What is the relationship between Chow motives and motivic Galois groups?

Here is P. O'Sullivan's answer. We assume the standard \( \sim_{\text{hom}} = \sim_{\text{num}} \) and finite-dim. of Chow motives (in the sense of Kimura-O'Sullivan).

But of course, the following applies to some \( \otimes \)-subcategories for which one knows the \( \sim \) holds via unconditional results.

We choose a field of coefficients \( F = \mathbb{Q} \).

\[ M_{\text{num}}(F) \text{ is tamagawa neutral} \]

\[ M_{\text{num}}(F) \cong \text{Rep}_F G \quad G = G_{\text{mot}} : \text{absolute pure motivic Galois gp} \]

(a pro-reductive \( G/F \))

\[ M_{\text{num}}(k)_F = \text{Rep}_F(G, -\text{id}) \]

acting as -1 on odd reps.

\[ M_{\text{mot}}(k)_F \cong M_{\text{num}}(k)_F \]
Under the above $q^*_{i}$: Thm (O’Sullivan). There exists an affine $G$-super-scheme Spec $A$ and a closed point $0 \in \text{Spec} A$ fixed by $G$, such that $A^G = F$ and

$$
\text{CHM}(k)_F \stackrel{\sim}{\rightarrow} \text{Vec}(\text{Spec} A; G, -id) \downarrow \text{fiber at } 0
$$

$$
M_{num}(k)_F \stackrel{\sim}{\rightarrow} \text{Rep}_F(G, -id)
$$

Here $\text{Vec}(\text{Spec} A; G, -id)$ is the category of $G$-equivariant super-vector bundles over $\text{Spec} A$ (for which the action of $-id \in G$ defines parity).

So understanding Chow motives amounts to understanding $G$ and $A \in \text{Ind} M_{num}(k)_F$, objects which belong to the "numerical world"!

$p: 0 \rightarrow a: A \rightarrow F$

- Voevodsky’s nilp $q^*_{i} \Leftrightarrow$ kernel nil-ideal
- weight grading $A = \bigoplus_{i} A_i$.
- Bloch-Beilinson-Murre’s conj. $A_i = 0$ for $i < 0 \Leftrightarrow A_0 = 1$ and $\text{generated by } A_1.$
**Construction of "tannakian" categories of mixed motives.**

Via conjectural $t$-structure on $\text{DM}_{gm}(k)_\mathbb{A}$

$$\text{DM}_{gm}(k)_\mathbb{A} \cap \text{DM}_{gm}(k)_\mathbb{A}^{t_0} = \text{MH}(k)_\mathbb{A}$$

$\tau_H^i : \text{DM}_{gm}(k)_\mathbb{A} \to \text{MH}(k)_\mathbb{A}$

$\mathcal{H}_t^{h+} \circ \tau_H^i$ tannakian after change of comm. constraint à la Koszul.

$\text{CHM}(k)_\mathbb{A} \leftarrow \text{DM}(k)_\mathbb{A}$

$\text{H}_{\text{mix}}(k)_\mathbb{A} \subset \text{MH}(k)_\mathbb{A}$

full subcategory of semi-simple objects.

absolute mixed motivic Galois group (attached to any realization).

Extension of the pure by a unipotent group.
Unconditional ex: mixed Tate motives over \( k \) = number field.

\[ D^b(\text{TM}(k)_q) \subset DM_{gm}(k)_q \]

Tannakian group is an extension of \( G_\mathfrak{m} \) by a pro-unipotent group.

\( \text{B)} \quad \text{Nori's category} \quad \mathcal{C}/k \subset C \)

\( \mathbb{Q} \text{ quiver} \) ("category without composition of morphisms")

"representation" \( T : Q \to \text{Ab} \)

\( \text{where} \quad Q \to (\text{End } T) - \text{-Mod} \quad \text{(where End } T \text{ is the ring of natural transf. of } T \text{)} \)

whenever \( Q \) is finite,

and \( Q \to C(T) - \text{-Mod}\) in general,

where \( C(T) = \lim \text{ of } (\text{End } T |_{Q \subset},) - \text{-Mod} \)

over all finite subquivers.

Apply to \( Q \): objects: \((x, y, i)\)

morphisms: \((x, y, i) \to (x', y', i)\) (obvious ones)

\[ + \quad (x, y, i) \to (y, z, i) \quad Z \subset Y \subset X \]

\[ T : (x, y, i) \to H : (X, Y, i) \quad \text{with "Tate object" in } C(T) \narrow \text{MM}(k)_q \]

tannakian neutral.
Back to Grothendieck's period $cij$.

