CAYLEY GROUPS

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ABSTRACT. The classical Cayley map, $X \mapsto (I_n - X)(I_n + X)^{-1}$, is a birational isomorphism between the special orthogonal group $\mathbf{SO}_n$ and its Lie algebra $\mathfrak{so}_n$, which is $\mathbf{SO}_n$-equivariant with respect to the conjugating and adjoint actions respectively. We ask whether or not maps with these properties can be constructed for other algebraic groups. We show that the answer is usually "no", with a few exceptions. In particular, we show that a Cayley map for the group $\mathbf{SL}_n$ exists if and only if $n \leqslant 3$, answering an old question of LUNA.

Contents

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1. Introduction

The exponential map is a fundamental instrument of Lie theory that yields local linearization of various problems involving Lie groups and their actions, see [Bou$_1$]. Let L be a real Lie group with Lie algebra $\mathfrak{l}$. As the differential at 0 of the exponential $\exp : \mathfrak{l} \to L$ is bijective, exp yields a diffeomorphism of an open neighborhood of 0 in $\mathfrak{l}$ onto an open neighborhood U of the identity element e in L. The inverse diffeomorphism λ (logarithm) is equivariant with respect to the action of L on $\mathfrak{l}$ via the adjoint representation $\mathrm{Ad}_L : L \to \mathrm{Aut}\,\mathfrak{l}$ and on L by conjugation, i.e., $\lambda(gug^{-1}) = \mathrm{Ad}_L g\,(\lambda(u))$ if $g \in L$, $u \in U$ and $gug^{-1} \in U$. This shows that the conjugating action of L on its underlying manifold is linearizable in a neighborhood of e.

In this paper we study what happens if L is replaced with a connected linear algebraic group G over an algebraically closed field k: what is a natural algebraic counterpart of λ for such G and for which G does it exist?

In the sequel we assume that $\operatorname{char} k = 0$ (in fact in many places this assumption is either redundant or can be bypassed by modifying the relevant proof).

1.1. The classical Cayley map. Let $\mathfrak{g}$ be the Lie algebra of G. One way to look at the problem is to replace the Hausdorff topology in the Lie group setting by the étale topology, i.e., to define the algebraic counterpart of λ as a G-equivariant morphism $G \to \mathfrak{g}$ étale at e. Then, at least for reductive groups, there is no existence problem: such morphisms always exist, see Corollary of Lemma 10.3 below. Properties of some of them has been studied by Kostant and Michor in [KM], see Example 10.4 below. In the general case, there always exists a rational dominant G-equivariant map $G \dashrightarrow \mathfrak{g}$, see Proposition 10.5 below.

In the present paper we look at the problem differently. Our point of view stems from a discovery made by Cayley in 1846, [Ca], cf. [Weyl], [Pos]. It suggests that the most direct approach, i.e., replacing the Hausdorff topology by the Zariski one, leads to something really interesting. Namely, let G be the special orthogonal group,

$$G = \mathbf{SO}_n := \{X \in \mathrm{Mat}_{n\times n} \mid X^{\mathsf{T}} X = I_n\},$$

where I_n is the identity $n \times n$-matrix. Then

$$\mathfrak{g} = \mathfrak{o}_n := \{Y \in \mathrm{Mat}_{n\times n} \mid Y^{\mathsf{T}} = -Y\},$$

and the adjoint representation $\mathrm{Ad}_G : G \to \mathrm{Aut}\,\mathfrak{g}$ is given by

$$\mathrm{Ad}_G g\,(Y) = gYg^{-1}, \quad g \in G,\ Y \in \mathfrak{g}. \tag{1.2}$$

Cayley discovered that there exists a birational isomorphism

$$\lambda\colon G \overset{\simeq}{\dashrightarrow} \mathfrak{g} \tag{1.3}$$

equivariant with respect to the conjugating and adjoint actions of G on the underlying varieties of G and $\mathfrak{g}$ respectively, i.e., such that

$$\lambda(gXg^{-1}) = \mathrm{Ad}_G g\,(\lambda(X)) \tag{1.4}$$

if g and $X \in G$ and both sides of (1.4) are defined. His proof is given by the explicit formula defining such λ:

$$\lambda\colon X \mapsto (I_n - X)(I_n + X)^{-1} \tag{1.5}$$

(one immediately deduces from (1.5) that $Y \mapsto (I_n - Y)(I_n + Y)^{-1}$ is the inverse of λ, and from (1.2) that (1.4) holds).

1.6. Basic definitions, main problem and examples. Inspired by this example, we introduce the following definition for an arbitrary connected linear algebraic group G.

Definition 1.7. A *Cayley map* for G is a birational isomorphism (1.3) satisfying (1.4). A group G is called a *Cayley group* if it admits a Cayley map. If G is defined over a subfield K of k, then a Cayley map defined over K is called a *Cayley K-map.* If G admits a Cayley K-map, G is called a *Cayley K-group.*

Our starting point was a question, posed in 1975 to the second-named author by Luna, [Lun_3]. Using Definition 1.7, it can be reformulated as follows:

Question 1.8. *For what n is the special linear group $\mathbf{SL}_n$ a Cayley group?*

It is easy to show, see Example 1.18 below, that $\mathbf{SL}_2$ is a Cayley group. Popov in [Pop_2] has proved that, contrary to what was expected, see [Lun_1, Remarque, p. 14], $\mathbf{SL}_3$ is a Cayley group as well.

More generally, given Definition 1.7, it is natural to pose the following problem:

Problem 1.9. *Which connected linear algebraic groups are Cayley groups?*

Before stating our main results, we will discuss several examples. Set

$$\boldsymbol{\mu}_d := \{a \in \mathbf{G}_m \mid a^d = 1\}.$$

This is a cyclic subgroup of order d of the multiplicative group $\mathbf{G}_m$. Below we use the same notation $\boldsymbol{\mu}_d$ for the central cyclic subgroup $\{aI_n \mid a \in \boldsymbol{\mu}_d\}$ of $\mathbf{GL}_n$.

Example 1.10. If $G_1, \ldots, G_n$ are Cayley, then $G := G_1 \times \ldots \times G_n$ is Cayley (the converse is false, see Subsection 4.10). Indeed, if $\mathfrak{g}_i$ is the Lie algebra of G_i and $\lambda_i : G_i \overset{\simeq}{\dashrightarrow} \mathfrak{g}_i$ a Cayley map, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ and $\lambda_1 \times \ldots \times \lambda_n \colon G \overset{\simeq}{\dashrightarrow} \mathfrak{g}$ is a Cayley map. $\square$

Example 1.11. Consider a finite-dimensional associative algebra A over k with identity element 1. Let $\mathcal{L}_A$ be the Lie algebra whose underlying vector space is that of A and whose Lie bracket is given by

$$[X_1, X_2] := X_1X_2 - X_2X_1. \tag{1.12}$$

The group

$$G := A^*$$

of invertible elements of A is a connected linear algebraic group whose underlying variety is an open subset of that of A. This implies that $\mathfrak{g}$ is naturally identified with $\mathcal{L}_A$, and the adjoint action is given by formula (1.2). Hence the natural embedding $\lambda : A^* \hookrightarrow \mathcal{L}_A$, $X \mapsto X$, is a Cayley map. Therefore G is a Cayley group.

Taking $A = \text{Mat}_{n\times n}$, we obtain that $G := \mathbf{GL}_n$ is Cayley for every $n \geqslant 1$. $\square$

Example 1.13. Maintain the notation of Example 1.11. For any element $a \in A$, denote by $\operatorname{tr} a$ the trace of the operator L_a of left multiplication of A by a. Since the algebra A is associative, $a \mapsto L_a$ is a homomorphism of A to the algebra of linear operators on the underlying vector space of A. From this and (1.12), we deduce that $k{\cdot}1$ is an ideal of $\mathcal{L}_A$, the map

$$\tau : \mathcal{L}_A \to k{\cdot}1, \quad a \mapsto \operatorname{tr} a{\cdot}1,$$

is a surjective homomorphism of Lie algebras, and

$$\mathcal{L}_A = \operatorname{Ker}\tau \oplus k{\cdot}1. \tag{1.14}$$

The subgroup $k^*{\cdot}1$ of A^* is normal; set

$$G := A^*/k^*{\cdot}1. \tag{1.15}$$

As the Lie algebras of A^* and $k^*{\cdot}1$ are respectively $\mathcal{L}_A$ and $k{\cdot}1$, it follows from (1.14) that one can identify $\mathfrak{g}$ with $\operatorname{Ker}\tau$. Let $A^* \to G$, $a \mapsto [a]$, be the natural projection. Then the formula

$$[a] \mapsto \tfrac{\operatorname{tr} 1}{\operatorname{tr} a} a - 1 \tag{1.16}$$

defines a rational map $\lambda : G \dashrightarrow \mathfrak{g} = \operatorname{Ker}\tau$. Since $\operatorname{tr} xax^{-1} = \operatorname{tr} a$ for any $a \in A$, $x \in A^*$, it follows from (1.16) that (1.4) holds. On the other hand, (1.16) clearly implies that

$$a \mapsto [a+1] \tag{1.17}$$

is the inverse of λ. Thus G is a Cayley group.

If A is defined over a subfield K of k, then the group G and birational isomorphisms (1.16), (1.17) are defined over K as well. Hence G is a Cayley K-group.

For $A = \mathrm{Mat}_{n\times n}$ this shows that $\mathbf{PGL}_n$ is a Cayley group for every $n \geqslant 1$. Note that in this case, $\frac{\operatorname{tr} 1}{\operatorname{tr} a} = \frac{n}{\operatorname{Tr} a}$, where $\operatorname{Tr} a$ is the trace of matrix a. Let K be a subfield of k. Since every inner K-form G of $\mathbf{PGL}_n$ is given by (1.15) for $A = D \otimes_K k$, where D is an n^2-dimensional central simple algebra over K and the K-structure of A is defined by D, cf. [Kn], all inner K-forms of $\mathbf{PGL}_n$ are Cayley K-groups.

Setting $A = \bigoplus_{i=1}^s \mathrm{Mat}_{n_i\times n_i}$ we conclude that $\prod_{i=1}^s \mathbf{GL}_{n_i}/k^* I_{n_1+\ldots+n_s}$, is a Cayley group. Here $\prod_{i=1}^s \mathbf{GL}_{n_i}$ is block-diagonally embedded in $\mathbf{GL}_{n_1+\ldots+n_s}$. $\square$

Example 1.18. The following construction was noticed by Weil in [Weil, p. 599]. Namely, maintain the notation of Example 1.11 (Weil assumed that A is semisimple, but his construction, presented below, does not use this assumption). Let ι be an involution (i.e., an involutory k-antiautomorphism) of the algebra A. Set

$$G := \{a \in A^* \mid a^\iota a = 1\}^\circ \tag{1.19}$$

(as usual, S° denotes the identity component of an algebraic group S). The Lie algebra of G is the subalgebra of odd elements of $\mathcal{L}_A$ for ι,

$$\mathfrak{g} = \{a \in \mathcal{L}_A \mid a^\iota = -a\}.$$

The formula

$$a \mapsto (1-a)(1+a)^{-1} \tag{1.20}$$

defines an equivariant rational map $\lambda\colon G \dashrightarrow \mathfrak{g}$, and the formula

$$b \mapsto (1-b)(1+b)^{-1} \tag{1.21}$$

defines its inverse, $\lambda^{-1}\colon \mathfrak{g} \dashrightarrow G$. Thus λ is a Cayley map and G is a Cayley group.

If A and ι are defined over a subfield K of k, then the group G and birational isomorphisms (1.20), (1.21) are defined over K as well. Hence G is a Cayley K-group.

For $A = \mathrm{Mat}_{n\times n}$ and the involution $X \overset{\iota}{\mapsto} X^{\top}$, this turns into the classical Cayley construction for $G = \mathbf{SO}_n$ yielding Cayleyness of this group for every $n \geqslant 1$. In particular, this shows that the following groups are Cayley: $\mathbf{G}_m \simeq \mathbf{SO}_2$, see Examples 1.11 and 1.22, $\mathbf{PGL}_2 \simeq \mathbf{SL}_2/\boldsymbol{\mu}_2 \simeq \mathbf{SO}_3$, see Example 1.13, $(\mathbf{SL}_2 \times \mathbf{SL}_2)/\boldsymbol{\mu}_2 \simeq \mathbf{SO}_4$ (here $\mathbf{SL}_2 \times \mathbf{SL}_2$ is block-diagonally embedded in $\mathbf{SL}_4$), $\mathbf{Sp}_4/\boldsymbol{\mu}_2 \simeq \mathbf{SO}_5$ and $\mathbf{SL}_4/\boldsymbol{\mu}_2 \simeq \mathbf{SO}_6$.

For $A=\mathrm{Mat}_{2n\times 2n}$ and the involution $X \overset{\iota}{\mapsto} J_{2n}{}^{-1}X^{\top}J_{2n}$, where $J_{2n} := \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$, we have

$$G = \mathbf{Sp}_{2n} := \{X \in \mathrm{Mat}_{2n\times 2n} \mid X^{\top} J_{2n} X = J_{2n}\},$$
$$\mathfrak{g} = \mathfrak{sp}_{2n} := \{Y \in \mathrm{Mat}_{2n\times 2n} \mid Y^{\top} J_{2n} = -J_{2n} Y\},$$

so the contruction shows that (1.5) is a Cayley map for $\mathbf{Sp}_{2n}$, cf. [Pos, Examples 6, 7]. Thus $\mathbf{Sp}_{2n}$ is Cayley for every $n \geqslant 1$. In particular, $\mathbf{SL}_2 \simeq \mathbf{Spin}_3 \simeq \mathbf{Sp}_2$, $\mathbf{Spin}_4 \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$ and $\mathbf{Spin}_5 \simeq \mathbf{Sp}_4$ are Cayley. Below we shall prove that $\mathbf{Spin}_n$ is not Cayley for $n \geqslant 6$.

Let K be a subfield of k. Since every K-form G of $\mathbf{SO}_n$ or $\mathbf{Sp}_{2n}$ is given by (1.19) for some algebra A and its involution ι, both defined over K, see [Weil], [Kn], all K-forms of $\mathbf{SO}_n$ and $\mathbf{Sp}_{2n}$ are Cayley K-groups. □

Example 1.22. Every connected commutative linear algebraic group G is Cayley. In fact, in this case, Condition (1.4) is vacuous, so the existence of (1.3) is equivalent to the property that the underlying variety of G is rational. Chevalley in [Ch$_1$] proved that over an algebraically closed field of characteristic zero this property holds for any connected linear algebraic group (not necessarily commutative). In particular, the algebraic torus $\mathbf{G}_m^d$, where

$$\mathbf{G}_m^d := \underbrace{\mathbf{G}_m \times \ldots \times \mathbf{G}_m}_{d} \quad \text{if } d \geqslant 1, \quad \mathbf{G}_m^0 = e,$$

is a Cayley group for every $d \geqslant 0$ (as $\mathbf{G}_m = \mathbf{GL}_1$, this also follows from Examples 1.10, 1.11).

Example 1.23. Every unipotent linear algebraic group G is Cayley (G is automatically connected because $\operatorname{char} k = 0$). Indeed, we may assume without loss of generality that $G \subset \mathbf{GL}_n$, so that elements of G are unipotent $n\times n$-matrices, elements of $\mathfrak{g}$ are nilpotent $n\times n$-matrices, and Ad_G is given by (1.2). So we have $(I_n - X)^n = Y^n = 0$ for any $X \in G$, $Y \in \mathfrak{g}$. Hence the exponential map is given by

$$\exp : \mathfrak{g} \longrightarrow G, \quad Y \mapsto \textstyle\sum_{i=0}^{n-1} \frac{1}{i!} Y^i.$$

Therefore exp is a G-equivariant morphism of algebraic varieties. Moreover, it is an isomorphism since the formula

$$\lambda := \ln \colon G \longrightarrow \mathfrak{g}, \quad X \mapsto -\textstyle\sum_{i=1}^{n-1} \frac{1}{i}(I_n - X)^i. \tag{1.24}$$

defines its inverse.

More generally, by the Corollary of Proposition 4.4 below, every connected solvable linear algebraic group is Cayley. □

1.25. Notational conventions. In order to formulate our main results we need some notation and definitions.

For any algebraic torus T, we denote by $\widehat{T}$ its character group,

$$\widehat{T} := \mathrm{Hom}_{\mathrm{alg}}(T, \mathbf{G}_m),$$

written additively. It is a lattice (i.e., a free abelian group of finite rank).

Let T be a maximal torus of G and let

$$\left\{\begin{aligned} N = N_{G,T} &:= \{g \in G \mid gTg^{-1} = T\},\\ C = C_{G,T} &:= \{g \in G \mid gtg^{-1} = t \text{ for all } t \in T\},\\ W = W_G = W_{G,T} &:= N/C \end{aligned}\right. \tag{1.26}$$

be respectively its normalizer, centralizer (which is the Cartan subgroup of G) and the Weyl group. The group C is the identity component of N, and if G is reductive, then $C = T$, see [Bor, 12.1, 13.17]. The finite group W naturally acts by automorphisms of $\widehat{T}$. Since all maximal tori in G are conjugate, W and the W-lattice $\widehat{T}$ do not depend, up to isomorphism, on the choice of T.

Definition 1.27. The W-lattice $\widehat{T}$ is called the *character lattice* of G and is denoted by $\mathcal{X}_G$.

Remark 1.28. The reader should be careful about this terminology: the elements of the character lattice of G are the characters of T, not of G.

Definition 1.29. A group G is called *stably Cayley* if $G \times \mathbf{G}_m^d$ is Cayley for some $d \geqslant 0$. If G is defined over a subfield K of k and $G \times \mathbf{G}_m^d$ is a Cayley K-group for some $d \geqslant 0$, then G is called a *stably Cayley K-group.*

We denote by $\mathbf{T}_G$ the generic torus of G, see its definition in Section 3.8, cf. [Vos], [CK].

1.30. Main results. Now we are ready to state our main results.

Theorem 1.31. *Let G be a connected reductive algebraic group. Then the following implications hold:*

$$\begin{array}{c}\mathcal{X}_G \textit{ is sign-}\\ \textit{permutation}\end{array} \overset{\text{(a)}}{\Longrightarrow} \begin{array}{c}G \textit{ is}\\ \textit{Cayley}\end{array} \overset{\text{(b)}}{\Longrightarrow} \begin{array}{c}\mathbf{T}_G \textit{ is}\\ \textit{rational}\end{array} \overset{\text{(c)}}{\Longrightarrow} \begin{array}{c}\mathbf{T}_G \textit{ is stably}\\ \textit{rational}\end{array} \overset{\text{(d)}}{\Longleftrightarrow} \begin{array}{c}\mathcal{X}_G \textit{ is quasi-}\\ \textit{permutation}\end{array} \overset{\text{(e)}}{\Longleftrightarrow} \begin{array}{c}G \textit{ is stably}\\ \textit{Cayley}\end{array}.$$

Moreover, the implications (a) *and* (b) *cannot be reversed. In particular, a stably Cayley group may not be Cayley.*

For the definitions of sign-permutation and quasi-permutation lattices, see Section 2.2. Note that it is a long-standing open question whether or not every stably rational torus is rational, see [Vos, p. 52]. In particular, we do not know whether or not implication (c) can be reversed. We also remark that (d) is well-known, see, e.g., [Vos, Theorem 4.7.2].

A proof of Theorem 1.31 will be given in Subsection 3.13. In Section 4 we will partially reduce Problem 1.8 to the case where G is a simple group.

We will then use Theorem 1.31 to translate results about stable rationality of generic tori into statements about the existence (and more often, the non-existence) of Cayley maps for various simple algebraic groups (i.e., groups having no proper connected normal subgroups). In particular, Lemire and Lorenz in [LL] and Cortella and Kunyavskiĭ in [CK] have recently proved that the character lattice of $\mathbf{SL}_n$ is quasi-permutation if and only if $n \leqslant 3$. (This result had been previously conjectured and proved for prime n by Le Bruyn in [LB$_1$], [LB$_2$].) Theorem 1.31 now tells us that $\mathbf{SL}_n$ is not stably Cayley (and thus not Cayley) for any $n \geqslant 4$. On the other hand, Example 1.18 shows that $\mathbf{SL}_2$ is Cayley, and Popov in [Pop$_2$] has proved that $\mathbf{SL}_3$ is Cayley as well (an outline of the arguments from [Pop$_2$] is reproduced in the Appendix; see also an explicit construction in Section 9). This settles Luna's original Question 1.8 about $\mathbf{SL}_n$.

In a similar manner, we proceed to classify the connected simple groups G with quasi-permutation character lattices $\mathcal{X}_G$. For simply connected and adjoint groups this was done by Cortella and Kunyavskiĭ in [CK]. In Sections 6 and 8 we extend their results to all other connected simple groups. Combining this classification with Theorem 1.31, we obtain the following result.

Theorem 1.32. *Let G be a connected simple algebraic group. Then the following conditions are equivalent:*

(a) *G is stably Cayley,*
(b) *G is one of the following groups:*

$$\mathbf{SL}_n \textit{ for } n \leqslant 3, \ \ \mathbf{SO}_n \textit{ for } n \neq 2, 4, \ \ \mathbf{Sp}_{2n}, \ \ \mathbf{PGL}_n, \ \ \mathbf{G}_2. \tag{1.33}$$

Remark 1.34. The groups $\mathbf{SO}_2$ and $\mathbf{SO}_4$ are stably Cayley (and even Cayley, see Example 1.18) but they are excluded because they are not simple. Note also that, due to exceptional isomorphisms, some groups are listed twice in (1.33). (For example, $\mathbf{Sp}_2 \simeq \mathbf{SL}_2$.)

It is now natural to ask which of the stably Cayley simple groups listed in Theorem 1.32(b) are in fact Cayley. Here is the answer:

Theorem 1.35. *Let G be a connected simple algebraic group.*

(a) *The following conditions are equivalent:*
  (i) *G is Cayley;*
  (ii) *G is one of the following groups:*

$$\mathbf{SL}_n \textit{ for } n \leqslant 3, \ \ \mathbf{SO}_n \textit{ for } n \neq 2, 4, \ \ \mathbf{Sp}_{2n}, \ \ \mathbf{PGL}_n. \tag{1.36}$$

(b) *The group $\mathbf{G}_2$ is not Cayley but the group $\mathbf{G}_2 \times \mathbf{G}_m^2$ is Cayley.*

The first assertion of part (b) is based on the recent work of Iskovskikh [Isk$_4$]. The groups $\mathbf{SO}_n$, $\mathbf{Sp}_{2n}$ and $\mathbf{PGL}_n$ were shown to be Cayley in Examples 1.18 and 1.13. The groups $\mathbf{SL}_3$ and $\mathbf{G}_2$ will be discussed in Section 9.

Remark 1.37. Question 1.8 was inspired by Luna's interest in the existence (for reductive G) of "algebraic linearization" of the conjugating action in a Zariski neighborhood of the identity element $e \in G$, i.e., in the existence of G-isomorphic neighborhoods of e and 0 in G and $\mathfrak{g}$ respectively, cf. [Lun$_1$]. In our terminology this is equivalent to the existence of a Cayley map (1.3) such that λ and λ^{-1} are defined at e and 0 respectively, and $\lambda(e) = 0$. Not all Cayley maps have this property. However, note that our proof of Theorem 1.35 (in combination with [Lun$_1$, p. 13, Proposition]) shows that each of the simple groups listed in (1.36) admits a Cayley map with this property (and so does any direct product of these groups); see Examples 1.10–1.23, Subsections 9.1, 9.9 and the Appendix.

Let K be a subfield of k. It follows from Theorems 1.32, 1.35 and Examples 1.13, 1.18 that classifying simple Cayley (respectively, stably Cayley) K-groups is reduced to classifying outer K-forms of $\mathbf{PGL}_n$ for $n \geqslant 3$ and K-forms of $\mathbf{SL}_3$ (respectively, outer K-forms of $\mathbf{PGL}_n$ for $n \geqslant 3$ and K-forms of $\mathbf{SL}_3$ and $\mathbf{G}_2$) that are Cayley (respectively, stably Cayley) K-groups. Note that not all of these K-forms are Cayley (respectively, stably Cayley) K-groups. Indeed, Definitions 1.7, 1.29 imply the following special property of Cayley (respectively, stably Cayley) K-groups: their underlying varieties are rational (respectively, stably rational) over K. For some of the specified K-forms this property does not hold:

Example 1.38. Berhuy, Monsurrò, and Tignol in [BMT] have shown that for every $n \equiv 0 \bmod 4$, the group $\mathbf{PGL}_n$ has a K-form G of outer type whose underlying variety is not stably rational over K. Hence G is not a stably Cayley K-group. □

Remark 1.39. The underlying varieties of all outer K-forms of $\mathbf{PGL}_n$ with odd n are rational over K; see [VK]. Note also that the underlying variety of any K-form of a linear algebraic group of rank at most 2 is rational over K, e.g., see [Me, p. 189], [Vos, 4.1, 4.9].

