GALOIS MODULE STRUCTURE OF pTH-POWER CLASSES OF CYCLIC EXTENSIONS OF DEGREE p^n

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Dedicated to the memory of Walter Feit

Abstract. In the mid-1960s Borevič and Faddeev initiated the study of the Galois module structure of groups of pth-power classes of cyclic extensions K/F of pth-power degree. They determined the structure of these modules in the case when F is a local field. In this paper we determine these Galois modules for all base fields F.

Contents

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Introduction and Main Theorems

In 1947 Šafarevič initiated the study of Galois groups of maximal p-extensions of fields with the case of local fields [Ša47], and this study has grown into what is both an elegant theory as well as an efficient tool in the arithmetic of fields. From the very beginning it became clear that the groups of pth-power classes of the various field extensions of a base field encode basic information about the structure of the Galois groups of maximal p-extensions. (See [Ko02, Se02].) Such groups of pth-power classes arise naturally in studies in arithmetic algebraic geometry, as for example in studies of elliptic curves.

In 1960 Faddeev began to study the Galois module structure of pth-power classes of cyclic p-extensions, again in the case of local fields, and during the mid-1960s he and Borevič established the structure of these Galois modules using basic arithmetic invariants attached to Galois extensions. (See [Fa60, Bo65].) In 2003 two of the authors ascertained the Galois module structure of pth-power classes in the case of cyclic extensions of degree p over all base fields F containing a primitive pth root of unity [MS03]. Very recently, this work paved the way for the determination of the entire Galois cohomology as a Galois module in the case of a cyclic extension of degree p of a base field containing a primitive pth root of unity, using Voevodsky's recent work on Galois cohomology ([LMS]; see [Vo03, Vo]).

In this paper we extend the results obtained in [MS03] in two directions. First, our results hold for cyclic extensions of any pth-power degree, rather than just p, and, furthermore, we no longer require that the base field contain a primitive pth root of unity. Thus our results provide a complete determination of pth-power classes as Galois modules for all cyclic extensions of pth-power degree.

We expect that, just as the results and techniques in [MS03] helped to determine the entire Milnor K-theory modulo p as a Galois module in the case of cyclic extensions of degree p, so will the results and methods developed in this paper lead to the determination of the entire Milnor K-theory modulo p as a Galois module in the case of cyclic extensions of pth-power degree. In fact, precisely such a generalization has already taken place in the case of characteristic p [BLMS].

Similarly, in the same way as the results and techniques developed in [MS03] led in [MS] to the solution of Galois embedding problems and the discovery of a new automatic realization of Galois groups, it is clear that the results in this paper will also have such Galois-theoretic applications. In a subsequent paper [MSSa] we will consider some of these applications.

Our basic approach to the problem is induction, and some of the results in [MS03] handle the base case. In the end, however, neither the results nor the techniques employed are straightforward generalizations of the work in [MS03]. First, the possible generalization of the innocent summand of dimension 1 or 2 considered in [MS03] turned out to be rather subtle to handle. These new summands of dimension $p^i + 1$ for some $i \in \mathbb{N}$ are very interesting invariants of cyclic extensions of pth-power degree. Another substantial challenge was to generate enough norms, and the resolution involves several thorny induction arguments. Finally, the case $p = 2$ presented a new problem for quartic extensions, and this problem is taken care of as a separate base induction case.

Fundamentally, the classification of pth-power classes as Galois modules depends upon arithmetic invariants, all of which originate from the images of the norms of the intermediate fields of K/F. The classification, in short, has the flavor of local class field theory, and although the arguments underlying the classification are not straightforward, the final results, just as in local class field theory, have a rather simple and elegant form, which we now describe.

Let p be a prime number, $n \geq 1$ an integer, F an arbitrary field, and K a Galois extension of F with group $G = \langle \sigma \rangle$ cyclic of order p^n. Let $F^\times$ denote the multiplicative group of nonzero elements of F. Let $J = J(K) = K^\times/K^{\times p}$ be the $\mathbb{F}_p[G]$-module of pth-power classes, denoted by $[\gamma]$ for $\gamma \in K^\times$. Similarly, let $J(F) = F^\times/F^{\times p}$ be the $\mathbb{F}_p$-module of pth-power classes of $F^\times$, denoted by $[f]_F$ for $f \in F^\times$. Let $N_{K/F}\colon K \to F$ be the norm map, and write $N\colon K^\times/K^{\times p} \to F^\times/F^{\times p}$ for the map induced by $N_{K/F}$. Also by abuse of notation we use the same symbol N to denote the endomorphism $N\colon K^\times/K^{\times p} \to K^\times/K^{\times p}$ induced by $N\colon K^\times/K^{\times p} \to F^\times/F^{\times p}$ defined above, followed by the map induced by the inclusion map $\epsilon = \epsilon_K\colon F^\times \to K^\times$.

Further let K_i, $i = 0, \ldots, n$, be the intermediate field of K/F such that $[K_i : F] = p^i$. Denote by H_i the Galois group $\mathrm{Gal}(K/K_i) \subset G$. Let $[K_i^\times]$ denote the submodule of J which is the image of the map induced by the inclusion map $K_i^\times \to K^\times : \left[K_i^\times\right] = K_i^\times K^{\times p}/K^{\times p}$. Similarly, for other G-submodules $A \subset K^\times$, such as $A = N_{K_i/F}(K_i^\times)$, let $[A] = AK^\times/K^{\times p}$.

Theorem 1. *Suppose that either*

- $\xi_p \notin F$, *or*
- $p = 2$, $n = 1$, *and* $-1 \notin N_{K/F}(K^\times)$.

