ whose objects are pairs (X, L) with X projective. Fix an algebraic group H. By definition, the category $\mathbf{W}^H$ has as objects couples (V, L) where V is endowed with an H-action and L a line bundle on V equipped with a left equivariant H-action (as in section 3). The set of morphisms (or W-*correspondances*) between two objects is a graded abelian group and is defined by $\mathrm{Hom}^i_{\mathbf{W}^H}((X, L), (Y, M)) = W^{i+dimX,H}(X \times Y, (L^{-1} \otimes \omega_X) \boxtimes M)$. For $a \in \mathrm{Hom}_{\mathbf{W}^H}((X, L), (Y, M))$ and $b \in \mathrm{Hom}_{\mathbf{W}^H}((Y, M), (Z, N))$ the composition ba is defined as

$$(\pi_{XZ}, Id_{L^{-1}\boxtimes\mathcal{O}_Y\boxtimes(N\otimes\omega_Z^{-1})})_*$$
$$(\mu((\pi_{XY}, Id_{(L^{-1}\otimes\omega_X)\boxtimes M\boxtimes\mathcal{O}_Z})^*(a), (\pi_{YZ}, Id_{\mathcal{O}_X\boxtimes(M^{-1}\otimes\omega_Y)\boxtimes N})^*(b))).$$

Proposition 4.25. *The above composition law in $\mathbf{W}^H$ is associative and any object admits an identity automorphism, so $\mathbf{W}^H$ really is a category.*

Proof: The proof of associativity is the usual proof of the associativity of correspondences, as in [22, §2, Lemma p. 446]. It just uses the composition of the pull-backs and push-forwards, the base change formula (Corollary 4.21) and the projection formula (Proposition 4.23). The identity of (X, L) is given by $(\Delta_X, Id_{\omega_X^{-1}})_*(1_X)$ (recall that 1_X is the class in $W_0(X, \mathcal{O}_X)$ of the one dimensional standard form $< 1 >$ on $\mathcal{O}_X$). Again, the proof that it is an identity is a generalization of the classical one. In fact, it is a particular case of the existence of graphs (see Proposition 4.27 below). □

Remark 4.26. There is an obvious category of Witt correspondences of degree zero defined by setting $\mathrm{Hom}_{\mathbf{W}^{0,H}}((X, L), (Y, M)) = \mathrm{Hom}^0_{\mathbf{W}^H}((X, L), (Y, M))$.

Now we can construct the graph functor.

Proposition 4.27. *There is a contravariant functor Γ from the category $\mathcal{PL}$ to $\mathbf{W}^H$. It is the identity on objects, and it sends a morphism $(f, \phi) : (X, L) \to (Y, M)$ to $(\gamma_f, (\phi^\vee)^{-1} \otimes Id_L \otimes Id_{\omega_X^{-1}})_*(1_X) \in W^{dimY}(Y \times X, (M^{-1} \otimes \omega_Y) \boxtimes L) = \mathrm{Hom}^0((Y, M), (X, L))$, where $\gamma_f : X \to Y \times X$ is the graph morphism (it is always proper as all considered varieties are separated). By $\phi^\vee$, we mean the morphism dual to ϕ, going from L^{-1} to $f^*(M)^{-1}$.*

Proof: This functor respects the composition. This follows from standard arguments, as in [22, §2, Proposition p. 447]. □

Of course, we can consider the full subcategory of $\mathbf{W}^H$ of objects of the form (X, O_X), but as we shall see, there are very few interesting motives that decompose in this category.

We now define a realization functor to the category of graded abelian groups.

Definition 4.28. We define the covariant functor R^H from $\mathbf{W}^H$ to the category of graded abelian groups by setting $R^H(X, L) = W^H(X, L)$ and $R^H(c) = (x \mapsto (p_Y)_*(p_X^*(x).c))$ for an element $c \in \mathrm{Hom}((X, L), (Y, M))$. For any subgroup H_1 of H, there is an obvious functor $\mathrm{Res}^H_{H_1}$ from $\mathbf{W}^H$ to $\mathbf{W}^{H_1}$ induced by the restriction of the action of H to H_1.

Remark 4.29. The functor R^H respects the composition because it coincides with the functor Hom(pt,). In particular we have thus obtained the Witt version (without twist) of [28, Key Lemma 6.5] wich is just a particular case of the fact that any motivic isomorphism induces an isomorphism on the realisations. Observe also that the composition $R^H \circ \Gamma$ sends a morphism (f, ϕ) to $(f, \phi)^*$.

The fact that we deal with categories with dualities is of course reflected by a duality already on the category of Witt motives. There is an involutive functor (of order 2) on $\mathbf{W}^H$ sending an object (X, L) to $(X, \omega_X \otimes L^{-1})$ and a morphism c in $\mathrm{Hom}^i((X, L), (Y, M)) = W_H^{i+dimX}(X \times Y, (L^{-1} \otimes \omega_X) \boxtimes M)$ to the corresponding element c^t in the group $W_H^{i+dimX}(Y \times X, M \boxtimes (\omega_X \otimes L^{-1})) = \mathrm{Hom}^{i+dimX-dimY}((Y, \omega_Y \otimes M^{-1}), (X, \omega_X \otimes L^{-1})$. Notice that it doesn't respect the graduation.

The composition $R^H \circ t \circ \Gamma$ sends a morphism (f, ϕ) to $(f, \phi)^*$ composed at the two ends by the isomorphism $W_H(X, L \otimes \omega_X) \simeq W_H(X, \omega_X \otimes L)$, and the similar one for (Y, M).

There is a pairing $W(X, M) \times W((X, M)^t) \to W(\mathrm{pt})$ given by the composition $(\pi, Id)_* \circ (\Delta_X, Id)^* \circ \lambda_{(X,M),(X,M)^t}$ where π is the structural morphism from X to the point.

Remark 4.30. Panin also constructs a category $\mathbf{A}^H$ where the objects are couples (X, B) with X smooth projective over F and B a central simple F-algebra, such that $\mathbf{K}^H$ is precisely the full subcategory of $\mathbf{A}^H$ of objects (X, F). The F-algebra B allows Panin to twist. We would like to do the same in our setting, considering of course F-algebras B with involution. Again the problem will be the existence of good transfers. There is a certain amount of literature about this. Looking at matrix algebras $M_n(F)$ over F, for instance, one might apply Morita equivalence for Witt groups [10, section 8]. This is sufficient for base fields with trivial Brauer groups (e.g. finite fields, algebraically closed fields or fields of transcendence degree one over those). Using the norm map of [24, Appendix 2] yields transfers for (matrix algebras over) quaternion algebras as well. (There are of course many computations concerning division algebras, see e.g. [21, section 5-7].) The most general transfer maps we know about can be found in [15]. In forthcoming work, we intend to explain how composing and iterating these maps one gets the transfer maps we will need.

4.5. **Effective Witt Motives.** We now define the category $\mathbf{W}^H_{eff}$ of effective Witt motives. It is just the pseudo-abelianised completion of the previous category. For a definition of the pseudo-abelian completion, see e. g. [22, §5]. Recall that the objects are just the pairs $((X, L), p)$ where p is an idempotent in $\mathrm{End}(X, L)$ and the morphisms between $((X, L), p)$ and $((Y, L), q)$ are given by the quotient of the subgroup $\mathrm{Hom}_{\mathbf{W}^H}((X, L), (Y, L))$ given by the elements f such that $fp = qf$ by the subgroup of elements f such that $fp = qf = 0$. It contains $\mathbf{W}^H$ as the full subcategory of objects for which $p = Id$.

Remark 4.31. We don't lose the graduation on the Hom sets because an idempotent has to be of degree zero so the relation $fp = qf = 0$ is homogeneous. We can extend the realisation functors R^H and R to $\mathbf{W}^H_{eff}$ because of the universal property of the pseudo-abelian completion. More precisely, we set $R^H((X, L), p) = \ker R^H(p)$ on objects.

We can define a tensor structure on this category by setting $(X, L, p) \otimes (Y, M, q) = (X \times Y, L \boxtimes M, p \times q)$.

Appendix A. Duality formalism and classical adjunctions

In this appendix, we obtain formal consequences of adjunctions of the type $\otimes$-$\mathcal{H}$, f^*-f_* and f_*-$f^!$ in tensor-triangulated categories that are useful for the theory of Balmer's higher Witt groups and have been applied in section 4. We present them in a general axiomatic framework which allows to distinguish between the geometric input and the formal arguments concerning adjunctions in triangulated categories. Moreover, we believe that the abstract frameworks and the results we prove about it do apply to different examples (e.g., other motivic categories or the stable homotopy category). For this reason, we have presented some aspects in slighty greater generality and provided some more results than we actually need for our applications in section 4.

Our philosophy is to exhibit a minimal axiomatic setting which can be verified without too much work in the examples of interest; and from which everything can be deduced in a formal way. The example of a triangulated category to keep in mind for this paper is of course the category $D^b_c(X)$ of bounded complexes of O_X-modules with coherent cohomology on a separated noetherian scheme X.

This appendix is rather long, and reading it might seem very unpleasant at first glance. Don't get discouraged: Writing it and checking all the details one is tempted to believe anyway was even more unpleasant. Of course, you can trust us (in the spirit of [16, pp. 117-119]) and stop reading the appendix now. If you don't, here is a survey of its subsections. After reviewing some generalities on triangulated categories (A.1), we refine the notion of a triangulated category with duality in A.2. In A.3, we axiomatize the derived tensor product, the derived Hom and the adjunction between them. Then we use this to construct dualities on triangulated categories in A.4. Section A.5 studies functors f^*, f_* and $f^!$ with adjunctions f^*-f_* and f_*-$f^!$ and discusses in which sense these have to be compatible with the tensor-triangulated structure. The projection formula appears in A.6. Combining everything yields to the general Theorems A.31 and A.32 in section A.7 about the existence of exact functors between triangulated categories which we then may apply in section 4 (see Proposition 4.7 and Theorem 4.10). In fact, before applying this to Witt groups some details remain to be checked (see A.8). In particular, we prove that there is a particular nice choice of signs. Don't forget that choosing correct signs is very important, as one wrong sign transforms the Witt group of symmetric forms to the Witt group of skew-symmetric forms.

A.1. **Generalities in triangulated categories.** We recall a few basic notions that we need in triangulated categories. Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories, with translation functors $T_{\mathcal{C}}$ and $T_{\mathcal{D}}$.

Definition A.1. (see for example [14, § 1.1]) Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant (resp. contravariant) functor. Let $\theta : FT_{\mathcal{C}} \to T_{\mathcal{D}}F$ (resp. $\theta : T^{-1}F \to FT$) be an isomorphism of functors. We say that the pair (F, θ) is δ-exact ($\delta = \pm 1$) if for any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

the triangle

$$FA \xrightarrow{Fu} FB \xrightarrow{Fv} FC \xrightarrow{\delta\theta_A \circ Fw} TFA$$

respectively

$$FC \xrightarrow{Fv} FB \xrightarrow{Fu} FA \xrightarrow{\delta T_{\mathcal{D}}(Fw \circ \theta_A)} TFC$$

is exact.

The following is a well-known lemma, but we include here a complete proof for lack of reference.