1.40. Application to Cremona groups. The Cremona group Cr_d, i.e., the group of birational automorphisms of the affine space $\mathbf{A}^d$, is a classical object in algebraic geometry; see [Isk$_2$] and the references therein. Classifying the subgroups of Cr_d up to conjugacy is an important research direction originating in the works of Bertini, Enriques, Fano, and Wiman. Most of the currently know results on Cremona groups relate to Cr_2 nd Cr_3 (the case $d = 1$ is trivial because $\mathrm{Cr}_1 = \mathbf{PGL}_2$). For $d \geq 4$ the groups Cr_d are poorly understood, and any results that shed light on their structure are prized by the experts.

Our results provide some information about subgroups of Cr_d by means of the following simple construction. Consider an action of an algebraic group G on a rational variety X of dimension d. Let G_0 be the kernel of this action. Any birational isomorphism between X and $\mathbf{A}^d$ gives rise to an embedding $\iota_X\colon G/G_0 \hookrightarrow \mathrm{Cr}_d$. A different birational isomorphism between X and $\mathbf{A}^d$ gives rise to a conjugate embedding, so i_X is is uniquely determined by X (as a G-variety) up to conjugacy in Cr_d. If Y is another rational variety on which G acts then the embeddings ι_X and ι_Y are conjugate if and only if X and Y are birationally isomorphic as G-varieties.

Now consider the special case of this construction, where G is a connected linear algebraic group, X is the underlying variety of G (with the conjugating G-action), $Y = \mathfrak{g}$ (with the adjoint G-action), and the kernel G_0 (for both actions) is the center of G; see [Bor, 3.15]. Definition 1.7 can now be rephrased as follows: a connected algebraic group G is Cayley if and only if the embeddings ι_G and $\iota_{\mathfrak{g}}\colon G/G_0 = \mathrm{Ad}_G G \hookrightarrow \mathrm{Cr}_{\dim G}$ are conjugate in $\mathrm{Cr}_{\dim G}$. In this paper we show that many connected algebraic groups are not Cayley; each non-Cayley group G gives rise to a pair of non-conjugate embeddings of the form ι_G, $\iota_{\mathfrak{g}}\colon \mathrm{Ad}_G G \hookrightarrow \mathrm{Cr}_{\dim G}$.

Definition 1.29 can be interpreted in a similar manner. For every $d \geq 1$ consider the embedding $\mathrm{Cr}_d \hookrightarrow \mathrm{Cr}_{d+1}$ given by writing $\mathbf{A}^{d+1}$ as $\mathbf{A}^d \times \mathbf{A}^1$ and sending an element $g \in \mathrm{Cr}_d$ to $g \times \mathrm{id}_{\mathbf{A}^1} \in \mathrm{Cr}_{d+1}$. Denote the direct limit for the tower of groups $\mathrm{Cr}_1 \hookrightarrow \mathrm{Cr}_2 \hookrightarrow \ldots$ obtained in this way by Cr_∞. Suppose G is a group acting on rational varieties X and Y (possibly of different dimensions) with the same kernel G_0. Then it is easy to see that the embeddings $\iota_X\colon G/G_0 \hookrightarrow \mathrm{Cr}_{\dim X}$ and $\iota_Y\colon G/G_0 \hookrightarrow \mathrm{Cr}_{\dim Y}$ are conjugate in Cr_∞ (or equivalently, in Cr_m for some $m \geq \max\{\dim X, \dim Y\}$) if and only if X and Y are stably isomorphic as G-varieties.

If V_1 and V_2 are vector spaces with faithful linear G-actions, then ι_{V_1} and ι_{V_2} are conjugate in Cr_∞ by the "no-name lemma", cf. Subsection 2.15. We call an embedding $G \hookrightarrow \mathrm{Cr}_d$ *stably linearizable* if it is conjugate, in Cr_∞, to i_V for some faithful linear G-action on a vector space V. Definition 1.29 and the "no-name lemma" now tell us that the following conditions are equivalent: (a) G is stably Cayley, (b) the embeddings ι_G and $\iota_{\mathfrak{g}}\colon \mathrm{Ad}_G G \hookrightarrow \mathrm{Cr}_{\dim G}$ are conjugate in Cr_∞ and (c) ι_G is stably linearizable. Once again, the results of this paper (and in particular, Theorem 1.32) can be used to produce many examples of pairs of embeddings of the form $\mathrm{Ad}_G G \hookrightarrow \mathrm{Cr}_{\dim G}$ that are not conjugate in Cr_∞.

Now suppose that Γ is a finite group and L and M are faithful Γ-lattices; see Section 2.2. Then Γ acts on their dual tori, which we will denote by X and Y. It now follows from Lemma 2.7 that the embeddings $\iota_X\colon \Gamma \hookrightarrow \mathrm{Cr}_{\operatorname{rank} L}$ and $\iota_Y\colon \Gamma \hookrightarrow \mathrm{Cr}_{\operatorname{rank} M}$ are conjugate in Cr_∞ if and only if L and M are equivalent in the sense of Definition 2.4. Taking M to be a faithful permutation lattice, we conclude that the embedding $\iota_X\colon \Gamma \hookrightarrow \mathrm{Cr}_{\operatorname{rank} X}$ is stably linearlizable if and only if L is quasi-permutation (cf. Definition 2.6 and Corollary to Lemma 2.7).

In the special case where $L = \mathcal{X}_G$ is the character lattice of algebraic group G, $\Gamma = W_G$ is the Weyl group, and $X = T$ is a maximal torus with Lie algebra $\mathfrak{t}$, we see that the following conditions are equivalent: (a) G is stably Cayley, (b) $\mathcal{X}_G$ is quasi-permutation, (c) the embeddings $\iota_{\mathfrak{t}}$ and $\iota_T\colon W \hookrightarrow \mathrm{Cr}_{\dim T}$ are conjugate in Cr_∞, and (d) ι_T is stably linearizable. (Note that (a) and (b) are equivalent by Theorem 1.31, and (c) and (d) are equivalent because the W-action on $\mathfrak{t}$ is linear.) Consequently, every reductive non-Cayley group G gives rise to a pair of embeddings i_T, $i_{\mathfrak{t}}\colon W \hookrightarrow \mathrm{Cr}_{\operatorname{rank} G}$ which are not conjugate in Cr_∞.

Example 1.41. Let G be a simple group of type A_{n-1} which is not stably Cayley, i.e., $G = \mathbf{SL}_n/\boldsymbol{\mu}_d$, where $d\,|\,n$, $d < n$, $n \geq 4$, and $(n,d) \neq (4,2)$. Then the embeddings ι_T and $\iota_{\mathfrak{t}}\colon \mathrm{S}_n \hookrightarrow \mathrm{Cr}_{n-1}$ are not conjugate in Cr_∞.

Assume further that $n \neq 6$. Then by Hölder's theorem (see [Höl]), S_n has no outer automorphisms. Thus the images $\iota_T(\mathrm{S}_n)$ and $\iota_{\mathfrak{t}}(\mathrm{S}_n)$ are isomorphic finite subgroups of Cr_{n-1} which are not conjugate in Cr_∞. $\square$

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2. Preliminaries

In this section we collect certain preliminary facts for subsequent use. Some of them are known and some are new. Throughout this section Γ will denote a group; starting from Subsection 2.2 it is assumed to be finite.

2.1. Γ-fields and Γ-varieties.

In the sequel we will use the following terminology. A Γ-*field* is a field K together with an action of Γ by automorphisms of K. Let K_1 and K_2 be Γ-fields containing a common Γ-subfield K_0. We say that K_1 and K_2 are *isomorphic as* Γ-*fields* (or Γ-*isomorphic*) *over* K_0 if there is a Γ-equivariant field isomorphism $K_1 \to K_2$ which is the identity on K_0. We say that K_1 and K_2 are *stably isomorphic as* Γ-*fields* (or *stably* Γ-*isomorphic*) *over* K_0 if, for suitable n and m, $K_1(x_1, \ldots, x_n)$ and $K_2(y_1, \ldots, y_m)$ are isomorphic as Γ-fields over K_0. Here, $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are algebraically independent variables over K_1 and K_2, respectively; these variables are assumed to be fixed by the Γ-action.

If Γ is an algebraic group, a Γ-*variety* is an algebraic variety X endowed with an algebraic (morphic) action of Γ. A Γ-equivariant morphism (respectively, rational map) of Γ-varieties is a Γ-*morphism* (respectively, *rational* Γ-*map*). If X_1 and X_2 are irreducible Γ-varieties, then $k(X_1)$ and $k(X_2)$ are Γ-fields with respect to the natural actions of Γ. These fields are stably F-isomorphic over k if and only if there is a birational Γ-isomorphism $X_1 \times \mathbf{A}^r \dashrightarrow X_2 \times \mathbf{A}^s$ for some r and s, where Γ acts on $X_1 \times \mathbf{A}^r$ and $X_2 \times \mathbf{A}^s$ via the first factors. In this case, X_1 and X_2 are called *stably birationally* Γ-*isomorphic*.

2.2. Γ-lattices. From now on we assume that Γ is a finite group.

A *lattice* L of rank r is a free abelian group of rank r. A Γ-*lattice* is a lattice equipped with an action of Γ by automorphisms. It is called *faithful* (respectively *trivial*) if the homomorphism $\Gamma \to \mathrm{Aut}_{\mathbb{Z}} L$ defining the action is injective (respectively trivial). If H is a subgroup of Γ, then L considered as an H-lattice is denoted by $L|_H$.

Given a group H and a ring R, we denote by $R[H]$ the group ring of H over R. If K is a field and L is a Γ-lattice, we denote by $K(L)$ the fraction field of $K[L]$; both $K[L]$ and $K(L)$ inherit a Γ-action from L. We usually think of these objects multiplicatively, i.e., we consider the set of symbols $\{x^a\}_{a\in L}$ as a basis of the K-vector space $K[L]$, and the multiplication being defined by $x^a x^b = x^{a+b}$. So $\sigma \cdot x^a = x^{\sigma \cdot a}$ for any $\sigma \in \Gamma$. If $a_1, \dots, a_r$ is a basis of L, and $x_i := x^{a_i}$, then $K[L] = K[x_1, x_1^{-1} \dots, x_r, x_r^{-1}]$ and $K(L) = K(x_1, \dots, x_r)$. Note that any group isomorphism $L \to \widehat{\mathbf{G}_m^r}$ induces the K-isomorphisms of algebras $K[L] \to K[\mathbf{G}_m^r]$ and fields $K(L) \to K(\mathbf{G}_m^r)$, and therefore it induces a K-defined algebraic action of Γ on the torus $\mathbf{G}_m^r$ by its automorphisms. Any such action is obtained in this way.

An important example is $L = \mathcal{X}_G$, the character lattice of a connected algebraic group G, and $\Gamma = W$, the Weyl group of G. In this case, $k(\mathcal{X}_G)$ is the field of rational functions on a maximal torus of G.

Definition 2.3. A Γ-lattice L is called *permutation* (respectively, *sign-permutation*) if it has a basis $\varepsilon_1, \dots, \varepsilon_r$ such that the set $\{\varepsilon_1, \dots, \varepsilon_r\}$ (respectively, $\{\varepsilon_1, -\varepsilon_1, \dots, \varepsilon_r, -\varepsilon_r\}$) is Γ-stable.

If X is a finite set endowed with an action of Γ, we denote by $\mathbb{Z}[X]$ the free abelian group generated by X and endowed with the natural action of Γ. Permutation lattices may be, alternatively, defined as those of the form $\mathbb{Z}[X]$. Since X is the union of Γ-orbits, any permutation lattice is isomorphic to some $\oplus_{i=1}^s \mathbb{Z}[\Gamma/\Gamma_i]$, where each Γ_i is a subgroup of Γ.

Definition 2.4. ([C–TS$_1$]) Two Γ-lattices M and N are called *equivalent*, written $M \sim N$, if they become Γ-isomorphic after extending by permutation lattices, i.e., if there are exact sequences of Γ-lattices

$$0 \longrightarrow M \longrightarrow E \longrightarrow P \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow N \longrightarrow E \longrightarrow Q \longrightarrow 0 \tag{2.5}$$

where P and Q are permutation.

For a direct proof that this does indeed define an equivalence relation and for further background see [C–TS$_1$, Lemma 8] or [Sw].

Definition 2.6. A Γ-lattice L is called *quasi-permutation* if $L \sim 0$ under this equivalence relation, i.e., L becomes permutation after extending by a permutation lattice. In other words, L is quasi-permutation if and only if there is an exact sequence of Γ-lattices

$$0 \longrightarrow L \longrightarrow P \longrightarrow Q \longrightarrow 0,$$

where P and Q are permutation lattices.

It is easily seen that the properties of being permutation, sign-permutation and quasi-permutation are preserved under passing to Γ-isomorphic Γ-lattices and that replacing equivalent Γ-lattices with Γ-isomorphic ones yields equivalent lattices as well.

Lemma 2.7. *Let M and N be faithful Γ-lattices and let K be a field. Then the following properties are equivalent:*

(i) *$K(M)$ and $K(N)$ are stably isomorphic as Γ-fields over K,*
(ii) *$M \sim N$.*

Proof. See [LL, Proposition 1.4]; this assertion is also implicit in [Sw], [C–TS$_1$] and [Vos, 4.7]. □

Lemma 2.7 and Definition 2.6 immediately imply the following.

Corollary. *Let L be a faithful Γ-lattice and let K be a field. Then the following properties are equivalent:*

(i) *$K(L)$ is stably isomorphic to $K(P)$ (as a Γ-field over K) for some faithful permutation F-lattice P,*
(ii) *L is quasi-permutation.*

2.8. Stable equivalence and flasque resolutions. In addition to the equivalence relation $\sim$ on Γ-lattices, we will also consider a stronger equivalence relation $\approx$ of stable equivalence. Two Γ-lattices L_1 and L_2 are called *stably equivalent* if $L_1 \oplus P_1 \simeq L_2 \oplus P_2$ for suitable permutation Γ-lattices P_1 and P_2.

A Γ-lattice L is called *flasque* if $\mathrm{H}^{-1}(S, L) = 0$ for all subgroups S of Γ. Every Γ-lattice L has a *flasque resolution*

$$0 \longrightarrow L \longrightarrow P \longrightarrow Q \longrightarrow 0 \tag{2.9}$$

with P a permutation Γ-lattice and Q a flasque Γ-lattice. Moreover, Q is determined by L up to stable equivalence: If $0 \to L \to P' \to Q' \to 0$ is another flasque resolution of L, then $Q \approx Q'$. Following [C–TS$_1$], we will denote the stable equivalence class of Q in the flasque resolution (2.9) by

$$\rho(L).$$

Note that by [C–TS$_1$, Lemme 8], for Γ-lattices M, N,

$$M \sim N \iff \rho(M) = \rho(N). \tag{2.10}$$

Dually, every Γ-lattice L has a *flasque resolution*

$$0 \longrightarrow R \longrightarrow P \longrightarrow L \longrightarrow 0 \tag{2.11}$$

with P a permutation Γ-lattice and R a *coflasque* Γ-lattice, that is, $\mathrm{H}^1(S, R) = 0$ holds for all subgroups S of Γ. Similarly, R is determined by L up to stable equivalence. Note that the dual of a flasque resolution for L is a *coflasque resolution* for L^* since $\mathrm{H}^1(S, L) \simeq \mathrm{H}^{-1}(S, L^*)$. For details, see [C–TS$_1$, Lemme 5]. Note that since $\mathrm{H}^{\pm 1}$ is trivial for permutation modules, $\mathrm{H}^{\pm 1}(\Gamma, L)$ depends only on the stable equivalence class $[L]$ of L and therefore is denoted by $\mathrm{H}^{\pm 1}(\Gamma, [L])$.

Following Colliot–Thélène and Sansuc, [C–TS$_1$, C–TS$_2$], we define

$$Ш^i(\Gamma, M) = \bigcap_{a \in \Gamma} \mathrm{Ker}\big(\mathrm{Res}^{\Gamma}_{\langle a\rangle} \colon \mathrm{H}^i(\Gamma, M) \longrightarrow \mathrm{H}^i(\langle a\rangle, M)\big).$$

for any $\mathbb{Z}[\Gamma]$-module M. Of particular interest for us will be the case where M is a Γ-lattice L and $i = 1$ or 2.

The following lemma is extracted from [C–TS$_2$, pp. 199–202]. For a proof, see also [LL, Lemma 4.2].

Lemma 2.12. (a) *For any exact sequence of $\mathbb{Z}[\Gamma]$-modules*

$$0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

with P a permutation projective Γ-lattice, $Ш^2(\Gamma, M) \simeq Ш^1(\Gamma, N)$.

(b) $\mathrm{H}^1(\Gamma, \rho(L)) \simeq Ш^2(\Gamma, L)$ *for any Γ-lattice L.*

(c) *If L is equivalent to a direct summand of a quasi-permutation Γ-lattice, then $\text{Ш}^2(S, L) = 0$ holds for all subgroups S of Γ.*

In particular, $\text{Ш}^2(\Gamma, \cdot\,)$ is constant on $\sim$-classes.

The following technical proposition will help us show that certain Γ-lattices are equivalent.

Proposition 2.13. *Let X and Y be Γ-lattices satisfying the exact sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0$$

where Γ acts trivially on $\mathbb{Z}/d\mathbb{Z}$.

(a) *If $(d, |\Gamma|) = 1$, then $X \oplus \mathbb{Z} \simeq Y \oplus \mathbb{Z}$ so that $X \approx Y$ and $X^* \approx Y^*$.*

(b) *If the fixed point sequence*

$$0 \longrightarrow X^S \longrightarrow Y^S \longrightarrow (\mathbb{Z}/d\mathbb{Z})^S \longrightarrow 0$$

is exact for all subgroups S of Γ, then $X^* \sim Y^*$ as Γ-lattices.

Proof. (a) This follows directly from Roiter's form of Schanuel's Lemma [CR, 31.8] applied to the sequence of the proposition and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times d} \mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0.$$

(b) We claim that any coflasque resolution

$$0 \longrightarrow C_1 \longrightarrow P \longrightarrow X \longrightarrow 0$$

for X can be extended to a coflasque resolution

$$0 \longrightarrow C_2 \longrightarrow P \oplus Q \longrightarrow Y \longrightarrow 0$$

for Y so that the following diagram commutes and has exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_2 \arrow[r] \arrow[d] & P \oplus Q \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \mathbb{Z}/d\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \quad . \tag{2.14}$$

Here C_1, C_2 are Γ-coflasque and P, Q are Γ-permutation. Indeed, as is described in [C–TS$_1$, Lemme 3], given a surjective homomorphism π from a permutation Γ-lattice P_0 to a given Γ-lattice X, to form a coflasque resolution of X, we need only adjust P_0 to $P = P_0 \oplus \sum_S \mathbb{Z}[\Gamma/S] \otimes X^S$ where the sum is taken over all subgroups S of Γ for which $\pi : P^S \to X^S$ is not a surjection, and adjust π to $\hat{\pi}$ such that $\hat{\pi}|_{\mathbb{Z}[\Gamma/S]\otimes X^S} = \epsilon_S \otimes \mathrm{id}$ with ϵ_S the augmentation map. Then $\hat{\pi}$ maps P^S surjectively onto X^S for all subgroups S of Γ so that $\mathrm{H}^1(S, \operatorname{Ker} \hat{\pi}) = 0$ as required. To obtain a compatible coflasque resolution for Y, extend the surjection from the permutation lattice P onto X to a surjection from the permutation lattice $P \oplus Q_0$ onto Y and then adjust this surjection $P \oplus Q_0 \to Y$ to one with a coflasque kernel $P \oplus Q \to Y$ as above. Then the top two rows are exact and commutative. The bottom row is obtained via the Snake Lemma.

Let S be a subgroup of Γ. Taking S-fixed points in (2.14), we obtain

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C_1^S & \longrightarrow & P^S & \longrightarrow & X^S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C_2^S & \to & P^S \oplus Q^S & \longrightarrow & Y^S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & U^S & \longrightarrow & Q^S & \longrightarrow & (\mathbb{Z}/d\mathbb{Z})^S & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} .$$

Since C_1, C_2 are coflasque and P, Q are permutation, we find that the first two rows and columns are exact. By hypothesis, the third column is exact. Then a diagram chase shows that the bottom row is exact. But then this means that U is coflasque since

$$0 \longrightarrow U^S \longrightarrow Q^S \longrightarrow (\mathbb{Z}/d\mathbb{Z})^S \longrightarrow \mathrm{H}^1(S, U) \longrightarrow \mathrm{H}^1(S, Q) = 0$$

is exact. Applying [LL, Lemma 1.1] to

$$0 \longrightarrow U \longrightarrow Q \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0,$$

we find that U is also quasi-permutation as it satisfies

$$0 \longrightarrow U \longrightarrow Q \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

So as U is coflasque, this sequence splits and U is in fact stably permutation with $U \oplus \mathbb{Z} \simeq Q \oplus \mathbb{Z}$. The first column of the first commutative diagram then shows us that $C_1 \oplus U \oplus \mathbb{Z} \simeq C_2 \oplus \mathbb{Z}$ so that $C_1 \oplus Q \oplus \mathbb{Z} \simeq C_2 \oplus \mathbb{Z}$. Since

$$0 \longrightarrow X^* \longrightarrow P \longrightarrow C_1^* \longrightarrow 0, \qquad 0 \longrightarrow Y^* \longrightarrow P \oplus Q \longrightarrow C_2^* \longrightarrow 0$$

are flasque resolutions of X^* and Y^*, this implies $\rho(X^*) = \rho(Y^*)$ (i.e., that the corresponding flasque lattices are stably equivalent). By [C–TS$_1$, Lemme 8], we conclude that $X^* \sim Y^*$. $\square$

2.15. Speiser's Lemma. Let $\pi : Y \to X$ be an algebraic vector bundle. We call it an algebraic *vector* Γ*-bundle* if Γ acts on X and Y, the morphism π is Γ-equivariant and $g\colon \pi^{-1}(x) \to \pi^{-1}(g(x))$ is a linear map for every $x \in X$ and $g \in \Gamma$.

The first of the following related rationality results is an immediate consequence the classical Speiser's Lemma; the others follow from the first. In a broader context, when Γ is any algebraic group, results of this type appear in the literature under the names of "no-name method" ([Do]) and "no-name lemma" (see [C–T]).

Lemma 2.16. (a) *Suppose E is a Γ-field and K is a Γ-subfield of E such that Γ acts on K faithfully, $E = K(x_1, \ldots, x_m)$ and $Kx_1 + \ldots + Kx_m$ is Γ-stable. Then $E = K(t_1, \ldots, t_m)$, where $t_1, \ldots, t_m$ are Γ-invariant elements of $Kx_1 + \ldots + Kx_m$.*

(b) *Let $\pi\colon Y \to X$ be an algebraic vector Γ-bundle. Suppose that X is irreducible and the action of Γ on X is faithful. Then π is birationally Γ-trivial, i.e., there exists a birational Γ-isomorphism $\varphi : Y \overset{\simeq}{\dashrightarrow} X \times k^m$, where Γ acts on $X \times k^m$ via the first factor, such that* *the diagram*

$$\begin{tikzcd} Y \arrow[rr, dashed, "\varphi"] \arrow[dr, "\pi"'] & & X \times k^m \arrow[dl, "\pi_1"] \\ & X & \end{tikzcd},$$

is commutative (π_1 *denotes projection to the first factor*).

(c) *Let* V_1 *and* V_2 *be finite dimensional vector spaces over* k *endowed with faithful linear actions of* Γ. *Then* V_1 *and* V_2 *are stably* Γ*-isomorphic.*

(d) *Suppose* L *is a field and*

$$0 \longrightarrow S \xrightarrow{\iota} N \xrightarrow{\tau} P \longrightarrow 0$$

is an exact sequence of Γ*-lattices, where* S *is faithful and* P *is permutation. Then the* Γ*-field* $L(N)$ *is* Γ*-isomorphic over* L *to the* Γ*-field* $L(S)(t_1,\ldots,t_r)$*, where the elements* $t_1,\ldots,t_r$ *are* Γ*-invariant and algebraically independent over* $L(S)$.

Proof. (a) follows from Speiser's Lemma, [Spe], cf. [HK, Theorem 1] or [Sh, Appendix 3].