Then the $\mathbb{F}_p[G]$*-module* J *decomposes as*

$$J = Y_n \oplus Y_{n-1} \oplus \cdots \oplus Y_0,$$

where Y_i *is a direct sum of cyclic* $\mathbb{F}_p[G]$*-modules of dimension* p^i *and*

$$[K_i^\times] = J^{H_i}, \quad 0 \le i \le n.$$

It is easy to show that this decomposition of J is unique. (In fact this also follows from a well-known result of Azumaya. See [AnFu73, page 144].) In the following corollary we determine the sizes of the modules Y_i in terms of norms. Observe that direct sums of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i are free $\mathbb{F}_p[G/H_i]$-modules. Let

$$e_i = \dim_{\mathbb{F}_p} \left(\left[N_{K_i/F}\left(K_i^\times\right)\right] / \left[N_{K_{i+1}/F}\left(K_{i+1}^\times\right)\right]\right), \quad 0 \le i < n,$$

and let $e_n = \dim_{\mathbb{F}_p}[N_{K/F}(K^\times)]$.

Corollary 1. *For each* $0 \le i \le n$,

$$[N_{K_i/F}(K_i^\times)] = (Y_i + Y_{i+1} + \cdots + Y_n)^G,$$

and

$$\operatorname{rank}_{\mathbb{F}_p[G/H_i]} Y_i = e_i.$$

For K/F not satisfying the conditions of the theorem above, we adopt the conventions $K_{-\infty}^\times = \{1\}$ and $p^{-\infty} = 0$ and make the following definition.

Definition (Exceptional Element)**.** Suppose that $\xi_p \in F$ and, if $p = 2$, that either $n > 1$ or $-1 \in N_{K/F}(K^\times)$. We set

$$\begin{aligned} i(K/F) := \min\{\ i \in \{-\infty, 0, 1, \dots, n\} \mid \exists \delta \in K^\times \text{ such that } \\ [N_{K/F}(\delta)]_F \neq [1]_F \text{ and } \\ [\delta]^{\tau-1} \in [K_i^\times]\ \forall \tau \in \mathrm{Gal}(K/F)\}. \end{aligned}$$

We say that $\delta \in K^\times$ is an *exceptional element* of K/F if $[N_{K/F}(\delta)]_F \neq [1]_F$ and $[\delta]^{\tau-1} \in [K_{i(K/F)}^\times]$ for all $\tau \in \mathrm{Gal}(K/F)$. Elements of $K^\times$ that are not exceptional are said to be *unexceptional*. For simplicity, we often write m instead of $i(K/F)$.

Observe that $[\delta]^{(\tau-1)} \in [K_i^\times]$ for all $\tau \in G$ if and only if $[\delta]^{(\sigma-1)} \in [K_i^\times]$ for a fixed generator $\sigma \in G$. In what follows we will use this formulation for our given generator σ.

Note that if δ is an exceptional element then $m = i(K/F) = -\infty$ if and only if $[\delta]^\sigma = [\delta]$ and $[N_{K/F}(\delta)]_F \neq [1]_F$.

Because the exceptionality of an element $\gamma \in K^\times$ is independent of the particular representative γ of $[\gamma]$, we define $[\gamma]$ to be exceptional if γ is exceptional. It is also useful to observe that if an $\mathbb{F}_p[G]$-generator $[\gamma]$ of a module $M_\gamma \subset J$ is exceptional, then so is any other $\mathbb{F}_p[G]$-generator $[\omega]$ of M_γ. Indeed, using additive notation for J for the moment, any such generator $[\omega]$ has the form

$$[\omega] = c_0[\gamma] + c_1(\sigma-1)[\gamma] + c_2(\sigma-1)^2[\gamma] + \dots, \quad c_0, c_1, \dots \in \mathbb{F}_p, \quad c_0 \neq 0.$$

Then $[N_{K/F}(\omega)]_F = [N_{K/F}(\gamma)]_F^{c_0} \neq [1]$ and $[\omega]^{\sigma-1} \in [K_m^\times]$.

In Proposition 2 we show that exceptional elements always exist for K/F satisfying the hypothesis in the Definition above, and in Proposition 7 we show that, in fact, $m \leq n-1$. Finally, note that since $N_{K/F}(K_{n-1}^\times) \subset F^{\times p}$, each exceptional element $\delta \in K_n^\times \setminus K_{n-1}^\times$.

Moreover, for these K/F, we have Kummer theory, because $\xi_p \in F$. Hence $K_1 = F(\sqrt[p]{a})$ for some $a \in F$. In section 4 we prove some more specific results about exceptional elements in terms of a: exceptional elements satisfy $[N_{K/F}(\delta)]_F = [a]_F^s$ for $s \not\equiv 0 \bmod p$ and that for all K/F as above, an exceptional element $\delta \in K^\times$ exists satisfying $[N_{K/F}(\delta)]_F = [a]_F$.

Theorem 2. *Suppose that $\xi_p \in F$ and, if $p = 2$, that either $n > 1$ or $-1 \in N_{K/F}(K^\times)$.*

Let $\delta \in K^\times$ be any exceptional element of K/F. Then the $\mathbb{F}_p[G]$-module J decomposes as

$$J = X \oplus Y, \quad Y = Y_n \oplus Y_{n-1} \oplus \cdots \oplus Y_0,$$

where

(1) *X is the cyclic $\mathbb{F}_p[G]$-module generated by $[\delta]$, with dimension $p^m + 1$;*
(2) *Y_i is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i; and*

(3) *for* $i \in \{0, \dots, n\}$,

$$[K_i^\times] = \begin{cases} X^{(\sigma-1)} \oplus Y^{H_i}, & m \le i; \\ X^{(\sigma-1)(\sigma^{p^i}-1)^{p^{m-i}-1}} \oplus Y^{H_i}, & i < m. \end{cases}$$

(Here $X^{(\sigma-1)}$ *and* $X^{(\sigma-1)(\sigma-1)^{p^{m-i}-1}}$ *denote images of* X *under the action of* $(\sigma - 1)$ *and* $(\sigma-1)(\sigma^{p^i}-1)^{p^{m-i}-1}$ *respectively.)*

As before, let

$$e_i = \dim_{\mathbb{F}_p} \left(\left[N_{K_i/F} \left(K_i^\times \right) \right] / \left[N_{K_{i+1}/F} \left(K_{i+1}^\times \right) \right] \right), \quad 0 \le i < n,$$

and let $e_n = \dim_{\mathbb{F}_p} [N_{K/F}(K^\times)]$.