Lemma A.2. *Let (F, f) be a covariant (resp. contravariant) δ-exact functor from $\mathcal{C}$ to $\mathcal{D}$, such that F admits a right adjoint R on the level of the underlying additive categories. Then there is a canonical way to define an isomorphism of functors $r : RT_{\mathcal{D}} \to T_{\mathcal{C}}R$ (resp. $r : T^{-1}R \to RT$) such that (R, r) is δ-exact. The same is true for a left adjoint.*

Proof: We prove the lemma in the contravariant case, for a right adjoint and for $\delta = 1$. The other cases are proved alike. The morphism $r_A : RTA \to TRA$ is the image of Id_{RTA} by the chain of isomorphisms

$$\mathrm{Hom}(RTA, RTA) \to \mathrm{Hom}(FRTA, TA) \to \mathrm{Hom}(FTT^{-1}RTA, TA)$$

$$\stackrel{(f^{-1})^\sharp}{\to} \mathrm{Hom}(TFT^{-1}RTA, TA) \to \mathrm{Hom}(FT^{-1}RTA, A) \to \mathrm{Hom}(T^{-1}RTA, RA) \\ \to \mathrm{Hom}(RTA, TRA).$$

Its inverse is obtained from Id_{RA} by the chain

$$\mathrm{Hom}(RA, RA) \to \mathrm{Hom}(FRA, A) \to \mathrm{Hom}(TFRA, TA)$$

$$\stackrel{f^\sharp}{\to} \mathrm{Hom}(FTRA, TA) \to \mathrm{Hom}(TRA, RTA).$$

It is easy to check (using the standard procedure as e.g. in Proposition A.14) that these two elements are inverse to each other. We now have to show that the pair (R, r) is exact. Let

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

be an exact triangle. We want to prove that the triangle

$$RA \xrightarrow{u} RB \xrightarrow{v} RC \xrightarrow{r_A \circ Rw} TRA$$

is exact. We first complete $RA \xrightarrow{u} RB$ as an exact triangle

$$RA \xrightarrow{u} RB \xrightarrow{v'} C' \xrightarrow{w'} TRA$$

and we prove that this triangle is in fact isomorphic to the previous one. To do so, one completes the uncomplete morphism of triangles

$$\begin{tikzcd}
FRA \arrow[r, "FRu"] \arrow[d] & FRB \arrow[r, "FRv"] \arrow[d] & FC' \arrow[r, "f_{RA} \circ FRw"] \arrow[d, dotted, "h"] & TFRA \arrow[d] \\
A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & TA
\end{tikzcd}$$

Looking at the adjoint diagram, we see that $ad(h) : C' \to RC$ is an isomorphism by the five lemma for triangulated categories. $\square$

A.2. **Weak duality.** We now explain the notion of triangulated category with weak duality. It is obtained from Balmer's definition of a triangulated category with duality by weakening the axiom $DT = T^{-1}D$ for the contravariant endofunctor (called *duality functor*) D on $\mathcal{C}$ of [3, definition 2.2]. Namely, we just assume that we have an isomorphism of functors

$$d : T^{-1}D \to DT.$$

such that the (contravariant) pair (D, d) is δ-exact, for some $\delta \in \pm 1$. By the definition of a morphism of functors and composition of those, we have the formula

$$dd = (dT^{-1}D \circ T^{-1}Dd) = (DTd \circ dDT)$$

for the natural isomorphism $dd : T^{-1}DDT \to DTT^{-1}D$. As in [14, Remark 1.1], we also get iterated versions ($d^{(2)} \neq dd$)

$$d^{(i)} : T^{-i}D \to DT^i$$

for all $i \in \mathbf{Z}$ which e. g. for $i > 0$ is given by $d^{(i)} = dT^{i-1} \circ T^{-1}d^{(i-1)}$ (or equivalently $d^{(i)} = d^{(i-1)}T \circ T^{-(i-1)}d$). One easily checks that if (D, d) is δ-exact, then (T^iDT^j, T^idT^j) is $(-1)^{i+j}\delta$-exact.

We also assume that we have an isomorphism of functors

$$\varpi : Id \to D^2$$

with the usual compatibility formula

$$D\varpi \circ \varpi D = id_D$$

and the *modified* usual compatibility formula

$$TdT^{-1}D \circ Dd \circ \varpi T = T\varpi.$$

Definition A.3. We say that $(\mathcal{C}, D, d, \varpi)$ is a triangulated category with weak δ-duality if all the above conditions are satisfied. If $T^{-1}D = DT$ and $d = id$, we recover Balmer's usual definition (subsequently called strict duality, not to be confused with the condition $\varpi = id$).

Definition A.4. Let $(\mathcal{C}, D_\mathcal{C}, d_\mathcal{C}, \varpi_\mathcal{C})$ (resp. $(\mathcal{D}, D_\mathcal{D}, d_\mathcal{D}, \varpi_\mathcal{D})$) be a triangulated category with weak $\delta_\mathcal{C}$-(resp. $\delta_\mathcal{D}$-) duality. Let (F, θ) be a 1-exact pair from $\mathcal{C}$ to $\mathcal{D}$ and let $\rho : FD_\mathcal{C} \to D_\mathcal{D}F$ be an isomorphism of functors. We then say that the triple (F, θ, ρ) is a duality preserving functor if the following diagrams are commutative:

(1)
$$\begin{tikzcd} F \arrow[r, "F\varpi_\mathcal{C}"] \arrow[d, "\varpi_\mathcal{D}F"'] & FD_\mathcal{C}D_\mathcal{C} \arrow[d, "\rho D_\mathcal{C}"] \\ D_\mathcal{D}D_\mathcal{D}F \arrow[r, "D_\mathcal{D}\rho"'] & D_\mathcal{D}FD_\mathcal{C} \end{tikzcd}$$

(2)
$$\begin{tikzcd} FT_\mathcal{C}D_\mathcal{C} \arrow[d, "(\delta_\mathcal{C}\delta_\mathcal{D})\theta D_\mathcal{C}"'] & FD_\mathcal{C}T_\mathcal{C}^{-1} \arrow[l, "FT_\mathcal{C}d_\mathcal{C}T_\mathcal{C}^{-1}"'] \arrow[r, "\rho T_\mathcal{C}^{-1}"] & D_\mathcal{D}FT_\mathcal{C}^{-1} \arrow[d, "D_\mathcal{D}T_\mathcal{D}^{-1}\theta T_\mathcal{C}^{-1}"] \\ T_\mathcal{D}FD_\mathcal{C} \arrow[r, "T_\mathcal{D}\rho"'] & T_\mathcal{D}D_\mathcal{D}F & D_\mathcal{D}T_\mathcal{D}^{-1}F \arrow[l, "T_\mathcal{D}d_\mathcal{D}T_\mathcal{D}^{-1}F"] \end{tikzcd}$$

Note that the first condition is the classical one (see for example [14, Definition 1.8, 1.]), and that the second is just a refinement of [14, Definition 1.8, 2.] where the special case of a strict duality is considered. The proof of the following proposition is straightforward.

Proposition A.5. *If $(\mathcal{C}, D, d, \varpi)$ is a triangulated category with weak δ-duality, then $(\mathcal{C}, TD, Td, -\delta(TdD) \circ \varpi)$ is a triangulated category with weak $(-\delta)$-duality. Iterating, we get that $(\mathcal{C}, T^iDT^j, T^idT^j, (-1)^{(i-j)(i-j+1)/2}\delta^{i-j}(T^id^{(i)}T^jd^{(j)}) \circ \varpi)$ is a category with weak $(-1)^{i-j}\delta$-duality.*

Definition A.6. Following Balmer's convention (see [3, Definition 2.8]), given a triangulated category with weak δ-duality $(\mathcal{C}, D, d, \varpi)$, we define a triangulated category with shifted (or translated) duality by

$$T(\mathcal{C}, D, d, \varpi) = (\mathcal{C}, TD, Td, -\delta TdD \circ \varpi).$$

Definition A.7. For convenience, when (F, ρ) is a duality preserving functor from $(\mathcal{C}, D_\mathcal{C}, d_\mathcal{C}, \varpi_\mathcal{C})$ to $(\mathcal{D}, D_\mathcal{D}, d_\mathcal{D}, \epsilon\varpi_\mathcal{D})$, $\epsilon = \pm 1$, we say that it is a duality ϵ-preserving functor from $(\mathcal{C}, D_\mathcal{C}, d_\mathcal{C}, \varpi_\mathcal{C})$ to $(\mathcal{D}, D_\mathcal{D}, d_\mathcal{D}, \varpi_\mathcal{D})$. The composition of such functors trivially multiplies the signs.

Remark A.8. As in the strict case, a duality ϵ-preserving functor induces duality ϵ-preserving functors on the translated categories (same ϵ).

It is possible to define symmetric spaces as usual, and to extend Balmer's definition of Witt groups to this more general setting with weak dualities.

A.3. **The functors $\mathcal{H}$ and $\otimes$.** We assume that the triangulated category $\mathcal{C}$ is endowed with an internal Hom functor, denoted by $\mathcal{H}$, and an internal tensor product denoted $\otimes$. This will be used below to make $\mathcal{C}$ into a triangulated category with duality.

We assume the $\mathcal{H}$ and $\otimes$ functors satisfies the following axioms.

Compatibility of $\mathcal{H}$ with the translation T.
TH1. There is a functorial (in both variables A and B) isomorphism $th_{1,A,B}$: $\mathcal{H}(TA, B) \to T^{-1}\mathcal{H}(A, B)$.
TH2. There is a functorial (in both variables A and B) isomorphism $th_{2,A,B}$: $\mathcal{H}(A, TB) \to T\mathcal{H}(A, B)$.
TH1TH2. The following diagram is **anti**commutative.

$$\begin{tikzcd}
\mathcal{H}(TA,TB) \arrow[r, "th_{1,A,TB}"] \arrow[d, "th_{2,TA,B}"'] & T^{-1}\mathcal{H}(A,TB) \arrow[d, "T^{-1}th_{2,A,B}"] \\
T\mathcal{H}(TA,B) \arrow[r, "Tth_{1,A,B}"'] & \mathcal{H}(A,B)
\end{tikzcd}$$

Compatibility of $\otimes$ with the translation T.
TP1. There is a functorial (in both variables A and B) isomorphism $tp_{1,A,B}$: $TA \otimes B \to T(A \otimes B)$.
TP2. There is a functorial (in both variables A and B) isomorphism $tp_{2,A,B}$: $A \otimes TB \to T(A \otimes B)$.
TP1TP2. The following diagram is **anti**commutative.

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "tp_{1,A,TB}"] \arrow[d, "tp_{2,TA,B}"'] & T(A \otimes TB) \arrow[d, "Ttp_{2,A,B}"] \\
T(TA \otimes B) \arrow[r, "Ttp_{1,A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

Adjunction of $\otimes$ and $\mathcal{H}$.
ATH. We have a functorial (in A, B and C) bijection $ath_{A,B,C} : \mathrm{Hom}(A \otimes B, C) \to \mathrm{Hom}(A, \mathcal{H}(B, C))$.