(b) Recall that, by definition, algebraic bundles are locally trivial in the étale topology, but algebraic vector bundles are automatically locally trivial in the Zariski topology, see [Se]. This implies that after replacing X by a Γ-stable dense open subset U and Y by $\pi^{-1}(U)$ we may assume that $Y = X \times k^m$ (but we do not claim that Γ acts via the first factor!) and π is projection to the first factor.

Using the projections $Y \to X$ and $Y \to k^m$, we shall view $k(X)$ and $k(k^m)$ as subfields of $k(Y)$. Put $E := k(Y)$, $K := k(X)$ and let $x_1,\ldots,x_m$ be the standard coordinate functions on k^m. If $g \in \Gamma$ and $b \in X$, then the definition of Γ-bundle implies that $g(x_i)|_{\pi^{-1}(b)} \in k\,x_1|_{\pi^{-1}(b)} + \ldots + k\,x_m|_{\pi^{-1}(b)}$. In turn, this implies that the assumptions of (a) hold. Part (b) now follows from part (a).

(c) Applying part (b) to the projections $V_1 \leftarrow V_1 \times V_2 \to V_2$, we see that both V_1 and V_2 are stably Γ-isomorphic to $V_1 \times V_2$.

(d) Identify S with $\iota(S)$; then $K := L(S)$ is an Γ-subfield of $E := L(N)$. Put $x_1 = 1 \in E$ and choose $x_2\ldots,x_m \in N \subset E$ such that $\tau(x_2),\ldots,\tau(x_m)$ is a basis of P permuted by Γ. The elements $x_2,\ldots,x_m$ are algebraically independent over K. If $g \in \Gamma$, then for every i there is a j such that $a_{ij} := g(x_i) - x_j \in \operatorname{Ker}\tau = S \subset K$; so $g(x_i) = a_{ij}x_1 + x_j$. This shows that the assumptions of (a) hold. The claim (with $r = m-1$) now follows from part (a). $\square$

2.17. Homogeneous fiber spaces. Let H be an algebraic group and let S be a closed subgroup of H. Consider an algebraic variety X endowed with an algebraic (morphic) action of S and the algebraic action of S on $H \times X$ defined by

$$s(h,x) = (hs^{-1}, s(x)), \quad s \in S,\ (h,x) \in H \times X. \tag{2.18}$$

Assume that there exists a geometric quotient, [MFK], [PV, 4.2],

$$H \times X \longrightarrow (H \times X)/S. \tag{2.19}$$

This is always the case if every finite subset of X is contained in an affine open subset of X (note that this property holds if the variety X is quasi-projective), [Se, 3.2], cf. [PV, 4.8]. The variety $(H \times X)/S$, called a *homogeneous fiber space over* H/S *with fiber* X, is denoted by $H \times^S X$. If H is connected and X is irreducible, then $H \times^S X$ is irreducible. We denote by $[h,x]$ the image of a point $(h,x) \in H \times X$ under the morphism (2.19).

The group H acts on $H \times X$ by left translations of the first factor. As this action commutes with the S-action (2.18), the universal property of geometric quotients implies that the corresponding H-action on $H \times^S X$,

$$h'[h, x] = [h'h, x], \quad h', h \in H, \ x \in X,$$

is algebraic. It also implies that since the composition of the projection $H \times X \to H$ with the canonical morphism $H \to H/S$ is constant on S-orbits of the action (2.18), this composition induces a morphism

$$\pi = \pi_{H,S,X} : H \times^S X \longrightarrow H/S, \quad [h, x] \mapsto hS. \tag{2.20}$$

This morphism is H-equivariant and its fiber over the point $o \in H/S$ corresponding to S is S-stable and S-isomorphic to X; in the sequel we identify X with this fiber. Since H acts transitively on H/S and π is H-equivariant, the H-orbit of any point of $H \times^S X$ intersects X. If Z is an open (respectively closed) H-stable subset of X, and $\iota : Z \hookrightarrow X$ is the identity embedding, then $H \times^S Z \to H \times^S X$, $[h, z] \mapsto [h, \iota(z)]$, is the embedding of algebraic varieties whose image is an H-stable closed (respectively open) subset of $H \times^S X$. Every H-stable closed (respectively open) subset of $H \times^S X$ is obtained in this way.

If the action of S on X is trivial, then $H \times^S X = H/S \times X$ and π is the projection to the first factor.

The morphism π is a locally trivial fibration in the étale topology, i.e., each point of H/S has an open neighborhood U such that the pull back of $\pi^{-1}(U) \xrightarrow{\pi} U$ over a suitable étale covering $\widetilde{U} \to U$ is isomorphic to the trivial fibration $\widetilde{U} \times X \to \widetilde{U}$, $(y, x) \mapsto x$, see [Se, §2], [PV, 4.8]. If X is a k-vector space and the action of S on X is linear, then (2.20) is an algebraic vector H-bundle, so π is locally trivial in the Zariski topology, i.e., $\pi^{-1}(U) \xrightarrow{\pi} U$ is isomorphic to $U \times X \to U$, $(u, x) \mapsto x$, for a suitable U, see [Se].

If ψ is a (not necessarily H-equivariant) morphism (respectively rational map) of $H \times^S X$ to $H \times^S Y$ such that

$$\pi_{H,S,X} = \pi_{H,S,Y} \circ \psi, \tag{2.21}$$

then we say that ψ is a morphism (respectively rational map) *over* H/S.

Lemma 2.22. (a) *If* $\psi : H \times^S X \to H \times^S Y$ *is an* H*-morphism over* H/S*, then* $\psi|_X$ *is an* S*-morphism* $X \to Y$*. The map* $\psi \mapsto \psi|_X$ *is a bijection between* H*-morphisms* $H \times^S X \to H \times^S Y$ *over* H/S *and* S*-morphisms* $X \to Y$*. Moreover,* ψ *is dominant* (*respectively, an isomorphism*) *if and only if* $\psi|_X$ *is dominant* (*respectively, an isomorphism*).

(b) *Let* H *be connected and let* X *and* Y *be irreducible. Then the statements in* (a) *hold with "morphism" and "isomorphism" replaced by respectively "rational map" and "birational isomorphism".*

Proof. (a) Since $X = \pi^{-1}_{H,S,X}(o)$, $Y = \pi^{-1}_{H,S,Y}(o)$, the first statement follows from (2.21). As every H-orbit in $H \times^S X$ intersects X and ψ is H-equivariant, ψ is uniquely determined by $\psi|_X$. If $\varphi : X \to Y$ is an S-morphism, then $H \times X \to H \times Y$, $(h, x) \mapsto (h, \varphi(x))$, is a morphism commuting with the actions of S (defined for $H \times X$ by (2.18) and analogously for $H \times Y$) and H. By the universal property of geometric quotients, the H-map ψ : $H \times^S X \to H \times^S Y$, $[h, x] \mapsto [h, \varphi(x)]$, is a morphism over H/S. We have $\psi|_X = \varphi$. The same argument proves the last statement.

(b) Since ψ is H-equivariant, its indeterminacy locus is H-stable. As every H-orbit in $H \times^S X$ intersects X, this locus cannot contain X. Consequently, $\psi|_X : X \dashrightarrow H \times^S Y$ is a well-defined rational S-map. In view of (2.21), its image lies in Y. Now (b) follows from

(a) because rational maps are the equivalence classes of morphisms of dense open subsets, and all H-stable open subsets in $H \times^S X$ are of the form $H \times^S Z$ where Z is an S-stable open subset of X. $\square$

3. Cayley maps, generic tori, and lattices

3.1. Restricting Cayley maps to Cartan subgroups. Let G be a connected linear algebraic group and let T be its maximal torus. Consider the Cartan subgroup C, its normalizer N and the Weyl group W defined by (1.26). Let $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{c}$ be the Lie algebras of G, T and C respectively.

Since C is the identity component of N and the Cartan subgroups of G are all conjugate to each other, [Bor, 12.1], assigning to a point of G/N the identity component of its G-stabilizer (respectively, the Lie algebra of this G-stabilizer) yields a bijection between G/N and the set of all Cartan subgroups in G (respectively, all Cartan subalgebras in $\mathfrak{g}$). So G/N can be considered as the *variety of all Cartan subgroups in* G (respectively, the *variety of all Cartan subalgebras in* $\mathfrak{g}$).

Moreover the Cartan subgroups in G (respectively the Cartan subalgebras in $\mathfrak{g}$) parametrized in this way by the points of G/N naturally "merge" to form a homogeneous fiber space over G/N with fiber C (respectively, $\mathfrak{c}$). More precisely, consider the homogeneous fiber space $G \times^N C$ over G/N defined by the conjugating action of N on C (respectively, the homogeneous fiber space $G \times^N \mathfrak{c}$ over G/N defined by the adjoint action of N on $\mathfrak{c}$). Then for any $g \in G$, the map $\pi^{-1}_{G,N,C}(g(o)) \to gCg^{-1}$, $[g,c] \mapsto gcg^{-1}$ (respectively, the map $\pi^{-1}_{G,N,\mathfrak{c}}(g(o)) \to \mathrm{Ad}_G\, g(\mathfrak{c})$, $[g,x] \mapsto \mathrm{Ad}_G\, g(x)$), is a well defined isomorphism (we use the notation of Subsection 2.17 for $H = G$, $S = N$).

Consider the conjugating and adjoint actions of G respectively on G and $\mathfrak{g}$. Then the definition of homogeneous fiber space implies that

$$\gamma_C : G \times^N C \longrightarrow G,\ [g,c] \mapsto gcg^{-1}, \qquad \gamma_\mathfrak{c} : G \times^N \mathfrak{c} \longrightarrow \mathfrak{g},\ [g,x] \mapsto \mathrm{Ad}_G\, g(x), \tag{3.2}$$

are well defined G-equivariant maps, and the universal property of geometric factor implies that they are morphisms.

Lemma 3.3. (a) *The morphisms* γ_C *and* $\gamma_\mathfrak{c}$ *in* (3.2) *are birational* G*-isomorphisms.*

(b) *Any rational* G*-maps* $G \times^N C \dashrightarrow G \times^N \mathfrak{c}$ *and* $G \times^N \mathfrak{c} \dashrightarrow G \times^N C$ *are rational maps over* G/N*.*

Proof. (a) Since the Cartan subgroups of G are all conjugate and every element of a dense open set U in G belongs to a unique Cartan subgroup, [Bor, §12], every fiber $\gamma_C^{-1}(u)$, where $u \in U$, is a single point. As $\operatorname{char} k = 0$, this means that γ_C is a birational isomorphism. For $\gamma_\mathfrak{c}$ the arguments are analogous because $\mathfrak{c}$ is a Cartan subalgebra in $\mathfrak{g}$, Cartan subalgebras in $\mathfrak{g}$ are all $\mathrm{Ad}_G G$-conjugate and a general element of $\mathfrak{g}$ is contained in a unique Cartan subalgebra, [Bou$_3$, Ch. VII].

(b) Since a general element of T (respectively $\mathfrak{t}$) is regular, C (respectively $\mathfrak{c}$) is the unique Cartan subgroup (respectively subalgebra) containing T (respectively $\mathfrak{t}$), [Bor, §13], see [Bou$_3$, Ch. VII]. This implies that C and $\mathfrak{c}$ are the fixed point sets of the actions of T on $G \times^N C$ and $G \times^N \mathfrak{c}$ respectively. Since the maps under consideration are G-equivariant, this immediately implies the claim. $\square$

Remark 3.4. The group varieties of C and $\mathfrak{c}$ are the "standard relative sections" of respectively G and $\mathfrak{g}$ induced by the rational G-map $\pi_{G,N,C} \circ \gamma_C^{-1} : G \dashrightarrow G/N$ and

$\pi_{G,N,\mathfrak{c}} \circ \gamma_{\mathfrak{c}}^{-1} : \mathfrak{g} \dashrightarrow G/N$; in particular, this yields the following isomorphisms of invariant fields:

$$k(G)^G \xrightarrow{\simeq} k(C)^N, f \mapsto f|_C, \qquad k(\mathfrak{g})^G \xrightarrow{\simeq} k(\mathfrak{c})^N, f \mapsto f|_{\mathfrak{c}}; \tag{3.5}$$

see [Pop$_3$, Definition (1.7.6) and Theorem (1.7.5)].

Lemma 3.6. (a) *G is Cayley if and only if C and $\mathfrak{c}$ are birationally N-isomorphic.*
(b) *G is stably Cayley if and only if C and $\mathfrak{c}$ are stably birationally N-isomorphic.*

Proof. (a) By Lemma 2.22, the existence of a birational N-isomorphism $\varphi\colon C \overset{\simeq}{\dashrightarrow} \mathfrak{c}$ implies the existence of a birational G-isomorphism $\psi : G \times^N C \overset{\simeq}{\dashrightarrow} G \times^N \mathfrak{c}$. Then Lemma 3.3 shows that $\gamma_{\mathfrak{c}} \circ \psi \circ \gamma_C^{-1} : G \overset{\simeq}{\dashrightarrow} \mathfrak{g}$ is a Cayley map.

Conversely, let $\lambda : G \overset{\simeq}{\dashrightarrow} \mathfrak{g}$ be a Cayley map. Then $\psi := \gamma_{\mathfrak{c}}^{-1} \circ \lambda \circ \gamma_C : G \times^N C \overset{\simeq}{\dashrightarrow} G \times^N \mathfrak{c}$ is a birational G-isomorphism. By Lemma 3.3, ψ is a rational map over G/N. Hence, by Lemma 2.22, $\psi|_C : C \overset{\simeq}{\dashrightarrow} \mathfrak{c}$ is a birational N-isomorphism.

(b) If C and $\mathfrak{c}$ are stably birationally N-isomorphic, it follows from rationality of the underlying variety of any linear algebraic torus that for some natural d there exists a birational N-isomorphism

$$C \times \mathbf{G}_m^d \overset{\simeq}{\dashrightarrow} \mathfrak{c} \oplus k^d, \tag{3.7}$$

where k^d is the Lie algebra of $\mathbf{G}_m^d$ and N acts on $C \times \mathbf{G}_m^d$ and $\mathfrak{c} \oplus k^d$ via C and $\mathfrak{c}$ respectively. Clearly $C \times \mathbf{G}_m^d$ is the Cartan subgroup of $G \times \mathbf{G}_m^d$ with normalizer $N \times \mathbf{G}_m^d$ and Lie algebra $\mathfrak{c} \oplus k^d$, and the birational isomorphism (3.7) is $N \times \mathbf{G}_m^d$-equivariant. Now (a) implies that $G \times \mathbf{G}_m^d$ is Cayley and hence G is stably Cayley.

Conversely, assume that $G \times \mathbf{G}_m^d$ is Cayley for some d. Then the above arguments and (a) show that there exists a birational N-isomorphism (3.7). Since the group varieties of $\mathbf{G}_m^d$ and k^d are rational, this means that C and $\mathfrak{c}$ are stably birationally N-isomorphic. □

For reductive groups, Lemma 3.6 translates into the statement resulting also from [Lun$_1$, p. 13, Proposition]:

Corollary. *Let G be a connected reductive linear algebraic group.*

(a) *G is Cayley if and only if T and $\mathfrak{t}$ are birationally W-isomorphic.*
(b) *G is stably Cayley if and only if T and $\mathfrak{t}$ are stably birationally W-isomorphic.*

Proof. Since G is reductive, $C = T$ and $\mathfrak{c} = \mathfrak{t}$. As T is commutative, this implies that the actions of N on T and $\mathfrak{t}$ descend to the actions of W. The claim now follows from Lemma 3.6. □

3.8. Generic tori. We now recall the definition of generic tori in a form suitable for our purposes; see [Vos, 4.1] or [CK, p. 772]. We maintain the notation of Subsections 2.17, 3.1.

Assume that G is a connected reductive linear algebraic group; then $C = T$ and $\mathfrak{c} = \mathfrak{t}$. According to the discussion in the previous subsection, G/N may be interpreted in two ways: first, as the *variety of all maximal tori in G*, and second, as the *variety of all maximal tori in $\mathfrak{g}$*. The maximal torus in G (respectively, in $\mathfrak{g}$) assigned to a point $g(o) \in G/N$ is gTg^{-1} (respectively, $\mathrm{Ad}_G\, g(\mathfrak{t})$); it is naturally identified with the fiber over $g(o)$ of the morphism $\pi_{G,N,T} : G \times^N T \longrightarrow G/N$ (respectively, $\pi_{G,N,\mathfrak{t}} : G \times^N \mathfrak{t} \longrightarrow G/N$).

Definition 3.9. The triples

$$\mathbf{T}_G := \big(G \times^N T, \pi_{G,N,T}, G/N\big) \quad \text{and} \quad \mathfrak{t}_{\mathfrak{g}} := \big(G \times^N \mathfrak{t}, \pi_{G,N,\mathfrak{t}}, G/N\big)$$

are called respectively the *generic torus of* G and the *generic torus of* $\mathfrak{g}$.

We identify the field $k(G/N)$ with its image in $k(G \times^N T)$ under the embedding $\pi^*_{G,N,T}$.

Definition 3.10. The generic torus $\mathbf{T}_G$ is called *rational* if $k(G \times^N T)$ is a purely transcendental extension of $k(G/N)$. If $\mathbf{T}_{G\times \mathbf{G}_m^d}$ is rational for some d, then $\mathbf{T}_G$ is called *stably rational.*

Equivalently, $\mathbf{T}_G$ is called rational if there exists a birational isomorphism

$$G \times^N T \overset{\simeq}{\dashrightarrow} G/N \times \mathbf{A}^r \tag{3.11}$$

over G/N (then $r = \dim T$). The arguments used in the proof of Lemma 3.6 (b) show that stable rationality of $\mathbf{T}_G$ is equivalent to the property that there exists a purely transcendental field extension E of $k(G \times^N T)$ such that E is a purely transcendental extension of $k(G/N)$. There are groups G such that the generic torus $\mathbf{T}_G$ is not stably rational (and hence not rational), [Vos], [CK].

Of course, for the generic torus $\mathfrak{t}_{\mathfrak{g}}$ in $\mathfrak{g}$, one could also introduce the notions analogous to that in Definition 3.10. However in the Lie algebra context the rationality problem of generic tori is quite easy: since $\pi_{G,N,\mathfrak{t}} : G \times^N \mathfrak{t} \longrightarrow G/N$ is a vector bundle, it is locally trivial in the Zariski topology, and hence $\mathfrak{t}_{\mathfrak{g}}$ is always rational, i.e., there exists a birational isomorphism

$$G \times^N \mathfrak{t} \overset{\simeq}{\dashrightarrow} G/N \times \mathbf{A}^r \tag{3.12}$$

over G/N.

3.13. Proof of Theorem 1.31.

Implication (a): By the Corollary of Lemma 3.6, it is enough to construct a W-equivariant birational isomorphism $\varphi\colon T \overset{\simeq}{\dashrightarrow} \mathfrak{t}$.

Using the sign-permutation basis of $\widehat{T}$, we can W-equivariantly identify the maximal torus T with $\mathbf{G}_m^r$, where r is the rank of G and every $w \in W$ acts on $\mathbf{G}_m^r$ by

$$(t_1, \ldots, t_r) \mapsto (t_{\sigma(1)}^{\varepsilon_1}, \ldots, t_{\sigma(r)}^{\varepsilon_r}), \tag{3.14}$$

for some $\sigma \in \mathrm{S}_r$ and some $\varepsilon_1, \ldots, \varepsilon_r \in \{\pm 1\}$ (depending on w). The Lie algebra $\mathfrak{t}$ is the tangent space to $\mathbf{G}_m^r$ at $e = (1, \ldots, 1)$; it follows from (3.14) that we can identify it with k^r where w acts by

$$(x_1, \ldots, x_r) \mapsto (\varepsilon_1 x_{\sigma(1)}, \ldots, \varepsilon_r x_{\sigma(r)}). \tag{3.15}$$

From (3.14) and (3.15) we easily deduce that the formula

$$(t_1, \ldots, t_r) \mapsto \big((1-t_1)(1+t_1)^{-1}, \ldots, (1-t_r)(1+t_r)^{-1}\big)$$

defines a desired birational W-isomorphism $\varphi\colon T \overset{\simeq}{\dashrightarrow} \mathfrak{t}$. This completes the proof of implication (a).

To see that implication (a) cannot be reversed, consider the group $G := \mathbf{SL}_3$. First note that this group is Cayley; see Proposition 9.2. On the other hand, $W \simeq \mathrm{S}_3$ and since the character lattice $\mathcal{X}_G$ has rank 2, it can not be sign-permutation. Indeed, if it were, then S_3 would embed into $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathrm{S}_2$, which is impossible.

Implication (b): By the Corollary of Lemma 3.6, there is a birational N-isomorphism $T \overset{\simeq}{\dashrightarrow} \mathfrak{t}$. By Lemma 2.22, this implies that there is a birational G-isomorphism $G \times^N T \overset{\simeq}{\dashrightarrow} G \times^N \mathfrak{t}$ over G/N. Its composition with the birational isomorphism (3.12) is a birational isomorphism (3.11) over G/N. Hence $\mathbf{T}_G$ is rational.

To see that implication (b) cannot be reversed, consider the exceptional group $\mathbf{G}_2$. The generic torus of $\mathbf{G}_2$ is rational; see [Vos, 4.9]. On the other hand, $\mathbf{G}_2$ is not a Cayley group; see Proposition 9.11.

Implication (c): This is obvious from the definition.

Equivalence (d): This is well-known, see, e.g., [Vos, Theorem 4.7.2].

Equivalence (e): Let V be any finite dimensional faithful permutation W-module over k (for instance, the one determined by the regular representation of W). Then clearly $k(V) = k(P)$ for some permutation W-lattice P. Since the action of W on $\mathfrak{t}$ is faithful, [Bor], we deduce from Lemma 2.16(c) that $k(\mathfrak{t})$ and $k(P)$ are stably W-isomorphic over k. Therefore, since $k(T) = k(\widehat{T})$, applying the Corollary of Lemma 3.6 implies that G is stably Cayley if and only if $k(\widehat{T})$ and $k(P)$ are stably W-isomorphic over k. On the other hand, the latter property holds if and only if the W-lattice $\widehat{T}$ is quasi-permutation, see the Corollary of Lemma 2.7, whence the claim. $\square$

Example 3.16. The character lattice $\mathbb{Z}\mathsf{A}_{n-1}$ of $\mathbf{PGL}_n$ is defined by the exact sequence

$$0 \longrightarrow \mathbb{Z}\mathsf{A}_{n-1} \longrightarrow \mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where ϵ is the augmentation map and the Weyl group $W = \mathrm{S}_n$ acts trivially on $\mathbb{Z}$ and naturally on $\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}]$, see Subsection 6.1. Thus $\mathbb{Z}\mathsf{A}_{n-1}$ is quasi-permutation. By Theorem 1.31, we conclude that $\mathbf{PGL}_n$ is stably Cayley. We know that in fact $\mathbf{PGL}_n$ is even Cayley; see Example 1.13. $\square$

4. Reduction theorems

The purpose of this section is to show that to a certain extent classifying arbitrary Cayley groups is reduced to classifying simple ones.

As before, let G be a connected linear algebraic group. Denote by R and R_u respectively the radical and the unipotent radical of G. Recall that a *Levi subgroup* of G is a connected subgroup L, necessarily reductive, such that $G = L \ltimes R_u$; since $\operatorname{char} k = 0$, Levi subgroups exist and are conjugate, [Bor, 11.22].

In this section we will address the following questions:

(a) If a Levi subgroup of G is (stably) Cayley, is G (stably) Cayley?
(b) Let G be reductive. If G/R is (stably) Cayley, is G (stably) Cayley?
(c) Let G be reductive and let $H_1, \ldots, H_n$ be a complete list of its connected normal simple subgroups. What is the relation between (stable) Cayleyness of G and that of $H_1, \ldots, H_n$?

4.1. Unipotent normal subgroups. We will need a generalization of Example 1.23. Let U be a normal unipotent subgroup of G. Denote by $\mathfrak{u}$ the Lie algebra of U. The group G acts on U by conjugation and on $\mathfrak{u}$ by $\mathrm{Ad}_G|_{\mathfrak{u}}$.