Corollary 2. *For each* $m < i \le n$,

$$[N_{K_i/F}(K_i^\times)] = (Y_i + Y_{i+1} + \cdots + Y_n)^G,$$

and, if $m \ge 0$, *for each* $0 \le i \le m$,

$$[N_{K_i/F}(K_i^\times)] = (X + Y_i + Y_{i+1} + \cdots + Y_n)^G.$$

For $i \ne m$,

$$\operatorname{rank}_{\mathbb{F}_p[G/H_i]} Y_i = e_i,$$

while if $m \ge 0$,

$$1 + \operatorname{rank}_{\mathbb{F}_p[G/H_m]} Y_m = e_m.$$

Finally, we present several interesting conditions equivalent to $m = i(K/F)$ being a particular element of the subset of field indices $\mathcal{E} = \{-\infty, 0, \dots, n-1\}$. To express these conditions, we define $-\infty \dot{+} 1 = 0$ and, for $e \in \mathcal{E}$ with $e \ge 0$, we define $e \dot{+} 1 = e + 1$. We also set $N_{K_{n-1}/F}(K_{-\infty}^\times)$ to be $\{1\}$.

Theorem 3. *Suppose that* $\xi_p \in F$ *and, if* $p > 2$, *that either* $n > 1$ *or* $-1 \in N_{K/F}(K^\times)$.

Then

$$\begin{aligned} i(K/F) &= \min \left\{ s \mid \xi_p \in N_{K/F}(K^\times) N_{K_{n-1}/F}(K_s^\times) \right\} \\ &= \min \left\{ s \mid \xi_p \in N_{K/K_{s \dot{+} 1}}(K^\times) \right\} \\ &= \min \left\{ s \mid \exists [\delta] \in J^{H_{s \dot{+} 1}}, [N_{K/K_{s \dot{+} 1}} \delta]_{K_{s \dot{+} 1}} \ne [1]_{K_{s \dot{+} 1}} \right\}. \end{aligned}$$

One can also connect these equalities with the existence of solutions of particular Galois embedding problems. This connection will be pursued in a forthcoming paper. (See [MSSa].) X summands also lead naturally to the investigation of some cyclotomic cyclic algebras over F which, in turn, allow us to construct fields with prescribed X summands. This topic will also be pursued in a subsequent paper. (See [MSSb].)

The proofs of Theorems 1 and 2 are inductive, resting on the base case $n = 1$ for Theorem 1 and two base cases $n = 1$ and $p = 2$, $n = 2$ for Theorem 2. In these base cases as well as the inductive proof, we employ lemmas which establish the structure of the fixed submodule J^G of J—in particular, whether this fixed submodule is no more than the image of the pth-power classes of the base field F—and specify which of these elements are norms.

In fact, these lemmas reflect what has emerged, both in this work as well as in the work on determining the entire Milnor K-theory modulo p as a Galois module (see [BLMS] and [LMS]), as two essential foundational ingredients in the proof. The first is Hilbert's Theorem 90, which in our situation may be viewed as a principle saying that we have enough norms. Indeed, Hilbert 90 tells us that the kernel of the norm map is as small as possible. In order to use Hilbert 90 effectively, we need again and again the technical refinements of this principle telling us that certain elements in a group of pth-power classes are norms. In this work these refinements, for example, begin with Lemmas 10, 11, and 12 (identifying some fixed elements as norms), and are completed in the full proofs of Theorems 1 and 2.

The second essential ingredient is control of the image of pth-power classes of the base field in the group of pth-power classes of our field extension, which in this work is obtained from Lemma 6 (the Exact Sequence Lemma) and its technical relative Lemma 5 (the Fixed Submodule Lemma). In this paper, both of these principles are elementary, but they are more sophisticated in the higher Milnor K-theory case. It is remarkable that one requires only repetitions of these two principles in order to determine fully the Galois module structure of the modules in question. Drawing out the structure from only these two first principles, however, does not come without cost, and a number of technical observations turn out to be necessary for us to fit the puzzle pieces together.

When $\xi_p \in F$, we need additional information to determine when an element $[\gamma] \in J^{H_i}$ lies in $[K_i^\times]$ or is instead an exceptional element. We begin by standardizing choices of the a_i in the presentations of subfields $K_{i+1} = K_i(\sqrt{a_i})$ in section 1.2. Then, in section 1.4, we collect several results used in identifying elements of $[K_i^\times]$. These are the Submodule-Subfield Lemma (7) for free components, the Norm Lemma (8) for comparisons among norms from K to various K_i (in order to determine when an exceptional element for K/F is an exceptional element for K/K_i), and the Proper Subfield Lemma (9) for elements that generate sufficiently small cyclic submodules.

In section 1.1, we present lemmas which we use to manipulate $\mathbb{F}_p[G]$-representations formally: the Inclusion Lemma (1), the Exclusion Lemma (2), and the Free Complement Lemma (3).