Compatibility of the adjunction ath and the translation T
Let $m : A \to A'$ be a morphism. For simplicity, we also denote by m the induced application from $\mathrm{Hom}(A', B)$ to $\mathrm{Hom}(A, B)$ (resp. from $\mathrm{Hom}(B, A)$ to $\mathrm{Hom}(B, A')$).
TATH12. The following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}(TA \otimes B, C) \arrow[r, "ath"] & \mathrm{Hom}(TA, \mathcal{H}(B,C)) \\
\mathrm{Hom}(T(A \otimes B), C) \arrow[u, "tp"] \arrow[d, "tp"'] & \mathrm{Hom}(A, T^{-1}\mathcal{H}(B,C)) \arrow[u, "T"'] \\
\mathrm{Hom}(A \otimes TB, C) \arrow[r, "ath"'] & \mathrm{Hom}(A, \mathcal{H}(TB,C)) \arrow[u, "th"']
\end{tikzcd}$$

TATH23. The following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}(A\otimes TB, C) \arrow[r, "ath"] & \mathrm{Hom}(A, \mathcal{H}(TB, C)) \arrow[d, "th"] \\
\mathrm{Hom}(T(A\otimes B), C) \arrow[u, "tp"] & \mathrm{Hom}(A, T^{-1}\mathcal{H}(B, C)) \arrow[d, "T^{-1}th"] \\
\mathrm{Hom}(A\otimes B, T^{-1}C) \arrow[u, "T"] \arrow[r, "ath"'] & \mathrm{Hom}(A, \mathcal{H}(B, T^{-1}C))
\end{tikzcd}$$

The following are consequences of the previous axioms.

TATH13. The following diagram is commutative (combine **TATH12** and **TATH23**.

$$\begin{tikzcd}
\mathrm{Hom}(TA\otimes B, C) \arrow[r, "ath"] & \mathrm{Hom}(TA, \mathcal{H}(B, C)) \\
\mathrm{Hom}(T(A\otimes B), C) \arrow[u, "tp"] & \mathrm{Hom}(A, T^{-1}\mathcal{H}(B, C)) \arrow[u, "T"'] \arrow[d, "T^{-1}th"] \\
\mathrm{Hom}(A\otimes B, T^{-1}C) \arrow[u, "T"] \arrow[r, "ath"'] & \mathrm{Hom}(A, \mathcal{H}(B, T^{-1}C))
\end{tikzcd}$$

TATH12b. The following diagram is **anti**commutative (glue the diagram induced by **TP1P2** on the left of **TATH12**.

$$\begin{tikzcd}
\mathrm{Hom}(TA\otimes B, C) \arrow[r, "ath"] \arrow[d, "T^{-1}tp"'] & \mathrm{Hom}(TA, \mathcal{H}(B, C)) \\
\mathrm{Hom}(T^{-1}(TA\otimes TB), C) & \mathrm{Hom}(A, T^{-1}\mathcal{H}(B, C)) \arrow[u, "T"'] \\
\mathrm{Hom}(A\otimes TB, C) \arrow[u, "T^{-1}tp"] \arrow[r, "ath"'] & \mathrm{Hom}(A, \mathcal{H}(TB, C)) \arrow[u, "th"']
\end{tikzcd}$$

TATH23b. The following diagram is **anti**commutative (glue the diagram induced by **TH1H2** on the right of **TATH23**.

$$\begin{tikzcd}
\mathrm{Hom}(A\otimes TB, C) \arrow[r, "ath"] & \mathrm{Hom}(A, \mathcal{H}(TB, C)) \arrow[d, "th"] \\
\mathrm{Hom}(T(A\otimes B), C) \arrow[u, "tp"] & \mathrm{Hom}(A, T\mathcal{H}(TB, T^{-1}C)) \arrow[d, "th"] \\
\mathrm{Hom}(A\otimes B, T^{-1}C) \arrow[u, "T"] \arrow[r, "ath"'] & \mathrm{Hom}(A, \mathcal{H}(B, T^{-1}C))
\end{tikzcd}$$

Symmetric commutativity constraint of $\otimes$

We assume that the following holds.

CP. There is a functorial isomorphism $c_{A,B} : A\otimes B \to B\otimes A$.

SCP. The isomorphism c satisfies $c_{B,A} \circ c_{A,B} = Id_{A\otimes B}$.

TCP. The following diagram is commutative.

$$\begin{tikzcd} TA\otimes B \arrow[r, "c_{TA,B}"] \arrow[d, "tp_{1,A,B}"'] & B\otimes TA \arrow[d, "tp_{2,B,A}"] \\ T(A\otimes B) \arrow[r, "Tc_{A,B}"'] & T(B\otimes A) \end{tikzcd}$$

A.3.1. *Another possible definition of all this.* It would have been possible to define the morphisms above in a different way, which would make some axioms become definitions. Suppose that the tensor product $\otimes$, tp_1 and tp_2 are given and that they satisfy **TP1TP2**. Assume $\mathcal{H}$ is also given, but not necessarily th_1 and th_2. Suppose an adjunction ath is given without any compatibility property. Then, one can define $th1$ using the diagram **TATH12** in the following way. Replace A in the diagram by $\mathcal{H}(TB,C)$, and start with $Id_{\mathcal{H}(TB,C)}$ in the lower right set. All the morphisms but the lower right one are defined, we therefore get the image of the identity in the middle right group by circling clockwise. This defines an element $th_{1,B,C}$, and by definition, the diagram

$$\begin{tikzcd} \mathrm{Hom}(T\mathcal{H}(TB,C)\otimes B,C) \arrow[r, "ath"] & \mathrm{Hom}(T\mathcal{H}(TB,C),\mathcal{H}(B,C)) \\ \mathrm{Hom}(T(\mathcal{H}(TB,C)\otimes B),C) \arrow[u, "tp"] \arrow[d, "tp"] & \mathrm{Hom}(\mathcal{H}(TB,C),T^{-1}\mathcal{H}(B,C)) \arrow[u, "T"'] \\ \mathrm{Hom}(\mathcal{H}(TB,C)\otimes TB,C) \arrow[r, "ath"'] & \mathrm{Hom}(\mathcal{H}(TB,C),\mathcal{H}(TB,C)) \arrow[u, "th"'] \end{tikzcd}$$

is commutative. Now it is easy to prove that **TATH12** is commutative for any A; to do this, we use the following easy trick. We have to show that any element f in $\mathrm{Hom}(A,\mathcal{H}(TB,C))$ is sent to the same element using both sides of the diagram. Putting the above diagram under **TATH12** and sending $Id_{\mathcal{H}(TB,C)}$ to f by the map composing by f we are done by functoriality.

Trick A.9. Since the above proof can be applied to other commutative diagrams involving morphisms obtained by adjunctions and can be adapted to all sorts of similar versions (exchanging left and right, adding isomorphisms of functors, etc...), each time we will need such a diagram, we will just refer to the previous proof, and leave it to the reader to make the suitable modifications.

Of course, one can define th_2 by a similar technique using **TATH23** and that tp and th_1 are already defined, considering $A = T^{-1}\mathcal{H}(B,C)$. Thus the commutativity of **TATH12** and **TATH23** is true by definition and trick A.9. It is then easy to show that th_1 and th_2 satisfy **TH1TH2**, using **TP1TP2**.

One can use the same trick to define tp_2 starting with tp_1 using c (symmetric) and **TCP**. This shows that one may just start with

- $\otimes$, tp_1
- c
- $\mathcal{H}$
- ath

and use them to define

- tp_2
- th_1
- th_2

such that all the constraints are satisfied.

A.3.2. *Evaluation morphism.* Let A and K be objects of $\mathcal{C}$. We define the evaluation morphism

$$\mathrm{ev}_{A,K} : \mathcal{H}(A,K) \otimes A \to K$$

as the image of the identity by the adjunction **ATH**.

$$\mathrm{Hom}(\mathcal{H}(A,K), \mathcal{H}(A,K)) \simeq \mathrm{Hom}(\mathcal{H}(A,K) \otimes A, K)$$

Proposition A.10. *The evaluation satisfies the equality*

$$\text{(EVT1)} \qquad \mathrm{ev}_{A,TK} = T(\mathrm{ev}_{A,K}) \circ tp_{1,\mathcal{H}(A,K),A} \circ (th_{2,A,K} \otimes Id_A)$$

Proof: Consider the following diagram of isomorphisms

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{H}(A,TK), \mathcal{H}(A,TK)) & \mathrm{Hom}(\mathcal{H}(A,TK) \otimes A, TK) \arrow[l, "ath"'] \\
\mathrm{Hom}(T\mathcal{H}(A,K), \mathcal{H}(A,TK)) \arrow[u, "th"] \arrow[d, "th"'] & \mathrm{Hom}(T\mathcal{H}(A,K) \otimes A, TK) \arrow[l, "ath"] \arrow[u, "th"'] \\
\mathrm{Hom}(T\mathcal{H}(A,K), T\mathcal{H}(A,K)) & \mathrm{Hom}(T(\mathcal{H}(A,K) \otimes A), TK) \arrow[u, "tp"'] \\
\mathrm{Hom}(\mathcal{H}(A,K), \mathcal{H}(A,K)) \arrow[u, "T"] & \mathrm{Hom}(\mathcal{H}(A,K) \otimes A, K) \arrow[l, "ath"] \arrow[u, "T"']
\end{tikzcd}$$

which is commutative by the functoriality of the adjunction and **TATH13** applied to $\mathcal{H}(A,K)$, A and TK.

Start with the identity in the top left set. It is sent to the identity in the bottom left set (this is completely formal, solely due to the fact that T is a functor). These identities are respectively map to $\mathrm{ev}_{A,TK}$ in the upper left corner and $\mathrm{ev}_{A,K}$ in the lower left corner. Now the result follows from the right column. □

Proposition A.11. *The evaluation satisfies the equality*

$$\text{(EVT2)} \quad \mathrm{ev}_{TA,K} = \mathrm{ev}_{A,K} \circ (tp_{1,T^{-1}\mathcal{H}(A,K),A})^{-1} \circ tp_{2,T^{-1}\mathcal{H}(A,K),A} \circ (th_{1,A,K} \otimes Id_{TA})$$

Proof: Similar to the previous proof, but replace **TATH13** by **TATH12** applied to $T^{-1}\mathcal{H}(A,K)$, A and K. □

A.3.3. *Bidual morphism.* We now define what we call the bidual morphism

$$\varpi_{A,K} : A \to \mathcal{H}(\mathcal{H}(A,K), K)$$

as the image of the evaluation $\mathrm{ev}_{A,K}$ under the chain of bijections

$$\mathrm{Hom}(\mathcal{H}(A,K) \otimes A, K) \simeq \mathrm{Hom}(A \otimes \mathcal{H}(A,K), K) \simeq \mathrm{Hom}(A, \mathcal{H}(\mathcal{H}(A,K), K)$$

where the first one is induced by $c_{\mathcal{H}(A,K),A}$ and the second one is the adjunction.

In most applications, K will be chosen so that this morphisms is an isomorphism for all A, but formally, it has no reason to be so.

Proposition A.12. *The bidual morphism $\varpi_{A,K}$ satisfies the formula*

$$(\varpi T1) \qquad \varpi_{TA,K} = \mathcal{H}(th_{1,A,K}, K) \circ Tth_{1,T^{-1}\mathcal{H}(A,K),K} \circ T\varpi_{A,K}.$$

Proof: We consider the following diagram.