Lemma 4.2. *There exists a G-isomorphism of G-varieties $U \to \mathfrak{u}$.*

Proof. We may assume without loss of generality that $G \subset \mathbf{GL}_n$. Since Ad_G is given by (1.2), it follows from (1.24) that $\ln\colon U \to \mathfrak{u}$ is a G-morphism. By Example 1.23, it is an isomorphism, whence the claim. $\square$

4.3. The Levi decomposition.

Proposition 4.4. *Let L be a Levi subgroup of G.*

(a) *If L is Cayley, then so is G.*
(b) *G is stably Cayley if and only if L is stably Cayley.*

Proof. Let T be a maximal torus of L. It is a maximal torus of G as well, [Bor, 11.20]. Using the notation of (1.26) and Subsection 3.1, we have $C = T \times U$ where U is a unipotent group, [Bor, 12.1]. Let $\mathfrak{u}$ be the Lie algebra of U and let $d = \dim U$. As T and U are respectively the semisimple and unipotent parts of the nilpotent group C, they are stable under the conjugating action of N, and C, as an N-variety, is the product of N-varieties T and U. Correspondingly, $\mathfrak{t}$ and $\mathfrak{u}$ are stable under the adjoint action of N, and $\mathfrak{c}$, as an N-variety, is the product of N-varieties $\mathfrak{t}$ and $\mathfrak{u}$. By Lemma 4.2, there exists an isomorphism of N-varieties

$$\tau\colon U \longrightarrow \mathfrak{u}. \tag{4.5}$$

(a) Assume that L is Cayley. Then by Corollary of Lemma 3.6, there is a birational $W_{L,T}$-isomorphism $\varphi\colon T \overset{\simeq}{\dashrightarrow} \mathfrak{t}$. Since the action of $W_{L,T}$ on T (respectively, $\mathfrak{t}$) is faithful, $W_{L,T}$ can be considered as a transformation group of T (respectively, $\mathfrak{t}$). By [Bor, 11.20], it coincides with the transformation group $\{T \to T,\ t \mapsto ntn^{-1} \mid n \in N\}$ (respectively, $\{\mathfrak{t} \to \mathfrak{t},\ x \mapsto \mathrm{Ad}_G n(x) \mid n \in N\}$). Therefore the map φ is N-equivariant. Hence

$$\varphi \times \tau\colon C = T \times U \dashrightarrow \mathfrak{t} \oplus \mathfrak{u} = \mathfrak{c}.$$

is a birational N-isomorphism. Lemma 3.6 now implies that G is Cayley.

(b) Since $L \times \mathbf{G}_m^d$ is the Levi subgroup of $G \times \mathbf{G}_m^d$, it follows from (a) that if L is stably Cayley, then G is stably Cayley.

To prove the converse, it suffices to show that if G is Cayley, then L is stably Cayley. In turn, Lemma 3.6 and its Corollary reduce this to proving that if there exists a birational N-isomorphism

$$\alpha\colon C = T \times U \overset{\simeq}{\dashrightarrow} \mathfrak{t} \times \mathfrak{u} = \mathfrak{c},$$

then T and $\mathfrak{t}$ are stably birationally $W_{L,T}$–isomorphic. We shall prove this last statement.

Since T is the identity component of $N_{L,T} = N \cap L$ and T acts trivially on C and $\mathfrak{c}$, the actions of $N_{L,T}$ on C, $\mathfrak{c}$, T, $\mathfrak{t}$, U and $\mathfrak{u}$ descend to actions of $W_{L,T} = N_{L,T}/T$. Moreover, C (respectively, $\mathfrak{c}$), as an $W_{L,T}$-variety, is the product of $W_{L,T}$-varieties T and U (respectively, $\mathfrak{t}$ and $\mathfrak{u}$), and α is a birational $W_{L,T}$-isomorphism.

Since $W_{L,T}$ acts linearly on $\mathfrak{u}$, Lemma 2.16(b) implies that there are birational $W_{L,T}$-isomorphisms

$$\beta\colon T \times \mathbf{A}^d \overset{\simeq}{\dashrightarrow} T \times \mathfrak{u} \quad \text{and} \quad \gamma\colon \mathfrak{t} \times \mathfrak{u} \overset{\simeq}{\dashrightarrow} \mathfrak{t} \times \mathbf{A}^d,$$

where $W_{L,T}$ acts on $T \times \mathbf{A}^d$ and $\mathfrak{t} \times \mathbf{A}^d$ via the first factors. Considering the composition of the following birational $W_{L,T}$-isomorphisms

$$T \times \mathbf{A}^d \overset{\beta}{\dashrightarrow} T \times \mathfrak{u} \xrightarrow{\mathrm{id} \times \tau^{-1}} T \times U \overset{\alpha}{\dashrightarrow} \mathfrak{t} \times \mathfrak{u} \overset{\gamma}{\dashrightarrow} \mathfrak{t} \times \mathbf{A}^d\ ,$$

we now see that T and $\mathfrak{t}$ are indeed stably birationally $W_{L,T}$-isomorphic. $\square$

Remark 4.6. The converse to Proposition 4.4(a) fails for $G := \mathbf{G}_2 \times \mathbf{G}_a^2$. Indeed, the first factor is the Levi subgroup of G. By Proposition 9.11, it is not Cayley. Consider the group $H := \mathbf{G}_2 \times \mathbf{G}_m^2$. Both G and H have the same Lie algebra $\mathfrak{g}$. By Proposition 9.13, H is Cayley; let $\lambda : H \overset{\simeq}{\dashrightarrow} \mathfrak{g}$ be a Cayley map. Fix a birational isomorphism of group varieties

$\delta : \mathbf{G}_a^2 \overset{\simeq}{\dashrightarrow} \mathbf{G}_m^2$. Since the second factors of G and H lie in the kernels of conjugating and adjoint actions, $\lambda \circ (\mathrm{id} \times \delta) \colon G \overset{\simeq}{\dashrightarrow} \mathfrak{g}$ is a Cayley map. Thus G is Cayley.

Corollary. *Every connected solvable linear algebraic group G is Cayley.*

Proof. A Levi subgroup L of G is a torus, [Bor, 10.6]. By Example 1.22, L is Cayley. Hence by Proposition 4.4(a), G is Cayley as well. $\square$

4.7. From reductive to semisimple.

Proposition 4.8. *Let G be a connected reductive group and let Z be a connected closed central subgroup of G.*

(a) *If G/Z is Cayley, then so is G.*
(b) *G is stably Cayley if and only if G/Z is stably Cayley.*

Proof. Since G is reductive, R is a torus and the identity component of the center of G, see [Bor, 11.21]. Thus Z is a subtorus of R. Let T be a maximal torus of G. We have $R \subset T$, see [Bor, 11.11], T/Z is a maximal torus of G/Z and the natural epimorphism $G \to G/Z$ identifies W with $W_{G/Z,T/Z}$ (we use the notation of (1.26) and Subsection 3.1), see [Bor, 11.20]. Since Z is central, it is pointwise fixed with respect to the action of W. Thus we have the following exact sequence of W-homomorphisms of tori

$$e \longrightarrow Z \longrightarrow T \longrightarrow T/Z \longrightarrow e$$

which in turn yields the exact sequence of W-lattices of character groups

$$0 \longrightarrow \widehat{T/Z} \longrightarrow \widehat{T} \longrightarrow \widehat{Z} \longrightarrow 0.$$

Note that W acts trivially on $\widehat{Z}$. In particular, $\widehat{Z}$ is a permutation W-lattice, and the last exact sequence tells us that the character lattices $\widehat{T}$ and $\widehat{T/Z}$ are equivalent, see Definition 2.4. Thus, by Lemma 2.7 and its Corollary, if one of them is quasi-permutation, then so is the other. Part (b) now follows from Theorem 1.31.

Since the W-fields $k(T)$ and $k(T/Z)$ are W-isomorphic to $k(\widehat{T})$ and $k(\widehat{T/Z})$ respectively, we deduce from Lemma 2.16(d) that T is birationally W-isomorphic to $T/Z \times \mathbf{A}^m$, where W acts on $T/Z \times \mathbf{A}^m$ via the first factor and $m = \dim Z$.

On the other hand, let $\mathfrak{f}$ and $\mathfrak{z}$ be the Lie algebras of T/Z and Z respectively. Then, since the Lie algebras $\mathfrak{t}$ and $\mathfrak{f} \oplus \mathfrak{z}$ are W-equivariantly isomorphic and W acts on $\mathfrak{z}$ trivially, we see that $\mathfrak{t}$, as a W-variety, is isomorphic to $\mathfrak{f} \times \mathbf{A}^m$, where W acts on $\mathfrak{f} \times \mathbf{A}^m$ via the first factor.

Now to prove part (a), assume that G/Z is Cayley. Then by the Corollary of Lemma 3.6, there is a birational W-isomorphism $\varphi : T/Z \overset{\simeq}{\dashrightarrow} \mathfrak{f}$. This gives a birational W-isomorphism $T/Z \times \mathbf{A}^m \overset{\varphi \times \mathrm{id}}{\dashrightarrow} \mathfrak{f} \times \mathbf{A}^m$. Applying the Corollary of Lemma 3.6 once again, we conclude that G is Cayley. This completes the proof of part (a). $\square$

Setting $Z = R$, we obtain

Corollary. *Let G be a connected reductive group and $G_{ss} := G/R$.*

(a) *If G_{ss} is Cayley, then so is G.*
(b) *G is stably Cayley if and only if G_{ss} is stably Cayley.* $\square$

Remark 4.9. The converse to statement (a) of Corollary fails for $G = \mathbf{G}_2 \times \mathbf{G}_m^2$. Indeed, G is Cayley by Proposition 9.13 and $G/R \simeq \mathbf{G}_2$ is not Cayley by Proposition 9.11.

4.10. From semisimple to simple. Let $G_1, \dots, G_n$ be connected linear algebraic groups and let $\mathfrak{g}_i$ be the Lie algebra of G_i. If each G_i is Cayley, then so is $G_1 \times \ldots \times G_n$, see Example 1.10. The converse fails for $n = 2$, $G_1 = \mathbf{G_2}$, $G_2 = \mathbf{G}_m^2$ by Proposition 9.11, Example 1.22 and Proposition 9.13.

Lemma 4.11. *$G_1 \times \ldots \times G_n$ is stably Cayley if and only if each G_i is stably Cayley.*

Proof. The "if" direction follows from Definition 1.29 and Example 1.10. To prove the converse, we use the fact that the underlying variety of each G_i is rational over k, see [Ch$_1$]. This implies that the underlying variety of $G_1 \times \ldots \times G_n$, as a G_i-variety, is birationally isomorphic to $G_i \times \mathbf{G}_m^{d_i}$ with the conjugating action via the first factor and $d_i = \sum_{j \neq i} \dim G_j$. The "only if" direction now follows from Definition 1.29 and the fact that the underlying variety of the Lie algebra of $G_1 \times \ldots \times G_n$, as G_i-variety, is isomorphic to $\mathfrak{g}_i \oplus k^{d_i}$ with the adjoint action via the first summand. $\square$

As usual, given subgroups X and Y of G, we denote by (X, Y) the subgroup generated by the commutators $xyx^{-1}y^{-1}$ with $x \in X$, $y \in Y$.

Proposition 4.12. *Assume G is a connected reductive group and let $H_1, \dots, H_m$ be the connected closed normal subgroups of G such that*

(i) $(H_i, H_j) = e$ *for all* $i \neq j$,
(ii) $G = H_1 \ldots H_m$.

Let $\widetilde{H}_i$ be the subgroup of G generated by all H_j's with $j \neq i$. If G is stably Cayley, then each $G/\widetilde{H}_i \simeq H_i/(H_i \cap \widetilde{H}_i)$ is stably Cayley.

Proof. Since $H_1, \dots, H_m$ are connected, each $\widetilde{H}_i$ is connected, see [Bor, 2.2]. Since G is reductive, all H_i and $\widetilde{H}_i$ are reductive.

It follows from (i) and (ii) that

$$H_1 \times \ldots \times H_m \to G, \quad (h_1, \dots, h_m) \mapsto h_1 \ldots h_m,$$

is an epimorphism of algebraic groups. Let T_i be a maximal torus of H_i. Then $T_1 \times \ldots \times T_m$ is a maximal torus of $H_1 \times \ldots \times H_m$. Therefore its image $T := T_1 \ldots T_m$ under the above epimorphism is a maximal torus of G, see [Bor, 11.14]. The same argument shows that the group S_i of T generated by all T_j's with $j \neq i$ is a maximal torus of $\widetilde{H}_i$.

It follows from (i) that $N_i := N_{H_i, T_i}$ is a subgroup of $N = N_{G,T}$ and S_i is pointwise fixed under the conjugating action of N_i on T. Since the subgroup T_i of N_i acts on T trivially, this action descends to an action of $W_i := W_{H_i, T_i} = N_i/T_i$. Since H_i is connected reductive, any maximal torus of H_i coincides with its centralizer in H_i, see [Bor, 13.17]. As T_i is such a torus, this yields the equality $T \cap H_i = T_i$. It shows that W_i, considered as a transformation group of T, is the image of N_i under the natural projection $N \to N/T = W$.

Let $\pi_i : H_i \to H_i/(H_i \cap \widetilde{H}_i)$ be the natural epimorphism. Then $\pi_i(T_i)$ is a maximal torus of $H_i/(H_i \cap \widetilde{H}_i)$. It follows from (i) and [Bor, 11.20, 11.11] that π_i identifies W_i with $W_{H_i/(H_i \cap \widetilde{H}_i), \pi_i(T_i)}$, so that the natural isomorphism $T_i/(T_i \cap \widetilde{H}_i) \to \pi_i(T_i)$ is W_i-equivariant. The above argument applied to $\widetilde{H}_i$ and S_i instead of H_i and T_i shows that $T \cap \widetilde{H}_i = S_i$. This, in turn, implies that

$$T_i \cap \widetilde{H}_i = T_i \cap S_i.$$

Thus a maximal torus of $H_i/(H_i \cap \widetilde{H}_i)$ is W_i-isomorphic to $T_i/(T_i \cap S_i)$. In turn, $T_i/(T_i \cap S_i)$ is W_i-isomorphic to T/S_i because $T = T_i S_i$. Therefore there is an exact sequence of W_i-homomorphisms of tori

$$e \longrightarrow S_i \longrightarrow T \longrightarrow T_i/(T_i \cap S_i) \longrightarrow e.$$

Passing to the character groups, we deduce from it the following exact sequence of W_i-lattices

$$0 \longrightarrow T_i/\widehat{(T_i \cap S_i)} \longrightarrow \widehat{T} \longrightarrow \widehat{S}_i \longrightarrow 0.$$

As the action of W_i on S_i is trivial, $\widehat{S}_i$ is a trivial and, in particular, a permutation W_i-lattice. Hence the above exact sequence shows that $T_i/\widehat{(T_i \cap S_i)}$ and $\widehat{T}$ are equivalent W_i-lattices.

Assume now that G is stably Cayley. Then Theorem 1.31 implies that $\widehat{T}$ is quasi-permutation as a W-lattice, and hence as a W_i-lattice because W_i is a subgroup of W. Therefore the equivalent W_i-lattice $T_i/\widehat{(S_i \cap T_i)}$ is quasi-permutation as well. Since the latter is the character lattice of $H_i/(H_i \cap \widetilde{H}_i)$, Theorem 1.31 implies that $H_i/(H_i \cap \widetilde{H}_i)$ is stably Cayley. □

Corollary. *Let G be a connected semisimple group. Let $H_1, \ldots, H_m$ be the minimal elements among its connected closed normal subgroups. Define $\widetilde{H}_i$ as in Proposition* 4.12. *If G is stably Cayley, then each $H_i/(H_i \cap \widetilde{H}_i)$ is stably Cayley.*

Proof. By [Bor, 14.10], the assumptions of Proposition 4.12 hold. □

Remark 4.13. In Proposition 4.12, if G is stably Cayley, H_i is not necessarily stably Cayley. For example, take $G = \mathbf{GL}_n$, $m = 2$, $H_1 = \mathbf{G}_m$ diagonally embedded in $\mathbf{GL}_n$ and $H_2 = \mathbf{SL}_n$. Then G is Cayley by Example 1.11, and H_2 is not stably Cayley for $n > 3$ by Theorem 1.32.

5. **Proof of Theorem 1.32: an overview**

In this section we outline a strategy for proving Theorem 1.32; the technical parts of the proof will be carried out in Sections 6–8.

By Theorem 1.31, it will suffice to determine which connected simple groups have a stably rational generic torus (or, equivalently, a quasi-permutation character lattice). Cortella and Kunyavskiĭ in [CK, Theorem 0.1] have classified all simply connected and all adjoint connected simple groups that have quasi-permutation character lattice. These are precisely $\mathbf{SO}_{2n+1}$, $\mathbf{Sp}_{2n}$, $\mathbf{PGL}_n$, $\mathbf{SL}_3$, and $\mathbf{G}_2$. Therefore in order to complete the proof of Theorem 1.32, we need to determine which intermediate (i.e., neither simply connected nor adjoint) connected simple groups have a quasi-permutation character lattice.

Recall that intermediate connected simple groups exist only for types A_n and D_n. Connected simple groups of type A_{n-1} are precisely the groups $\mathbf{SL}_n/\boldsymbol{\mu}_d$, where d is a divisor of n. Among them, intermediate groups are those with $1 < d < n$. In Section 7 we will prove the following.

Proposition 5.1. *Let d be a divisor of n, where $1 < d < n$ and $(n, d) \neq (4, 2)$. Then the character lattice of the group $\mathbf{SL}_n/\boldsymbol{\mu}_d$ is not quasi-permutation.*

As we saw in Example 1.18, the group $\mathbf{SL}_4/\boldsymbol{\mu}_2$ is Cayley; in particular, by Theorem 1.31, its character lattice is quasi-permutation.

The intermediate connected simple groups of type D_n are $\mathbf{SO}_{2n}$ for any $n \geqslant 3$ and the half-spinor groups $\mathbf{Spin}_{2n}^{1/2}$ for even $n \geqslant 4$. The latter are defined as follows. Consider the spinor group $\mathbf{Spin}_{2n}$ for even $n \geqslant 4$. Its the center is isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, see [Ch$_2$], [KMRT, §25], and consequently contains precisely three subgroups of order 2. One of them is the kernel of the vector representation, so the quotient of $\mathbf{Spin}_{2n}$ modulo it is $\mathbf{SO}_{2n}$. Two others are the kernels of the half-spinor representations of $\mathbf{Spin}_{2n}$. They are mapped to each other by an outer automorphism of $\mathbf{Spin}_{2n}$, so the images of the half-spin representations are isomorphic to the same group that is $\mathbf{Spin}_{2n}^{1/2}$.

By Example 1.18, the groups $\mathbf{SO}_{2n}$ are Cayley. If $n = 4$, the group of outer automorphisms of $\mathbf{Spin}_{2n}$ is isomorphic to S_3 (for $n > 4$, it is isomorphic to S_2) and acts transitively on the set of all subgroups of order 2 of the center of $\mathbf{Spin}_{2n}$. Therefore $\mathbf{Spin}_8^{1/2} \simeq \mathbf{SO}_8$, whence it is Cayley. Thus we only need to consider the half-spin groups $\mathbf{Spin}_{2n}^{1/2}$ for even $n > 4$. In Section 8 we will prove the following.

Proposition 5.2. *The character lattice of the half-spinor group* $\mathbf{Spin}_{2n}^{1/2}$ *for even* $n > 4$ *is not quasi-permutation.*

Thus in order to complete the proof of Theorem 1.32, we need to prove Propositions 5.1 and 5.2. This will be done in the next three sections.

6. The groups $\mathbf{SL}_n/\boldsymbol{\mu}_d$ and their character lattices

6.1. Lattices $Q_n(d)$. For any divisor d of n, the Weyl group W of the group $G = \mathbf{SL}_n/\boldsymbol{\mu}_d$ is isomorphic to the permutation group S_n of the set of integers $\{1, \ldots, n\}$. The character lattice $\mathcal{X}_G$ is described as follows.

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the standard basis for the permutation S_n-lattice $\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}]$ on which $\sigma \in \mathrm{S}_n$ acts via

$$\sigma(\varepsilon_i) = \varepsilon_{\sigma(i)} \quad \text{for all } i = 1, \ldots, n. \tag{6.2}$$

We naturally embed $\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}]$ into the $\mathbb{Q}$-vector space $\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}] \otimes_{\mathbb{Z}} \mathbb{Q}$ endowed with the Euclidean structure such that $\varepsilon_1, \ldots, \varepsilon_n$ is the orthonormal basis and we naturally extend the action of S_n to this space.

The root system of type A_{n-1} is the subset

$$\mathsf{A}_{n-1} := \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i \neq j \leqslant n\}.$$

of $\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}] \otimes_{\mathbb{Z}} \mathbb{Q}$. The Weyl group $W(\mathsf{A}_{n-1})$ of A_{n-1} is S_n acting by (6.2), and the standard base of A_{n-1} is $\alpha_1, \ldots, \alpha_{n-1}$, where

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, \ldots, n-1, \tag{6.3}$$

see [Bou$_2$]. The kernel of augmentation map

$$\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}] \xrightarrow{\epsilon} \mathbb{Z}, \quad \textstyle\sum_{i=1}^n a_i\varepsilon_i \mapsto \sum_{i=1}^n a_i,$$

is the root S_n-lattice $\mathbb{Z}\mathsf{A}_{n-1}$ of A_{n-1},

$$\mathbb{Z}\mathsf{A}_{n-1} := \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{n-1} = \left\{\textstyle\sum_{i=1}^n a_i\varepsilon_i \mid \sum_{i=1}^n a_i = 0\right\}. \tag{6.4}$$

The character lattice of $\mathbf{SL}_n/\boldsymbol{\mu}_d$ is isomorphic to the following S_n-lattice

$$Q_n(d) := \mathbb{Z}\mathsf{A}_{n-1} + \mathbb{Z}d\varpi_1, \text{ where } \varpi_1 = \varepsilon_1 - \tfrac{1}{n}\textstyle\sum_{i=1}^n \varepsilon_i. \tag{6.5}$$

The vector ϖ_1 is the first fundamental dominant weight of the root system A_{n-1} with respect to the base $\alpha_1, \ldots, \alpha_{n-1}$.

Observe that the character lattice of $\mathbf{SL}_n/\boldsymbol{\mu}_n = \mathbf{PGL}_n$ is the root S_n-lattice $Q_n(n) = \mathbb{Z}\mathsf{A}_{n-1}$, the character lattice of $\mathbf{SL}_n/\boldsymbol{\mu}_1 = \mathbf{SL}_n$ is the weight S_n-lattice Λ_n of type A_{n-1}, and that the following sequences of homomorphisms of S_n-lattices are exact:

$$0 \longrightarrow \mathbb{Z}\mathsf{A}_{n-1} \longrightarrow Q_n\,(n/d) \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0, \tag{6.6}$$

$$0 \longrightarrow Q_n(d) \longrightarrow \Lambda_n \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0. \tag{6.7}$$

Here $\mathbb{Z}/d\mathbb{Z}$ denotes the cyclic group of order d with trivial S_d-action. Note that

$$Q_n(d)^* \simeq Q_n(n/d). \tag{6.8}$$

In this section we will prove a number of preliminary results about the lattices $Q_n(d)$. In the next section we will use these results to prove Proposition 5.1.

6.9. Properties of $\boldsymbol{Q_n(d)}$. We begin by recalling a simple lemma which computes the cohomology $\mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1})$ for all subgroups Γ of S_n. The first part is extracted from [LL, Lemma 4.3].

Lemma 6.10. *For any subgroup Γ of S_n, we have*

$$\mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}) \simeq \mathbb{Z}/\textstyle\sum_{\mathcal{O}} |\mathcal{O}|\mathbb{Z},$$

where $\mathcal{O}$ runs over the orbits of Γ in $\{1, \ldots, n\}$. More explicitly, the connecting homomorphism of the cohomology sequence induced by the augmentation sequence

$$0 \longrightarrow \mathbb{Z}\mathsf{A}_{n-1} \longrightarrow \mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}] \xrightarrow{\ \epsilon\ } \mathbb{Z} \longrightarrow 0, \tag{6.11}$$

is given by

$$\mathbb{Z} = \mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}]/\mathbb{Z}\mathsf{A}_{n-1} \xrightarrow{\ \partial\ } \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}), \quad m\varepsilon_1 + \mathbb{Z}\mathsf{A}_{n-1} \mapsto [\sigma \mapsto m(\varepsilon_{\sigma(1)} - \varepsilon_1)],$$

where the image is the class of the given 1-cocycle from Γ to $\mathbb{Z}\mathsf{A}_{n-1}$.

Proof. From the cohomology sequence that is associated with (6.11), one obtains the exact sequence $\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}]^\Gamma \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\partial} \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}) \to 0$ which implies the asserted description of $\mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1})$. The calculation of the connecting homomorphism ∂ follows directly from the identification of $\mathbb{Z}$ with $\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}]/\mathbb{Z}\mathsf{A}_{n-1}$ and an application of the Snake Lemma. □

Lemma 6.12. *For any subgroup Γ of S_n, the exact sequence* (6.6) *induces the following connecting homomorphism in cohomology:*

$$\mathbb{Z}/d\mathbb{Z} = Q_n(n/d)/\mathbb{Z}\mathsf{A}_{n-1} \xrightarrow{\ \partial\ } \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}), \quad m + d\mathbb{Z} \mapsto \tfrac{mn}{d} + \textstyle\sum_{\mathcal{O}} |\mathcal{O}|\mathbb{Z},$$

where the sum on the right runs over the orbits $\mathcal{O}$ of Γ in $\{1, \ldots, n\}$. In particular, if $|\mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1})|$ divides n/d, then ∂ is the zero map.