$k \in \mathbb{C}$ 

$M \in \text{MM}(k)_E \rightarrow \text{periods } \Omega_M$

(matrix of the composition w.r.t. "rat." bases)

$H_{\text{DR}}(M) \otimes_{\mathbb{Q}} \mathbb{C} \Rightarrow H_{\beta}(M) \otimes_{\mathbb{Q}} \mathbb{C}$.

(mixed) motivic Galois gp $G_{\text{mot}}(M)$

period tensor $G_{\beta}(M)$

$G_{\text{mot}}(M)_k$

One can extend Grothendieck's period $cij$ to this setting:

"all alg. relations between periods (with coeff. in $k$) one of motivic origin"

$\text{tr. deg}_{\mathbb{Q}} k(\Omega_M) \Rightarrow \dim G_{\text{mot}}(M)$.

Kontsevich viewpoint: if one uses Nori's category, this $cij$ takes the following form:

"all alg. relations between periods come from alg. changes of variables and Stokes formula for integrals."

More precisely:
Consider a $Q$-space gen. by

$$[(x, D, w, y)] \subset \alpha \text{ affine smooth } Q \ni D \subset X \text{ NCD} \quad \omega \in \Omega^{d-1}(x) \quad y \in H_{d-1}^Q(x, D, Q)$$

modulo relations:

i) linearity in $w$ and $y$,

ii) $\forall f : (x, D) \to (x', D')$, $\forall \omega \in \Omega^{d-1}(x')$

$$y \in H_{d-1}^Q(x, D, Q),$$

$$[(x, D, f^{*}w, y)] = [(x, D, w, f^{*}y)]$$

iii) $D^{(i)}$ (resp. $D^{(i+2)}$) normalization of $D$

(normalization stratum), $\forall \gamma \in 2^Q$

$$[(x, D, d\gamma, y)] = [D^{(i)}, D^{(i+2)}, \gamma|_{D^{(i+2)}} \oplus y]$$

this is a $Q$-alg. $\to$ alg. $D$ after invariance of $[(a^{i}, 0, d\gamma, y)]$ $\oplus y$ "formal mixed period"

$$\hat{D} \to C : [(x, D, w, y)] \to \int w$$

period of: this is injective.
Comment. Toward a Galois theory for transcendental numbers?

Alg. num \( \alpha \leftrightarrow \bar{\alpha} \) conjugates, which are permuted by the Galois gp of the normal closure of \( \mathbb{Q}[\alpha] \) over \( \mathbb{Q} \).

Is there anything similar for transcendental complex numbers \( \alpha \)? If \( \alpha \) is a period, motivic Galois theory suggests a positive answer.

Normal closure of \( \mathbb{Q}[\alpha] \): \( \mathbb{Q}[\overline{\mathbb{H}}(M)] \) for \( M \) minimal (s.t. \( \mathbb{Q}[\overline{\mathbb{H}}] \) contains \( \mathbb{Q}[\alpha] \)).

\[ G = \mathbb{G}_{mot}(M)(\mathbb{Q}) \]

corresponds to \( \alpha \): orbit of \( \alpha \) under this action of \( G \).

Ex: \( \mathbb{Q}[\alpha] \) the field \( M \): corresponding Artin motive \( \mathbb{Q}[\overline{\mathbb{H}}] = \) normal closure of \( \mathbb{Q}[\alpha] \)

and \( G \) is the usual Galois gp.

- \( \alpha = 2\pi i \), \( M = \mathbb{C}(\zeta) \), \( G = \mathbb{Q}^* \)
  conj. of \( \alpha \): non-zero rational multiples.

- \( \alpha = \omega \), period of the 1st kind without complex multiplication

\( M = h'(x) \), \( G = GL_2(\mathbb{Q}) \),
  conj. of \( \alpha = \omega \), are \( \mathfrak{e} \) of \( \mathbb{Q} \mathfrak{w}, \mathfrak{w}_1, \mathfrak{w}_2 \).
Polyzetas, mixed Tate motives and their motivic Galois groups.

A. Polyzetas

\[ \sum_{n_1 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \]

\[ = \int_{t_1 > \cdots > t_k > 0} \omega_{s_1} \cdots \omega_{s_k} \]

Chen iterated integral

period of the ind.-Tate motive

\[ h(\pi^{uni}_{\mathbb{Q}}(\mathbb{P}^{1,0,1,0,1}_{\mathbb{Q}}, \{0,1,\infty_1, \infty_1\}) \in \text{Ind } TM(\mathbb{Q}) \]

(Goncharov, Deligne...)