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{H}(TA,K)\otimes TA,K) \arrow[r, "c"] & \mathrm{Hom}(TA\otimes\mathcal{H}(TA,K),K) \arrow[r, "ath"] & \mathrm{Hom}(TA,\mathcal{H}(\mathcal{H}(TA,K),K)) \\
\mathrm{Hom}(T^{-1}\mathcal{H}(A,K)\otimes TA,K) \arrow[r, "c"] \arrow[u, "th"] & \mathrm{Hom}(TA\otimes T^{-1}\mathcal{H}(A,K),K) \arrow[r, "ath"] \arrow[u, "th"'] & \mathrm{Hom}(TA,\mathcal{H}(T^{-1}\mathcal{H}(A,K),K)) \arrow[u, "th"'] \\
\mathrm{Hom}(T(T^{-1}\mathcal{H}(A,K)\otimes A),K) \arrow[r, "c"] \arrow[u, "tp"] \arrow[d, "tp"'] & \mathrm{Hom}(T(A\otimes T^{-1}\mathcal{H}(A,K)),K) \arrow[u, "tp"'] \arrow[d, "tp"] & \mathrm{Hom}(TA,T\mathcal{H}(\mathcal{H}(A,K),K)) \arrow[u, "Tth"'] \\
\mathrm{Hom}(\mathcal{H}(A,K)\otimes A,K) \arrow[r, "c"'] & \mathrm{Hom}(A\otimes\mathcal{H}(A,K),K) \arrow[r, "ath"'] & \mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,K),K)) \arrow[u, "T"']
\end{tikzcd}$$

We have already seen in the proof of Proposition A.11 that the left column send $\mathrm{ev}_{A,K}$ to $\mathrm{ev}_{TA,K}$. The top and bottom rows send respectively $\mathrm{ev}_{TA,K}$ and $\mathrm{ev}_{A,K}$ to $\varpi_{TA,K}$ and $\varpi_{A,K}$, by definition. Following what happens in the right column, we thus just have to show that the outer diagram is commutative. In fact, each inner diagram is commutative:

- the two bottom left squares because of **TCP**
- the bottom right rectangle because of **TATH12**
- the top left square because of the functoriality of the morphism c
- the top right square because of the functoriality of the adjunction ath.

This proves the stated equality. □

Proposition A.13. *The bidual morphism $\varpi_{A,K}$ satisfies the formulas:*

$$(\varpi T2a) \qquad \varpi_{A,TK} = \mathcal{H}(th_{2,A,K},TK)\circ th^{-1}_{1,\mathcal{H}(A,K),TK}\circ T^{-1}th^{-1}_{2,\mathcal{H}(A,K),K}\circ\varpi_{A,K}$$

$$(\varpi T2b) \qquad \varpi_{A,TK} = -\mathcal{H}(th_{2,A,K},TK)\circ th^{-1}_{2,T\mathcal{H}(A,K),K}\circ Tth^{-1}_{1,\mathcal{H}(A,K),K}\circ\varpi_{A,K}$$

Proof: To get the first formula, we proceed as for the previous proposition with the following diagram of isomorphism. The commutative rectangle involved is **TATH23**.

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{H}(A,TK)\otimes A,TK) \arrow[r, "c"] & \mathrm{Hom}(A\otimes\mathcal{H}(A,TK),TK) \arrow[r, "ath"] & \mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,TK),TK)) \\
\mathrm{Hom}(T\mathcal{H}(A,K)\otimes A,TK) \arrow[r, "c"] \arrow[u, "th"] & \mathrm{Hom}(A\otimes T\mathcal{H}(A,K),TK) \arrow[r, "ath"] \arrow[u, "th"] & \mathrm{Hom}(A,\mathcal{H}(T\mathcal{H}(A,K),TK)) \arrow[u, "th"'] \arrow[d, "th"] \\
\mathrm{Hom}(T(\mathcal{H}(A,K)\otimes A),TK) \arrow[r, "c"] \arrow[u, "tp"] & \mathrm{Hom}(T(A\otimes\mathcal{H}(A,K)),TK) \arrow[u, "tp"] & \mathrm{Hom}(A,T^{-1}\mathcal{H}(\mathcal{H}(A,K),TK)) \arrow[d, "T^{-1}th"] \\
\mathrm{Hom}(\mathcal{H}(A,K)\otimes A,K) \arrow[r, "c"'] \arrow[u, "T"] & \mathrm{Hom}(A\otimes\mathcal{H}(A,K),K) \arrow[r, "ath"'] \arrow[u, "T"] & \mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,K),K))
\end{tikzcd}$$

The second formula is then a trivial consequence of the first one, using **TH1TH2**.
□

Proposition A.14. *The bidual morphism $\varpi_{A,K}$ satisfies the formula:*

$$(\varpi D) \qquad \mathcal{H}(\varpi_{A,K},K)\circ\varpi_{\mathcal{H}(A,K),K} = Id_{\mathcal{H}(A,K)}$$

Proof: Consider the following diagram, in which all vertical maps are isomorphisms. Let $f : F \to F'$. We use the notation $f^\sharp : \mathrm{Hom}(F', G) \to \mathrm{Hom}(F, G)$ and $f_\sharp : \mathrm{Hom}(G, F) \to \mathrm{Hom}(G, F')$ for the maps iduced by f.

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{H}(A,K),\mathcal{H}(A,K)) & \mathrm{Hom}(\mathcal{H}(A,K),\mathcal{H}(\mathcal{H}(\mathcal{H}(A,K),K),K)) \arrow[l, "(\mathcal{H}(\varpi_{A,K},K))_\sharp"'] \\
\mathrm{Hom}(\mathcal{H}(A,K)\otimes A,K) \arrow[u, "ath_{\mathcal{H}(A,K),A,K}"] & \mathrm{Hom}(\mathcal{H}(A,K)\otimes\mathcal{H}(\mathcal{H}(A,K),K),K) \arrow[l, "(\mathcal{H}(A,K)\otimes\varpi_{A,K})^\sharp"] \arrow[u, "ath_{\mathcal{H}(A,K),\mathcal{H}(\mathcal{H}(A,K),K),K}"'] \\
\mathrm{Hom}(A\otimes\mathcal{H}(A,K),K) \arrow[u, "c_{A,\mathcal{H}(A,K)}"] \arrow[d, "ath_{A,\mathcal{H}(A,K),K}"'] & \mathrm{Hom}(\mathcal{H}(\mathcal{H}(A,K),K)\otimes\mathcal{H}(A,K),K) \arrow[l, "(\varpi_{A,K}\otimes\mathcal{H}(A,K))^\sharp"] \arrow[u, "c_{\mathcal{H}(\mathcal{H}(A,K),K),\mathcal{H}(A,K)}"'] \arrow[d, "ath_{\mathcal{H}(\mathcal{H}(A,K),K),\mathcal{H}(A,K),K}"] \\
\mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,K),K)) & \mathrm{Hom}(\mathcal{H}(\mathcal{H}(A,K),K),\mathcal{H}(\mathcal{H}(A,K),K)) \arrow[l, "(\varpi_{A,K})^\sharp"]
\end{tikzcd}$$

Everything commutes by functoriality of ath and c. Now $Id_{\mathcal{H}(\mathcal{H}(A,K),K)}$ in the lower right set is sent to $\varpi_{\mathcal{H}(A,K),K}$ in the upper right set, which is in turn sent to $\mathcal{H}(\varpi_{A,K},K) \circ \varpi_{\mathcal{H}(A,K),K}$ in the upper left set. But $Id_{\mathcal{H}(\mathcal{H}(A,K),K)}$ is also sent to $\varpi_{A,K}$ in the lower left set, which is sent to $Id_{\mathcal{H}(A,K)}$ in the upper left set by definition of $\varpi_{A,K}$. This proves the formula. □

A.4. **Natural structures of triangulated categories with dualities.** We now assume that K is an object such that $(\mathcal{H}(-,K), \varpi_{-,K})$ is δ_K-exact. In this assumption is included the fact that ϖ_K is an isomorphism. We introduce some notations for the following functors and morphisms of functors.

$$\begin{array}{rlcl}
D_K: & \mathcal{C} & \to & \mathcal{C} \\
 & A & \mapsto & \mathcal{H}(A,K) \\
\varpi_K: & Id & \to & D_K \circ D_K \\
 & (\varpi_K)_A & = & \varpi_{A,K} \\
d_K: & T^{-1}D_K & \to & D_K T \\
 & (d_K)_A & = & th^{-1}_{1,A,K} \\
\rho_K: & TD_K & \to & D_{TK} \\
 & (\rho_K)_A & = & th^{-1}_{2,A,K}
\end{array}$$

Theorem A.15. *The 4-tuple $(\mathcal{C}, D_K, d_K, \varpi_K)$ is a triangulated category with weak duality. If furthermore $T^{-1}\mathcal{H}(-,K) = \mathcal{H}(T(-),K)$ and $th_{1,-,K} = Id_{\mathcal{H}(T(-),K)}$, then $(\mathcal{C}, D_K, \varpi_K)$ is a triangulated category with strict duality.*

Proof: We have to prove the relations $D_K\varpi_K \circ \varpi_K D_K = id_{D_K}$ and $Td_KT^{-1}D_K \circ D_Kd_K \circ \varpi_K T = T\varpi_K$. The first one is Proposition A.14 and the second one is Proposition A.12. □

Proposition A.16. *$(\mathcal{C}, D_{TK}, d_{TK}, \varpi_{TK})$ is a triangulated category with weak δ_K-duality. If furthermore $T^{-1}\mathcal{H}(-,K) = \mathcal{H}(T(-),K)$ and $th_{1,A,K} = Id_{\mathcal{H}(TA,K)}$, then $(\mathcal{C}, D_{TK}, \varpi_{TK})$ is a triangulated category with strict duality.*

Proof: All the relations required are obtained by replacing K by TK in the previous theorem, so we just have to prove that (D_{TK}, d_{TK}) is δ_K-exact. Recall that $(\mathcal{C}, TD_K, Td_K, (Td_KD_K) \circ \varpi_K)$ is a triangulated category with duality by Proposition A.5, so if

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

is an exact triangle, then the triangle

$$D_K(C) \xrightarrow{D_K(v)} D_K(B) \xrightarrow{D_K(u)} D_K(A) \xrightarrow{\delta_K T(D_K(w)\circ(d_K)_A)} TD_K(C)$$

is exact. Applying the functor T, we get that the triangle

$$TD_K(C) \xrightarrow{TD_K(v)} TD_K(B) \xrightarrow{TD_K(u)} TD_K(A) \xrightarrow{-\delta_K T^2(D_K(w)\circ(d_K)_A)} T^2D_K(C)$$

is exact. We now use the isomorphism of triangles

$$\begin{tikzcd}
TD_K(C) \arrow[r, "TD_K(v)"] \arrow[d, "(\rho_K)_C"'] & TD_K(B) \arrow[r, "TD_K(u)"] \arrow[d, "(\rho_K)_B"'] & TD_K(A) \arrow[r, "-\delta_K T^2(D_K(w)\circ(d_K)_A)"] \arrow[d, "(\rho_K)_A"'] & T^2D_K(C) \arrow[d, "T(\rho_K)_C"] \\
D_{TK}(C) \arrow[r, "D_{TK}(v)"'] & D_{TK}(B) \arrow[r, "D_{TK}(u)"'] & D_{TK}(A) \arrow[r, "\delta_K T(D_{TK}(w)\circ(d_{TK})_A)"'] & TD_{TK}(C)
\end{tikzcd}$$

