INTRODUCTION:
MODULI PROBLEMS

"DEFINITION"
A moduli problem is a class of geometric objects which one tries to view as another geometric object.

OUR BASIC EXAMPLE will be the "moduli problem of smooth n-pointed curves of genus g":
For every scheme $S$, we put

$\mathcal{M}_{g,m}(S) =$ category with

- objects: $(f: X \to S, x_1, \ldots, x_m)$ where $f: X \to S$ is smooth, proper, with geometric fibres 1-dimensional, connected, of genus $g$, and $x_1, \ldots, x_m: S \to X$ are disjoint sections

- morphisms: $S$-isomorphisms respecting the sections.

$\mathcal{M}_{g,n}(S) =$ the set of isomorphism classes of objects of $\mathcal{M}_{g,m}(S)$.

The latter is just a set, while by construction $\mathcal{M}_{g,m}(S)$ is a groupoid (= category with all maps invertible).
For every morphism $S' \to S$, we have base change functors

$$\mathcal{M}_{g,m}(S) \to \mathcal{M}_{g,m}(S')$$

whence natural maps

$$\underline{M}_{g,m}(S) \to \underline{M}_{g,n}(S')$$

making each $\underline{M}_{g,n}$ into a functor

$$\underline{M}_{g,n} : (\text{Schemes})^0 \to (\text{Sets})$$
In general, $M_{g,m}$ is NOT (representable by) a scheme.

(In fact, it is a scheme iff $n > 2g + 2$).

For instance, one can find two curves $X \xrightarrow{f} X'$

which are not isomorphic, but locally isomorphic over $S$ (for the étale topology, or even the Zariski topology).

Thus, $M_{g,0}$ (any $g$) is not a sheaf for these topologies, hence not representable.

(And, of course, if we take the associated sheaf we lose even more information.)
Main Idea:

\( M_{g,m} \) is a better object to look at than \( M_{g,n} \):

- It is the natural object one tries to study (no loss of information)
- It has good local-to-global properties
- It has good approximations by schemes

Of course, in a sense it is a more complicated object (\( M_{g,m} (S) \) is a groupoid, not a set).
FIBERED GROUPOIDS

Let $C$ be a category (typically: the category of schemes, possibly over a fixed "base scheme")

A fibered groupoid $\mathcal{M}$ over $C$ (C-groupoid consists of the following data:

- For each $U \in \text{ob } C$, a groupoid $\mathcal{M}(U)$
- For each map $V \rightarrow U$ in $C$, a functor $f^*: \mathcal{M}(U) \rightarrow \mathcal{M}(V)$
- For each composite map $W \rightarrow V \rightarrow U$, an isomorphism $g^* f^* \cong (fg)^*$

of functors $\mathcal{M}(U) \rightarrow \mathcal{M}(W)$

+ Compatibility with the associativity of composition in $C$.
Examples:

1. $C = \text{(Schemes)}$
   1. $U \mapsto M_{g,m}(U)$
   2. $U \mapsto \text{cat. of all } U\text{-schemes } + U\text{-isomorphisms}$
   3. $U \mapsto \text{Qcoh}(U) := \text{cat. of quasi-coherent } O_U\text{-modules (}\text{+isomorphisms)}$
   4. $U \mapsto \text{Bun}_m(U) := \text{cat. of locally free } (m \in \mathbb{N}) \text{ } O_U\text{-modules of rank } m \\ \text{ (}\text{+isomorphisms)}$

5. For any $C$, any presheaf on $C$, i.e., any functor $F: C^\circ \to \text{(Sets)}$
   defines a $C$-groupoid (denoted by $F$):
   $F(U) := \text{the discrete category } F(U)$
   - set of objects = $F(U)$
   - maps = identities.
MORPHISMS OF GROUPOIDS

If \( M, N \) are \( C \)-groupoids, a morphism 
\[ \Phi: M \to N \]
consists of the following data:

- For each \( U \in \text{ob} \, C \), a functor 
\[ \Phi(U): M(U) \to N(U) \]
- For each \( V \xrightarrow{f} U \) in \( C \), consider the diagram

\[
\begin{array}{ccc}
M(U) & \xrightarrow{\Phi(U)} & N(U) \\
\downarrow^{f^*} & & \downarrow^{f^*} \\
M(V) & \xrightarrow{\Phi(V)} & N(V)
\end{array}
\]

We require (as part of the data) an isomorphism
\[ f^* \cdot \Phi(U) \cong \Phi(V) \cdot f^* \]
of functors \( M(U) \to N(V) \).

