Comment. Different motivic Galois groups are attached to different realizations; what is the relation between them?

Abstract answer: $M$ tamahian category of motives $/F = \mathbb{Q}$, say (e.g. $M = M_{\text{num}}(k)$ under $\sim_{\text{hom}} = \sim_{\text{num}}$, or $M$, built in terms of motived correspondences, unconditionally if char$=0$)

$M \cong \text{Vec}$?

$\text{Aut}^\otimes_H$

e.g. $H_{\text{B}}$, $H_{dR}$, $H_{\varepsilon}$ ($l$ prime $\neq$ char$K$)

(if $k \subseteq C$) (if char$=0$)

Internal motivic Galois group $\pi(M)$: pro-object in $M^P$, independent of $H$.

For any $H$, $H(\pi(M)) = \text{Aut}^\otimes_H$

Concrete answer for char$=0$:

comparison $\circledast$.

$k \in C$ $\Rightarrow$ $C$

$H_{\text{B}}(M) \circledast C = H_{dR}(M) \circledast C$

$H_{\varepsilon}(M) \circledast \mathbb{Q}_l = H_{\varepsilon}(M)$.

The motivic Galois group $G_{\text{mot}}^{(H)}(M) = \text{Aut}^\otimes_H/\langle H \rangle$ in $\text{GL}(H(\mathbb{Q}))$ correspond to each other via these comp. $\circledast$.
Enriched realizations of pure motives.

a) Hodge realization

\[ \text{HS}_q = \{ \text{Q- Hodge structures } V \} \]

\[ \text{Q- space + bigrading on } V \varepsilon \text{ st } V^M = V^W \]

\[ \mu : G^2_m \rightarrow GL(V) \]

Mumford-Tate group

\[ MT(V) := \text{smallest closed alg. subgp of } GL(V) \]

where qplex points contain \( \text{Im } \mu \).

connected reductive gp. \((Y, V \text{ pol.})\)

\[ k \in \overline{C} \]

Hodge realization: \( \text{M}_{\text{hom}}(k)_\alpha \rightarrow \text{HS}_q \)

Hodge \( \alpha \) \iff \( k = \overline{k} \), it is \text{fully faithful}.

(under standard \( \alpha \), \( \iff k = \overline{k} \)

\( MT(H^2(V)) = G_{\text{mot}}(M) \)

Without standard \( \alpha \).

\[ M \rightarrow \text{HS}_q \]

fully faithful \( \iff \) every Hodge class on any \( X \in U \) is motivated

\( \iff MT(H^2(V)) = G_{\text{mot}}(M) \)

Example:

in \( H^2(V) \), any Hodge class is alg. (reflected)

- on abelian varieties, any Hodge class is motivated (provided \( V \) is compact ab. perf (A.)).
6) Tate realization.

\[ \text{Rep}_k(\text{Gal}(\overline{F}/F)) = \text{! continuous flat}\ \text{groups of } \text{Gal}(\overline{F}/F) \]

Tate realization: \( M_{\text{hom}}(k) \rightarrow \text{Rep}_k(\text{Gal}(\overline{F}/F)) \)

\[ G_2(M) = \text{Zariski closure of } \]

\[ \text{in} (\text{Gal}(\overline{F}/F) \rightarrow \text{GL}(H_2(M))) \]

Tate & Tate realization is fully faithful.

Without \( k \)

\[ M_{\text{alg}} \rightarrow \text{Rep}_k(\text{Gal}(\overline{F}/F)) \rightarrow \bigoplus\limits_{\nu \subset M} G_2(M) \subseteq G_{\text{alg}}(M) \]

\[ \text{of } k \]

\[ M_{\text{alg}} \]

and Tate realization.

fully faithful \( \Leftrightarrow \) every Tate class on any \( x \) is \( G_2 \)-linear combination of motivated elements

\[ \Rightarrow M_{\text{alg}} \]

abelian and Tate realization.

\[ \Rightarrow \text{G}_2(M) = \text{G}_{\text{alg}}(M) \]

\[ \forall M \subseteq k \]

So \( G_{\text{alg}}(M) \) "interpolates" the \( G_2(M) \) for various \( k \).