The two squares on the left are commutative by simple functoriality, and the square on the right is in fact T applied to the diagram

$$\begin{tikzcd}
\mathcal{H}(A,K) \arrow[r, "(-\delta_K)Tth^{-1}_{1,A,K}"] \arrow[d, "T^{-1}th^{-1}_{2,A,K}"'] & T\mathcal{H}(TA,K) \arrow[r, "T\mathcal{H}(w,K)"] \arrow[d, "th^{-1}_{2,TA,K}"'] & T\mathcal{H}(C,K) \arrow[d, "th^{-1}_{2,C,K}"] \\
T^{-1}\mathcal{H}(A,TK) \arrow[r, "\delta_K th^{-1}_{1,A,TK}"'] & \mathcal{H}(TA,TK) \arrow[r, "\mathcal{H}(w,TK)"'] & \mathcal{H}(C,TK)
\end{tikzcd}$$

in which the first square is commutative by **TH1TH2** and the second is commutative by functoriality of th. $\square$

Proposition A.17. *The pair $(Id_{\mathcal{C}}, id, \rho_K)$ defines an isomorphism of triangulated categories with dualities from $(\mathcal{C}, TD_K, Td_K, -(Td_K D_K)\circ\varpi_K)$ to $(\mathcal{C}, D_{TK}, d_{TK}, \varpi_{TK})$.*

Proof: Let us prove the relations of Definition A.4:

(1) $\rho_K TD_K \circ (-Td_K D_K) \circ \varpi_K = D_{TK}\rho_K \circ \varpi_{TK}$
(2) $T\rho_K \circ T^2 d_K T^{-1} = -Td_{TK}T^{-1} \circ \rho_K T^{-1}$ (since TD_K has sign $-\delta_K$ and D_{TK} has sign δ_K)

The first one is exactly equality $(\varpi T2b)$ in Proposition A.13 and the second one is, after translation, **TH1TH2** applied to $T^{-1}A$ and K $\square$

Note that the first category in the previous proposition is (resp. is not) $T(\mathcal{C}, D_K, d_K, \varpi_K)$ of Definition A.6 when $\delta_K = 1$ (resp. when $\delta_K = -1$). For convenience, we now set $\mathcal{C}_K = (\mathcal{C}, D_K, d_K, \varpi_K)$. With this notation, (Id, id, ρ_K) is a duality δ_K-preserving functor (recall Definition A.7) from $T\mathcal{C}_K$ to $\mathcal{C}_{TK}$.

Corollary A.18. *By induction on i, we obtain higher versions of $(Id_{\mathcal{C}}, \rho_K)$*

$$\Gamma^{(i)}_K : T^i\mathcal{C}_K \to \mathcal{C}_{T^iK}$$

by setting $\Gamma^{(1)}_K = (Id_{\mathcal{C}}, \rho_K)$ and $\Gamma^i_K = \Gamma_{T^{i-1}K} \circ T(\Gamma^{(i-1)}_K)$. By multiplications of signs, $\Gamma^{(i)}_K$ is a duality δ^i_K-preserving functor.

Proof: Follows from the previous discussion and Remark A.8. $\square$

Of course, all this only affects the duality transformation ρ (the underlying functor is always $Id_{\mathcal{C}}$). So these duality transformations induce isomorphisms from Witt groups to Witt groups or to skew Witt groups (changing the bidual isomorphism by (-1)) according to the sign of Γ.

Remark A.19. If with start with a K such that $\mathcal{C}_K$ is 1-exact, then no duality (-1)-preserving functor can appear in the higher versions.

A.5. **The functors f^*, f_* and $f^!$.** We now assume that we have two categories as above, $\mathcal{C}$ and $\mathcal{D}$, that each of them is equipped with internal Hom and $\otimes$ satisfying the axioms of section A.3. Whenever possible, we use the same notation for this data in both categories.

We assume furthermore that we have additive functors $E, G : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathcal{C}$ such that E is a left adjoint to F and G is a right adjoint to F. Of course, we are interested in the example corresponding to Corollary 4.3, that is $E = \mathbf{L}f^*$, $F = \mathbf{R}f_*$ and $G = f^!$ (which - contrary to the adjunction between the tensor product and the internal Hom - does not exist on the level of additive categories already, but only when passing to derived categories). These adjunctions are denoted aef and afg. We also assume the following.

TE. There is a functorial isomorphism $te_A : ETA \to TEA$, such that (E, te) is a 1-exact functor.

Using Lemma A.2, one can define an isomorphism of functors $tf_A : FTA \to TFA$ (resp. $tg_A : GTA \to TGA$) such that (F, tf) (resp. (G, tg)) is a 1-exact functor. It is easy to show using trick A.9 that the two following diagrams are commutative, because they are exactly the ones used to find the inverse of tf (resp. tg) with $A = FB$ (resp. $A = GB$).

AEF.

$$\begin{tikzcd}
\mathrm{Hom}(EA, B) \arrow[r, "T"] \arrow[d, "aef_{A,B}"'] & \mathrm{Hom}(TEA, TB) & \mathrm{Hom}(ETA, TB) \arrow[l, "(te_A)^\sharp"'] \arrow[d, "aef_{TA,TB}"] \\
\mathrm{Hom}(A, FB) \arrow[r, "T"'] & \mathrm{Hom}(TA, TFB) \arrow[r, "(tf_B)_\sharp"'] & \mathrm{Hom}(TA, FTB)
\end{tikzcd}$$

AFG.

$$\begin{tikzcd}
\mathrm{Hom}(FA, B) \arrow[r, "T"] \arrow[d, "afg_{A,B}"'] & \mathrm{Hom}(TFA, TB) & \mathrm{Hom}(FTA, TB) \arrow[l, "(tf_A)^\sharp"'] \arrow[d, "afg_{TA,TB}"] \\
\mathrm{Hom}(A, GB) \arrow[r, "T"'] & \mathrm{Hom}(TA, TGB) \arrow[r, "(tg_B)_\sharp"'] & \mathrm{Hom}(TA, GTB)
\end{tikzcd}$$

We then assume the following.

EP. There is a functorial (in both variables) isomorphism $ep_{A,B} : EA \otimes EB \to E(A \otimes B)$.

TEP1. The following diagram is commutative.

$$\begin{tikzcd}
EA \otimes ETB \arrow[r, "id_{EA} \otimes te_B"] \arrow[d, "ep_{A,TB}"'] & EA \otimes TEB \arrow[r, "tp_{2,EA,EB}"] & T(EA \otimes EB) \arrow[d, "Tep_{A,B}"] \\
E(A \otimes TB) \arrow[r, "Etp_{2,A,B}"'] & ET(A \otimes B) \arrow[r, "te_{A \otimes B}"'] & TE(A \otimes B)
\end{tikzcd}$$

TEP2. The following diagram is commutative.

$$\begin{tikzcd}
ETA \otimes EB \arrow[r, "te_A \otimes id_{EB}"] \arrow[d, "ep_{TA,B}"'] & TEA \otimes EB \arrow[r, "tp_{1,EA,EB}"] & T(EA \otimes EB) \arrow[d, "Tep_{A,B}"] \\
E(TA \otimes B) \arrow[r, "Etp_{1,A,B}"'] & ET(A \otimes B) \arrow[r, "te_{A \otimes B}"'] & TE(A \otimes B)
\end{tikzcd}$$

EPC. The following diagram is commutative.

$$\begin{tikzcd} EA\otimes EB \arrow[r, "ep_{A,B}"] \arrow[d, "c_{EA,EB}"'] & E(A\otimes B) \arrow[d, "Ec_{A,B}"] \\ EB\otimes EA \arrow[r, "ep_{B,A}"'] & E(B\otimes A) \end{tikzcd}$$

Remark A.20. Note that **TEP1** and **EPC** imply **TEP2**, using **TCP**.

Definition A.21. Let x_A be the counit of aef, that is the image of Id_{FA} by the adjunction $\mathrm{Hom}(FA,FA) \to \mathrm{Hom}(EFA,A)$. Let $x_{A,B}$ be the element $x_A\otimes x_B$ in $\mathrm{Hom}(EFA\otimes EFB, A\otimes B)$. We denote by $fp_{A,B}$ the image of $x_{A,B}$ by the chain of morphisms

$$\mathrm{Hom}(EFA\otimes EFB, A\otimes B) \overset{(ep^{-1}_{FA,FB})^\sharp}{\longrightarrow} \mathrm{Hom}(E(FA\otimes FB), A\otimes B) \overset{aef}{\to} \mathrm{Hom}(FA\otimes FB, F(A\otimes B))$$

Notice that we have used the fact that ep is an isomorphism, so we cannot go on with the same procedure to define a similar morphism from $GA\otimes GB$ to $G(A\otimes B)$ since there is no reason for fp to be an isomorphism (and it is of course not an isomorphism in the classical examples).

The following is a consequence of **EPC** and the adjunction of E and F:
FPC. The following diagram is commutative.

$$\begin{tikzcd} FA\otimes FB \arrow[r, "fp_{A,B}"] \arrow[d, "c_{FA,FB}"'] & F(A\otimes B) \arrow[d, "Fc_{A,B}"] \\ FB\otimes FA \arrow[r, "fp_{B,A}"'] & F(B\otimes A) \end{tikzcd}$$

We will see below that this definition also implies that the same diagrams as **TEP1** and **TEP2** are still commutative after having replaced E by F, te by tf and ep by fp. Explicitly, we get the following commutative diagrams.
TFP1. The following diagram is commutative.

$$\begin{tikzcd} FA\otimes FTB \arrow[r, "id_{FA}\otimes tf_B"] \arrow[d, "fp_{A,TB}"'] & FA\otimes TFB \arrow[r, "tp_{2,FA,FB}"] & T(FA\otimes FB) \arrow[d, "Tfp_{A,B}"] \\ F(A\otimes TB) \arrow[r, "Ftp_{2,A,B}"'] & FT(A\otimes B) \arrow[r, "tf_{A\otimes B}"'] & TF(A\otimes B) \end{tikzcd}$$

TFP2. The following diagram is commutative.

$$\begin{tikzcd} FTA\otimes FB \arrow[r, "tf_A\otimes id_{FB}"] \arrow[d, "fp_{TA,B}"'] & TFA\otimes FB \arrow[r, "tp_{1,FA,FB}"] & T(FA\otimes FB) \arrow[d, "Tfp_{A,B}"] \\ F(TA\otimes B) \arrow[r, "Ftp_{1,A,B}"'] & FT(A\otimes B) \arrow[r, "tf_{A\otimes B}"'] & TF(A\otimes B) \end{tikzcd}$$

To establish the commutativity of **TFP1** is equivalent to showing that $fp_{TA,B} \in Hom(FA\otimes FTB, F(A\otimes TB))$ resp. $fp_{A,B} \in Hom(FA\otimes FB, F(A\otimes B))$ are mapped to the same element in $Hom(FA\otimes FTB, TF(A\otimes B))$ under $tf_\sharp \circ tp_\sharp$ resp. $T\circ tf^\sharp \circ tp^\sharp$. By definition we have that $aef\circ(ep^{-1})^\sharp(x_{A,TB}) = fp_{A,TB}$ and $aef\circ(ep^{-1})^\sharp(x_{A,B}) = fp_{A,B}$. Using all kind of functorialities and **AEF** and **TEP1**, we are reduced to show that $x_{A,B}$ maps to $x_{A,TB}$ under $T\circ tf^\sharp \circ te^\sharp \circ tp^\sharp \circ (tp^{-1})_\sharp$. This follows using the definition of tf and again functorialities. The commutativity of **TFP2** can be proved in a similar way.