We begin the proof by proving the base cases for an induction in section 2. Our inductive strategy is first to show that J contains a sufficiently large direct sum of $\mathbb{F}_p[G]$-submodules of pth-power dimensions. We do so in section 3 in Proposition 6, the result of which is already enough to prove Theorem 1. When $\xi_p \in F$ and, if $p = 2$, $n > 1$, we also need to establish the dimension of the X component and connect notions of exceptional elements for subextensions K/K_i. We do so in section 4. In section 5, we prove an analogue of Proposition 6 which establishes Theorem 2 without the independence of X and Y, and then we prove Theorem 2 fully. Finally, in section 7, we prove Theorem 3.

For the reader's convenience, we have made our paper self-contained; in particular, it is independent from [MS03].

1. Notation and Lemmas

1.1. $\mathbb{F}_p[G]$-modules.

Let G be a cyclic group of order p^n with generator σ. For an $\mathbb{F}_p[G]$-module U, let U^G denote the submodule of U fixed by G, and for an arbitrary element $u \in U$, let $l(u)$ denote the dimension of the $\mathbb{F}_p[G]$-submodule of U generated by u. Denote by N the operator $(\sigma-1)^{p^n-1}$ acting on U. For an $\mathbb{F}_p[G]$-module V and an element $\gamma \in V$, let $\langle\gamma\rangle$ denote the $\mathbb{F}_p$-subspace of V spanned by γ, and let M_γ denote the cyclic $\mathbb{F}_p[G]$-module generated by γ. If $[\gamma]$ is an element of $K^\times/K^{\times p}$ represented by $\gamma \in K^\times$, we write M_γ instead of $M_{[\gamma]}$.

We will usually use additive notation for general $\mathbb{F}_p[G]$-modules, switching to multiplicative notation when considering the specific module $J = K^\times/K^{\times p}$. However, occasionally even in this case we employ additive notation, in particular writing $\{0\}$ to denote $\{[1]\}$.

Lemma 1 (Inclusion Lemma)**.** *Let U and V be $\mathbb{F}_p[G]$-modules contained in an $\mathbb{F}_p[G]$-module W. Suppose that $(U+V)^G \subset U$ and for each $w \in (U+V) \setminus (U+V)^G$ there exists $u \in U$ such that*

$$(\sigma - 1)^{l(w)-1}(w) = N(u).$$

Then $V \subset U$.

Proof. Let $\{T_i\}_{i=1}^s$ be the socle series of $U+V$: $T_1 = (U+V)^G$ and $T_{i+1}/T_i = ((U+V)/T_i)^G$, and let s be the least natural number such that $T_s = U+V$. Observe that since $(\sigma-1)^{p^n} = 0$, we have $s \le p^n$. We prove the lemma by induction on the socle series.

By hypothesis, $T_1 \subset U$. Assume now that $T_i \subset U$ for some $i < s$. Then for each $w \in T_{i+1} \setminus T_i$ we have $l(w) = i+1$ and $(\sigma-1)^{l(w)-1}(w) = N(u) = (\sigma-1)^{p^n-1}(u)$ for some $u \in U$. Therefore

$$(\sigma-1)^{l(w)-1}\big(w - (\sigma-1)^{p^n-l(w)}(u)\big) = 0.$$

Therefore $w - (\sigma-1)^{p^n-l(w)}(u) \in T_i \subset U$. Hence $w \in U$ and $T_{i+1} \subset U$. Therefore $U+V = U$ and $V \subset U$ as required. □

Lemma 2 (Exclusion Lemma)**.** *Let U and V be $\mathbb{F}_p[G]$-modules contained in an $\mathbb{F}_p[G]$-module W. Suppose that $U^G \cap V^G = \{0\}$. Then $U+V = U \oplus V$.*

Proof. Let $Z = U \cap V$ and suppose that $y \in Z \setminus \{0\}$. Let

$$z = (\sigma-1)^{l(y)-1}(y) \neq 0.$$

Then $z \in U^G \cap V^G$, a contradiction. Hence $U \cap V = \{0\}$ and $U+V = U \oplus V$. □

The following lemma follows from the fact that each free $\mathbb{F}_p[G]$-module is injective. (See [Ca96, Theorem 11.2].) We shall, however, provide a direct proof.

Lemma 3 (Free Complement Lemma)**.** *Let $V \subset U$ be free $\mathbb{F}_p[G]$-modules. Then there exists a free $\mathbb{F}_p[G]$-submodule $\tilde{V}$ of U such that $V \oplus \tilde{V} = U$.*

Proof. Let Z be a complement of V^G in U^G as $\mathbb{F}_p$-vector spaces, and let $\mathcal{Z}$ be an $\mathbb{F}_p$-base of Z. For each $z \in \mathcal{Z}$, there exists $u(z)$ such that $z = N(u(z))$. Let $M(z)$ be the $\mathbb{F}_p[G]$-submodule of U generated by $u(z)$. Then $M(z)$ is a free $\mathbb{F}_p[G]$-submodule. Moreover, its fixed submodule $M(z)^G$ is the $\mathbb{F}_p$-vector subspace generated by z.

We claim that the $M(z)$, $z \in \mathcal{Z}$, are independent. First we show by induction on the number of modules that a finite set of modules $M(z)$ is independent. The base case is trivial. Now let $W = M(z) \cap \sum_{z' \neq z} M(z')$. Now by the inductive assumption on independence, $(\sum_{z' \neq z} M(z'))^G = \sum_{z' \neq z} M(z')^G$, and for each z, $M(z)^G = \langle z \rangle$. Since the z form an $\mathbb{F}_p$-base for Z, we obtain $W^G = \{0\}$. The Exclusion Lemma (2) then gives that $M(z)+\sum_{z' \neq z} M(z') = M(z) \oplus \sum_{z' \neq z} M(z')$.

The case of an infinite sum follows from the same argument, since the fact that $m \in M(z)^G \cap \sum_{z' \neq z} M(z')^G$ forces m to be a finite sum of elements $m(z')$. Hence the $M(z)$, $z \in \mathcal{Z}$, are independent.