Actually \( \in \text{Ind } TM(\mathbb{Z})_{\mathbb{Q}} \)

mixed Tate motives

mixed Tate motives unramified at every prime \( p \).

\[ \Xi \subset \mathbb{Q} \text{-subspace generated by } 5(s) \text{ for } s \in \mathbb{Z} \]

\[ \Xi = \sum \Xi \]

B. \( TM(\mathbb{Z})_{\mathbb{Q}} \)

\[ \text{in } TM(\mathbb{Q})_{\mathbb{Q}}, \quad \text{Ext}^i(4, 4(1)) = 0 \quad (i > 1, r > 0) \]

\[ \text{Ext}^i(4, 4(r)) = H^i(\text{Spec } \mathbb{Q}, \mathbb{Q}(r)) \]

\[ \text{Ext}^i(4, 4(r)) = \begin{cases} 0 & \text{even} \\ \mathbb{Q} & \text{odd} \end{cases} \]

\[ = \text{Ker}^{i-1}(\mathbb{Q}) \otimes _{\mathbb{Q}} \mathbb{Q} = \begin{cases} 0 & r = 1 \\ \mathbb{Q} & r = 0 \end{cases} \]

\[ \text{in } TM(\mathbb{Z})_{\mathbb{Q}}, \text{ same but } \text{Ker}^{i-1}(\mathbb{Z}) \otimes _{\mathbb{Q}} \mathbb{Q} = \begin{cases} 0 & r = 1 \\ \mathbb{Q} & r = 0 \end{cases} \]
Theorem (Deligne, Goncharov). The motivic Galois group attached to $\text{TM}(\mathbb{Z})_\mathcal{Q}$ is of the form $G_{\text{TM}(\mathbb{Z})} = G_m \times G'_{\text{pro-unipotent}}$

1) Lie $G'_{\text{TM}(\mathbb{Z})}$, graded by the $G_m$ action, is the free graded Lie algebra with one generator in each odd degree $\leq -3$.

2) $\text{TM}(\mathbb{Z})_\mathcal{Q} \cong \{ f.d. \ (\text{Lie} \ G'_{\text{TM}(\mathbb{Z})}) \text{- graded} \text{ modules} \}$

$\text{TM}'(\mathbb{Z})_\mathcal{Q} \subset \text{TM}(\mathbb{Z})_\mathcal{Q}$, sub-Tannakian category generated by f.d. pieces of $h(\mathbb{P}^\infty_{\mathcal{Q}}(\mathbb{P}^1_{\mathcal{Q}} - \{0,1,\infty\}, \bar{\Omega}))$.

so that $\tau$ (which is a $\mathcal{Q}$-algebra as we shall see) is the $\mathcal{Q}$-alg of real periods of objects of $\text{TM}'(\mathbb{Z})_\mathcal{Q}$.

Cor. (Goncharov, Terasaoma)

$\dim_{\mathcal{Q}} \tau \leq d_s$, where $d_s = d_{s-2} + \varepsilon$

$d_0 = d_2 = 1$, $d_1 = \varepsilon$

Remark: there is no non-motivic proof of this inequality!
Hint: as a graded comm. alg., $(U(\text{Lie } G_{TM}(\mathbb{C})))$
\[\cong \mathbb{Q}[T_1] \otimes U(L(T_{-2}, T_{-5}, ...))\]

$= \text{graded Hopf alg. of functions on } G_{TM}^1(\mathbb{C})$

via $G_{TM}(\mathbb{C}) \longrightarrow G_{TM}^1(\mathbb{C})$,

$U(\text{Lie } G_{TM}^1(\mathbb{C}))^\vee \subseteq \mathbb{Q}[T_1] \otimes U(L(T_{-2}, T_{-5}, ...))^\vee$

period torsor

$\varphi: U(\text{Lie } G_{TM}(\mathbb{C}))^\vee \rightarrow \mathbb{Z}/(2\pi i)$

$\mathbb{Z} = \text{image of } \varphi \cap \mathbb{Q}[T_1^2] \otimes U(L(T_{-2}, ...))^\vee$

graded piece $s$ of dim $d_s$.

