Homogeneous varieties - zero cycles of degree one versus rational points

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Abstract *Examples of projective homogeneous varieties over the field of Laurent series over p-adic fields which admit zero cycles of degree one and which do not have rational points are constructed.*

Let k be a field and X a quasi-projective variety over k. Let $Z_0(X)$ denote the group of zero cycles on X and deg : $Z_0(X) \to \mathbb{Z}$ the degree homomorphism which associates to a closed point x of X, the degree $[k(x) : k]$ of its residue field.

We study the question: for what classes $\mathfrak{X}$ of varieties (respectively, what classes of fields k) is it true that for $X \in \mathfrak{X}$, if X admits a 0-cycle of degree one, then X has a rational point.

If X is a curve of genus zero or one over any field k, then X has a rational point once it admits a zero cycles of degree one. However, the question is more reasonable to ask for classes of rational varieties or homogeneous varieties. In the setting of rational varieties, there are examples, due to Colliot-Thélène and Coray [CTC] of conic bundles over the projective line over a p-adic field with a zero cycle of degree one, which have no rational points. We shall list from literature some questions in this direction for homogeneous varieties. $Q(HP_r)$ **(Veisfeiler)**[V] Let X be a projective homogeneous variety under a connected linear algebraic group defined over a field k. If X has a zero cycle of degree one, does X have a rational point?

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$Q(PHS)$[**Se**] Let G be a connected linear algebraic group defined over a field k. Let X be a principal homogeneous space for G over k. If X has a zero cycle of degree one, does X have a rational point?

The following questions combine the above two in a more general setting.

$Q(H)$ (**Colliot-Thélène**)[**To**] Let X be a quasi-projective homogeneous variety under a connected linear algebraic group defined over k. If X has a zero cycle of degree one, does X have a rational point?

$Q(Hd)$ (**Totaro**)[**To**] Let X be a quasi-projective variety under a connected linear algebraic group defined over k. If X has a zero cycle of degree $d > 0$, does X have a closed point of degree dividing d?

Totaro mentions that the most reasonable cases of his question are where X is a principal homogeneous space or if X is projective.

The first example where $Q(H)$ has a negative answer is due to [F]; the stabilizer of a rational point over the algebraic closure for these homogeneous spaces is a finite group.

In this paper, we give examples to show that $Q(HPr)$ has a negative answer in general.

§1 Connectedness

We remark that connectedness is essential with respect to these questions. There exist automorphisms (Coleman automorphisms [HK]) of finite groups with the properties:

1. f is not inner.

2. for every sylow subgroup H of G, $f|_H : H \to G$ is given by $f(x) = y_H x y_H^{-1}$ for some $y_H \in G$.

The class of $[f]$ in $H^1(G, G)$ for the trivial action of G on G is non-trivial, but restricted to each p-sylow subgroup, it is trivial. We thank P. Gille for bringing to our attention these examples.

§2 Principal homogeneous spaces

The case of $Q(PHS)$ is wide open, and in special cases $Q(PHS)$ is proved to have an affirmative answer. The cases of PGL_n and O_n are classical; for O_n the result goes back to a theorem of Springer [Sp]. The case of unitary groups is settled in the affirmative by Eva-Bayer and Lenstra [BL]. A positive answer to $Q(PHS)$ when k is a number field is due to Sansuc [Sa]. A main ingredient in the proof is the Hasse principle for principal homogeneous spaces under semisimple simply connected linear algebraic groups defined over number fields. There are conjectures concerning Hasse principle for fields of virtual cohomological dimension 2, due to Colliot-Thélène which have been proved for classical groups [BP]. One can show that $Q(PHS)$ has an affirmative answer for principal homogeneous spaces under connected linear algebraic groups defined over a field of virtual cohomological dimension 2, provided the corresponding simply connected group satisfies Hasse Principle conjecture [P1]

§3 Projective homogeneous varieties

The question $Q(HPr)$ also admits a positive answer for number fields and this can be deduced from the theorem of Harder on Hasse principle for projective homogeneous varieties defined over number fields [H]. If one follows the proof of Borovoi of Harder's theorem, using the non-abelian H^2 of Springer to study homogeneous varieties, one can show, using the results of [CGP] that if k is a 2-dimensional strict henselian field, $Q(HPr)$ has a positive answer.

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It is good to study $Q(HPr)$ in the case of 2-dimensional fields. A positive answer can be derived for classical groups over C_2 fields [P2].

§4 Example

In this section, we construct an example to show that $Q(HPr)$ has a negative answer in general. This example is a refinement of an example given in [PSS].