Set $\tilde{V} := \oplus_{z \in \mathcal{Z}} M(z)$. Then $\tilde{V}$ is a free $\mathbb{F}_p[G]$-submodule of U and $\tilde{V}^G = Z$. By the Exclusion Lemma (2), we have that $V + \tilde{V} = V \oplus \tilde{V}$ and $(V \oplus \tilde{V})^G = V^G \oplus \tilde{V}^G = U^G$.

Now let $u \in U$ be arbitrary and let M be the cyclic $\mathbb{F}_p[G]$-submodule of U generated by u. Then $(M + V + \tilde{V})^G \subset U^G \subset V + \tilde{V}$. Moreover, for any $m \in (M + V + \tilde{V}) \setminus (M + V + \tilde{V})^G$,

$$(\sigma - 1)^{l(m)-1}(m) \in (M + V + \tilde{V})^G \subset U^G = (V + \tilde{V})^G = N(V + \tilde{V})$$

by the freeness of V and $\tilde{V}$. By the Inclusion Lemma (1), then, $M \subset V + \tilde{V}$. Hence $U = V \oplus \tilde{V}$. □

Remark. At several points later, we use the same argument as that contained in the proof above to show that a possibly infinite set of modules is independent, and we use the Exclusion Lemma (2) as an abbreviation for this argument.

1.2. **Kummer Subfields of K/F and Exceptional Elements.**

Suppose that $\xi_p \in F$. In this case we have Kummer theory and may organize presentations of the extensions K_{i+1}/K_i as follows.

Proposition 1 (Subfield Generators)**.** *We may choose $a_i \in K_i^\times$, $0 \leq i < n$ such that*

- $K_{i+1} = K_i(\sqrt[p]{a_i})$ *and*
- $N_{K_i/K_j} a_i = a_j$ *for all* $0 \le j < i < n$.

In what follows we will assume that the choices of a_i have been made according to Proposition 1, and we set $a = a_0$.

We prove this result by means of the following

Lemma 4. *Suppose that* $\xi_p \in K$ *and let* L'/K *be a cyclic extension of degree* p^2 *with* L/K *the intermediate extension of degree* p. *Then, for every* $b \in L$ *with* $L' = L(\sqrt[p]{b})$, *we have* $L = K(\sqrt[p]{N_{L/K}(b)})$.

Proof. Let σ be a generator of $\mathrm{Gal}(L'/K)$. For each $i \in \{1, 2, \dots, p-1\}$, we have

$$\left(\sqrt[p]{b}\right)^{\sigma^i} = \sqrt[p]{b^{\sigma^i}}$$

for a suitable choice of a pth root of b^{σ^i}. Hence

$$\left(\sqrt[p]{b}\right)^{1+\sigma+\cdots+\sigma^{p-1}} = \sqrt[p]{b^{1+\sigma+\cdots+\sigma^{p-1}}} = \sqrt[p]{N_{L/K}(b)} \in L'$$

for a suitable choice of a pth root of $N_{L/K}(b)$.

Observe that since $\xi_p \in K$ the equality

$$\left(\sqrt[p]{b}\right)^{(1+\sigma+\cdots+\sigma^{p-1})(\sigma-1)} = \sqrt[p]{b}^{\sigma^p - 1} = \sqrt[p]{N_{L/K}(b)}^{\sigma-1}$$

is independent of the choice of pth roots. Moreover, since $L' = L(\sqrt[p]{b})$ and σ^p generates $\mathrm{Gal}(L'/L)$, we see that $\sqrt[p]{b}^{\sigma^p-1} \neq 1$. Hence we conclude that $L = K(\sqrt[p]{N_{L/K}(b)})$. □

Proof of Proposition 1. By Kummer theory, there exists $a_{n-1} \in K_{n-1}^\times$ such that $K_n = K_{n-1}(\sqrt[p]{a_{n-1}})$. Then inductively define

$$a_{n-i} = N_{K_{n-i+1}/K_{n-i}}(a_{n-i+1})$$

for $i \in \{2, \dots, n\}$. Applying the lemma to extensions K_{n-i+2}/K_{n-i}, we have the results. □

Our definition of exceptional elements makes use of a subset of the set $\{\delta \in K^\times \mid [N_{K/F}(\delta)]_F \neq [1]_F\}$. In general, however, this latter set may be empty. Consider, for example, the extension $\mathbb{C}/\mathbb{R}$, for which $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) \subset \mathbb{R}^{\times 2}$. The next proposition shows that under the conditions we require in the definition of exceptional elements, this set is never empty and therefore exceptional elements exist.

Proposition 2. *Let $\xi_p \in F$ and, if $p = 2$, that $n > 1$ or $-1 \in N_{K/F}(K^\times)$. Then an exceptional element δ exists.*

Proof. Consider $\delta = \sqrt[p]{a_{n-1}}$. If $p > 2$ then $N_{K/K_{n-1}}(\delta) = a_{n-1}$ and hence $N_{K/F}(\delta) = a_0 = a$. Now if $p = 2$ then $N_{K/K_{n-1}}(\delta) = -a_{n-1}$ and for $n > 1$ we similarly have $N_{K/F}(\delta) = a_0 = a$. If $p = 2$ and $n = 1$, then $-a = N_{K/F}(\sqrt{a})$ and hence $-1 \in N_{K/F}(K^\times)$ if and only if $a \in N_{K/F}(K^\times)$. Consequently, under our hypothesis, exceptional elements always exist. □

1.3. The Fixed Submodule J^G of J.

Recall that we write $[F^\times]$ for $F^\times K^{\times p}/K^{\times p} \subset J$.