(+ compatibility with the associativity data).
Examples:

- $\mathcal{M}_{g,m} \rightarrow \mathcal{M}_{g,m-1}$ \hspace{1cm} (m > 0)
  
  "forget the m-th marked point"

- $\mathcal{M}_{1,1} \rightarrow \mathbb{A}^1$ (viewed as a presheaf)
  
  $(x \rightarrow s) \mapsto j(x/s) \in \Gamma(S, \mathcal{O}_S) = \mathbb{A}^1(S)$
  
  elliptic curve

- Let us define two morphisms

  $\Phi, \Psi : \mathcal{M}_{1,1} \rightarrow \text{BUN}_1$

  by

  $\Phi(x \xrightarrow{p} s) := \pi_* \Omega^2_{x/s}$

  $\Psi(x \xrightarrow{p} s) := \pi^* \Omega^2_{x/s}$

These are different morphisms, but $\Phi(x)$ and $\Psi(x)$ are known to be canonically isomorphic.

So there should be a notion of (iso)morphism between morphisms!
Easy exercise: we obtain in this way an equivalence of categories

\[ \{1\text{-morphisms } U \to N \} \sim N(U). \]

For instance: if $S$ is a scheme, a 1-morphism

\[ S \to \mathcal{M}_{g,m} \]

is "the same thing" as an object of $\mathcal{M}_{g,m}(S)$. 
Given a diagram of \( C \)-groupoids

\[
\begin{array}{c}
N \\
\downarrow \Phi \\
M \xrightarrow{\Phi} P
\end{array}
\]

there is a "fibre product" groupoid, assigning to each \( U \in \mathcal{C} \) the fibre product category

\[
M(U) \times_{P(U)} N(U)
\]

whose objects are triples

\[(X, Y, \alpha)\]

with

\[
\begin{cases} 
X \in \text{ob} M(U) \\
Y \in \text{ob} N(U) \\
\alpha: \Phi(X) \cong \Phi(Y) \quad \text{(isomorphism in } P(U) \text{)}
\end{cases}
\]
Example:
Assume $\mathcal{M}$ is a $C$-groupoid, $U_1, U_2$ objects of $C$
$X_i \in \text{ob } \mathcal{M}(U_i)$ (i = 1, 2)

Viewing $X_i$ as a 1-morphism $U_i \to \mathcal{M}$, we get a diagram:

$U_1 \xrightarrow{X_1} X_2 \xrightarrow{X_2} \mathcal{M}$

What is $U_1 \times X_1, U_1 \times X_2$?

Answer: it is the presheaf on $C$ given by

$T \mapsto \{(u_1, u_2, \alpha) : u_2 = \text{a morphism } T \to U_2,$
$u_1 = \text{a morphism } T \to U_1,$
$\alpha : u_1^* X_1 \to u_2^* X_2 \text{ in } \mathcal{M}(T)\}$

or, in standard notations,

$\text{Isom}_{U_1 \times U_2} (pr_1^* X_1, pr_2^* X_2)$
Example:

Consider the 1-morphism "forget the marked point"
\[ \Phi: \mathcal{M}_{g,1} \to \mathcal{M}_{g,0} \]

If \( S \) is a scheme and \( X \) is an \( S \)-curve of genus \( g \), then the fibre product \( S \times_{\mathcal{M}_{g,0}, \Phi} \mathcal{M}_{g,1} \) is the \( S \)-curve \( X \). In other words, we have a Cartesian diagram

\[
\begin{array}{ccc}
X & \longrightarrow & \mathcal{M}_{g,1} \\
\downarrow & & \downarrow \Phi \\
S \times_{\mathcal{M}_{g,0}, \Phi} \mathcal{M}_{g,1} & \longrightarrow & \mathcal{M}_{g,0}
\end{array}
\]

which shows that \( \mathcal{M}_{g,1} \) can be seen as the universal curve over \( \mathcal{M}_{g,0} \).
REPRESENTABLE MORPHISMS
(For safety, assume $C$ has fibre products)

A 1-morphism $\Phi : \mathcal{M} \to \mathcal{N}$
is representable if for each $U \in \mathcal{C}$ and $X : U \to \mathcal{N}$ (i.e. object of $\mathcal{N}(U)$)
the fibre product $U \times_{X, \mathcal{N}, \Phi} \mathcal{M}$ is a presheaf, representable by an object of $\mathcal{C}$.