Let k be a p-adic field containing a primitive p-th root of unity ξ with $p \geq 5$. Let $K = k((t))$. Let $\ell|k$ be a degree two extension which is totally ramified over k. Then $|\ell^*/\ell^{*p}| > |k^*/k^{*p}|$. Let $L = \ell((t))$. Let $\mu \in \ell^*$ be such that $[\mu] \in \ker(N_{\ell/k} : \ell^*/\ell^{*p} \to k^*/k^{*p})$ and $[\mu] \neq 1$ in ℓ^*/ℓ^{*p}. Let D be the cyclic algebra of degree p over L defined by:

$$X^p = \mu, Y^p = t, XY = \xi Y X.$$

It is represented by $(\mu) \cup (t) \in H^2(L, \mu_p)$. We have, by choice, $\mathrm{cores}_{L/K}(D) = 1$ so that D supports an involution of second kind. Let τ be an $L|K$ involution on D. Let $\lambda \in k^*$ be such that $\lambda \notin N_{\ell/k}(\ell^*)$. Let h be the rank 3 hermitian form $\langle 1, -\lambda, t \rangle$ over (D, τ). Then we have the following:

Lemma The hermitian form h is anisotropic over (D, τ).

Proof Let $\Delta = \{a \in D : \mathrm{Nrd}_{D|L}(a) \in l[[t]]\}$ be the unique maximal $\ell[[t]]$-order in D. Every element a of Δ can be written as $a = \pi^n b$, where b is a unit in Δ and π a generator of the unique maximal right ideal in Δ. Suppose there exist $v_1, v_2, v_3 \in D$, not all zero, such that:

$$v_1 \tau(v_1) - \lambda v_2 \tau(v_2) + t v_3 \tau(v_3) = 0 \tag{1}$$

Without loss of generality, we may assume that each $v_i \in \Delta$. We write

$v_i = \pi^{n_i} u_i$, where u_i is a unit in Δ for $i = 1, 2, 3$. Thus (1) becomes:

$$\pi^{n_1} u_1 \tau(\pi^{n_1} u_1) - \lambda \pi^{n_2} u_2 \tau(\pi^{n_2} u_2) + t \pi^{n_3} u_3 \tau(\pi^{n_3} u_3) = 0 \qquad (2)$$

We first consider the case when n_1 is smallest of all n_i. In this case setting $m_2 = n_2 - n_1$ and $m_3 = n_3 - n_1$, (2) can be rewritten as:

$$u_1 \tau(u_1) = \lambda \pi^{m_2} u_2 \tau(\pi^{m_2} u_2) - t \pi^{m_3} u_3 \tau(\pi^{m_3} u_3) \qquad (3)$$

Since $t = \pi^p u_0$, u_0 is a unit in Δ and p odd, the valuation of $\lambda \pi^{m_2} u_2 \tau(\pi^{m_2} u_2) - t \pi^{m_3} u_3 \tau(\pi^{m_3} u_3)$ is the minimum of $\{2m_2, p+2m_3\}$, which is zero; this implies that $m_2 = 0$. Thus

$$\lambda u_2 \tau(u_2) = u_1 \tau(u_1) \left(1 + t \tau(u_1)^{-1} u_1^{-1} \pi^{m_3} u_3 \tau(\pi^{m_3} u_3)\right) \qquad (4)$$

Set $w = \tau(u_1)^{-1} u_1^{-1} \pi^{m_3} u_3 \tau(\pi^{m_3} u_3)$. Then taking reduced norm on the both sides of (4) we get:

$$\mathrm{Nrd}_D\left(\lambda u_2 \tau(u_2)\right) = \mathrm{Nrd}_D\left(u_1 \tau(u_1)\right) \mathrm{Nrd}_D(1 + tw)$$

which gives:

$$\lambda^p N_{L|K}\left(\mathrm{Nrd}_D(u_2)\right) = N_{L|K}\left(\mathrm{Nrd}_D(u_1)\right) \mathrm{Nrd}_D(1 + tw)$$

Reading the above equality modulo t, we conclude that $\lambda^p \in N_{\ell|k}(\ell^*)$. Since ℓ is a quadratic extension over k and p is odd, this implies that $\lambda \in N_{\ell|k}(\ell^*)$. But this is a contradiction to the choice of λ and therefore the form h is anisotropic in this case. The case when n_2 is smallest can be treated in a similar manner.

Now we consider the case when n_3 is the smallest among all n_i's. Let $r_1 = n_1 - n_3$ and $r_2 = n_2 - n_3$. Then (2) becomes:

$$\pi^{r_1} u_1 \tau(\pi^{r_1} u_1) - \lambda \pi^{r_2} u_2 \tau(\pi^{r_2} u_2) = -t u_3 \tau(u_3) \tag{5}$$

Suppose $r_1 \neq r_2$. Valuation of $\pi^{r_1} u_1 \tau(\pi^{r_1} u_1) - \lambda \pi^{r_2} u_2 \tau(\pi^{r_2} u_2)$ is minimum of $\{2r_1, 2r_2\}$, which is even, while the valuation of $t u_3 \tau(u_3)$ is p which is odd leading to a contradiction. Therefore $r_1 = r_2$ and we have

$$u_1 \tau(u_1) = \lambda u_2 \tau(u_2) - t \pi^{-r_1} u_3 \tau(\pi^{-r_1} u_3) \tag{6}$$