The following lemmas generalize [MS03, Lemma 2 and Remark 2]:

Lemma 5 (Fixed Submodule Lemma)**.**

(1) *If $\xi_p \notin N_{K/F}(K^\times)$,*
$$J^G = [F^\times].$$
(2) *If $\xi_p \in N_{K/F}(K^\times)$,*
$$J^G = \langle[\delta]\rangle \oplus [F^\times],$$
where $\delta \in K^\times$ with $\delta^{\sigma-1} = \lambda^p$, $N_{K/F}(\lambda)$ is a primitive pth root of unity, and $[N_{K/F}(\delta)]_F = [a]_F$. In particular, δ is an exceptional element of K/F.

Proof. Suppose that $\theta \in K^\times$ such that $[\theta] \in J^G$. Then $\theta^{\sigma-1} = \lambda^p$ for some $\lambda \in K^\times$, and hence $N_{K/F}(\lambda)^p = 1$. Therefore $N_{K/F}(\lambda)$ is a pth root of unity.

Now consider the first case, $\xi_p \notin N_{K/F}(K^\times)$. Then $N_{K/F}(\lambda) = 1$, because otherwise ξ_p would be the norm of a suitable power of λ. From Hilbert 90 we see that $\theta^{\sigma-1} = (k^p)^{\sigma-1}$ for some $k \in K^\times$. We conclude that $\theta/k^p \in F^\times$ and hence $[\theta] = [f]$ for some $f \in F^\times$. Therefore if $\xi_p \notin N_{K/F}(K^\times)$ then $J^G = [F^\times]$ as required.

Now assume that $\xi_p \in N_{K/F}(K^\times)$. Then $\xi_p = N_{K/F}(\lambda)$ for some $\lambda \in K^\times$ and by Hilbert 90 there exists an element $\delta \in K^\times$ such that $\delta^{\sigma-1} = \lambda^p$. Then the $\mathbb{F}_p[G]$-submodule of J generated by $[\delta]$ and $\epsilon(F^\times)$ is isomorphic to $[F^\times] \oplus \langle[\delta]\rangle$.

By [Al35, Theorem 3], $K(\sqrt[p]{\delta})$ is a cyclic extension of F of degree p^{n+1}. Then repeated application of Lemma 4 gives that

$$K_{n-i} = K_{n-i-1}\left(\sqrt[p]{N_{K_n/K_{n-i-1}}(\delta)}\right)$$

for $i \in \{0, 1, \ldots, n-1\}$. Hence $K_1 = F(\sqrt[p]{N_{K/F}(\delta)})$. By Kummer theory, $\langle [N_{K/F}(\delta)]_F\rangle = \langle [a]_F\rangle$ as subgroups of $F^\times/F^{\times p}$. By replacing δ with another power if necessary, then, $[N_{K/F}(\delta)]_F = [a]_F$ and $\delta^{\sigma-1} = \lambda^p$, where $N_{K/F}(\lambda)$ is a primitive pth root of unity. We have that $[\delta]^{(\sigma-1)} = [1]$ and so by definition δ is exceptional for K/F.

Now for each $[\theta] \in J^G$, $\theta^{\sigma-1} = \nu^p$ with $N_{K/F}(\nu) = N_{K/F}(\lambda)^c$ for some $c \in \mathbb{Z}$. Then we have $(\theta\delta^{-c})^{\sigma-1} = \nu^p\lambda^{-pc}$. Because $N(\nu\lambda^{-c}) = 1$, from Hilbert 90 we see that there exists $\omega \in K^\times$ such that $\omega^{\sigma-1} = \nu\lambda^{-c}$. Hence $(\theta\delta^{-c})^{\sigma-1} = (\omega^p)^{\sigma-1}$ and we see that $[\theta] \in [F^\times] + [\delta]^c$. Hence $J^G \cong [F^\times] \oplus \langle[\delta]\rangle$, as required. □

Lemma 6 (Exact Sequence Lemma)**.** *There is an exact sequence*

$$1 \to A \to F^\times/F^{\times p} \xrightarrow{\epsilon} J^G \xrightarrow{N} A$$

where $A = (F^\times \cap K^{\times p})/F^{\times p}$, ϵ is the natural homomorphism induced by the inclusion $F^\times \to K^\times$, and N is the homomorphism induced by the norm map $N_{K/F}\colon K^\times \to F^\times$.

- *If $\xi_p \notin F$, then $A = 1$.*
- *If $\xi_p \in F$, $A = \langle[a]\rangle$, in which case the map N is surjective if and only if $\xi_p \in N_{K/F}(K^\times)$.*

Proof. If $\xi_p \in F$, then Kummer theory implies that the first occurrence of A in the exact sequence above is equal to $A = \langle [a]_F\rangle$. Otherwise, suppose that $\xi_p \notin F$. If $\mathrm{char}(F) = p$ then no primitive pth root of unity lies in the algebraic closure of F, whence $\xi_p \notin K$. If $\mathrm{char}(F) \neq p$, then since $2 \leq [F(\xi_p) : F] \leq p-1$ and $[K : F] = p^n$, we similarly obtain $\xi_p \notin K$. In any case, then, $\xi_p \notin K$. Assume that $k^p = f \in F^\times$. Then $(k^p)^{\sigma-1} = (k^{\sigma-1})^p = 1$, whence $k^{\sigma-1}$ is a pth root of unity, which must be 1. Hence $k^{\sigma-1} = 1$, and we deduce $k \in F$ and $f \in F^{\times p}$. Therefore $A = 1$.

The Fixed Submodule Lemma (5) then gives exactness at J^G and that N is surjective if and only if either $\xi_p \notin F$ or $\xi_p \in N_{K/F}(K^\times)$. Exactness at $F^\times/F^{\times p}$ follows from Kummer theory. □

1.4. $\mathbb{F}_p[G]$-Submodules of J.