Definition A.22. Let $eh_{A,B} : E\mathcal{H}(A,B) \to \mathcal{H}(EA,EB)$ denote the image of $Id_{\mathcal{H}(A,B)}$ by the chain of morphisms

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{H}(A,B),\mathcal{H}(A,B)) & & \mathrm{Hom}(E\mathcal{H}(A,B),\mathcal{H}(EA,EB)) \\
\mathrm{Hom}(\mathcal{H}(A,B)\otimes A,B) \arrow[u, "ath"] \arrow[r, "E"'] & \mathrm{Hom}(E(\mathcal{H}(A,B)\otimes A),EB) \arrow[r, "(ep_{\mathcal{H}(A,B),A})^\sharp"'] & \mathrm{Hom}(E\mathcal{H}(A,B)\otimes EA,EB) \arrow[u, "ath"']
\end{tikzcd}$$

We define $fh_{A,B} : F\mathcal{H}(A,B) \to \mathcal{H}(FA,FB)$ the same way, replacing E by F.

Applying trick A.9, we immediately get that the following diagram is commutative

EHP.

$$\begin{tikzcd}
\mathrm{Hom}(A,\mathcal{H}(B,C)) \arrow[r, "E"] & \mathrm{Hom}(EA,E\mathcal{H}(B,C)) \arrow[r, "(eh_{B,C})_\sharp"] & \mathrm{Hom}(EA,\mathcal{H}(EB,EC)) \\
\mathrm{Hom}(A\otimes B,C) \arrow[u, "ath"] \arrow[r, "E"'] & \mathrm{Hom}(E(A\otimes B),EC) \arrow[r, "(ep_{A,B})^\sharp"'] & \mathrm{Hom}(EA\otimes EB,EC) \arrow[u, "ath"']
\end{tikzcd}$$

We also get the same commutative diagram with E, eh and ep replaced by F, fh and fp.

FHP.

$$\begin{tikzcd}
\mathrm{Hom}(A,\mathcal{H}(B,C)) \arrow[r, "F"] & \mathrm{Hom}(FA,F\mathcal{H}(B,C)) \arrow[r, "(fh_{B,C})_\sharp"] & \mathrm{Hom}(FA,\mathcal{H}(FB,FC)) \\
\mathrm{Hom}(A\otimes B,C) \arrow[u, "ath"] \arrow[r, "F"'] & \mathrm{Hom}(F(A\otimes B),FC) \arrow[r, "(fp_{A,B})^\sharp"'] & \mathrm{Hom}(FA\otimes FB,FC) \arrow[u, "ath"']
\end{tikzcd}$$

It is also easy to show that the following diagrams are commutative, using **TEP1** (resp. **TEP2**) and **ATH** and proceeding similar to the argument that establishes **TFP1** using **TEP1** and **AEF**.

TEH1.

$$\begin{tikzcd}
\mathcal{H}(ET^{-1}A,EB) \arrow[r] & \mathcal{H}(T^{-1}EA,EB) \arrow[r] & T\mathcal{H}(EA,EB) \\
E\mathcal{H}(T^{-1}A,B) \arrow[u] \arrow[r] & ET\mathcal{H}(A,B) \arrow[r] & TE\mathcal{H}(A,B) \arrow[u]
\end{tikzcd}$$

TEH2.

$$\begin{tikzcd}
\mathcal{H}(EA,ETB) \arrow[r] & \mathcal{H}(EA,TEB) \arrow[r] & T\mathcal{H}(EA,EB) \\
E\mathcal{H}(A,TB) \arrow[u] \arrow[r] & ET\mathcal{H}(A,B) \arrow[r] & TE\mathcal{H}(A,B) \arrow[u]
\end{tikzcd}$$

Similarly, using **TFP1** and **TFP2** we obtain

TFH1.

$$\begin{tikzcd}
\mathcal{H}(FT^{-1}A,FB) \arrow[r] & \mathcal{H}(T^{-1}FA,FB) \arrow[r] & T\mathcal{H}(FA,FB) \\
F\mathcal{H}(T^{-1}A,B) \arrow[r] \arrow[u] & FT\mathcal{H}(A,B) \arrow[r] & TF\mathcal{H}(A,B) \arrow[u]
\end{tikzcd}$$

TFH2.

$$\begin{tikzcd}
\mathcal{H}(FA,FTB) \arrow[r] & \mathcal{H}(FA,TFB) \arrow[r] & T\mathcal{H}(FA,FB) \\
F\mathcal{H}(A,TB) \arrow[r] \arrow[u] & FT\mathcal{H}(A,B) \arrow[r] & TF\mathcal{H}(A,B) \arrow[u]
\end{tikzcd}$$

Proposition A.23. *The morphisms $eh_{A,B}$ make the diagram*

$$\begin{tikzcd}
EA \arrow[r, "E\varpi_B"] \arrow[d, "\varpi_{EB}"'] & E\mathcal{H}(\mathcal{H}(A,B),B) \arrow[d, "eh_{\mathcal{H}(A,B),B}"] \\
\mathcal{H}(\mathcal{H}(EA,EB),EB) \arrow[r, "\mathcal{H}(eh_{A,B},EB)"'] & \mathcal{H}(E\mathcal{H}(A,B),EB)
\end{tikzcd}$$

commutative, in other words

$$eh_{\mathcal{H}(A,B),B} \circ E\varpi_B = \mathcal{H}(eh_{A,B},B) \circ \varpi_{EB}$$

Proof: This amounts, after applying the functor $\mathcal{H}(EA,-)$, to proving that Id_{EA} is sent to the same element by the two paths in the diagram

$$\begin{tikzcd}
\mathrm{Hom}(EA,EA) \arrow[r, "(E\varpi_B)_\sharp"] \arrow[d, "(\varpi_{EB})_\sharp"'] & \mathrm{Hom}(EA,E\mathcal{H}(\mathcal{H}(A,B),B)) \arrow[d, "(eh_{\mathcal{H}(A,B),B})_\sharp"] \\
\mathrm{Hom}(EA,\mathcal{H}(\mathcal{H}(EA,EB),EB)) \arrow[r, "\mathcal{H}(eh_{A,B},B)_\sharp"'] & \mathrm{Hom}(EA,\mathcal{H}(E\mathcal{H}(A,B),EB))
\end{tikzcd}$$

Glueing under this last diagram the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(EA,\mathcal{H}(\mathcal{H}(EA,EB),EB)) \arrow[r, "eh"] & \mathrm{Hom}(EA,\mathcal{H}(E\mathcal{H}(A,B),EB)) \\
\mathrm{Hom}(EA\otimes\mathcal{H}(EA,EB),EB) \arrow[r, "eh"] \arrow[u, "ath"] & \mathrm{Hom}(EA\otimes E\mathcal{H}(A,B),EB) \arrow[u, "ath"'] \\
\mathrm{Hom}(\mathcal{H}(EA,EB)\otimes EA,EB) \arrow[r, "eh"] \arrow[u, "c"] \arrow[d, "ath"'] & \mathrm{Hom}(E\mathcal{H}(A,B)\otimes EA,EB) \arrow[u, "c"'] \arrow[d, "ath"] \\
\mathrm{Hom}(\mathcal{H}(EA,EB),\mathcal{H}(EA,EB)) \arrow[r, "eh"'] & \mathrm{Hom}(E\mathcal{H}(A,B),\mathcal{H}(EA,EB))
\end{tikzcd}$$

in which all vertical maps are isomorphisms, one sees that id_{EA} is sent to $eh_{A,B}$ in the lower right set by definition of ϖ_{EB}. We now prove that id_{EA} is again sent to this element $eh_{A,B}$ using the other path in the first diagram. Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,B),B)) \arrow[r, "E"] & \mathrm{Hom}(EA,E\mathcal{H}(\mathcal{H}(A,B),B)) \arrow[r, "(eh_{\mathcal{H}(A,B),B})_\sharp"] & \mathrm{Hom}(EA,\mathcal{H}(E\mathcal{H}(A,B),EB)) \\
\mathrm{Hom}(A\otimes\mathcal{H}(A,B),B) \arrow[u, "ath"] \arrow[r, "E"] & \mathrm{Hom}(E(A\otimes\mathcal{H}(A,B)),EB) \arrow[r, "(ep_{A,\mathcal{H}(A,B)})^\sharp"] & \mathrm{Hom}(EA\otimes E\mathcal{H}(A,B),EB) \arrow[u, "ath"'] \\
\mathrm{Hom}(\mathcal{H}(A,B)\otimes A,B) \arrow[u, "c"] \arrow[d, "ath"'] \arrow[r, "E"] & \mathrm{Hom}(E(\mathcal{H}(A,B)\otimes A),EB) \arrow[u, "c"'] \arrow[r, "(ep_{\mathcal{H}(A,B),A})^\sharp"] & \mathrm{Hom}(E\mathcal{H}(A,B)\otimes EA,EB) \arrow[u, "c"'] \arrow[d, "ath"] \\
\mathrm{Hom}(\mathcal{H}(A,B),\mathcal{H}(A,B)) \arrow[r, "E"'] & \mathrm{Hom}(E\mathcal{H}(A,B),E\mathcal{H}(A,B)) \arrow[r, "(eh_{A,B})_\sharp"'] & \mathrm{Hom}(E\mathcal{H}(A,B),\mathcal{H}(EA,EB))
\end{tikzcd}$$

in which the top and bottom rectangles are commutative because of **EHP**, the middle left one because of the functoriality of E and the middle right one because of **EPC**. Starting with $id_{\mathcal{H}(A,B)}$ in the lower left corner, we end up with $\varpi_{A,B}$ in the upper left corner, with $eh_{A,B}$ in the lower right corner, wich proves our claim. □

Definition A.24. We define $\alpha_{A,B} : F\mathcal{H}(A,GB) \to \mathcal{H}(FA,B)$ as the image of $id_{\mathcal{H}(A,GB)}$ by the chain of morphisms

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{H}(A,GB),\mathcal{H}(A,GB)) & & \mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,B)) \\
\mathrm{Hom}(\mathcal{H}(A,GB)\otimes A,GB) \arrow[u, "ath"] \arrow[r, "afg^{-1}"] & \mathrm{Hom}(F(\mathcal{H}(A,GB)\otimes A),B) \arrow[r, "(fp_{\mathcal{H}(A,GB),A})^\sharp"] & \mathrm{Hom}(F\mathcal{H}(A,GB)\otimes FA,B) \arrow[u, "ath"']
\end{tikzcd}$$

Verdier [36, Proposition 3] defines a natural isomorphism with similar source and target. In our setting, his definition becomes the following.

Definition A.25. We define $\tilde{\alpha}$ by the composition

$$\tilde{\alpha} : F\mathcal{H}(A,GB) \xrightarrow{fh} \mathcal{H}(FA,FGB)) \xrightarrow{Tr} \mathcal{H}(FA,B)$$

where $Tr : FG \to id$ is the counit of the adjunction afg between F and G.

Verdier uses the projection formula to show that $\tilde{\alpha}$ is an isomorphism. Below, we will use the projection formula to show that α is also an isomorphism. But anyway, we have the following.