Proof. Since $Q_n(n/d)$ has $\mathbb{Z}$-basis $\frac{n}{d}\varpi_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-2} - \varepsilon_{n-1}$ where ϖ_1 is given by (6.5), we conclude that $Q_n(n/d)/\mathbb{Z}\mathsf{A}_{n-1}$ is generated by $\frac{n}{d}\varpi_1 + \mathbb{Z}\mathsf{A}_{n-1}$. Using the Snake Lemma, one sees that the connecting homomorphism $\mathbb{Z}/d\mathbb{Z} = Q_n(n/d)/\mathbb{Z}\mathsf{A}_{n-1} \xrightarrow{\partial} \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1})$ sends $\frac{n}{d}\varpi_1 + \mathbb{Z}\mathsf{A}_{n-1}$ to the class of the 1-cocycle $[\sigma \mapsto \frac{n}{d}(\varepsilon_{\sigma(1)} - \varepsilon_1)]$ in $\mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1})$. An application of Lemma 6.10 and the identification $\mathbb{Z}/d\mathbb{Z} = Q_n(n/d)/\mathbb{Z}\mathsf{A}_{n-1}$ completes the proof of the first statement. The second statement follows directly from the first. □

Lemma 6.13. *Let Γ be a subgroup of S_n which fixes at least one integer $i \in \{1, \ldots, n\}$. Then $\mathrm{H}^1(\Gamma, Q_n(d)) = 0$.*

Proof. Note that in this case, $\{\varepsilon_t-\varepsilon_i \mid t\neq i\}$ is a permutation basis for $\mathbb{Z}\mathsf{A}_{n-1}$ so that both $\mathbb{Z}\mathsf{A}_{n-1}$ and $\Lambda_n=(\mathbb{Z}\mathsf{A}_{n-1})^*$ are permutation Γ-lattices. This implies that $\mathrm{H}^1(\Gamma,\mathbb{Z}\mathsf{A}_{n-1})=0=\mathrm{H}^1(\Gamma,\Lambda_n)$. Observe that $\nu_i=\varepsilon_i-\frac{1}{n}\sum_{t=1}^n\varepsilon_t\in\Lambda_n^\Gamma$ and that $\nu_i+Q_n(d)=\varpi_1+Q_n(d)$ since $\nu_i-\varpi_1=\varepsilon_i-\varepsilon_1\in\mathbb{Z}\mathsf{A}_{n-1}\subseteq Q_n(d)$. Then applying cohomology to the exact sequence (6.7), we obtain

$$\Lambda_n^\Gamma\longrightarrow\mathbb{Z}/d\mathbb{Z}\longrightarrow\mathrm{H}^1(\Gamma,Q_n(d))\longrightarrow\mathrm{H}^1(\Gamma,\Lambda_n)=0.$$

Since $\Lambda_n/Q_n(d)=\mathbb{Z}/d\mathbb{Z}$ is generated by $\varpi_1+Q_n(d)$, the above argument shows that the map $\Lambda_n^\Gamma\to\mathbb{Z}/d\mathbb{Z}$ is surjective so that $\mathrm{H}^1(\Gamma,Q_n(d))=0$, as required. $\square$

For a sequence of integers $1\leqslant i_1<\ldots<i_r\leqslant n$, set

$$\mathrm{S}_{\{i_1,\ldots,i_r\}}:=\big\{\sigma\in\mathrm{S}_n\mid\sigma(j)=j\ \text{ for every }\ j\notin\{i_1,\ldots,i_r\}\big\}.$$

This is a subgroup of S_n; in particular, $\mathrm{S}_{\{1,\ldots,n\}}=\mathrm{S}_n$. The map

$$\iota_{\{i_1,\ldots,i_r\}}:\ \mathrm{S}_r\longrightarrow\mathrm{S}_{\{i_1,\ldots,i_r\}},\quad \iota_{\{i_1,\ldots,i_r\}}(\sigma)(i_s)=i_{\sigma(s)}\quad\text{for all }\sigma\text{ and }s,$$

is an isomorphism. In the sequel, the subgroup $\mathrm{S}_{\{1,\ldots,m\}}\times\mathrm{S}_{\{m+1,\ldots,2m\}}$ of S_{2m} is denoted simply by $\mathrm{S}_m\times\mathrm{S}_m$. For a sequence of integers

$$1\leqslant i_1<\ldots<i_r<j_1<\ldots<j_r<\ldots<l_1<\ldots<l_r\leqslant n,$$

the image of the embedding

$$\mathrm{S}_r\longrightarrow\mathrm{S}_n,\quad\sigma\mapsto\iota_{\{i_1,\ldots,i_r\}}(\sigma)\iota_{\{j_1,\ldots,j_r\}}(\sigma)\ldots\iota_{\{l_1,\ldots,l_r\}}(\sigma),$$

is called the *copy of* S_r *diagonally embedded in* $\mathrm{S}_{\{i_1,\ldots,i_r,j_1,\ldots,j_r,\ldots,l_1,\ldots,l_r\}}$.

Lemma 6.14. *Let $n=td$. Then the following properties hold:*

(a) *Let X_d be the copy of S_d diagonally embedded in S_n. Then*

$$\mathbb{Z}\mathsf{A}_{n-1}|_{X_d}\simeq\mathbb{Z}\mathsf{A}_{d-1}\oplus\mathbb{Z}[\mathrm{S}_d/\mathrm{S}_{d-1}]^{t-1}.$$

(b) *Let $Y_d:=\mathrm{S}_{\{1,\ldots,d\}}\times\widetilde{X}_d$ where $\widetilde{X}_d$ is the copy of S_d diagonally embedded in $\mathrm{S}_{\{d+1,\ldots,n\}}$. Then*

$$\mathbb{Z}\mathsf{A}_{n-1}|_{Y_d}\simeq\mathbb{Z}\mathsf{A}_{2d-1}|_{\mathrm{S}_d\times\mathrm{S}_d}\oplus\mathbb{Z}[(\mathrm{S}_d\times\mathrm{S}_d)/(\mathrm{S}_d\times\mathrm{S}_{d-1})]^{t-2}.$$

Proof. For the first statement, note that

$$\{\varepsilon_i-\varepsilon_{d+i}\mid i=1,\ldots,(t-1)d\}\cup\{\varepsilon_1-\varepsilon_2,\ldots,\varepsilon_{d-1}-\varepsilon_d\}$$

is a basis for $\mathbb{Z}\mathsf{A}_{n-1}$, since $\{\alpha_i=\varepsilon_i-\varepsilon_{i+1}\mid i=1,\ldots,n-1\}$ is a basis for $\mathbb{Z}\mathsf{A}_{n-1}$ and

$$\varepsilon_i-\varepsilon_{d+i}=\textstyle\sum_{t=i}^{d+i-1}\alpha_k$$

for $i=1,\ldots,(t-1)d$. But then

$$\mathbb{Z}\mathsf{A}_{n-1}|_{X_d}=\textstyle\sum_{r=1}^{t-1}\big(\sum_{i=(r-1)d+1}^{rd}\mathbb{Z}(\varepsilon_i-\varepsilon_{d+i})\big)\oplus\sum_{i=1}^{d-1}\mathbb{Z}(\varepsilon_i-\varepsilon_{i+1})\simeq\mathbb{Z}[\mathrm{S}_d/\mathrm{S}_{d-1}]^{t-1}\oplus\mathbb{Z}\mathsf{A}_{d-1}.$$

For the second statement, similarly note that

$$\{\varepsilon_i-\varepsilon_{d+i}\mid i=d+1,\ldots,(t-1)d\}\cup\{\varepsilon_1-\varepsilon_2,\ldots,\varepsilon_{2d-1}-\varepsilon_{2d}\}$$

is a basis for $\mathbb{Z}\mathsf{A}_{n-1}$ so that

$$\begin{aligned}\mathbb{Z}\mathsf{A}_{n-1}|_{Y_d}&=\textstyle\sum_{r=2}^{t-1}\big(\sum_{i=(r-1)d+1}^{rd}\mathbb{Z}(\varepsilon_i-\varepsilon_{d+i})\big)\oplus\sum_{i=1}^{2d-1}\mathbb{Z}(\varepsilon_i-\varepsilon_{i+1})\\&\simeq\mathbb{Z}[(\mathrm{S}_d\times\mathrm{S}_d)/(\mathrm{S}_d\times\mathrm{S}_{d-1})]^{t-2}\oplus\mathbb{Z}\mathsf{A}_{2d-1}|_{\mathrm{S}_d\times\mathrm{S}_d}.\end{aligned}\qquad\square$$

7. Stably Cayley groups of type A_n

7.1. Restricting $Q_n(d)$ to some subgroups. In this section we will prove Proposition 5.1. We will first show that $Q_n(d)$ restricted to certain appropriate subgroups of S_n is equivalent in each case to a smaller more manageable sublattice. We will then show that the smaller lattices are not quasi-permutation.

Proposition 7.2. *Suppose $d|n$ and let p be a prime divisor of n/d. Let X_p be the copy of S_p diagonally embedded in S_n, and let $Y_p = \mathrm{S}_{\{1,\dots,p\}} \times \widetilde{X}_p$, where $\widetilde{X}_p$ is the copy of S_p diagonally embedded in $\mathrm{S}_{\{p+1,\dots,n\}}$. Then the following equivalencies hold:*

(a) $Q_n(d)|_{X_p} \sim \Lambda_p$.
(b) $Q_n(d)|_{Y_p} \sim \Lambda_{2p}|_{\mathrm{S}_p\times\mathrm{S}_p}$.

Proof. Recall that we have the exact sequence (6.6). The definition of p implies that $n = lp$ for a positive integer l. By Lemma 6.14,

$$\mathbb{Z}\mathsf{A}_{n-1}|_{X_p} \simeq \mathbb{Z}\mathsf{A}_{p-1} \oplus \mathbb{Z}[\mathrm{S}_p/\mathrm{S}_{p-1}]^{l-1},$$
$$\mathbb{Z}\mathsf{A}_{n-1}|_{Y_p} \simeq \mathbb{Z}\mathsf{A}_{2p-1}|_{\mathrm{S}_p\times\mathrm{S}_p} \oplus \mathbb{Z}[(\mathrm{S}_p \times \mathrm{S}_p)/(\mathrm{S}_p \times \mathrm{S}_{p-1})]^{l-2}.$$

Using this and Lemma 6.10, we see that $\mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}) = \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{p-1}) = 0$ or $\mathbb{Z}/p\mathbb{Z}$ for all subgroups Γ of X_p and that $\mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}) = \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{2p-1}) = 0$ or $\mathbb{Z}/p\mathbb{Z}$ for all subgroups Γ of Y_p. Then by Lemma 6.12 and the fact that p divides n/d shows that the connecting homomorphism $(\mathbb{Z}/d\mathbb{Z})^\Gamma \to \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1})$ is zero for all subgroups Γ of X_p or of Y_p. But then the sequence above restricted to X_p or Y_p satisfies the conditions of Proposition 2.13(b). This means that

$$Q_n(d)|_{X_p} = Q_n(n/d)^*|_{X_p} \sim (\mathbb{Z}\mathsf{A}_{n-1})^*|_{X_p} \sim (\mathbb{Z}\mathsf{A}_{p-1})^* = \Lambda_p,$$
$$Q_n(d)|_{Y_p} = Q_n(n/d)^*|_{Y_p} \sim (\mathbb{Z}\mathsf{A}_{n-1})^*|_{Y_p} \sim (\mathbb{Z}\mathsf{A}_{2p-1})^*|_{\mathrm{S}_p\times\mathrm{S}_p} = \Lambda_{2p}|_{\mathrm{S}_p\times\mathrm{S}_p}. \quad \square$$

7.3. Lattices Λ_p and Λ_{2p}. The following lemma is essentially a rephrasing of a result proved by Bessenrodt and Le Bruyn in [BLB]:

Lemma 7.4. *Let $p > 3$ be prime. Then Λ_p is not a quasi-permutation S_p-lattice.*

Proof. Tensoring the augmentation sequence for $\mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}]$ with $\mathbb{Z}\mathsf{A}_{n-1}$, we obtain the exact sequence

$$0 \longrightarrow (\mathbb{Z}\mathsf{A}_{n-1})^{\otimes 2} \longrightarrow \mathbb{Z}\mathsf{A}_{n-1} \otimes \mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}] \xrightarrow{\tau} \mathbb{Z}\mathsf{A}_{n-1} \longrightarrow 0. \tag{7.5}$$

We have

$$\mathbb{Z}\mathsf{A}_{n-1} \otimes \mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}] \simeq \mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-2}].$$

One can show that $\{(\varepsilon_i - \varepsilon_j) \otimes \varepsilon_j \mid i \neq j\}$ is the set of elements of a permutation basis for $\mathbb{Z}\mathsf{A}_{n-1} \otimes \mathbb{Z}[\mathrm{S}_n/\mathrm{S}_{n-1}]$. The map τ then sends $(\varepsilon_i - \varepsilon_j) \otimes \varepsilon_j$ to $\varepsilon_i - \varepsilon_j$.

For p prime, Bessenrodt and Le Bruyn in [BLB] show that

$$0 \longrightarrow (\mathbb{Z}\mathsf{A}_{p-1})^{\otimes 2} \longrightarrow \mathbb{Z}[\mathrm{S}_p/\mathrm{S}_{p-2}] \longrightarrow \mathbb{Z}\mathsf{A}_{p-1} \longrightarrow 0$$

is a coflasque resolution of $\mathbb{Z}\mathsf{A}_{p-1}$ as an S_p-lattice. They also show that $(\mathbb{Z}\mathsf{A}_{p-1})^{\otimes 2}$ is permutation projective as an S_p-lattice but is only S_p-stably permutation if $p = 2, 3$. By duality, the stable equivalence class of $\big((\mathbb{Z}\mathsf{A}_{p-1})^{\otimes 2}\big)^*$ is $\rho(\Lambda_p)$, see Subsection 2.8). The statements above then imply that Λ_p is not a quasi-permutation S_p-lattice for any $p > 3$.
$\square$

Proposition 7.6. *Let p be a prime and let*

$$\Gamma := \langle (1,\dots,p),(p+1,\dots,2p)\rangle \leqslant \mathrm{S}_p\times\mathrm{S}_p \leqslant \mathrm{S}_{2p}.$$

Then

(a) $Ш^2(\Gamma,\Lambda_{2p})=0$. *In particular, a lattice in the stable equivalence class $\rho(\Lambda_{2p})$ is coflasque as an Γ-lattice.*
(b) *If p is odd, Λ_{2p} is not quasi-permutation as an Γ-lattice and hence is not quasi-permutation as an $\mathrm{S}_p\times\mathrm{S}_p$-lattice.*

Proof. (a) The second statement follows from the first. Note that any proper subgroup of Γ is cyclic, so that by the claim $Ш^2(S,\Lambda_{2p})=0$ for all subgroups S of Γ. Then if

$$0\longrightarrow \Lambda_{2p}\longrightarrow Q\longrightarrow M\longrightarrow 0$$

is a flasque resolution of Λ_{2p} considered as an S-lattice, then $\mathrm{H}^1(S,M)=Ш^2(S,\Lambda_{2p})=0$ by Lemma 2.12.

To prove the first statement, we need to first compute $\mathrm{H}^1(\Gamma,\Lambda_{2p})$ and $\mathrm{H}^2(\Gamma,\Lambda_{2p})$.

We have $\mathrm{H}^1(\Gamma,\Lambda_{2p})=\mathrm{H}^{-1}(\Gamma,\mathbb{Z}\mathsf{A}_{2p-1})$ by duality. Then

$$\mathrm{H}^{-1}(\Gamma,\mathbb{Z}\mathsf{A}_{2p-1})=\mathrm{Ker}_{\mathbb{Z}\mathsf{A}_{2p-1}}(N_\Gamma)/I_\Gamma\mathbb{Z}\mathsf{A}_{2p-1},$$

where N_Γ is the endomorphism $l\mapsto\sum_{a\in\Gamma}al$, and I_Γ is the augmentation ideal of $\mathbb{Z}[\Gamma]$, [Br]. We need to compute N_Γ on a basis for $\mathbb{Z}\mathsf{A}_{2p-1}$: we have $N_\Gamma(\varepsilon_i-\varepsilon_{i+1})=0$ for $i=1,\dots,p-1,p+1,\dots,2p-1$, and $N_\Gamma(\varepsilon_p-\varepsilon_{p+1})=p(\varepsilon_1+\dots+\varepsilon_p-\varepsilon_{p+1}-\dots-\varepsilon_{2p})$. Then

$$\operatorname{Ker}N_\Gamma=\operatorname{Span}\{\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{p-1}-\varepsilon_p,\varepsilon_{p+1}-\varepsilon_{p+2},\dots,\varepsilon_{2p-1}-\varepsilon_{2p}\}.$$

But $I_\Gamma\mathbb{Z}\mathsf{A}_{2p-1}=\operatorname{Ker}N_\Gamma$ as $((1,\dots,p)-\mathrm{id})(\varepsilon_{p+1}-\varepsilon_i)=\varepsilon_i-\varepsilon_{i+1}$, $i=1,\dots,p-1$, $((p+1,\dots,2p)-\mathrm{id})(\varepsilon_1-\varepsilon_i)=\varepsilon_i-\varepsilon_{i+1}$, $i=p+1,\dots,2p-1$. This shows that $\mathrm{H}^1(\Gamma,\Lambda_{2p})=\mathrm{H}^{-1}(\Gamma,\mathbb{Z}\mathsf{A}_{2p-1})=0$.

To determine $\mathrm{H}^2(\Gamma,\Lambda_{2p})$, we use the restriction of the sequence

$$0\longrightarrow\mathbb{Z}\longrightarrow\mathbb{Z}[\mathrm{S}_{2p}/\mathrm{S}_{2p-1}]\longrightarrow\Lambda_{2p}\longrightarrow 0$$

to Γ. Let

$$C_1=\langle(1,\dots,p)\rangle,\ \ C_2=\langle(p+1,\dots,2p)\rangle\quad\text{and}\quad P_1=\mathbb{Z}[\Gamma/C_2],\ \ P_2=\mathbb{Z}[\Gamma/C_1]. \tag{7.7}$$

Then we have the following exact sequence of Γ-lattices

$$0\longrightarrow\mathbb{Z}\longrightarrow P_1\oplus P_2\longrightarrow\Lambda_{2p}\longrightarrow 0.$$

Taking cohomology of this sequence, we get

$$\begin{aligned}0=\mathrm{H}^1(\Gamma,\Lambda_{2p})\longrightarrow\mathrm{H}^2(\Gamma,\mathbb{Z})&\longrightarrow\mathrm{H}^2(\Gamma,P_1)\oplus\mathrm{H}^2(\Gamma,P_2)\\&\longrightarrow\mathrm{H}^2(\Gamma,\Lambda_{2p})\longrightarrow\mathrm{H}^3(\Gamma,\mathbb{Z})\longrightarrow\mathrm{H}^3(\Gamma,P_1\oplus P_2).\end{aligned}$$

But by Shapiro's Lemma, we have $\mathrm{H}^2(\Gamma,P_i)=\mathrm{H}^2(\mathbb{Z}/p\mathbb{Z},\mathbb{Z})=\mathbb{Z}/p\mathbb{Z}$ and $\mathrm{H}^3(\Gamma,P_i)=\mathrm{H}^3(\mathbb{Z}/p\mathbb{Z},\mathbb{Z})=0$ for $i=1,2$. Also, by the Künneth formula, [Weib, p. 166],

$$\begin{aligned}\mathrm{H}^n(\Gamma,\mathbb{Z})=&\textstyle\bigoplus_{i+j=n}\mathrm{H}^i(\mathbb{Z}/p\mathbb{Z},\mathbb{Z})\otimes\mathrm{H}^j(\mathbb{Z}/p\mathbb{Z},\mathbb{Z})\\&\oplus\textstyle\bigoplus_{i+j=n+1}\mathrm{Tor}^1_{\mathbb{Z}}\big(\mathrm{H}^i(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}),\mathrm{H}^j(\mathbb{Z}/p\mathbb{Z},\mathbb{Z})\big),\end{aligned}$$

so that, in particular, $\mathrm{H}^3(\Gamma,\mathbb{Z})=\mathbb{Z}/p\mathbb{Z}$ and $\mathrm{H}^2(\Gamma,\mathbb{Z})=(\mathbb{Z}/p\mathbb{Z})^2$. This all yields an exact sequence

$$0\longrightarrow(\mathbb{Z}/p\mathbb{Z})^2\longrightarrow(\mathbb{Z}/p\mathbb{Z})^2\longrightarrow\mathrm{H}^2(\Gamma,\Lambda_{2p})\longrightarrow\mathbb{Z}/p\mathbb{Z}\longrightarrow 0,$$

and so $\mathrm{H}^2(\Gamma, \Lambda_{2p}) = \mathbb{Z}/p\mathbb{Z}$.

To show that $Ш^2(\Gamma, \Lambda_{2p}) = 0$, it would suffice to find a cyclic subgroup C of Γ for which $\mathrm{Res}^\Gamma_C : \mathrm{H}^2(\Gamma, \Lambda_{2p}) \to \mathrm{H}^2(C, \Lambda_{2p})$ is injective.

Take $C = C_1$. Since $\mathrm{H}^1(\Gamma, \Lambda_{2p}) = 0$, we have that the sequence

$$0 \longrightarrow \mathrm{H}^2(\Gamma/C, \Lambda_{2p}^C) \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(\Gamma, \Lambda_{2p}) \xrightarrow{\mathrm{Res}} \mathrm{H}^2(C, \Lambda_{2p})$$

is exact. So it suffices to show that $\mathrm{H}^2(\Gamma/C, \Lambda_{2p}^C) = 0$.

The fundamental dominant weights for Λ_{2p} are

$$\varpi_t = \textstyle\sum_{i=1}^t \varepsilon_i - \frac{t}{2p}\sum_{i=1}^{2p} \varepsilon_i, \quad t = 1, \dots, 2p-1.$$

Let $\nu_i = \varepsilon_i - \frac{1}{2p}\sum_{i=1}^{2p} \varepsilon_i$, $i = 1, \dots, 2p$. Note that

$$\nu_1 = \varpi_1, \quad \nu_t = \varpi_t - \varpi_{t-1}, t = 2, \dots, 2p-1, \quad \nu_{2p} = -\varpi_{2p-1}.$$

This shows that $\nu_1, \dots, \nu_p, \varpi_{p+1}, \dots, \varpi_{2p-1}$ is another basis for Λ_{2p} and that

$$\Lambda_{2p}|_C = \textstyle\bigoplus_{i=1}^p \mathbb{Z}\nu_i \oplus \bigoplus_{i=p+1}^{2p-1} \mathbb{Z}\varpi_i \simeq \mathbb{Z}[C] \oplus \mathbb{Z}^{p-1}.$$

This shows that

$$\Lambda_{2p}^C = \mathbb{Z}(\textstyle\sum_{i=1}^p \nu_i) \oplus \bigoplus_{i=p+1}^{2p-1} \mathbb{Z}\varpi_i = \bigoplus_{i=p}^{2p-1} \mathbb{Z}\varpi_i = \bigoplus_{i=p+1}^{2p} \mathbb{Z}\nu_i.$$

But Γ/C permutes $\nu_{p+1}, \dots, \nu_{2p}$ cyclically so that $\Lambda_{2p}^C \simeq \mathbb{Z}[\Gamma/C]$. This implies that $\mathrm{H}^2(\Gamma/C, \Lambda_{2p}^C) = 0$ as required.

(b) To prove that Λ_{2p} is not Γ-quasi-permutation, we will construct a coflasque Γ-resolution of $\mathbb{Z}\mathsf{A}_{2p-1}$ (dually, a flasque resolution of Λ_{2p}) and show that the flasque lattice is not permutation projective as an Γ-lattice.