Lemma 7 (Submodule-Subfield Lemma)**.** *Let U be a free $\mathbb{F}_p[G]$-submodule of J and $i \in \{0, 1, \dots, n\}$. Then*

$$U^{H_i} = U^{(\sigma-1)^{p^n-p^i}} = U \cap [N_{K_n/K_i} K_n^\times] = U \cap [K_i^\times].$$

Proof. Suppose $[u] \in U^{H_i}$. Then $[u]^{(\sigma^{p^i}-1)} = [u]^{(\sigma-1)^{p^i}} = [1]$, so $l(u) \leq p^i$. Since U is free, $[u] = [\tilde{u}]^{(\sigma-1)^{p^n-l(u)}}$ for some $[\tilde{u}] \in U$. In particular,

$$[u] = ([\tilde{u}]^{(\sigma-1)^{p^i-l(u)}})^{(\sigma-1)^{p^n-p^i}}.$$

Hence $U^{H_i} \subset U^{(\sigma-1)^{p^n-p^i}}$. Now suppose $[u] = [\tilde{u}]^{(\sigma-1)^{p^n-p^i}}$. Then since

$$[\tilde{u}]^{(\sigma-1)^{p^n-p^i}} = [N_{K_n/K_i}(\tilde{u})],$$

$U^{(\sigma-1)^{p^n-p^i}} \subset U \cap [N_{K_n/K_i} K_n^\times] \subset U \cap [K_i^\times]$.

Finally suppose that $[u] \in U \cap [K_i^\times]$. Then $[u] \in U^{H_i}$ and we see that all of our inclusions above are actually equalities. □

Remark. If U is a free $\mathbb{F}_p[G]$-module, then U is also a free $\mathbb{F}_p[H_i]$-module. But then $H^2(H_i, U) = \{0\}$. Hence $U^{H_i} = N_i(U) :=$ the image of the norm operator N_i. Thus $U^{H_i} = U^{(\sigma-1)^{p^n-p^i}}$ as required.

Just as with $F = K_0$, denote elements of the $\mathbb{F}_p[G/H_i]$-module $J(K_i) = K_i^\times / K_i^{\times p}$ by $[\gamma]_{K_i}$, $\gamma \in K_i^\times$.

Lemma 8 (Norm Lemma)**.** *For all elements $[\gamma] \in J$ with $l(\gamma) < p^n$, $[N_{K/F}(\gamma)]_F \in \langle [a]_F \rangle$.*

Now suppose additionally that $l(\gamma) \leq p^n - p^i$ for some $0 \leq i < n$. Then $[N_{K/K_i}(\gamma)]_{K_i} \in \langle [a_i]_{K_i} \rangle$, and $[N_{K/F}(\gamma)]_F = [a]_F^s$ if and only if $[N_{K/K_i}(\gamma)]_{K_i} = [a_i]_{K_i}^s$.

Proof. For the first statement, observe that $(1 + \sigma + \cdots + \sigma^{p^n-1}) \equiv (\sigma - 1)^{p^n-1}$ on J, and hence $[N_{K/F}(\gamma)] = [\gamma]^{(\sigma-1)^{p^n-1}}$. Since $l(\gamma) < p^n$, $[N_{K/F}(\gamma)] = [1]$. Therefore $N_{K/F}(\gamma) \in F^\times \cap K^{\times p}$, which by Kummer theory is the union $\cup_{j=0}^{p-1} a^j F^{\times p}$. We obtain $[N_{K/F}(\gamma)]_F \in \langle [a]_F \rangle$.

For the second statement, observe first that if $[\gamma] = [1]$ then the lemma is trivial. Otherwise, consider J as an $\mathbb{F}_p[H_i]$-module and let $\tau = \sigma^{p^i}$. The $\mathbb{F}_p[H_i]$-module generated by $[\gamma]$ has dimension equal to

t, where $[\gamma]^{(\tau-1)^t} = [1]$ and $[\gamma]^{(\tau-1)^{t-1}} \neq [1]$. Since $\tau - 1 \equiv (\sigma-1)^{p^i}$ on J, this condition is equivalent to $(t-1)p^i < l(\gamma) \leq tp^i$. Since $l(\gamma) \leq (p^{n-i}-1)p^i$, the dimension t is strictly less than p^{n-i}. Hence $(\tau-1)^{p^{n-i}-1}$ annihilates the cyclic $\mathbb{F}_p[H_i]$-module generated by γ, and so its length, as an $\mathbb{F}_p[H_i]$-module, is less than p^{n-i}.

Applying the first statement in the case of the cyclic extension K/K_i, we have $[N_{K/K_i}(\gamma)]_{K_i} \in \langle [a_i]_{K_i}\rangle$. Now because $N_{K_i/F}(a_i) = a$ and

$$[N_{K/F}(\gamma)]_F = N_{K_i/F}([N_{K/K_i}(\gamma)]_{K_i}),$$

we have $[N_{K/K_i}(\gamma)]_{K_i} = [a_i]^s_{K_i}$ if and only if $[N_{K/F}(\gamma)]_F = [a]^s_F$. □

Remark. Occasionally, we will cite the Norm Lemma (8) as an abbreviation of the simple argument, at the end of the lemma's proof, which shows that

$$[N_{K/F}(\gamma)]_F = [a]^s_F \text{ if and only if } [N_{K/K_i}(\gamma)]_{K_i} = [a_i]^s_{K_i}.$$

Lemma 9 (Proper Subfield Lemma). *Let $[z] \in J^{H_i}$, $i < n$. Then $[z] \in [K_i^\times]$ if and only if $[N_{K/F}(z)]_F = [1]_F$.*

Proof. If $[z] \in J^{H_i}$, then $[z]^{(\sigma^{p^i}-1)} = [1]$. Since $(\sigma^{p^i}-1) \equiv (\sigma-1)^{p^i}$ on J, $l(z) \leq p^i$.