Proposition A.26. *The two natural isomorphisms $\alpha = \tilde{\alpha}$ are equal.*

Proof: We have to prove that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}(F\mathcal{H}(A,GB),F\mathcal{H}(A,GB)) \arrow[rr, "fh"] & & \mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,FGB)) \arrow[d, "Tr"] \\
\mathrm{Hom}(\mathcal{H}(A,GB),\mathcal{H}(A,GB)) \arrow[u, "F"] & & \mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,B)) \\
\mathrm{Hom}(\mathcal{H}(A,GB)\otimes A,GB) \arrow[u, "ath"] \arrow[r, "afg"'] & \mathrm{Hom}(F(\mathcal{H}(A,GB)\otimes A),B) \arrow[r, "fp"'] & \mathrm{Hom}(F\mathcal{H}(A,GB)\otimes FA,B) \arrow[u, "ath"']
\end{tikzcd}$$

Using that $Tr \circ F = afg$ and that Tr and fp are applied to different variables and thus commute, we are reduced to show the commutativity of

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{H}(A,GB),\mathcal{H}(A,GB)) \arrow[r, "F"] & \mathrm{Hom}(F\mathcal{H}(A,GB),F\mathcal{H}(A,GB)) \arrow[r, "fh"] & \mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,FGB)) \arrow[d, "ath"] \\
\mathrm{Hom}(\mathcal{H}(A,GB)\otimes A,GB) \arrow[u, "ath"] \arrow[r, "F"'] & \mathrm{Hom}(F(\mathcal{H}(A,GB)\otimes A),FGB) \arrow[r, "fp"'] & \mathrm{Hom}(F\mathcal{H}(A,GB)\otimes FA,FGB)
\end{tikzcd}$$

which is **FHP** applied to $\mathcal{H}(A,GB)$, A and GB. $\square$

Thus we no longer distinguish between α and $\tilde{\alpha}$ Applying trick A.9, we immediately get the commutative diagram from Definition A.24

GHP.

$$\begin{tikzcd}
\mathrm{Hom}(A,\mathcal{H}(B,GC)) \arrow[r, "F"] & \mathrm{Hom}(FA,F\mathcal{H}(B,GC)) \arrow[r, "(\alpha_{B,C})_\sharp"] & \mathrm{Hom}(FA,\mathcal{H}(FB,C)) \\
\mathrm{Hom}(A\otimes B,GC) \arrow[u, "ath"] \arrow[r, "afg"'] & \mathrm{Hom}(F(A\otimes B),C) \arrow[r, "(fp_{A,B})^\sharp"'] & \mathrm{Hom}(FA\otimes FB,C) \arrow[u, "ath"']
\end{tikzcd}$$

Diagram **TFH1** and Proposition A.26 imply that the following diagram is commutative.

TFG.

$$\begin{tikzcd}
FT\mathcal{H}(A,GB) \arrow[d, "tf"'] & F\mathcal{H}(T^{-1}A,GB) \arrow[l, "FTth^{-1}T^{-1}"'] \arrow[r, "\alpha"] & \mathcal{H}(FT^{-1}A,B) \arrow[d, "T^{-1}tfT^{-1}"] \\
TF\mathcal{H}(A,GB) \arrow[r, "\alpha"'] & T\mathcal{H}(FA,B) & \mathcal{H}(T^{-1}FA,B) \arrow[l, "Tth^{-1}T^{-1}"]
\end{tikzcd}$$

Proposition A.27. *The morphisms $\alpha_{A,B}$ make the diagram*

$$\begin{tikzcd}
FA \arrow[r, "F\varpi_{A,GB}"] \arrow[d, "\varpi_{FA,B}"'] & F\mathcal{H}(\mathcal{H}(A,GB),GB) \arrow[d, "\alpha_{\mathcal{H}(A,GB),B}"] \\
\mathcal{H}(\mathcal{H}(FA,B),B) \arrow[r, "\mathcal{H}(\alpha_{A,B},B)"'] & \mathcal{H}(F\mathcal{H}(A,GB),B)
\end{tikzcd}$$

commutative.

Proof: The proof is the same as for Proposition A.23, but replacing E by F, B by GB, eh by $\alpha = Tr \circ fh$, **EHP** by **GHP**, **EPC** by **FPC** and functoriality of E by functoriality of afg. $\square$

A.6. **The projection formula and its consequences.** Let u be the unit of the adjunction aef. We now assume that the morphism $q_{A,B} = u \circ fp : A\otimes FB \to F(EA\otimes B)$ obtained by adjunction from $id_{EA\otimes B}$ is an isomorphism for all A and B.

Definition A.28. We define $\beta_{A,B} : \mathcal{H}(FA,B) \to F\mathcal{H}(A,GB)$ as the image of $id_{\mathcal{H}(A,GB)}$ by the chain of morphisms

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{H}(FA,B),\mathcal{H}(FA,B)) \arrow[d, "ath^{-1}"'] & \mathrm{Hom}(\mathcal{H}(FA,B),F\mathcal{H}(A,GB)) \\
\mathrm{Hom}(\mathcal{H}(FA,B)\otimes FA,B) \arrow[d, "(q^{-1}_{\mathcal{H}(FA,B),A})^\sharp"'] & \mathrm{Hom}(E\mathcal{H}(FA,B),\mathcal{H}(A,GB)) \arrow[u, "aef"'] \\
\mathrm{Hom}(F(E\mathcal{H}(FA,B)\otimes A),B) \arrow[r, "afg"] & \mathrm{Hom}(E\mathcal{H}(FA,B)\otimes A,GB) \arrow[u, "ath"']
\end{tikzcd}$$

Proposition A.29. *The morphisms $\alpha_{A,B}$ and $\beta_{A,B}$ are inverse to each other.*

Proof: The proof that $\alpha_{A,B} \circ \beta_{A,B} = id_{\mathcal{H}(FA,B)}$ follows as in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(F\mathcal{H}(A,GB),F\mathcal{H}(A,GB)) \arrow[r, "(\beta_{A,B})^\sharp"] \arrow[d, "aef^{-1}"'] & \mathrm{Hom}(\mathcal{H}(FA,B),F\mathcal{H}(A,GB)) \arrow[d, "aef^{-1}"] \\
\mathrm{Hom}(EF\mathcal{H}(A,GB),\mathcal{H}(A,GB)) \arrow[r, "\beta"] \arrow[d, "ath^{-1}"'] & \mathrm{Hom}(E\mathcal{H}(FA,B),\mathcal{H}(A,GB)) \arrow[d, "ath^{-1}"] \\
\mathrm{Hom}(EF\mathcal{H}(A,GB)\otimes A,GB) \arrow[r, "\beta"] \arrow[d, "afg^{-1}"'] & \mathrm{Hom}(E\mathcal{H}(FA,B)\otimes A,GB) \arrow[d, "afg^{-1}"] \\
\mathrm{Hom}(F(EF\mathcal{H}(A,GB)\otimes A),B) \arrow[r, "\beta"] \arrow[d, "q"'] & \mathrm{Hom}(F(E\mathcal{H}(FA,B)\otimes A),B) \arrow[d, "q"] \\
\mathrm{Hom}(F\mathcal{H}(A,GB)\otimes FA,B) \arrow[r, "\beta"] \arrow[d, "ath"'] & \mathrm{Hom}(\mathcal{H}(FA,B)\otimes FA,B) \arrow[d, "ath"] \\
\mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,B)) \arrow[r, "(\beta_{A,B})^\sharp"'] & \mathrm{Hom}(\mathcal{H}(FA,B),\mathcal{H}(FA,B))
\end{tikzcd}$$

the element id in the upper left corner is mapped to α in the lower left (since $u^\sharp = aef^{-1} \circ F$ and all morphism are natural with respect to u) and to β in the upper right corner, which are both mapped to id in the lower right corner. The fact that $\beta_{A,B} \circ \alpha_{A,B} = id_{F\mathcal{H}(A,GB)}$ comes again from the same kind of reasoning applied to the same diagram with β instead of α.

□

Remark A.30. Looking at the proof of the previous proposition, one can in fact weaken the assumptions, and just require that $q_{F\mathcal{H}(A,GB),A}$ and $q_{\mathcal{H}(FA,B),A}$ are isomorphisms for a certain A and B, and the proposition will still hold for these particular A and B.

A.7. **Natural functors of triangulated categories with duality.** Recall that (E, te) is a 1-exact functor from $\mathcal{C}$ to $\mathcal{D}$ and that (F, tf) is a 1-exact functor from $\mathcal{D}$ to $\mathcal{C}$. We denote by $\lambda_K = e_{\ ,K}$ the morphism of functors

$$\begin{array}{rlcl}
\lambda_K : & ED_K & \to & D_{EK}E \\
eh_{A,K} : & E\mathcal{H}(A,K) & \mapsto & \mathcal{H}(EA,EK).
\end{array}$$

Name	context
$th_{1,A,B}$	$\mathcal{H}(TA,B) \to T^{-1}\mathcal{H}(A,B)$
$th_{2,A,B}$	$\mathcal{H}(A,TB) \to T\mathcal{H}(A,B)$
$tp_{1,A,B}$	$TA\otimes B \to T(A\otimes B)$
$tp_{2,A,B}$	$A\otimes TB \to T(A\otimes B)$
$ath_{A,B,C}$	$\mathrm{Hom}(A\otimes B,C) \to \mathrm{Hom}(A,\mathcal{H}(B,C))$
$c_{A,B}$	$A\otimes B \to B\otimes A$
$\mathrm{ev}_{A,K}$	$\mathcal{H}(A,K)\otimes K \to K$
$\varpi_{A,K}$	$A \to \mathcal{H}(\mathcal{H}(A,K),K)$
$aef_{A,B}$	$\mathrm{Hom}(EA,B) \to \mathrm{Hom}(A,FB)$
$afg_{A,B}$	$\mathrm{Hom}(FA,B) \to \mathrm{Hom}(A,GB)$
te_A	$ETA \to TEA$
tf_A	$FTA \to TFA$
tg_A	$GTA \to TGA$
$ep_{A,B}$	$EA\otimes EB \to E(A\otimes B)$
$fp_{A,B}$	$FA\otimes FB \to F(A\otimes B)$
$eh_{A,B}$	$E\mathcal{H}(A,B) \to \mathcal{H}(EA,EB)$
$fh_{A,B}$	$F\mathcal{H}(A,B) \to \mathcal{H}(FA,FB)$
$\alpha_{A,B}$	$F\mathcal{H}(A,GB) \to \mathcal{H}(FA,B)$
$\beta_{A,B}$	$\mathcal{H}(FA,B) \to F\mathcal{H}(A,GB)$
$q_{A,B}$	$A\otimes FB \to F(EA\otimes B)$

TABLE 1. Summary of the different definitions

Theorem A.31. *Let K be an object of $\mathcal{C}$ such that ϖ_K and λ_K are isomorphisms of functors, and such that (D_K, d_K) and (D_{EK}, d_{EK}) are δ-exact (same δ). Then the triple (E, te, λ_K) is a functor of triangulated categories with dualities from $(\mathcal{C}, D_K, d_K, \varpi_K)$ to $(\mathcal{D}, D_{EK}, d_{EK}, \varpi_{EK})$.*

Proof: We have to check that λ_K satisfies the diagrams of Definition A.4. The first one is Proposition A.23 and the second one is **TEH1**. □

Theorem A.32. *Let L be an object of $\mathcal{D}$ such that ϖ_{GL} and ϖ_L are isomorphisms, (D_L, d_L) and $(D_{GL}, d_G L)$ are δ-exact (same δ), and such that $q_{FD_{GL}(A),A}$ and $q_{D_L F(A),A}$ are isomorphism for all A and set $\alpha_L = \alpha_{\ ,L}$. Then the triple (F, tf, α_L) is a functor of triangulated categories with dualities from $(\mathcal{D}, D_{GL}, d_{GL}, \varpi_{GL})$ to $(\mathcal{C}, D_L, d_L, \varpi_L)$.*

Proof: Proposition A.29 and Remark A.30 ensure that α_L is an isomorphism of functors. We then have to check that α_L satisfies the diagrams of Definition A.4. The first one is Proposition A.27 and the second one is **TFG**. □

A.8. An example for section A.3.