As $\alpha_1, \dots, \alpha_{p-1}$ and $\alpha_{p+1}, \dots \alpha_{2p-1}$ are the standard bases of the root subsystems of type A_{p-1}, we denote the Γ-sublattice of $\mathbb{Z}\mathsf{A}_{2p-1}$ generated by them simply by $\mathbb{Z}\mathsf{A}_{p-1} \oplus \mathbb{Z}\mathsf{A}_{p-1}$. Let ι be its natural embedding into $\mathbb{Z}\mathsf{A}_{2p-1}$. It is easily seen that $\alpha_p + \mathbb{Z}\mathsf{A}_{p-1} \oplus \mathbb{Z}\mathsf{A}_{p-1}$ is Γ-stable. This implies that there is an exact sequence of Γ-lattices

$$0 \longrightarrow \mathbb{Z}\mathsf{A}_{p-1} \oplus \mathbb{Z}\mathsf{A}_{p-1} \xrightarrow{\iota} \mathbb{Z}\mathsf{A}_{2p-1} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

A coflasque resolution of the Γ-lattice $\mathbb{Z}\mathsf{A}_{p-1} \oplus \mathbb{Z}\mathsf{A}_{p-1}$ is given by

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow P_1 \oplus P_2 \longrightarrow \mathbb{Z}\mathsf{A}_{p-1} \oplus \mathbb{Z}\mathsf{A}_{p-1} \longrightarrow 0$$

where P_1 and P_2 are defined by (7.7) and the generator of the Γ-lattice P_1 (respectively P_2) is sent to α_1 (respectively α_{p+1}).

We now extend ι to a coflasque resolution of the Γ-lattice $\mathbb{Z}\mathsf{A}_{2p-1}$. Let

$$P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \xrightarrow{\varrho} \mathbb{Z}\mathsf{A}_{2p-1}$$

be a map of Γ-lattices where $\varrho_{P_1 \oplus P_2} = \iota$, ϱ sends $1 \in \mathbb{Z}[\Gamma]$ to α_p and ϱ sends the $1 \in \mathbb{Z}$ to $\sum_{i=1}^p \varepsilon_i - \sum_{i=p+1}^{2p} \varepsilon_i = 2\varpi_p$. It is easily verified that ϱ is surjective (in fact $\varrho|_{\mathbb{Z}\Gamma}$ is surjective).

Let $L = \mathrm{Ker}\,\varrho$. To check that L is coflasque and hence that

$$0 \longrightarrow L \longrightarrow P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \xrightarrow{\varrho} \mathbb{Z}\mathsf{A}_{2p-1} \longrightarrow 0$$

is a coflasque resolution of $\mathbb{Z}\mathsf{A}_{2p-1}$, we need only verify that for $R := P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z}$, we have $\varrho(R^K) = (\mathbb{Z}\mathsf{A}_{2p-1})^K$ for all subgroups K of Γ.

For $K = \Gamma$ or a cyclic subgroup generated by a disjoint product of two p-cycles, $(\mathbb{Z}\mathsf{A}_{2p-1})^K = \mathbb{Z}2\varpi_p$ so that $\varrho(\mathbb{Z}^K) = \varrho(\mathbb{Z}) = (\mathbb{Z}\mathsf{A}_{2p-1})^K$ and so $\varrho(R^K) = (\mathbb{Z}\mathsf{A}_{2p-1})^K$.

The only other subgroups are C_1 and C_2. As the arguments are similar, we just consider C_1: the lattice $(\mathbb{Z}\mathsf{A}_{2p-1})^{H_1}$ has basis $2\varpi_p, \alpha_{p+1}, \ldots, \alpha_{2p-1}$, and we have $\varrho(\mathbb{Z}) = \mathbb{Z}2\varpi_2$ and $\varrho(P_2^{H_1}) = \varrho(P_2) = \oplus_{i=p+1}^{2p-1}\mathbb{Z}\alpha_i$. This shows that

$$0 \longrightarrow L \longrightarrow P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \xrightarrow{\varrho} \mathbb{Z}\mathsf{A}_{2p-1} \longrightarrow 0$$

is a coflasque resolution. Dualizing, we obtain a flasque resolution for Λ_{2p}:

$$0 \longrightarrow \Lambda_{2p} \longrightarrow P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \longrightarrow L^* \longrightarrow 0.$$

We have $Ш^2(\Gamma, \Lambda_{2p}) = \mathrm{H}^1(\Gamma, L^*) = 0$. This shows that L is flasque and coflasque as a Γ-lattice.

We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}^2 \arrow[r] \arrow[d] & P_1 \oplus P_2 \arrow[r, "\iota"] \arrow[d] & \mathbb{Z}\mathsf{A}_{p-1} \oplus \mathbb{Z}\mathsf{A}_{p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \arrow[r, "\varrho"] \arrow[d] & \mathbb{Z}\mathsf{A}_{2p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U(p) \arrow[r] \arrow[d] & \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \arrow[r, "\theta"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{7.8}$$

where $U(p)$ is the kernel of the induced map θ. Now $2\varpi_p = \sum_{i=1}^{p-1} i(\alpha_i + \alpha_{2p-i}) + p\alpha_p$. So θ sends $1 \in \mathbb{Z}\Gamma$ to $\overline{\alpha_p}$ and sends $1 \in \mathbb{Z}$ to $p\overline{\alpha_p}$. This shows that

$$\{(h-1, 0) \mid h \in \Gamma\} \cup \{(-p, 1)\}$$

is a set of elements of a $\mathbb{Z}$-basis for $U(p)$. Note that $U(p)$ also satisfies

$$0 \longrightarrow U(p) \longrightarrow \mathbb{Z}[\Gamma] \longrightarrow \mathbb{Z}/p\mathbb{Z},$$

so that $\mathbb{Q}U(p) \simeq \mathbb{Q}[\Gamma]$.

From the above diagram, we then see that $\mathbb{Q}L \simeq \mathbb{Q}[\Gamma] \oplus \mathbb{Q}^2$. By [CW, Lemmas 2, 3], to determine whether or not L is permutation projective is equivalent to checking whether $\mathbb{F}_pL$ is a permutation module for $\mathbb{F}_p[\Gamma]$.

Tensoring the diagram (7.8) with $\mathbb{F}_p$ leaves it exact so we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{F}_p^2 \arrow[r] \arrow[d] & \mathbb{F}_pP_1 \oplus \mathbb{F}_pP_2 \arrow[r, "\mathrm{id}\otimes\iota"] \arrow[d] & \mathbb{F}_p\mathsf{A}_{p-1} \oplus \mathbb{F}_p\mathsf{A}_{p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{F}_pL \arrow[r] \arrow[d] & \mathbb{F}_pP_1 \oplus \mathbb{F}_pP_2 \oplus \mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p \arrow[r, "\mathrm{id}\otimes\varrho"] \arrow[d] & \mathbb{F}_p\mathsf{A}_{2p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{F}_pU(p) \arrow[r] \arrow[d] & \mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p \arrow[r, "\mathrm{id}\otimes\theta"] \arrow[d] & \mathbb{F}_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}.$$

Suppose that $\mathbb{F}_pL$ is permutation. Then since L is coflasque, the sequence

$$0 \longrightarrow L^\Gamma \xrightarrow{p} L^\Gamma \longrightarrow (L/pL)^\Gamma \longrightarrow 0$$

is exact so that $(\mathbb{F}_pL)^\Gamma = L^\Gamma/pL^\Gamma$. Since $\mathbb{Q}[L] \simeq \mathbb{Q}[\Gamma] \oplus \mathbb{Q}^2$, $\operatorname{rank} L^\Gamma = 3$. But then $\dim_{\mathbb{F}_p}(\mathbb{F}_pL)^\Gamma = 3$. This means that $\mathbb{F}_pL$ must then have three transitive components. Since $\operatorname{rank} L = p^2+2$ and $p > 2$, this means that $\mathbb{F}_pL = \mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p^2$.

Looking at the $\mathbb{Z}$-basis for $U(p)$ given above, it is clear that $\mathbb{F}_pU(p) \simeq \mathbb{F}_p \oplus \mathbb{F}_pI_\Gamma$ where $\mathbb{F}_pI_\Gamma$ is the augmentation ideal of $\mathbb{F}_p[\Gamma]$. Then the left column of the last commutative diagram implies that we have a surjective map $\mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p^2 \to \mathbb{F}_p \oplus \mathbb{F}_pI_\Gamma$. Since $(\mathbb{F}_pI_\Gamma)^\Gamma = 0$, this would imply that we have a surjective map $\mathbb{F}_p[\Gamma] \to \mathbb{F}_pI_\Gamma$ or equivalently that $\mathbb{F}_pI_\Gamma$ is a cyclic $\mathbb{F}_p[\Gamma]$-module. But since $\mathbb{F}_p[\Gamma]$ is a local ring with unique maximal ideal $\mathbb{F}_pI_\Gamma$, Nakayama's Lemma implies that $\mathbb{F}_pI_\Gamma$ is a cyclic $\mathbb{F}_p[\Gamma]$-module if and only if $\mathbb{F}_pI_\Gamma/(\mathbb{F}_pI_\Gamma)^2$ is generated by one element over $\mathbb{F}_p$. As $\dim_{\mathbb{F}_p} \mathbb{F}_pI_\Gamma/(\mathbb{F}_pI_\Gamma)^2 = 2$, this is impossible. By contradiction, there is no such surjective map from $\mathbb{F}_p[\Gamma]$ to $\mathbb{F}_pI_\Gamma$. This implies that $\mathbb{F}_pL$ is not permutation and hence L is not permutation projective as an $\mathbb{Z}[\Gamma]$-module. This implies in turn that Λ_{2p} is not quasi-permutation as a Γ-lattice or as an $\mathrm{S}_p \times \mathrm{S}_p$-lattice. □

Remark 7.9. Note that this argument fails for $p = 2$. Indeed, we showed that $\operatorname{rank} L = p^2+2$ and if $\mathbb{F}_pL$ were permutation, it would have three transitive components. For $p > 2$, we used these facts to conclude that $\mathbb{F}_pL = \mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p^2$. For $p = 2$, this is not so; here $\mathbb{F}_2L$ may have three permutation components, each of rank 2. Indeed, if $\Gamma = \langle g, h\rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then one can define a surjective $\mathbb{F}_2[\Gamma]$-homomorphism

$$\mathbb{F}_2[\Gamma/\langle g\rangle] \oplus \mathbb{F}_2[\Gamma/\langle h\rangle] \oplus \mathbb{F}_2[\Gamma/\langle gh\rangle] \to \mathbb{F}_2I_\Gamma \oplus \mathbb{F}_2$$

by sending the generator of the first component to $(1+g, 0)$, the generator of the second component to $(1+h, 0)$ and that of the third component to $(0, 1)$.

In fact, by Proposition 7.2, we see that $Q_4(2)|_\Gamma \sim \Lambda_4|_\Gamma$. Since $Q_4(2)$ is the character lattice of the Cayley group $\mathbf{SL}_4/\boldsymbol{\mu}_2 \simeq \mathbf{SO}_6$, by Theorem 1.31 it must be quasi-permutation as an S_4-lattice and hence as an Γ-lattice. Alternatively, one could show directly that $Q_4(2)$ is a sign-permutation S_4-lattice and hence is quasi-permutation.

7.10. Completion of the proof of Proposition 5.1. It now suffices to prove the following proposition to complete the proof of Proposition 5.1:

Proposition 7.11. *Suppose n/d is divisible by a prime p.*

(a) *If $p > 2$, then the S_n-lattice $Q_n(d)$ is not quasi-permutation.*
(b) *If $n > p^2$, then the S_n-lattice $Q_n(d)$ is not quasi-permutation.*

Indeed, by part (a), the S_n-lattice $Q_n(d)$ is not quasi-permutation if the prime factorization of n/d includes a prime larger than 2. On the other hand, if $n/d = 2^k$, then, by part (b), the S_n-lattice $Q_n(d)$ is not quasi-permutation, for any $(n, d) \neq (4, 2)$, and Proposition 5.1 follows.

Proof. (a) Proposition 7.2 shows that $Q_n(d)|_{Y_p}$ is equivalent to $\Lambda_{2p}|_{\mathrm{S}_p\times\mathrm{S}_p}$ which is not quasi-permutation by Proposition 7.6. Thus $Q_n(d)$ is not quasi-permutation as an Y_p-lattice and hence as an S_n-lattice as well.

(b) We have $n = tp$ with $t > p$. Following the proof of Proposition 4.1(i) in [LL], we define a subgroup $\Gamma \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of S_n as follows. Arrange the numbers from 1 to n into a rectangular table with p columns and t rows, so that the first row is $1, \ldots, p$, the second row is $p+1, \ldots, 2p$, etc. Let σ_i be the p-cycle that cyclically permutes the ith row and leaves elements of all other rows fixed. Note that $\sigma_1, \ldots, \sigma_t$ are commuting p-cycles; explicitly

$$\sigma_i = \big((i-1)p+1, (i-1)p+2, \ldots, ip\big).$$

We now set $\Gamma := \langle \alpha, \beta \rangle$, where

$$\alpha := \prod_{i=1}^{t-1} \sigma_i \quad \text{and} \quad \beta := \prod_{i=1}^{p-1} \sigma_i^{-i} \cdot \prod_{i=p+1}^{t} \sigma_i.$$

The subgroup Γ has orbits $\mathcal{O}_i = \{(i-1)p+1, (i-1)p+2, \ldots, ip\}$, $i = 1, \ldots, t$, all of length p and every cyclic subgroup C of Γ has fixed points. This means that by Lemma 6.10

$$\mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}) \simeq \mathbb{Z}/p\mathbb{Z} \quad \text{but} \quad \mathrm{H}^1(C, \mathbb{Z}\mathsf{A}_{n-1}) = 0.$$

Also by Lemma 6.13, we find that

$$\mathrm{H}^1(C, Q_n\,(n/d)) = 0.$$

Then, Lemma 6.12 and the fact that p divides n/d shows that $\mathbb{Z}/d\mathbb{Z} \xrightarrow{\partial} \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1})$ is the zero map. The following commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}/d\mathbb{Z} & \xrightarrow{\;0\;} & \mathrm{H}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}) & \longrightarrow & \mathrm{H}^1\left(\Gamma, Q_n\,(n/d)\right) \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\
\prod_{a\in\Gamma} \mathbb{Z}/d\mathbb{Z} & \xrightarrow{\;0\;} & \prod_{a\in\Gamma} \mathrm{H}^1(\langle a \rangle, \mathbb{Z}\mathsf{A}_{n-1}) = 0 & \longrightarrow & \prod_{a\in\Gamma} \mathrm{H}^1\left(\langle a \rangle, Q_n\,(n/d)\right) = 0
\end{array}$$

shows that

$$\mathbb{Z}/p\mathbb{Z} \simeq \text{Ш}^1(\Gamma, \mathbb{Z}\mathsf{A}_{n-1}) \leqslant \text{Ш}^1(\Gamma, Q_n\,(n/d))\,.$$

Now if M were a flasque lattice with $\rho(Q_n(d)) =$ stable equivalence class of M, then M^* is a coflasque lattice satisfying

$$0 \longrightarrow M^* \longrightarrow P \longrightarrow Q_n\,(n/d) \longrightarrow 0,$$

so that by Lemma 2.12(a), $\text{Ш}^2(\Gamma, M^*) \simeq \text{Ш}^1(\Gamma, Q_n(n/d)) \neq 0$. Lemma 2.12(c) now shows that M^* cannot be a direct summand of a quasi-permutation lattice and hence not stably permutation. This implies that M cannot be stably permutation and so $Q_n(d)$ cannot be quasi-permutation. $\square$

8. Stably Cayley groups of type D_n

8.1. Root system of type D_n. Let $\varepsilon_1, \ldots \varepsilon_n$ be the same as in Subsection 6.1. The root system of type D_n is the set

$$\mathsf{D}_n = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}.$$

It has a base $\alpha_1, \ldots, \alpha_n$, where $\alpha_1, \ldots, \alpha_{n-1}$ are given by (6.3) and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. The fundamental dominant weights of D_n with respect to this base are $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $i = 1, \ldots, n-2$,

$$\varpi_{n-1} = \tfrac{1}{2}\textstyle\sum_{i=1}^{n-1} \varepsilon_i - \tfrac{1}{2}\varepsilon_n \quad \text{and} \quad \varpi_n = \tfrac{1}{2}\textstyle\sum_{i=1}^{n-1} \varepsilon_i + \tfrac{1}{2}\varepsilon_n.$$

The Weyl group $W(\mathsf{D}_n)$ of D_n is $(\mathbb{Z}/2\mathbb{Z})^{n-1} \ltimes \mathrm{S}_n$, where $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ consists of all even numbers of sign changes on $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and S_n acts via (6.2). The root and weight $W(\mathsf{D}_n)$-lattices of D_n are respectively $\mathbb{Z}\mathsf{D}_n$ and $\Lambda(\mathsf{D}_n) := \mathbb{Z}\varpi_1 \oplus \ldots \oplus \mathbb{Z}\varpi_n$.

8.2. Lattices Y_{2m} and Z_{2m}. As we explained in Section 5, we are interested in the case where n is even, $n = 2m$, $m > 2$. There are precisely the following three lattices between $\Lambda(\mathsf{D}_{2m})$ and $\mathbb{Z}\mathsf{D}_{2m}$:

$$\mathrm{X}_{2m} := \mathbb{Z}\mathsf{D}_{2m} + \mathbb{Z}\varpi_1, \quad \mathrm{Y}_{2m} := \mathbb{Z}\mathsf{D}_{2m} + \mathbb{Z}\varpi_{2m-1} \quad \text{and} \quad \mathrm{Z}_{2m} := \mathbb{Z}\mathsf{D}_{2m} + \mathbb{Z}\varpi_{2m}. \tag{8.3}$$

The character lattice of $\mathbf{Spin}_{4m}^{1/2}$ (see Section 5) is isomorphic to either of the lattices Y_{2m} and Z_{2m} while X_{2m} is isomorphic to the character lattice of $\mathbf{SO}_{4m}$. Note that $\varepsilon_1, \dots, \varepsilon_n$ is the sign-permutation basis for X_{2m}; this is consistent with the fact that $\mathbf{SO}_{4m}$ is Cayley, see Theorem 1.31(a). Also note that

$$\{\tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4), \tfrac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4), \tfrac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4), \tfrac{1}{2}(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)\}$$

is the sign-permutation basis for Y_4, and

$$\{\tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4), \tfrac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4), \tfrac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4), \tfrac{1}{2}(-\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)\}$$

is that for Z_4; this is consistent with the fact that $\mathbf{Spin}_8^{1/2}$ is Cayley, see Section 5.

Our goal is to prove Proposition 5.2. In view of the aforesaid, this is equivalent to proving the following.

Proposition 8.4. *The $W(\mathsf{D}_{2m})$-lattices Y_{2m} and Z_{2m} are not quasi-permutation for any $m > 2$.*

Proof. For the subgroup S_{2m} of $W(\mathsf{D}_{2m})$ acting by (6.2), we consider the S_{2m}-lattices $\mathrm{Y}_{2m}|_{\mathrm{S}_{2m}}$ and $\mathrm{Z}_{2m}|_{\mathrm{S}_{2m}}$ and compare them to the S_{2m}-lattice $Q_{2m}(m)$ defined by (6.5) and (6.4),

$$Q_{2m}(m) = \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_{2m-1} + \mathbb{Z}\beta, \quad \text{where} \quad \beta := m\varepsilon_1 - \tfrac{1}{2}\textstyle\sum_{i=1}^{2m}\varepsilon_i, \tag{8.5}$$

that is isomorphic to the character lattice of $\mathbf{SL}_{2m}/\boldsymbol{\mu}_m$, see Subsection 6.1.

First we observe that

$$\alpha_1, \dots, \alpha_{2m-2}, \gamma, \varepsilon_{2m-2}+\varepsilon_{2m-1}, \quad \text{where} \quad \gamma := \tfrac{1}{2}\textstyle\sum_{i=1}^{m}\varepsilon_i - \tfrac{1}{2}\sum_{i=m+1}^{2m}\varepsilon_i,$$

is a basis for Y_{2m} if m is odd, and for Z_{2m} if m is even. Since $\alpha_1, \dots, \alpha_{2m-2}, \varepsilon_{2m-2}+\varepsilon_{2m-1}$ is a basis for $\mathbb{Z}\mathsf{D}_{2m-1}$, (8.3) implies that proving this claim is equivalent to proving the equality

$$\mathbb{Z}\mathsf{D}_{2m-1} + \mathbb{Z}\gamma = \begin{cases} \mathbb{Z}\mathsf{D}_{2m} + \mathbb{Z}\varpi_{2m-1} & \text{if } m \text{ is odd,} \\ \mathbb{Z}\mathsf{D}_{2m} + \mathbb{Z}\varpi_{2m} & \text{if } m \text{ is even.} \end{cases} \tag{8.6}$$

Note that

$$\varpi_{2m-1} - \gamma = \textstyle\sum_{i=m+1}^{2m-1}\varepsilon_i \in \mathbb{Z}\mathsf{D}_{2m-1} \quad \text{if } m \text{ is odd,}$$
$$\varpi_{2m} + \gamma = \textstyle\sum_{i=1}^{m}\varepsilon_i \in \mathbb{Z}\mathsf{D}_{2m-1} \quad \text{if } m \text{ is even.}$$

Therefore proving (8.6) is equivalent to proving the inclusion

$$\mathbb{Z}\mathsf{D}_{2m} \subseteq \mathbb{Z}\mathsf{D}_{2m-1} + \mathbb{Z}\gamma,$$

which in turn is equivalent to proving the inclusions

$$\varepsilon_{2m-1} \pm \varepsilon_{2m} \in \mathbb{Z}\mathsf{D}_{2m-1} + \mathbb{Z}\gamma.$$

Finally, the last inclusions indeed hold as we have

$$2\gamma + (\varepsilon_{2m-1}+\varepsilon_{2m}) = \textstyle\sum_{i=1}^{m}\varepsilon_i - \sum_{i=m+1}^{2m-2}\varepsilon_i \in \mathbb{Z}\mathsf{D}_{2m-1},$$
$$2\gamma - (\varepsilon_{2m-1}-\varepsilon_{2m}) = \textstyle\sum_{i=1}^{m-1}(\varepsilon_i - \varepsilon_{m+i}) + (\varepsilon_m - \varepsilon_{2m-1}) \in \mathbb{Z}\mathsf{D}_{2m-1}.$$

Thus the claim is proved.

Further, the easily checked equalities

$$\beta = \gamma + \textstyle\sum_{i=1}^{m-1}(m-i)\alpha_i,$$
$$\alpha_{2m-1} = 2\gamma - \textstyle\sum_{i=1}^{m} i\alpha_i - \sum_{i=1}^{m-2}(m-i)\alpha_{m+i}$$

and (8.5) imply that $\alpha_1, \ldots, \alpha_{2m-2}, \gamma$ is a $\mathbb{Z}$-basis for $Q_{2m}(m)$.

We thus obtain the following exact sequences of S_{2m}-lattices:

$$0 \longrightarrow Q_{2m}(m) \longrightarrow \mathrm{Y}_{2m}|_{\mathrm{S}_{2m}} \longrightarrow \mathbb{Z} \longrightarrow 0$$

if m is odd and

$$0 \longrightarrow Q_{2m}(m) \longrightarrow \mathrm{Z}_{2m}|_{\mathrm{S}_{2m}} \longrightarrow \mathbb{Z} \longrightarrow 0$$

if m is even. Here the S_{2m}-lattice $\mathbb{Z}$ is generated by $\varepsilon_{2m-2} + \varepsilon_{2m-1}$, modulo $Q_{2m}(m)$. We claim that the S_{2m}-action on this lattice is trivial. Indeed, on the one hand, the alternating subgroup of S_{2m} has to act on this lattice trivially because it has no non-trivial one-dimensional representations. On the other hand, as $m > 2$, the transposition $(1,2)$ acts trivially on $\varepsilon_{2m-2} + \varepsilon_{2m-1}$. Since the alternating subgroup and the transposition $(1,2)$ generate S_{2m}, this proves the claim.

The above exact sequences thus tell us that $\mathrm{Y}_{2m}|_{\mathrm{S}_{2m}} \sim Q_{2m}(m)$ if m is odd, and $\mathrm{Z}_{2m}|_{\mathrm{S}_{2m}} \sim Q_{2m}(m)$ if m is even. By Proposition 5.1, the S_{2m}-lattice $Q_{2m}(m)$ is not quasi-permutation for any $m > 2$. Thus for $m > 2$, the $W(\mathsf{D}_{2m})$-lattice Y_{2m} is not quasi-permutation if m is odd, and the $W(\mathsf{D}_{2m})$-lattice Z_{2m} is not quasi-permutation if m is even, as their restrictions to S_{2m} are not quasi-permutation. Since $Y_{2m} \simeq Z_{2m}$ as $W(\mathsf{D}_{2m})$-lattices, this completes the proof. $\square$

9. **Which stably Cayley groups are Cayley?**

In this section we will prove Theorem 1.35. The groups $G = \mathbf{SO}_n$, $\mathbf{Sp}_{2n}$ and $\mathbf{PGL}_n$ are shown to be Cayley in Examples 1.18 and 1.13. It thus remains to consider $\mathbf{SL}_3$ and $\mathbf{G}_2$.