Ex. on abelian varieties over finite fields, every Tate class is \( G_2 \)-linear combination of motivated elements.
c) "period realization" (or "B. Hodge realisation")

\[ \text{Vec}_{\mathfrak{g}, \alpha} = \{ (v, w, \pi) \mid v, w \in \text{Vec}_{\alpha}, \pi : w \otimes e = v \otimes e \}\]

transition \( \alpha \).

period realization: \( M_{\text{hyp}}(H)^{\alpha} \rightarrow \text{Vec}_{\mathfrak{g}, \alpha} \)

\[ M \rightarrow (H, H, H_{\alpha}(H)) \]

conversely, \( \alpha \). \( \otimes_{\mathbb{C}} \mathbb{C} = H^*(H) \otimes_{\mathbb{C}} \mathbb{C} \).

Concretely, \( \alpha \) is given by a matrix whose coefficients are called periods.

**Period, e.g. (weak form)**

if \( k = \mathbb{C} \), the period realization is fully faithful.

**Examples:** for \( \langle \mathbb{Q}(1) \rangle_{\mathbb{C}} \), this amounts to the transcendence of \( \pi \).

**For elliptic curves,** this follows from known results in transcendence theory.

Grothendieck's period, e.g. (strong form) (i.e. all alg. relations with coeff. in \( k \) between periods of underlying origin.)
period torsor: $\mathcal{P}(M) = \text{Iso}^\oplus (H_{\text{DR}}, H_B \otimes k)$

(torsor under $G_{\text{mot}}(M) \otimes k$).

$\omega_M: \text{Spec } C \rightarrow \mathcal{P}(M)$

Grothendieck's period $\omega_M$

$\omega_M$ is the generic point of $\mathcal{P}(M)$

$\mathcal{P}(M)$ is connected and

$\text{tr. deg}_k \Omega^2_M = \dim G_{\text{mot}}(M)$.

Rmk: as before, one can get rid of the

st. $\omega_M$ using motivated classes instead of

alg. classes.

Ex: (strong) period $\omega_M$ is known for

elliptic curves with complex multiplication

(Chevalley).

- linear alg. relations between

periods of an $H^2$ are of motivic

origin (Wüstholz).
Comment.

\[ \text{char } k = 0 \]

\[ \text{CHM}(k)_\alpha = \text{M}_{\text{tot}}(k)_\alpha \]

\[ \text{M}_{\text{hom}}(k)_\alpha \]

\[ \text{HS}_\alpha \quad \text{Rep}_\alpha \text{Gal}(E/k) \quad \text{Vec}_h, \alpha \]

\[ M_1, M_2 \in \text{CHM}(k)_\alpha \]

\[ M_1 = M_2 \]

under either "Schur finiteness" or "finite dimensionality"

"same" underlying homological motives

"same" Hodge structure

"same" Galois rep.

period of \( i \) if \( k \subset \bar{k} \)

"same" periods "up to \( \bar{k} \)"
Techniques of computation of motivic Galois groups.

M motive/k, e.g. $A(X)$, $X \in \mathcal{O}(k)$.

A) char $k = 0$ case

I. $k = \mathbb{C}$ First compute the Mumford-Tate gp $\text{MT}(M)$

Reason: it is connected, $\text{MT}(M)$ ↔ Lie $\text{MT}(M)$.

and use $\text{MT} = \mathbb{Z} \cdot \text{MT}^s$

$G_m \twoheadrightarrow \mathbb{Z} \subset (\text{End} M)^\times$

weight cochar.

$V = H^0_B(M)$ by polarizability, $\mathbb{Z} / \text{im} w$ is a compact torus.