If $p < 2r_1$, $u_3 \tau(u_3) = t^{-1} \pi^{r_1} \left(-u_1 \tau(u_1) + \lambda u_2 \tau(u_2) \right) \tau(\pi^{r_1})$ with the valuation of right hand side positive, leading to a contradiction. Thus $p > 2r_1$. Taking reduced norm on the both sides of (6) and then reading modulo t we conclude, as before that $\lambda \in N_{\ell|k}(\ell^*)$, which is a contradiction. Thus the hermitian form h is anisotropic. $\qquad\qquad\square$

Remark (i) The algebra $D_{K((\sqrt{\lambda}))}$ is division and $h_{K(\sqrt{\lambda})} \simeq \langle 1, -1, t \rangle$ is isotropic over $\left(D_{K(\sqrt{\lambda})}, \tau_{K(\sqrt{\lambda})} \right)$ with a rank one isotropic subspace (over $\left(D_{K(\sqrt{\lambda})} \right)$).

(ii) Let $M = K(t^{1/p})$. Then $[M : K] = p, D_M$ is split and h_M is Morita equivalent to a $3p$-dimensional hermitian form $\tilde{h}_M$ over $LM|M$. Since M is a Laurent series field over a p-adic field, every 9-dimensional quadratic form over M is isotropic and every 5-dimensional hermitian form over $LM|M$ is isotropic. Since $p \geq 5, \tilde{h}_M$ has a totally isotropic subspace of rank at least p over LM; hence h_M has a rank one isotropic subspace over D_M.

Let X be the variety of rank one (rank over D) zero subspaces of h over (D, τ). Then X is a projective variety homogeneous under the action of $SU(h)$. Further $X(K) = \phi, X\left(K\left(\sqrt{\lambda}\right)\right) \neq \phi$ and $X(M) \neq \phi$. Thus X admits a zero-cycle of degree one and X has no K-rational point.

Remark One may replace the Laurent series fields over p-adic fields in the above examples by the rational function field in one variable over p-adic fields. One needs to use results of [HV] and [PS] stating that quadratic forms over such fields in sufficiently many variables have a nontrivial zero.

We shall now describe the parabolic subgroup defining the stabilizer of a rational point of $X(\bar{K})$.

Let $\triangle$ be the set of simple roots with respect to a pair (T, B) for $G(\overline{k}) = SU(h)(\overline{k})$ for a choice of a maximal torus T and a Borel subgroup B containing T. Let $S = \triangle \setminus \{p, 2p\}$ where the ordering on vertices of $\triangle$ are as in [T]. Let X be the variety of parabolic subgroups of $G(\overline{k})$ defined by the conjugacy class associated to S. Since S is invariant under Galois action, X is defined over k. We have the following criterion due to Tits [T] for the existence of a rational point for X: $X(E) \neq \phi$ for any extension $E|K$ if and only if S contains all the non-distinguished vertices of the Tits index of $SU(h)(E)$; this is equivalent to h having a rank 1 isotropic subspace over (D_E, τ_E). The variety X is the variety of rank 1 isotropic subspaces in h. We have the following Tits indices for $SU(h)_{K\left(\sqrt{\lambda}\right)}$ and $SU(h)_M$ (Witt index of h over M is $p + r$).
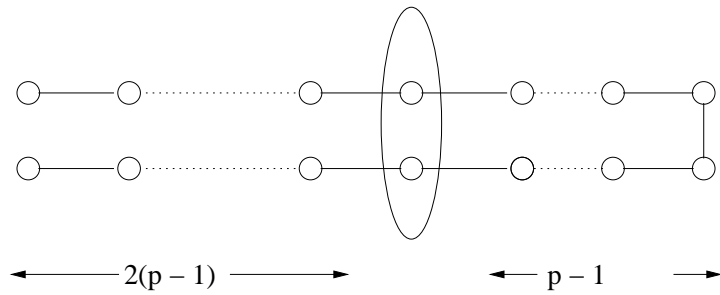
Tits index for $SU(h)_{K(\sqrt{\lambda})}$
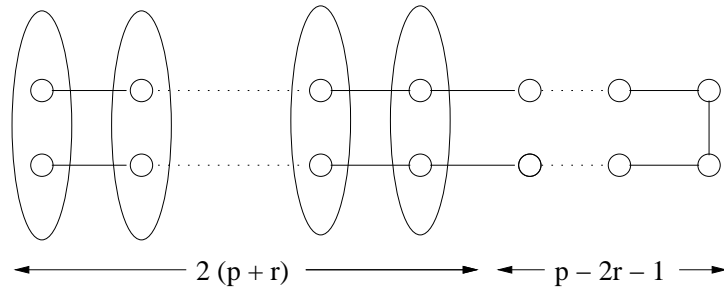
Tits index for $SU(h)_M$

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