9.1. The group $\mathbf{SL}_3$.

Proposition 9.2. *The group* $\mathbf{SL}_3$ *is Cayley.*

The proof below is based on analysis of the explicit formulas in [Vos, 4.9] and the geometric ideas of the proof of Proposition 9.2 given in [Pop$_2$]. We present it in a form that will also help us prove that $\mathbf{G}_2 \times \mathbf{G}_m^2$ is Cayley, see Proposition 9.13 below. On the other hand, the spirit of arguments in [Pop$_2$] is close to that in [Isk$_4$]. Since [Isk$_4$] is the main ingredient we will use in showing that $\mathbf{G}_2$ is not Cayley, see Lemma 9.11 and Proposition 9.12 below, we will give an outline of the proof of Proposition 9.2 from [Pop$_2$] in the Appendix.

Proof. The Weyl group W of $\mathbf{SL}_3$ is S_3. Consider the following subalgebra D of $\mathrm{Mat}_{3\times 3}$:

$$D := \{\mathrm{diag}(a_1, a_2, a_3) \in \mathrm{Mat}_{3\times 3} \mid a_i \in k\} \tag{9.3}$$

and the action of S_3 on D given by

$$\sigma(\mathrm{diag}(a_1, a_2, a_3)) := \mathrm{diag}(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) \quad \text{where } \sigma \in \mathrm{S}_3. \tag{9.4}$$

The S_3-stable subvarieties

$$T = \{X \in D \mid \det X = 1\} \quad \text{and} \quad \mathfrak{t} = \{Y \in D \mid \mathrm{tr}\, Y = 0\} \tag{9.5}$$

are respectively the maximal torus of $\mathbf{SL}_3$ and its Lie algebra, considered as W-varieties. By the Corollary of Lemma 3.6, it suffices to show that T and $\mathfrak{t}$ are birationally S_3-isomorphic.

Let $D \setminus \{0\} \to \mathbf{P}(D)$, $X \mapsto [X]$, be the natural projection. Denote by $\mathbf{P}^2_{\mathrm{S}_3\text{-natural}}$ and $\mathbf{P}^2_{\mathrm{S}_3\text{-twisted}}$ the projective plane $\mathbf{P}(D)$ endowed respectively with the natural and "twisted" rational actions of S_3 given by

$$\sigma([X]) := [\sigma(X)] \quad \text{and} \quad \sigma([X]) := [\sigma(X)^{\operatorname{sign}\sigma}], \quad \text{where } \sigma \in \mathrm{S}_3, X \in D.$$

Let $\pi : \mathbf{SL}_3 \to \mathbf{PGL}_3$ be the natural projection. Since $d_e\pi$ is an isomorphism between the Lie algebras of $\mathbf{SL}_3$ and $\mathbf{PGL}_3$, and $\mathbf{PGL}_3$ is a Cayley group, see Example 1.13, the Corollary of Lemma 3.6 tells us that $\mathfrak{t}$ is birationally S_3-isomorphic to the maximal torus $\pi(T)$ of $\mathbf{PGL}_3$. In turn, we have the following birational S_3-isomorphisms of S_3-varieties:

$$\pi(T) \overset{\simeq}{\dashrightarrow} \mathbf{P}^2_{\mathrm{S}_3\text{-natural}}, \quad \pi(X) \mapsto [X],$$
$$\mathbf{P}^2_{\mathrm{S}_3\text{-twisted}} \overset{\simeq}{\dashrightarrow} T, \quad [\operatorname{diag}(a_1, a_2, a_3)] \mapsto \operatorname{diag}\big(a_2/a_3, a_3/a_1, a_1/a_2\big).$$

Thus we only need to show that $\mathbf{P}^2_{\mathrm{S}_3\text{-natural}}$ and $\mathbf{P}^2_{\mathrm{S}_3\text{-twisted}}$ are birationally S_3-isomorphic. We shall establish this in three steps.

Step 1. Consider the action of S_3 on $\mathfrak{t} \times \mathfrak{t}$ given by

$$\sigma(Y, Z) := \begin{cases} \big(\sigma(Y), \sigma(Z)\big) & \text{if } \sigma \text{ is even,} \\ \big(\sigma(Z), \sigma(Y)\big) & \text{if } \sigma \text{ is odd,} \end{cases} \quad \text{where } \sigma \in \mathrm{S}_3,\ Y, Z \in \mathfrak{t}. \tag{9.6}$$

It determines the action of S_3 on the surface $\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t})$. Denote by $(\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t}))_{\mathrm{S}_3\text{-twisted}}$ the surface endowed with this action.

We claim that the S_3-varieties $\mathbf{P}^2_{\mathrm{S}_3\text{-twisted}}$ and $(\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t}))_{\mathrm{S}_3\text{-twisted}}$ are birationally S_3-isomorphic. Indeed, it is immediately seen that the rational map

$$\varphi : \mathbf{P}^2_{\mathrm{S}_3\text{-twisted}} \dashrightarrow (\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t}))_{\mathrm{S}_3\text{-twisted}}, \quad [X] \mapsto \big(\big[X - \tfrac{\operatorname{tr}(X)}{3} I_3\big], \big[X^{-1} - \tfrac{\operatorname{tr}(X^{-1})}{3} I_3\big]\big),$$

is S_3-equivariant and we shall now construct a rational map inverse to φ. Note that for $Y, Z \in \mathfrak{t}$ in general position, Y, Z, I_3 form a basis of the vector space D. Thus there are unique $\alpha, \beta, \gamma \in k$ such that

$$\alpha Z + \beta Y + \gamma I = -YZ$$

Note that α, β, and γ are, in fact, bihomogeneous rational functions of Y and Z of bidegree $(1, 0)$, $(0, 1)$ and $(1, 1)$, respectively. We now consider the map

$$\psi\colon (\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t}))_{\mathrm{S}_3\text{-twisted}} \dashrightarrow \mathbf{P}^2_{\mathrm{S}_3\text{-twisted}}, \quad ([Y], [Z]) \mapsto [Y + \alpha I_3].$$

To compute $\psi \circ \varphi$, note that if $Y = X - \frac{\operatorname{tr}(X)}{3} I_3$ and $Z = X^{-1} - \frac{\operatorname{tr}(X^{-1})}{3} I_3$, then expanding

$$I_3 = \big(Y + \tfrac{\operatorname{tr}(X)}{3} I_3\big)\big(Z + \tfrac{\operatorname{tr}(X^{-1})}{3} I_3\big),$$

we see that $\alpha = \frac{\operatorname{tr}(X)}{3}$ and thus $\psi([Y], [Z]) = [X]$. Thus $\psi \circ \varphi = \mathrm{id}$, and hence φ is a birational S_3-isomorphism.

Step 2. We now consider the linear action of S_3 on $\mathfrak{t} \otimes \mathfrak{t}$ determined by the action (9.6) and the corresponding action of S_3 on $\mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$. Then the Segre embedding

$$(\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t}))_{\mathrm{S}_3\text{-twisted}} \hookrightarrow \mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$$

is S_3-equivariant. Its image is a quadric Q in $\mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$ described as follows. Choose a basis $D_1 := \operatorname{diag}(1, \zeta, \zeta^2)$, $D_2 := \operatorname{diag}(1, \zeta^2, \zeta)$ of $\mathfrak{t}$, where ζ is a primitive cube root of unity. Set $D_{ij} = D_i \otimes D_j$. Then

$$Q = \{(\alpha_{11} : \alpha_{12} : \alpha_{21} : \alpha_{22}) \mid \alpha_{11}\alpha_{22} = \alpha_{12}\alpha_{21}\}, \tag{9.7}$$

where $(\alpha_{11} : \alpha_{12} : \alpha_{21} : \alpha_{22})$ is the point of $\mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$ corresponding to $\alpha_{11}D_{11} + \alpha_{12}D_{12} + \alpha_{21}D_{21} + \alpha_{22}D_{22} \in \mathfrak{t} \otimes \mathfrak{t}$.

Step 3. Decomposing $\mathfrak{t} \otimes \mathfrak{t}$ as a sum of S_3-submodules, we obtain

$$\mathfrak{t} \otimes \mathfrak{t} = V_1 \oplus V_2 \oplus V_3, \tag{9.8}$$

where $V_1 = kD_{11} + kD_{22}$ is a simple 2-dimensional submodule and $V_2 = kD_{12}$, $V_3 = kD_{21}$ are trivial 1-dimensional submodules. Since the S_3-fixed point $(0 : 0 : 1 : 0) \in \mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$ corresponding to V_3 lies on Q, the stereographic projection $Q \dashrightarrow \mathbf{P}(V_1 \oplus V_2)$ from this point is a birational S_3-isomorphism.

Finally, the S_3-module D is isomorphic to $V_1 \oplus V_2$. Hence $\mathbf{P}(V_1 \oplus V_2)$ and $\mathbf{P}^2_{\mathrm{S}_3\text{-natural}}$ are S_3-isomorphic.

To sum up, we have established the existence of the following birational S_3-isomorphisms:

$$\mathbf{P}^2_{\mathrm{S}_3\text{-twisted}} \overset{\text{Step1}}{\underset{\simeq}{\dashrightarrow}} (\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t}))_{\mathrm{S}_3\text{-twisted}} \overset{\text{Step2}}{\underset{\simeq}{\dashrightarrow}} Q \overset{\text{Step3}}{\underset{\simeq}{\dashrightarrow}} \mathbf{P}^2_{\mathrm{S}_3\text{-natural}}.$$

This completes the proof of Proposition 9.2. $\square$

9.9. The group $\mathbf{G}_2$.

The Weyl group of $\mathbf{G}_2$ is the dihedral group $\mathrm{S}_3 \times \mathrm{S}_2$ of order 12. The maximal torus of $\mathbf{G}_2$ and its Lie algebra are $\mathrm{S}_3 \times \mathrm{S}_2$-isomorphic respectively to T and $\mathfrak{t}$ given by (9.5), where the action of the first factor of $\mathrm{S}_3 \times \mathrm{S}_2$ is defined, as in the case of $\mathbf{SL}_3$, by (9.4), and that of the nontrivial element θ of the second factor by

$$\theta(X) := X^{-1} \text{ for } X \in T \quad \text{and} \quad \theta(Y) := -Y \text{ for } Y \in \mathfrak{t}. \tag{9.10}$$

We begin with the following surprising recent result due to Iskovskikh, [Isk$_4$].

Lemma 9.11. *The* $\mathrm{S}_3 \times \mathrm{S}_2$*-varieties T and $\mathfrak{t}$ are not birationally* $\mathrm{S}_3 \times \mathrm{S}_2$*-isomorphic.*

Proof outline. Since T and $\mathfrak{t}$ are rational surfaces, the theory of rational G-surfaces, due to Manin [Ma] and Iskovskikh [Isk$_1$], [Isk$_3$], can be applied; this is precisely what is done in [Isk$_4$]. Minimal rational $\mathrm{S}_3 \times \mathrm{S}_2$-surfaces are known, and any equivariant birational isomorphism between two such surfaces can be written as a composition of so-called "elementary links", which are completely enumerated in [Isk$_3$]. The argument in [Isk$_4$] amounts to constructing suitable minimal models for T and $\mathfrak{t}$, and explicitly checking that it is impossible to get from one to the other by a sequence of elementary links. $\square$

Proposition 9.12. $\mathbf{G}_2$ *is not a Cayley group.*

Proof. By the Corollary of Lemma 3.6, this follows from Lemma 9.11. $\square$

The following result illustrates how delicate the matter is.

Proposition 9.13. $\mathbf{G}_2 \times \mathbf{G}_m^2$ *is a Cayley group.*

Proof. By Corollary of Lemma 3.6, it suffices to show that $T \times \mathbf{A}^2$ and $\mathfrak{t} \times \mathbf{A}^2$ are birationally $\mathrm{S}_3 \times \mathrm{S}_2$-isomorphic, where in both cases $\mathrm{S}_3 \times \mathrm{S}_2$ acts via the first factor. We shall define a birational $\mathrm{S}_3 \times \mathrm{S}_2$-isomorphism between them in three steps.

Step 1. Let $(\mathfrak{t} \times \mathfrak{t})_{\mathrm{S}_3 \times \mathrm{S}_2\text{-twisted}}$ be the variety $\mathfrak{t} \times \mathfrak{t}$ endowed with the action of $\mathrm{S}_3 \times \mathrm{S}_2$ given by

$$(\sigma, \varepsilon)(Y, Z) := \begin{cases} (\sigma(Y), \sigma(Z)) & \text{if } \operatorname{sign}\sigma = \operatorname{sign}\varepsilon, \\ (\sigma(Z), \sigma(Y)) & \text{otherwise,} \end{cases} \quad \text{where } (\sigma, \varepsilon) \in \mathrm{S}_3 \times \mathrm{S}_2,\ Y, Z \in \mathfrak{t}. \tag{9.14}$$

The latter determines the action of $S_3\times S_2$ on $\mathbf{P}(\mathfrak{t})\times\mathbf{P}(\mathfrak{t})$. Denote by $(\mathbf{P}(\mathfrak{t})\times\mathbf{P}(\mathfrak{t}))_{S_3\times S_2\text{-twisted}}$ the surface endowed with this action. Then

$$(\mathfrak{t}\times\mathfrak{t})_{S_3\times S_2\text{-twisted}} \longrightarrow (\mathbf{P}(\mathfrak{t})\times\mathbf{P}(\mathfrak{t}))_{S_3\times S_2\text{-twisted}}, \quad (Y,Z)\mapsto([Y],[Z]),$$

is an algebraic vector $S_3\times S_2$-bundle of rank 2. Since $S_3\times S_2$ acts on $(\mathbf{P}(\mathfrak{t})\times\mathbf{P}(\mathfrak{t}))_{S_3\times S_2\text{-twisted}}$ faithfully, Lemma 2.16(b) shows that $(\mathfrak{t}\times\mathfrak{t})_{S_3\times S_2\text{-twisted}}$ and $(\mathbf{P}(\mathfrak{t})\times\mathbf{P}(\mathfrak{t}))_{S_3\times S_2\text{-twisted}}\times\mathbf{A}^2$, where $S_3\times S_2$ acts via the first factor, are birationally $S_3\times S_2$-isomorphic.

Step 2. Let $\mathbf{P}^2_{S_3\times S_2\text{-twisted}}$ be the projective plane $\mathbf{P}(D)$ endowed with the action of $S_3\times S_2$ given by

$$(\sigma,\varepsilon)([X]) := [\sigma(X)^{\operatorname{sign}\sigma\ \operatorname{sign}\varepsilon}], \quad \text{where } (\sigma,\varepsilon)\in S_3\times S_2,\ X\in D.$$

Then the rational maps

$$\mathbf{P}^2_{S_3\times S_2\text{-twisted}} \dashrightarrow T, \quad [\operatorname{diag}(a_1,a_2,a_3)]\mapsto \operatorname{diag}\big(a_2/a_3, a_3/a_1, a_1/a_2\big), \quad \text{and}$$
$$\mathbf{P}^2_{S_3\times S_2\text{-twisted}} \dashrightarrow (\mathbf{P}(\mathfrak{t})\times\mathbf{P}(\mathfrak{t}))_{S_3\times S_2\text{-twisted}}, \quad [X]\mapsto \Big(\big[X-\tfrac{\operatorname{tr}(X)}{3}I_3\big], \big[X^{-1}-\tfrac{\operatorname{tr}(X^{-1})}{3}I_3\big]\Big),$$

are birational $S_3\times S_2$-isomorphisms — the arguments are similar to those in the proof of Proposition 9.2.

Step 3. Since

$$(\mathfrak{t}\times\mathfrak{t})_{S_3\times S_2\text{-twisted}} \longrightarrow \mathfrak{t}, \quad (t_1,t_2)\mapsto t_1-t_2,$$

is an algebraic vector $S_3\times S_2$-bundle of rank 2 and $S_3\times S_2$ acts on $\mathfrak{t}$ faithfully, applying Lemma 2.16(b) once again we conclude that $(\mathfrak{t}\times\mathfrak{t})_{S_3\times S_2\text{-twisted}}$ is birationally $S_3\times S_2$-isomorphic to $\mathfrak{t}\times\mathbb{A}^2$, where $S_3\times S_2$ acts via the first factor.

To sum up, we have established the existence of the following birational $S_3\times S_2$-isomorphisms:

$$T\times\mathbf{A}^2 \overset{\text{Step2}}{\underset{\simeq}{\dashrightarrow}} (\mathbf{P}(\mathfrak{t})\times\mathbf{P}(\mathfrak{t}))_{S_3\times S_2\text{-twisted}}\times\mathbf{A}^2 \overset{\text{Step1}}{\underset{\simeq}{\dashrightarrow}} (\mathfrak{t}\times\mathfrak{t})_{S_3\times S_2\text{-twisted}} \overset{\text{Step3}}{\underset{\simeq}{\dashrightarrow}} \mathfrak{t}\times\mathbf{A}^2.$$

This completes the proof of Proposition 9.13. $\square$

Remark 9.15. We do not know whether or not $\mathbf{G}_2\times\mathbf{G}_m$ is a Cayley group.

10. **Generalization**

The notions of Cayley map and Cayley group naturally lead to generalizations which will be considered in this section.

10.1. Generalized Cayley maps. Let G be a connected linear algebraic group and let $\mathfrak{g}$ be its Lie algebra. We consider G and $\mathfrak{g}$ as G-varieties with respect to the conjugating and adjoint actions respectively and denote by $\operatorname{Rat}_G(G,\mathfrak{g})$ the set of all rational G-maps $G\dashrightarrow\mathfrak{g}$ endowed with the natural structure of a vector space over $k(G)^G$. Set $\operatorname{Mor}_G(G,\mathfrak{g}) := \{\varphi\in\operatorname{Rat}_G(G,\mathfrak{g}) \mid \varphi \text{ is a morphism}\}$.

Definition 10.2. An element $\varphi\in\operatorname{Rat}_G(G,\mathfrak{g})$ (respectively, $\varphi\in\operatorname{Mor}_G(G,\mathfrak{g})$) is called a *generalized Cayley map* (respectively, *generalized Cayley morphism*) *of* G if φ is a dominant map.

To prove the existence of generalized Cayley maps of G we need the following known fact, see [Lun_2, Lemme III.1], cf. [PV, 6.3].

Lemma 10.3. *Assume that the group G is reductive. Let X be an affine algebraic variety endowed with an algebraic action of G and let $x \in X$ be a nonsingular fixed point of G. Let T_x be the tangent space of X at x endowed with the natural action of G. Then there is a G-morphism $\varepsilon : X \to \mathrm{T}_x$ étale at x (hence dominant) and such that $\varepsilon(x) = 0$.*

Proof. We can consider X as a G-stable subvariety of a finite dimensional algebraic G-module V, see [PV, Theorem 1.5]. Since x is a fixed point of H, we can replace X by its image under the parallel translation $v \mapsto v - x$ and assume that $x = 0$. The tangent space T_x is identified with a submodule of V. Since G is reductive, the G-module V is semisimple. Hence $V = \mathrm{T}_x \oplus M$ for some submodule M. Now we can take $\varepsilon = \pi|_X$, where $\pi : V \to \mathrm{T}_x$ is the projection parallel to M. □

Taking $X = G$ with the conjugating action and $x = e$, we obtain the following.

Corollary. *Assume that G is reductive. Then there is a generalized Cayley morphism φ of G étale at e and such that $\varphi(e) = 0$.*

The following special case of this construction was considered by Kostant and Michor, [KM].

Example 10.4. Assume that G is reductive. Consider an algebraic homomorphism $\nu : G \to \mathbf{GL}(S)$, where S is a finite dimensional vector space over k. Then the k-vector space $V := \mathrm{End}(S)$ has a natural G-module structure defined by $g(h) := \nu(g)h\nu(g)^{-1}$ for every $g \in G$ and $h \in V$. If ν is injective, identify G with the image of $\iota \circ \nu$, where $\iota : \mathbf{GL}(S) \hookrightarrow V$ is the natural embedding. Then G is a G-stable subvariety of V and the restriction to $\mathfrak{g} = \mathrm{T}_e$ of the G-invariant inner product $(x, y) \mapsto \mathrm{tr}\, xy$ on V is nondegenerate. This yields the G-module decomposition $V = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$, where $\mathfrak{g}^{\perp}$ is the orthogonal complement to $\mathfrak{g}$ with respect to $(\ ,\ )$. The restriction to G of the projection $V \to \mathfrak{g}$ parallel to $\mathfrak{g}^{\perp}$ is a generalized Cayley morphism $\varphi : G \to \mathfrak{g}$ étale at e such that $\varphi(e) = 0$. □

Proposition 10.5. *Every connected linear algebraic group G admits a generalized Cayley map.*

Proof. We use the notation of Proposition 4.4 and its proof. The group $W_{L,T}$ is finite, hence reductive, and $e \in T$ is its fixed point. Therefore Lemma 10.3 implies that there is a dominant $W_{L,T}$-morphism $\varepsilon : T \to \mathfrak{t}$. The arguments in the proof of part (a) of Proposition 4.4 show that ε is N-equivariant. Consider an N-isomorphism (4.5). Then

$$\varepsilon \times \tau : C = T \times U \longrightarrow \mathfrak{t} \oplus \mathfrak{u} = \mathfrak{c},$$

is a dominant N-morphism. Hence by Lemma 2.22, there is a dominant G-morphism

$$\theta : G \times^N C \longrightarrow G \times^N \mathfrak{c}$$

such that $\theta|_C = \varepsilon \times \tau$. Now, since, by Lemma 3.3, the G-morphisms γ_C and $\gamma_{\mathfrak{c}}$ given by (3.2) are birational G-isomorphisms, $\gamma_{\mathfrak{c}} \circ \theta \circ \gamma_C^{-1} \in \mathrm{Rat}_G(G, \mathfrak{g})$ is a generalized Cayley map. □

Which groups admit generalized Cayley morphisms? By the Corollary of Lemma 10.3, reductive groups have this property. The following shows that this property is shared by a much wider class of groups (in particular, by all groups whose radical is nilpotent).

Lemma 10.6. *Assume that the group $\mathrm{Ad}_G\, G$ admits no nontrivial homomorphisms to $\mathbf{G}_m$. Then for any $\varphi \in \mathrm{Rat}_G(G, \mathfrak{g})$, there is $f \in k[G]^G$ such that*

(i) *$\{g \in G \mid f(g) = 0\}$ is the indeterminacy locus of φ,*

(ii) $f\varphi \in \mathrm{Mor}_G(G, \mathfrak{g})$.

If moreover φ *is a generalized Cayley map of* G*, then* (ii) *may be replaced with*

(ii)$'$ $f\varphi$ *is a generalized Cayley morphism* $G \to \mathfrak{g}$.

Proof. We may assume that φ is not a morphism. Then the indeterminacy locus of φ is an unmixed closed subset X of G of codimension 1. Since, by [Pop$_1$, Theorem 6], the Picard group of the underlying variety of G is finite, this implies that there is $t \in k[G]$ such that $\{g \in G \mid t(g) = 0\} = X$. As φ is G-equivariant, X is G-stable. Hence, by [PV, Theorem 3.1], t is a semi-invariant of G and therefore $t \in k[G]^G$ by the assumption on G. This implies that the function $f = t^m$ for a sufficiently big natural m satisfies the first statement. The second statement follows from Lemma 10.7 below. $\square$

Lemma 10.7. *Let* $\psi : X \dashrightarrow V$ *be a dominant rational map, where* X *is an irreducible algebraic variety,* V *a vector space over* k *and* $\dim X = \dim V$*. Then for every nonzero function* $t \in k(X)$*, at least one of the maps* $\alpha := t\psi$ *and* $\beta := t^2\psi$ *is dominant.*

Proof. Put $h_i := \psi^*(x_i) \in k(X)$, where $x_1, \ldots, x_n$ are the coordinate functions on V with respect to some basis. Then $K := \psi^*\big(k(V)\big) = k(h_1, \ldots, h_n)$, $K_1 := \alpha^*\big(k(\overline{\alpha(X)})\big) = k(th_1, \ldots, th_n)$ and $K_2 := \beta^*\big(k(\overline{\beta(X)})\big) = k(t^2h_1, \ldots, t^2h_n)$, where bar denotes the closure in V. All three fields contain the subfield $K_0 := k(\ldots, h_i/h_j, \ldots)$. We have $\mathrm{trdeg}_k K = n$. Therefore $\mathrm{trdeg}_k K_0 = n - 1$.