$\text{MT}^s \subset \text{SL}(H^0_B(M)) = \text{SL}_m$

Recall: there are only finitely many conjugacy classes of semi-simple subgroups of $\text{SL}_m$

(determined by tensor invariants of effectively (?) bounded degree).

More advanced techniques:

Lie $\text{MT}^s = \bigoplus g_i$ simple

$V = H^0_B(M)$

$V_c = \bigoplus V_i$, irred

$V_i = \bigoplus W_i$, irred rep of $g_i$
zarhin: bounds for level \( (= \max p - q, M \neq 0) \)
bounds for weights of \( W \)
e.g. if level = 1, all weights are minircule etc...

II. Using classical invariant theory,
determine generators of small coh. degree
for the algebra of MT-invariant tensors
(i.e. Hodge classes)
* if \( M \subset h(\mathbb{X}) \) and generators \( \in H^2(\mathbb{X}) \)
  \( \subset H(\mathbb{X})^{-\infty} \)
  Lefschetz' then \( \Rightarrow \) Hodge if \( \text{for} \langle M \rangle \)
  \( \Rightarrow \ G_{\text{mot}}(L) = \text{MT}(M) \).
* otherwise, try to deform \( M \) to a
  motive which satisfies Hodge \( \Psi \)
  (see below).

III. \( k \subset \overline{k} \subset \mathbb{C} \)
\( G_{\text{mot}}(L_K) = G_{\text{mot}}(M_{\overline{k}}) \subset G_{\text{mot}}(M) \)
finite index
"gap" determined by
Galois rep. on \( H_1(M) \).

\( \mathbb{F} \) char. \( k = p \)
try to replace \( I \) by study of Galois
rep ( replacing MT by Ge, Zarhin's work
by Pink's work etc....)
if $k$ is transcendental over its prime field, monodromy techniques are available (see below).

\[ \text{Ex: } X: \text{elliptic curve }/k \]
\[ h(x) = S(h^4(x)) \text{ in } \text{Mon}(k) \]
\[ (\text{i.e. } = \wedge(h^4(x)) \text{ in } \text{Mon}(k)) \]
so $G_{\text{mot}}(h(x)) = G_{\text{mot}}(h^4(x)) \subseteq \text{GL}(H^4(x))$

\[ w: \mathbb{G}_m \rightarrow G_{\text{mot}}(h(x)) \subseteq \text{GL}_2 \]
diagonal

\[ h = c: \text{MT connected nd. subg of } \text{GL}_2, \exists G_\text{m} \]
\[ V = H^1_\text{B}(x) \] → determined by $\text{End}_{\text{MT}} V$\n
\[ \text{MT} = \text{GL}_2 \]

\[ \text{MT} = \mathbb{R}_{\text{c}}(\text{End}_X) \]
complex multiplication

\[ \text{in both cases, covariant tensors} \]
\[ \text{are generated by } V^{\otimes 2}(v) \text{, i.e. } \text{End}_X \otimes \mathbb{Q} \]
\[ \text{hence Hodge } c_1 \text{ holds for all powers of } X \text{ and} \]
\[ G_{\text{mot}}(X) = \text{MT}(X). \]

If $\text{char} k = 0$, $G_{\text{mot}}(X) = \text{GL}_2$ if $X$ has no CM

= non-split Cartan if $X$ has no CM

= split Cartan if $X$ has CM.
$k$ finite. (M. Spiess)

$G_{mot} = G_2(M)$ and all invariant tensors are gen. in $\operatorname{deg} 2$.

**Ex:** Abelian var. with complex multiplication.

$x \in \operatorname{CM} \otimes \mathbb{F}_{\text{char} \neq 2}, k$ (number field)

$\operatorname{End} x \otimes \mathbb{Q} = E \text{ cm field } \overline{\mathbb{Q}} \subset E$

$[E : \overline{\mathbb{Q}}] = 2 \dim x$

$\Omega'(x) \text{ } k \otimes E$-module.