Remark: $TM(\mathbb{Z})_{\alpha} \not\cong TH'(\mathbb{Z})_{\alpha}$

+ Grothendieck's period $\phi$ for $TH'_{\alpha}(\mathbb{Z})$

$\iff \mathbb{Z} = \bigoplus Z_s$

and $\dim Z_s = d_s$. 
Explicit relations between polyzetas.

Two sets of known relations:
- Regularized double shuffle relation (RDS)
- Drinfeld's associator relations (Ass).

\[ \text{RDS : } 5(s) \cdot 5(s') \]

\[ \sum \sum \frac{1}{\prod_{n_i} \prod_{n_i'} \ldots} \]

decompose the index set

\[ = \text{lín. comb. of } S(s) \]

= another lin comb. of \( S(s) \)'s

\[ \to B \text{ is a } \mathbb{Q} \text{-algebra.} \]

Can be extended to \( s_1 = 1 \) (regularization)

\text{Thm ( Goncharov) } \text{ RDS relations are of motivic origin.}

\text{Thm ( Racinet) } \text{ they define a torsor under some affine gp scheme } G_{RDS} \text{ which contains } G_{MM'}(\mathbb{Z}).
Ass: \[ \frac{dG(z)}{dz} = \left( \frac{X_0}{z} + \frac{X_1}{1-z} \right) G(z) \]

sol. \( G_1(z) \sim z^{X_0}, \ G_2(z) \sim (1-z)^{-X_1} \)

\( G_1(z)^{-1} G_0(z) = \Phi(X_0, X_1) \) indp. of \( z \)

\[ \Phi_{Kz} = \Phi \left( \frac{X_0}{2\pi i}, -\frac{X_1}{2\pi i} \right) \text{ Drinfel'd}'s \text{ associator.} \]

- exp. of a lie series in \( X_0, X_1 \)
- \( \Phi_{Kz}(X_1, X_0) = \Phi_{Kz}(X_0, X_1)^{-1} \)
- \( e^{X_0/2} \Phi_{Kz}(X_1, X_0) e^{-X_1/2} \Phi_{Kz}(X_1, X_1) = 1 \)

with \( X_{-1} = -X_0 - X_1 \)

\[ \Phi_{Kz}(X_{01}, X_{12} + X_{13}). \Phi_{Kz}(X_{02} + X_{12}, X_{23}) \]

\[ = \Phi_{Kz}(X_{12}, X_{23}) \Phi_{Kz}(X_{01} + X_{02}) \]

\[ X_{i3} + X_{23}). \Phi_{Kz}(X_{01}, X_{i3}) \text{, } X_{ij}, 0 \leq i \neq j \leq 3 \text{ non-comm. var.} \]

\[ X_{ij} X_{kl} = X_{kl} X_{ij}, \text{ } (\forall X_{ij} + X_{ik}, X_{jk}) \]

\[ \text{Point: coeff. of } \Phi = 1 + 5(z) X_0 X_1, \ldots, \text{ are polyzetas} \]

Drinfel'd's rl \( \Rightarrow \) assoc. relations between polyzetas.
Fact: (Ass) relations are of motivic origin, and define a torsor under the so-called Grothendieck–Teichmüller gp \( GT \):

\[
G_m(\mathbb{Z}) \longrightarrow G_{TM}(\mathbb{Z}) \subseteq G_{\text{RDS}} \subseteq GT
\]

Conj: these gps coincide.

(\( \text{Ass} \)) and (\( \text{RDS} \)) are, independently, defining equations for polyzetas.

\[\boxed{\text{Hodge and Tate conjectures for } \text{TM}(\mathbb{Z})_{\alpha}}\]

\[
\text{TM}(\mathbb{Z})_{\alpha} \xrightarrow{H_3} \text{MHS}_\alpha
\]

\[
\text{TM}(\mathbb{Z})_{\alpha} \xrightarrow{H_6} \text{Rep}_\mathbb{Q} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})
\]

fully faithful.

\[
\text{Ext}^1(\lambda, \lambda(\mathbb{Z})) \subseteq \text{Ext}^1_{\text{MHS}}(\lambda, \lambda(\mathbb{Z})) = \mathbb{G}_{\text{m}}/\mathbb{G}_{\text{m}}(2)_{\mathbb{Q}}
\]

\[
\text{Ext}^4(\lambda, \lambda(\mathbb{Z})) = K_{\text{gr}}(\mathbb{Z})_{\text{unr. outside }}(\lambda)
\]

The END

(\( \text{Soulié} \)).