Assume the contrary: neither $t\psi$ nor $t^2\psi$ is dominant. Then $\mathrm{trdeg}_k K_1 = \mathrm{trdeg}_k K_2 = n-1$. Since $K_1 = K_0(th_i)$ and $K_2 = K_0(t^2h_i)$ for any i, this implies that both th_i and t^2h_i are algebraic over K_0. Hence $h_i = (th_i)^2/t^2h_i$ is algebraic over K_0. Thus K is algebraic over K_0. Hence $\mathrm{trdeg}_k K = \mathrm{trdeg}_k K_0 = n - 1$, a contradiction. $\square$

From Proposition 10.5 and Lemma 10.6 we deduce the following.

Proposition 10.8. *If the group* $\mathrm{Ad}_G\, G$ *has no nontrivial homomorphisms to* $\mathbf{G}_m$*, then the group* G *admits a generalized Cayley morphism.*

10.9. The Cayley degree. Note that any generalized Cayley map $\varphi : G \dashrightarrow \mathfrak{g}$ has finite degree, i.e., $\deg\varphi := [k(G) : \varphi^*(k(\mathfrak{g}))] < \infty$. Since we are working in characteristic zero, $\deg\varphi$ is equal to the number of points in the fiber $\varphi^{-1}(x)$ for a general point $x \in \mathfrak{g}$; see, e.g., [Sh, II.6.3, Theorem 4].

Example 10.10. Let G be reductive. Take a nonzero integer n. Using Jordan decomposition arguments and the fact that any semisimple element of G lies in a maximal torus, [Bor, Theorem 10.6], one can show that for a general element t of a maximal torus of G, the equation $x^n = t$ has exactly $|n|^{\mathrm{rk}\, G}$ solutions. Since G is reductive, its general element is semisimple, see Lemma 3.3(a). This implies that the G-morphism $\varrho_n : G \to G$, $g \mapsto g^n$, is a dominant map of degree $|n|^{\mathrm{rk}\, G}$. Therefore if φ is a generalized Cayley map of G, then $\varphi \circ \varrho_n$ is its generalized Cayley map of degree $|n|^{\mathrm{rk}\, G} \deg\varphi$. In particular, this shows that

$$\{\deg\varphi \mid \varphi \text{ is a generalized Cayley morphism of } G\}$$

is an infinite set. $\square$

Example 10.11. Let $G = G_1 \times \ldots \times G_n$, where G_i be a connected linear algebraic group with Lie algebra $\mathfrak{g}_i$. Let $\varphi_i : G_i \dashrightarrow \mathfrak{g}_i$ be a generalized Cayley map. The interpretation of degree as the number of points in a general fiber implies that $\varphi_1 \times \ldots \times \varphi_n : G \dashrightarrow \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus g_n$ is a generalized Cayley map and

$$\deg(\varphi_1 \times \ldots \times \varphi_n) = \deg \varphi_1 \ldots \deg \varphi_n. \quad \square \tag{10.12}$$

By Definition 1.7, Cayley maps are exactly generalized Cayley maps of degree 1. This naturally leads to the following definition of a "measure of non-Cayleyness" of G.

Definition 10.13. The *Cayley degree* of G is the number

$$\mathrm{Cay}(G) := \min_{\varphi} \deg \varphi, \tag{10.14}$$

where φ runs through all generalized Cayley maps of G. If G is defined over a subfield K of k, then the *Cayley K-degree* of G is the number $\mathrm{Cay}_K(G)$ defined by the right-hand side of (10.14), where φ runs through all generalized Cayley maps of G defined over K.

Thus G is a Cayley group (respectively, Cayley K-group) if and only if $\mathrm{Cay}(G) = 1$ (respectively, $\mathrm{Cay}_K(G) = 1$). Therefore Theorem 1.35 yields a classification of all connected simple algebraic groups of Cayley degree 1. This result may be considered as a contribution to the solution of the following general problem:

Problem 10.15. *Find the Cayley degrees of connected simple algebraic groups and, more generally, the Cayley K-degrees of connected simple algebraic K-groups.*

Example 10.16. Let G be the K-form of $\mathbf{PGL}_n$ from Example 1.38. Then $\mathrm{Cay}(G) = 1$ but $\mathrm{Cay}_K(G) > 1$. $\square$

From (10.12) and (10.14) we deduce that

$$\begin{aligned} \mathrm{Cay}(G_1 \times \ldots \times G_n) &\leqslant \mathrm{Cay}(G_1) \ldots \mathrm{Cay}(G_n), \\ \mathrm{Cay}_K(G_1 \times \ldots \times G_n) &\leqslant \mathrm{Cay}_K(G_1) \ldots \mathrm{Cay}_K(G_n) \end{aligned} \tag{10.17}$$

(in the last inequality, it is assumed that every G_i is defined over a subfield K of k).
In general, the equality in (10.17) does not hold.

Example 10.18. Theorem 1.35 yields $\mathrm{Cay}(\mathbf{G}_2 \times \mathbf{G}_m^2) = 1$ but $\mathrm{Cay}(\mathbf{G}_2)\mathrm{Cay}(\mathbf{G}_m^2) > 1$. $\square$

The following number can be considered as a natural "measure of stable non-Cayleyness" of G:

Definition 10.19. The *stable Cayley degree* of G is the number

$$\mathrm{sCay}(G) := \min_{n \geqslant 0} \mathrm{Cay}(G \times \mathbf{G}_m^n).$$

If G is defined over a subfield K of k, then the *stable Cayley K-degree* of G is the number

$$\mathrm{sCay}_K(G) := \min_{n \geqslant 0} \mathrm{Cay}_K(G \times \mathbf{G}_m^n).$$

Thus G is a stably Cayley group (respectively, stably Cayley K-group) if and only if $\mathrm{sCay}(G) = 1$ (respectively, $\mathrm{sCay}_K(G) = 1$). Therefore Theorem 1.32 yields a classification of all connected simple algebraic groups of stable Cayley degree 1. This result may be considered as a contribution to the solution of the following general problem:

Problem 10.20. *Find the stable Cayley degrees of connected simple algebraic groups and, more generally, the stable Cayley K-degrees of connected simple algebraic K-groups.*

It follows from Definition 10.19 that

$$\mathrm{sCay}(G) \leqslant \mathrm{Cay}(G), \quad \mathrm{sCay}_K(G) \leqslant \mathrm{Cay}_K(G). \tag{10.21}$$

In general, the equality in (10.21) does not hold.

Example 10.22. Theorem 1.35 and Definition 10.19 imply that $\mathrm{Cay}(\mathbf{G}_2) > 1$ (in Example 10.30 below we shall show that actually $\mathrm{Cay}(\mathbf{G}_2) = 2$) but $\mathrm{sCay}(\mathbf{G}_2) = 1$. □

Using our results, we can find Cayley degrees and stable Cayley degrees of some non-Cayley groups.

Example 10.23. Let $\pi\colon G \to H$ be an isogeny (so $\operatorname{Ker}\pi$ is finite and the Lie algebra of H is $\mathfrak{g}$). Let $\varphi : H \dashrightarrow \mathfrak{g}$ be a generalized Cayley map. Since $\operatorname{Ker}\pi$ is a central subgroup of G and $\deg\pi = |\operatorname{Ker}\pi|$, the composition $\varphi\circ\pi : G \dashrightarrow \mathfrak{g}$ is a generalized Cayley map of degree $|\operatorname{Ker}\pi|\deg\varphi$. From this and Definition 10.13 we deduce that

$$\mathrm{Cay}(G) \leqslant |\operatorname{Ker}\pi|\mathrm{Cay}(H). \tag{10.24}$$

In particular, (10.24) implies the following:

$$\text{If } G \text{ is not Cayley, } H \text{ is Cayley and } \operatorname{Ker}\pi \simeq \boldsymbol{\mu}_2, \text{ then } \mathrm{Cay}(G)\text{=}2. \tag{10.25}$$

If moreover G is not stably Cayley, then (10.25) and (10.21) imply that $\mathrm{sCay}(G) = 2$.

For instance, Example 1.18 and Theorem 1.35 show that the assumptions of (10.25) hold if $G = \mathbf{Spin}_n$, $H = \mathbf{SO}_n$, $n \geqslant 6$, and π the natural projection. This and Theorem 1.32 yield

$$\mathrm{sCay}(\mathbf{Spin}_n) = \mathrm{Cay}(\mathbf{Spin}_n) = \begin{cases} 2 & \text{for } n \geqslant 6,\\ 1 & \text{for } n \leqslant 5.\end{cases} \tag{10.26}$$

If $G = \mathbf{SL}_n/\boldsymbol{\mu}_d$, $H = \mathbf{PGL}_n$ and π is the natural projection, then Examples 1.13, 1.18, Proposition 5.1, Theorem 1.32, and (10.24), (10.25) yield

$$\mathrm{Cay}(\mathbf{SL}_n/\boldsymbol{\mu}_d) \leqslant n/d,$$

$$\mathrm{sCay}(\mathbf{SL}_{2m}/\boldsymbol{\mu}_m) = \mathrm{Cay}(\mathbf{SL}_{2m}/\boldsymbol{\mu}_m) = \begin{cases} 2 & \text{for } m \geqslant 3,\\ 1 & \text{for } m \leqslant 2.\end{cases} \tag{10.27}$$

In turn, taking $G = \mathbf{SL}_4$, $H = \mathbf{SL}_4/\boldsymbol{\mu}_2$ and π the natural projection, we deduce from (10.27), Theorems 1.32, 1.35 and (10.25) that

$$\mathrm{sCay}(\mathbf{SL}_4) = \mathrm{Cay}(\mathbf{SL}_4) = 2.$$

Note that as $\mathbf{SL}_4 \simeq \mathbf{Spin}_6$, the last equality follows also from (10.26). □

The problem of finding $\mathrm{Cay}(G)$ admits the following useful reduction. Let C be a Cartan subgroup of G with the Lie algebra $\mathfrak{c}$. We use the notation of (1.26). By Lemma 2.22, if $\varphi : G\times^N C \dashrightarrow G\times^N \mathfrak{c}$ is a dominant rational G-map, then $\varphi|_C : C \dashrightarrow \mathfrak{c}$ is a dominant rational N-map and $\varphi^{-1}(x) = \varphi|_C^{-1}(x)$ for a general point $x\in\mathfrak{c}$. Hence

$$\deg\varphi = |\varphi^{-1}(x)| = |\varphi|_C^{-1}(x)| = \deg\varphi|_C. \tag{10.28}$$

Thus Lemmas 3.3 and 2.22 yield a degree preserving bijection between generalized Cayley maps for G and dominant rational N-maps $C \dashrightarrow \mathfrak{c}$. Therefore

$$\mathrm{Cay}(G) = \min_{\psi} \deg\psi, \tag{10.29}$$

where ψ runs through all dominant rational N-maps $C \dashrightarrow \mathfrak{c}$. If G is reductive, then ψ in (10.29) runs through all dominant rational W-maps $T \dashrightarrow \mathfrak{t}$.

Example 10.30. Using (10.29), we can show that

$$\mathrm{Cay}(\mathbf{G}_2) = 2.$$

To that end it suffices, by (10.29) and Theorem 1.35, to prove the existence of a rational W-map $T \dashrightarrow \mathfrak{t}$ of degree 2. It can be constructed as follows.

Maintain the notation of Subsection 9.9. In the course of proving Proposition 9.13, see Step 2, we showed that T is birationally $\mathrm{S}_3 \times \mathrm{S}_2$-isomorphic to $(\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t}))_{\mathrm{S}_3\times\mathrm{S}_2\text{-twisted}}$. Consider the action of $\mathrm{S}_3\times\mathrm{S}_2$ on $\mathfrak{t}\otimes\mathfrak{t}$ determined by the action (9.14) and the corresponding action of $\mathrm{S}_3 \times \mathrm{S}_2$ on $\mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$. Then the Segre embedding

$$(\mathbf{P}(\mathfrak{t}) \times \mathbf{P}(\mathfrak{t}))_{\mathrm{S}_3\times\mathrm{S}_2\text{-twisted}} \hookrightarrow \mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$$

is $\mathrm{S}_3 \times \mathrm{S}_2$-equivariant. Its image is the quadric Q given by (9.7).

The points of V_1 in (9.8) are fixed by θ and $\theta(D_{12}) = D_{21}$. Hence $D_{12} + D_{21}$ is a fixed point and $V_0 := V_1+k(D_{12}-D_{21})$ is a submodule of the $\mathrm{S}_3\times\mathrm{S}_2$-module $\mathfrak{t}\otimes\mathfrak{t}$. Consequently, the point $p = (0 : 1 : 1 : 0)$ is fixed and the plane $\mathbf{P}(V_0) \subset \mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$ is stable for the action of $\mathrm{S}_3 \times \mathrm{S}_2$ on $\mathbf{P}(\mathfrak{t} \otimes \mathfrak{t})$. Since the line $\mathbf{P}(V_1) \subset \mathbf{P}(V_0)$ is stable as well, we obtain an action of $\mathrm{S}_3 \times \mathrm{S}_2$ on the affine plane $\mathbf{P}(V_0) \setminus \mathbf{P}(V_1)$. One easily sees that it is isomorphic to the adjoint action of $\mathrm{S}_3 \times \mathrm{S}_2$ on $\mathfrak{t}$. Therefore $\mathbf{P}(V_0)$ and $\mathfrak{t}$ are birationally $\mathrm{S}_3 \times \mathrm{S}_2$-isomorphic.

It now only remains to note that since $p \notin Q$ and Q is a quadric, the restriction to Q of the stereographic projection $\mathbf{P}(\mathfrak{t}\otimes\mathfrak{t}) \dashrightarrow \mathbf{P}(V_0)$ from p is a rational $\mathrm{S}_3\times\mathrm{S}_2$-map of degree 2. □

Proposition 10.31. *Let φ be a generalized Cayley map of a reductive group G. Then*

$$\deg\varphi = [k(G)^G : k(\mathfrak{g})^G].$$

If φ is a generalized Cayley morphism, Proposition 10.31 can be deduced from [Lun_2, Lemme Fondamental]. In the case where φ is the particular generalized Cayley morphism described in Example 10.4, a proof can be found in [KM, Corollary (3.3)].

Proof. Since W is a finite group acting on T and $\mathfrak{t}$ faithfully, we have $[k(T) : k(T)^W] = [k(\mathfrak{t}) : k(\mathfrak{t})^W] = |W|$. From this we deduce that $\deg\varphi|_T := [k(T) : k(\mathfrak{t})] = [k(T)^W : k(\mathfrak{t})^W]$. But, by (3.5), we have $[k(T)^W : k(\mathfrak{t})^W] = [k(G)^G : k(\mathfrak{g})^G]$. The claim now follows from (10.28). □

If X is an irreducible algebraic variety endowed with an action of an algebraic group H, and V is a vector space over k of dimension $\dim X$ endowed with a linear action of H, then rational dominant H-maps $X \dashrightarrow V$ are described as follows. Let M be a submodule of the H-module $k(X)$ such that

(i) M is isomorphic to the H-module V^*,
(ii) $k(X)$ is algebraic over the subfield $k(M)$ generated by M over k.

By (ii), $k(M)/k$ is a purely transcendental extension of degree $\dim X$. Since $k(V)$ is generated over k by V^*, any isomorphism of H-modules $V^* \to M$ can be uniquely extended up to an H-equivariant embedding $\iota\colon k(V) \hookrightarrow k(X)$ whose image is $k(M)$. This embedding determines a dominant rational H-map $\psi\colon X \dashrightarrow V$ such that $\psi^* = \iota$. We have $\deg\psi = [k(X) : k(M)]$. Any dominant rational H-map $X \dashrightarrow V$ is obtained in this way.

This description can be used for obtaining upper bounds for $\mathrm{Cay}(G)$.

Example 10.32. Let $G = \mathbf{G}_2$. Use the notation of Subsection 9.9. Let x_i be the restriction to T of the ith coordinate function on D, i.e., $x_i(\operatorname{diag}(a_1, a_2, a_3)) = a_i$. Then $x_1x_2x_3 = 1$ and $k(T) = k(x_1, x_2)$. Put

$$y_i := x_i - x_i^{-1}. \tag{10.33}$$

From (9.4), (9.10) and (9.5) it follows that

$$M := \{\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 \mid \alpha_1 + \alpha_2 + \alpha_3 = 0, \ \ \alpha_i \in k\}$$

is a submodule of the $\mathrm{S}_3 \times \mathrm{S}_2$-module $k(T)$ that is isomorphic to the $\mathrm{S}_3 \times \mathrm{S}_2$-module $\mathfrak{t}^*$. Let

$$z_1 := y_1 - y_2, \ z_2 := y_1 - y_3 \tag{10.34}$$

Then z_1, z_2 is a basis of M, so $k(M) = k(z_1, z_2)$. We have $k(x_1, z_1, z_2) = k(T)$ (because $x_2 = (x_1^2 - 1)(x_1^2 z_1 + x_1 z_2 - x_1^3 - x_1^2 + x_1 + 1)^{-1}$). It follows from (10.33), (10.34) that

$$\begin{cases} -x_2 + x_2^{-1} = z_1 - x_1 + x_1^{-1}, \\ x_1x_2 - x_1^{-1}x_2^{-1} = z_2 - x_1 + x_1^{-1}. \end{cases} \tag{10.35}$$

Eliminating x_2 and x_2^{-1} from (10.35), we obtain the following equation:

$$\begin{aligned} x_1^6 - (z_1 + z_2)x_1^5 + (z_1z_2 - 2z_1 - 2z_2 - 1)x_1^4 + (z_1^2 + z_2^2 - 5)x_1^3 \\ + (z_1z_2 + 2z_1 + 2z_2 + 1)x_1^2 + (z_1 + z_2 + 1)x_1 + 1 = 0. \end{aligned}$$

Thus for the conjugating and adjoint actions of $H := \mathrm{S}_3 \times \mathrm{S}_2$ on $X := T$ and $V := \mathfrak{t}$ respectively the above conditions (i), (ii) hold for M and $[k(T) : k(M)] \leqslant 6$. Hence $\mathrm{Cay}(\mathbf{G}_2) \leqslant 6$ (of course, by Example 10.30, we know that in fact $\mathrm{Cay}(\mathbf{G}_2) = 2$). $\square$

Appendix. Alternative proof of Proposition 9.2: an outline

Step 1. Consider D, see (9.3), as an open subset of $\mathbf{P}^3$ given by $x_0 \neq 0$, and extend the S_3-action (9.4) up to $\mathbf{P}^3$ by

$$\sigma(a_0 : a_1 : a_2 : a_3) = (a_0 : a_{\sigma(1)} : a_{\sigma(2)} : a_{\sigma(3)}), \quad \text{where } \sigma \in \mathrm{S}_3.$$

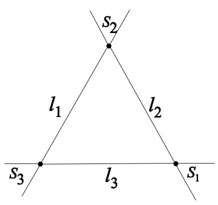
The closure X of T in $\mathbf{P}^3$, see (9.5), is the rational cubic surface given by $x_1x_2x_3 - x_0^3 = 0$. It has exactly three fixed points

$$a_i := (1 : \varepsilon^i : \varepsilon^i : \varepsilon^i), \ i = 1, 2, 3, \quad \varepsilon^3 = 1, \ \varepsilon \neq 1,$$

and three singular (double) points

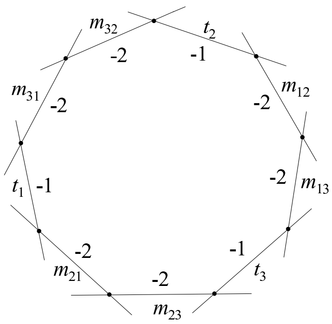
$$s_1 = (0 : 1 : 0 : 0), \ s_2 = (0 : 0 : 1 : 0), \ s_3 = (0 : 0 : 0 : 1).$$

The hyperplane section of X given by $x_0 = 0$ is $H := l_1 + l_2 + l_3$, where l_i is the line given by $x_0 = x_i = 0$.



Since H is S_3-invariant, the S_3-action on X lifts to the surface $\widetilde{X}$ obtained from X by the simultaneous blowing up $\mu : \widetilde{X} \to X$ of s_1, s_2, s_3. The surface $\widetilde{X}$ is smooth and T is its open S_3-stable subset.

Step 2. We have $\mu^*(H) = \sum_i t_i + \sum_{ij} m_{ij}$ where t_i is the proper inverse image of l_i and $\mu^{-1}(s_i) = m_{ij} \cup m_{ir}$, $\{i, j, r\} = \{1, 2, 3\}$. The curves t_i, m_{ij} are isomorphic to $\mathbf{P}^1$ and form a 9-gon as shown on the figure below. Their intersections are transversal and the self-intersection indices are $(t_i, t_i) = -1$, $(m_{ij}, m_{ij}) = -2$.



Computing the canonical classes gives $K_X = -H$ and $K_{\widetilde{X}} = \mu^*(-H)$. Hence

$$(K_{\widetilde{X}}, K_{\widetilde{X}}) = (-H, -H) = \deg X = 3. \tag{A1}$$

Step 3. By the Castelnuovo criterion, the curves t_i are exceptional, so they can be simultaneously blown down: $\nu \colon \widetilde{X} \to Y$. The surface Y is smooth, and the S_3-invariance of $t_1 + t_2 + t_3$ implies that the action of S_3 on $\widetilde{X}$ descends to Y. We can consider T as an open S_3-stable subset of Y.

It follows from (A1) that

$$(K_Y, K_Y) = 6, \tag{A2}$$

and Pic $T = 0$ implies that $(\text{Pic}\, Y)^{S_3}$ is generated by $D := \nu_*(\sum_{ij} m_{ij})$. Hence $K_Y = nD$ for some nonzero integer n. Rationality of Y implies $n < 0$. From this, the Nakai–Moishezon criterion and (A2) one deduces that $-K_Y$ is ample. In turn, using this fact, the Riemann–Roch theorem, the Castelnuovo rationality criterion and the Kodaira vanishing theorem, one shows that

$$\dim \mathrm{H}^0\big(Y, \mathcal{O}(-K_Y)\big) = 7. \tag{A3}$$

Applying the Riemann–Roch theorem again, one further deduces that the linear system $|-K_Y|$ has no fixed components. Using this, (A3), Bertini's theorem, the Riemann–Roch theorem and (A2) one shows that $|-K_Y|$ has no base points.

Step 4. Thus $|-K_Y|$ defines a morphism $Y \to \mathbf{P}^6$ equivariant with respect to a certain action of S_3 on $\mathbf{P}^6$. Using (A2), one shows that in fact it is an embedding. We keep the notation Y for its image.

Consider on Y the linear system $|R|$ of all hyperplane sections in $\mathbf{P}^6$ containing the fixed point $a_1 \in T \subseteq Y$ and singular at a_1. Such hyperplanes are tangent to Y at a_1, so

$$\dim |R| = 4. \tag{A4}$$

This system $|R|$ is an S_3-stable subsystem of $|-K_Y|$. Using Bertini's theorem, one deduces that its general element is an irreducible rational curve whose singular locus is the double point a_1. This system has no fixed components, and (A4) implies that a_1 is its unique base point.

Step 5. Let $\gamma\colon \widetilde{Y} \to Y$ be the blowing up of a_1. The action of S_3 lifts to $\widetilde{Y}$. The proper inverse image $|\widetilde{R}|$ of $|R|$ is a 4-dimensional S_3-stable linear system on $\widetilde{Y}$. It has no base points and separates points of an open subset of $\widetilde{Y}$. Hence $|\widetilde{R}|$ defines an S_3-equivariant morphism $\psi\colon \widetilde{Y} \to \mathbf{P}^3$ with respect to a certain S_3-action on $\mathbf{P}^3$. Its image $Z := \psi(\widetilde{Y})$ is an S_3-stable quadric in $\mathbf{P}^3$, and $\psi\colon \widetilde{Y} \to Z$ is a birational S_3-isomorphism.

Step 6. Since the point $a_2' := \psi \circ \gamma^{-1}(a_2) \in Z$ is fixed by S_3, it follows from the complete reducibility of representations of reductive groups that there is an S_3-stable plane $L \simeq \mathbf{P}^2$ in $\mathbf{P}^3$ not passing through a_2'. Consider the stereographic projection $\pi\colon Z \dashrightarrow L$ from a_2'; it is S_3-equivariant. The map π is defined at $\psi \circ \gamma^{-1}(a_3)$ and $a_3' := \pi \circ \psi \circ \gamma^{-1}(a_3) \in L$ is a fixed point of S_3. Using the complete reducibility argument again, we conclude that there is an S_3-stable line $l \subset L$ such that $a_3' \in L \setminus l$. Thus we obtain a faithful linear action of S_3 on $\mathbf{A}^2 \simeq L \setminus l$. But there is a unique 2-dimensional faithful linear representation of S_3, namely that on $\mathfrak{t}$ given by (9.4), (9.5). This completes the proof. $\square$

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