$\det_k (x \otimes ? | \Omega'(x)) : T_E \rightarrow T_k$

$x \mapsto \prod_{s : \overline{E} \otimes \mathbb{C}} \tau(s)^{\nu(s)}$

$x \mapsto \text{ "cm type of weight +" attached to } x$

$\left( \tau(s) + \tau(\overline{s}) = 1 \right)$.

On the other hand,

$\det_E (? \otimes 1 | \Omega'(x)) : T_k \rightarrow T_E$

$\gamma \mapsto \prod_{\overline{s} : \overline{E} \otimes \mathbb{C}} \tilde{s}(N_{k/\overline{E}}(\gamma))^\tilde{\tau}(\overline{s})$

where $\overline{E}$ (reflex cm field) is the smallest cm subfield of $k$ s.t.

$T_k \rightarrow T_E \text{ factors through } N_{k/\overline{E}} : T_k \rightarrow T_{\overline{E}}$.

The image of the induced homom. $T_{\overline{E}} \rightarrow T_E$

is $\operatorname{MT}(x) = G_{mot}(x)$. 

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(3) Parallel transport of algebraic classes

\( k \in \mathbb{C} \) for simplicity.

\( f : X \to S \) proj. smooth, \( S \) smooth connected.

Parallel transport:

\[
\text{Hom}(X_s, \pi_1(S(C), s)) \cong \text{Hom}(X_t, \pi_1(S(C), t))
\]

**Conj (Grothendieck):** \( \pi_1 \) respects alg. classes.

**Stronger conj:** \( \pi_1 \) is induced by an alg. correspondence

**prop (A.):** \( \text{Std} \text{ eff } \sim_{\text{hom}} \sim_{\text{num}} \rightarrow \text{this conjecture.} \)

(also true in char. p).

**prop (A.):** \( \pi_1 \) is motivated

(for \( U \) big enough).

Conseq: if for one fiber of the family, all Hodge cycles are motivated, it is the same for every fiber.

(8) Variation of Galois motivic groups

in families

Same setting. Variation of \( G_{\text{mot}}(X_s) \)

with \( s \) ?
Ex: non-trivial elliptic pencil \( X \to S \) in general \( G_{\text{mot}}(X_s) = \text{GL}_2 \), except for countably many pts \( s \) (complex multiplication).

\[
G_{\text{mono}}(X_s) := \left[ \text{Im} \left( \pi_2(S, s) \to \text{GL}(H(X_s)) \right) \right]_{\text{Zar}}
\]

\( G_{\text{mono}} \) semi-simple (Deligne).

\( k = \overline{k} \subseteq \mathbb{C} \)

Then \((A.) \) (for \( U \) big enough)

There exists a local system \((\Gamma_{\underline{s}})\) of reductive subgroups of \( \text{GL}(H(X_s)) \) s.t.

1) \( \forall s, \ G_{\text{mono}}(X) \triangleleft \Gamma_{\underline{s}}, \ G_{\text{mot}}(X_s) \subseteq \Gamma_{\underline{s}} \)

2) \( \exists \) countable union \( \Sigma \) of subvarieties of \( S \) s.t. \( \forall s \not\in \Sigma, \ G_{\text{mot}}(X_s) = \Gamma_{\underline{s}} \)

3) \( \exists \) countably many \( s \in S(h) \) s.t.

\( G_{\text{mot}}(X_s) = \Gamma_{\underline{s}} \).

In particular, if \( s \in S(\mathbb{C}) \) is "Weil-generic", \( G_{\text{mono}}(X_s) \triangleleft G_{\text{mot}}(X_s) \).

Ex: for a generic hypersurface \( Y \) in \( \mathbb{P}^{2n} \) moving in a nef-ample pencil,

\( G_{\text{mono}}(Y) = \text{Sp} \)

\( \text{for } u \neq 1 \quad \to \quad G_{\text{mot}}(Y) = \text{Sp} \).