For instance, the "forgetful" morphism

$\mathcal{M}_{g,1} \to \mathcal{M}_{g,0}$
is representable.
Another example: for \( g \geq 2 \), consider
\[
\text{3K:} \quad \mathcal{M}_{g, 0} \rightarrow \text{BUN}_r \quad (r = 5g - 5)
\]
\[
\left( X \xrightarrow{f} U \right) \mapsto f^* \omega_X^{\otimes 3}
\]

I claim that 3K is representable:

Pick a scheme \( U \) and a \( 1 \)-morphism \( U \rightarrow \text{BUN}_r \)
(that is, a locally free sheaf \( E \) on \( U \), of rank \( r \)):

\[
\begin{array}{ccc}
N & \rightarrow & \mathcal{M}_{g, 0} \\
\downarrow & & \downarrow 3K \\
U & \xrightarrow{E} & \text{BUN}_r
\end{array}
\]

Then, for a \( U \)-scheme \( T \), we have

\[
N(T) = \text{cat. of curves } X \xrightarrow{f} T \text{ of genus } g,
\]
plus isomorphism \( f^* \omega_X^{\otimes 3} \simeq \mathcal{E}_T \)

Such a curve is naturally (3-canonically) embedded in \( \mathbb{P}(\mathcal{E}_T) \). Putting \( P = \mathbb{P}(\mathcal{E}) \), we obtain an equivalence:

\[
N(T) \simeq \text{cat. of embedded smooth curves of genus } g: \quad X \hookrightarrow P \times_U T,
\]
plus isomorphism \( O(1)|_X \simeq \omega_X^{\otimes 3} \)

The representability then follows from Hilbert scheme theory.
PROPERTIES OF REPRESENTABLE MORPHISMS

If $P$ is a property (i.e. a class) of morphisms of $C$, which is stable by base change, it makes sense to say that a representable 1-morphism $\Phi : M \to N$ has property $P$.

For instance, in the above examples,

$\Phi : M_{g', 1} \to M_{g', 0}$ is proper and smooth.

$3K : M_{g, 0} \to \text{BUN}_{S_{g, 5}}$ is surjective and smooth.
USING THE DIAGONAL

(We assume C has fibre products)

Proposition: For a C-groupoid \( \mathcal{M} \), the following conditions are equivalent:

(i) The diagonal 1-morphism

\[
\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}
\]

\( X \mapsto (X, X) \)

is representable.

(ii) For all \( U \in \text{Ob} C \) and \( X, Y \in \text{Ob} \mathcal{M}(U) \), the presheaf \( \text{Isom}_{\mathcal{M}}(X, Y) \) is representable (by an object of \( C/U \)).

(iii) For each \( U \in \text{Ob} C \), every 1-morphism \( \mathcal{U} \to \mathcal{M} \) is representable.

WARNING: \( \Delta_{\mathcal{M}} \) is NOT in general a "monomorphism," i.e., a fully faithful functor.

In fact:

\[
\Delta_{\mathcal{M}}(U) : \mathcal{M}(U) \to (\mathcal{M} \times \mathcal{M})(U) \text{ is fully faithful for each } U
\]

\( \mathcal{M} \) is (associated to) a presheaf on \( C \)
Note: The properties in the above proposition are satisfied for $M_0 = M_{g,m}$.

In fact,

$$
\Delta_{M_{g,m}} : M_{g,m} \rightarrow M_{g,m} \times M_{g,m}
$$

is representable, separated, of finite type.

If $2g - 2 + n > 0$, it is finite unramified (objects of $M_{g,m}$ have no infinitesimal automorphisms).

If $n > 2g + 2$ then it is a monomorphism, in fact a closed immersion (objects of $M_{g,m}$ have no nontrivial automorphisms, and $M_{g,m}$ is a preorder in this case).

But for instance, for $g = n = 0$, consider $X : \text{Spec} \mathbb{Z} \rightarrow M_{0,0}$ defined by $B^2$. Then

$$
\text{Spec } \mathbb{Z} \times_{M_{0,0}} \text{Spec } \mathbb{Z} = \text{RGL } q, \mathbb{Z}
$$