A.8.1. *Category of complexes.* Let $\mathcal{A}$ be an additive category with an internal Hom (denoted by h) and an internal tensor product (denoted by $\bullet$) with a commutativity constraint and an adjunction between $\bullet$ and h additive and functorial in the three variables. We now show that the category of complexes of objects of $\mathcal{A}$ can be equipped with an internal Hom $\mathcal{H}$ and an internal tensor product $\otimes$ satisfying the axioms of section A.3. This is essentially a problem of choosing some signs, and as explained in section A.3.1, some choices determine the others.

We work with homological differentials, *i.e.*

$$d_i^A : A_i \to A_{i-1}.$$

The groups defining the translation functor are

$$(TA)_n = A_{n-1}.$$

The groups in the tensor product and the internal Hom are given by

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \bullet B_j$$

and

$$\mathcal{H}(A,B)_n = \prod_{j-i=n} h(A_i, B_j).$$

In table 2 can be read where we put which sign in our definitions of the different groups and morphisms. As a general rule, the indices in the sign correspond to the groups from which the morphism with this sign starts.

Definition of	Sign	Locus
TA	ϵ_i^T	$d_{i+1}^{TA} = \epsilon_i^T d_i A$
$A \otimes B$	$\epsilon_{i,j}^{1\otimes}$ $\epsilon_{i,j}^{2\otimes}$	$\epsilon_{i,j}^{1\otimes} d_i^A \bullet id_{B_j}$ $\epsilon_{i,j}^{2\otimes} id_{A_i} \bullet d_j^B$
$\mathcal{H}(A,B)$	$\epsilon_{i,j}^{1\mathcal{H}}$ $\epsilon_{i,j}^{2\mathcal{H}}$	$\epsilon_{i,j}^{1\mathcal{H}} (d_{i+1}^A)^\sharp$ $\epsilon_{i,j}^{2\mathcal{H}} (d_j^B)_\sharp$
$tp_{1,A,B}$	$\epsilon_{i,j}^{tp1}$	$\epsilon_{i,j}^{tp1} id_{A_i \bullet B_j}$
$tp_{2,A,B}$	$\epsilon_{i,j}^{tp2}$	$\epsilon_{i,j}^{tp2} id_{A_i \bullet B_j}$
$th_{1,A,B}$	$\epsilon_{i,j}^{th1}$	$\epsilon_{i,j}^{th1} id_{h(A_i,B_j)}$
$th_{2,A,B}$	$\epsilon_{i,j}^{th2}$	$\epsilon_{i,j}^{th2} id_{h(A_i,B_j)}$
$ath_{A,B,C}$	$\epsilon_{i,j}^{ath}$	$\epsilon_{i,j}^{ath} (\mathrm{Hom}(A_i \bullet B_j, C_{i+j}) \to \mathrm{Hom}(A_i, h(B_j, C_{i+j})))$
$c_{A,B}$	$\epsilon_{i,j}^{c}$	$\epsilon_{i,j}^{c} (A_i \bullet B_j \to B_j \bullet A_i)$

TABLE 2. Sign definitions

In table 3, we state the compatibility that these signs must satisfy for the axioms to be true.

As the discussion in section A.3.1 suggests, some equalities are consequences of other ones. It is also easy to see that $(1,4,6) \Rightarrow 7$ and $(2,10,12) \Rightarrow 8$. If you assume 19, then 17 and 18 are equivalent.

Balmer, Gille and Nenashev [3], [4],[11], [14] always consider strict dualities, that is $\epsilon^{th1} = 1$. The signs choosen in [4, §2.6] imply that $\epsilon_{i,0}^{1\mathcal{H}} = 1$. The choices made by [14, Example 1.4] are $\epsilon_{i,j}^{1\otimes} = 1$ and $\epsilon_{i,j}^{2\otimes} = (-1)^i$. In [11, p. 111] the signs $\epsilon_{i,j}^{1\mathcal{H}} = 1$ and $\epsilon_{i,j}^{2\mathcal{H}} = (-1)^{i+j+1}$ are choosen. Finally, the sign choosen for ϖ in [11, p. 112] corresponds via our definition of ϖ to the equality $\epsilon_{j-i,i}^{ath} \epsilon_{i,j-i}^{ath} \epsilon_{j-i,i}^{c} = (-1)^{j(j-1)/2}$. It is possible to choose the signs in a way compatible with all these choices and our formalism.

	compatibility	reason
1	$\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i,j-1}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i-1,j} = -1$	$A \otimes B$ is a complex
2	$\epsilon^{1\mathcal{H}}_{i,j}\epsilon^{1\mathcal{H}}_{i,j-1}\epsilon^{2\mathcal{H}}_{i,j}\epsilon^{2\mathcal{H}}_{i+1,j} = -1$	$\mathcal{H}(A,B)$ is a complex
3	$\epsilon^{T}_{i}\epsilon^{T}_{i+j}\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i+1,j}\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i-1,j} = 1$	$tp_{1,A,B}$ is a morphism
4	$\epsilon^{T}_{i+j}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i+1,j}\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i,j-1} = 1$	
5	$\epsilon^{T}_{j}\epsilon^{T}_{i+j}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i,j-1} = 1$	$tp_{2,A,B}$ is a morphism
6	$\epsilon^{T}_{i+j}\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i-1,j} = 1$	
7	$\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i+1,j} = -1$	**TP1TP2** is true
8	$\epsilon^{th1}_{i,j}\epsilon^{th1}_{i,j+1}\epsilon^{th2}_{i,j}\epsilon^{th2}_{i+1,j} = -1$	**TH1TH2** is true
9	$\epsilon^{T}_{i+1}\epsilon^{T}_{j-i-1}\epsilon^{1\mathcal{H}}_{i,j}\epsilon^{1\mathcal{H}}_{i+1,j}\epsilon^{th1}_{i,j}\epsilon^{th1}_{i+1,j} = 1$	$th_{1,A,B}$ is a morphism
10	$\epsilon^{T}_{j-i-1}\epsilon^{2\mathcal{H}}_{i,j}\epsilon^{2\mathcal{H}}_{i+1,j}\epsilon^{th1}_{i,j}\epsilon^{th1}_{i,j-1} = 1$	
11	$\epsilon^{T}_{j}\epsilon^{T}_{j-i}\epsilon^{2\mathcal{H}}_{i,j}\epsilon^{2\mathcal{H}}_{i,j+1}\epsilon^{th2}_{i,j}\epsilon^{th2}_{i,j-1} = 1$	$th_{2,A,B}$ is a morphism
12	$\epsilon^{T}_{j-i}\epsilon^{1\mathcal{H}}_{i,j}\epsilon^{1\mathcal{H}}_{i,j+1}\epsilon^{th2}_{i,j}\epsilon^{th2}_{i+1,j} = 1$	
13	$\epsilon^{1\otimes}_{i,j}\epsilon^{2\otimes}_{i,j}\epsilon^{1\mathcal{H}}_{j-1,i+j-1}\epsilon^{ath}_{i,j-1}\epsilon^{ath}_{i-1,j} = -1$	ath is well defined
14	$\epsilon^{1\otimes}_{i,j}\epsilon^{2\mathcal{H}}_{j,i+j}\epsilon^{ath}_{i-1,j}\epsilon^{ath}_{i,j} = 1$	
15	$\epsilon^{tp1}_{i,j}\epsilon^{tp2}_{i,j}\epsilon^{th1}_{j,i+j+1}\epsilon^{ath}_{i,j+1}\epsilon^{ath}_{i+1,j} = 1$	**TATH12** is true
16	$\epsilon^{tp2}_{i,j}\epsilon^{th1}_{j,i+j+1}\epsilon^{th2}_{j,i+j}\epsilon^{ath}_{i,j}\epsilon^{ath}_{i,j+1} = 1$	**TATH23** is true
17	$\epsilon^{1\otimes}_{i,j}\epsilon^{2\otimes}_{j,i}\epsilon^{c}_{i,j}\epsilon^{c}_{i-1,j} = 1$	$c_{A,B}$ is a morphism
18	$\epsilon^{1\otimes}_{j,i}\epsilon^{2\otimes}_{i,j}\epsilon^{c}_{i,j}\epsilon^{c}_{i,j-1} = 1$	
19	$\epsilon^{c}_{i,j}\epsilon^{c}_{j,i} = 1$	**SCP** is true
20	$\epsilon^{tp1}_{i,j}\epsilon^{tp2}_{j,i}\epsilon^{c}_{i,j}\epsilon^{c}_{i+1,j} = 1$	**TCP** is true

TABLE 3. Sign definitions

Theorem A.33. *Let* $a, b \in \{+1, -1\}$. *Then*

$$\begin{array}{ll} \epsilon^{1\otimes}_{i,j} = 1 & \epsilon^{tp1}_{i,j} = a \\ \epsilon^{2\otimes}_{i,j} = (-1)^{i} & \epsilon^{tp2}_{i,j} = a(-1)^{i} \\ \epsilon^{1\mathcal{H}}_{i,j} = 1 & \epsilon^{th1}_{i,j} = 1 \\ \epsilon^{2\mathcal{H}}_{i,j} = (-1)^{i+j+1} & \epsilon^{th2}_{i,j} = a(-1)^{i+j} \\ \epsilon^{ath}_{i,j} = b(-1)^{i(i-1)/2} & \epsilon^{c}_{i,j} = (-1)^{ij} \\ \epsilon^{T}_{i} = -1 & \end{array}$$

satisfies all equalities of Table 3 as well as $\epsilon^{ath}_{j-i,i}\epsilon^{ath}_{i,j-i}\epsilon^{c}_{j-i,i} = (-1)^{j(j-1)/2}$ *and is compatible with all the above sign choices of Balmer, Gille and Nenashev.*

Proof: Straightforward. □

A.8.2. *Derived category.* What we did so far implies that if $\mathcal{A}$ is as in A.8.1 and furthermore is an abelian (or more generally exact) category with enough injectives and projectives (or flat objects), then the derived category $D(\mathcal{A})$ can also naturally be equipped with all the data in section A.3 so that it satisfies the compatibilities by setting $\mathcal{H}_{D(\mathcal{A})}(A,B) = R_I R_{II} \mathcal{H}(A,B)$ as in Hartshorne, compare section 4. In particular, for any given abelian category $\mathcal{A}$, we obtain an adjunction between $\mathcal{H}_{D(\mathcal{A})}$ and $\otimes^{\mathbf{L}}$ satisfying all the formulae of subsection A.3. In particular, all statements and results of section 4 that are deduced from the appendix are true for the choice of signs made above. Beware that we sometimes tacitly restrict finctors that are a priori defined on unbounded complexes or complexes bounded above or below to bounded complexes. Doing this sometimes requires the existence of bounded flat resolutions which is often the reason why we need to assume that our schemes are regular.

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