Consider J as an $\mathbb{F}_p[H_i]$-module. Then from the Fixed Submodule Lemma (5) applied to the field extension K/K_i, we see that

$$[z] \in [K_i^\times] \text{ or } [z] \in \langle[\delta]\rangle \oplus [K_i^\times]$$

according to whether

$$\xi_p \notin N_{K/K_i}(K^\times) \text{ or } \xi_p \in N_{K/K_i}(K^\times).$$

(Here $\delta \in K^\times$ with $\delta^{\sigma-1} = \lambda^p$, $N_{K/K_i}(\lambda)$ is a primitive pth root of unity, and $[N_{K/K_i}(\delta)]_{K_i} = [a_i]_{K_i}$.)

Therefore if $[z] \notin [K_i^\times]$ then $[N_{K/K_i}(z)]_{K_i} = [a_i]^c_{K_i}$ for $c \not\equiv 0 \bmod p$, and by the Norm Lemma (8), $[N_{K/F}(z)]_F = [a]^c_F$, which contradicts our hypothesis. Hence if $[z] \in J^{H_i}$ and $[N_{K/F}(z)]_F = [1]_F$, then $[z] \in [K_i^\times]$.

Conversely, if $[z] \in [K_i^\times]$ then $[z] \in J^{H_i}$ and

$$[N_{K/F}(z)]_F = [N_{K_i/F}(z)]_F^{p^{n-i}} = [1]_F,$$

since $n > i$. □

1.5. **Fixed Submodules of Cyclic Submodules of J.**

Lemma 10 (First Fixed Elements are Norms Lemma)**.** *Suppose that $p > 2$, $n = 1$, $[\gamma] \in J$, $2 \le l(\gamma) < p$, and that one of the following holds:*

- $\xi_p \notin F$
- $\xi_p \in F$ *and* $l(\gamma) \ge 3$
- $\xi_p \in F$, $l(\gamma) = 2$, *and γ is unexceptional.*

Then there exists $[\alpha] \in J$ such that $M_\gamma^G = \langle N[\alpha]\rangle$.

Proof. First suppose $\xi_p \notin F$. We show by induction on i that there exists an element $\alpha_i \in K^\times$ such that $\langle [\alpha_i]^{(\sigma-1)^{i-1}}\rangle = M_\gamma^G$. Then since $(\sigma-1)^{p-1} \equiv 1 + \sigma + \cdots + \sigma^{p-1}$ we may set $\alpha := \alpha_p$ and the proof of the first item will be complete. If $i = l(\gamma)$ we set $a_i = \gamma$. Assume now that $l(\gamma) \le i < p$ and that our statement is true for i.

Set $c = N_{K/F}(\alpha_i)$. Since $[\alpha_i]^{(\sigma-1)^{p-1}} = [c]$ and $i < p$, we see that $[c] = [1]$. Then $c \in F^\times \cap K^{\times p}$, which by the Exact Sequence Lemma (6) is equal to $F^{\times p}$. Hence $c = f^p$ for some $f \in F^\times$. Then $N_{K/F}(\alpha_i/f) = 1$. By Hilbert 90 there exists an element $\omega \in K^\times$ such that $\omega^{\sigma-1} = \alpha_i/f$. Then $\omega^{(\sigma-1)^2} = \alpha_i^{(\sigma-1)}$. Since $l(\alpha_i) \ge 2$ and $\langle [\alpha_i]^{(\sigma-1)^{i-1}}\rangle = M_\gamma^G$, $\langle [\omega]^{(\sigma-1)^i}\rangle = M_\gamma^G$ and we may set $\alpha_{i+1} = \omega$. Our induction is complete.

Now suppose that $\xi_p \in F$, and assume $l(\gamma) \ge 3$. As before, we show by induction on i that there exists an element $\alpha_i \in K^\times$ such that $\langle [\alpha_i]^{(\sigma-1)^{i-1}}\rangle = M_\gamma^G$. If $i = l(\gamma)$ we set $\alpha_i = \gamma$. Assume now that $l(\gamma) \le i < p$ and that our statement is true for i.

By the Norm Lemma (8) we have $[N_{K/F}(\alpha_i)]_F \in \langle [a]_F\rangle$. Hence $c := N_{K/F}(\alpha_i) = a^s f^p$ for some $f \in F^\times$ and $s \in \mathbb{Z}$. Then $N_{K/F}(\alpha_i/f\delta^s) = 1$, where $\delta = \sqrt[p]{a}$. By Hilbert 90 there exists an element $\omega \in K^\times$ such that $\omega^{\sigma-1} = \alpha_i/f\delta^s$. Then $\omega^{(\sigma-1)^2} = \alpha_i^{(\sigma-1)}/\xi_p^s$. Since $i \ge 3$, $\langle [\omega]^{(\sigma-1)^i}\rangle = \langle [\alpha_i]^{(\sigma-1)^{i-1}}\rangle = M_\gamma^G$ and we can set $\alpha_{i+1} := \omega$.

Assume then that $l(\gamma) = 2$ and γ is an unexceptional element of K/F. By the Norm Lemma (8), $[N_{K/F}(\gamma)]_F \in \langle [a]_F\rangle$, and as before $c := N_{K/F}(\gamma) = a^s f^p$ for some $f \in F^\times$ and $s \in \mathbb{Z}$.

Since γ is unexceptional, either $s \equiv 0 \bmod p$, in which case $c = f^p$ for some $f \in F^\times$, or $[\gamma]^{\sigma-1} \notin [K_m^\times]$. In the former case, $N_{K/F}(\gamma/f) = 1$.

By Hilbert 90 there exists an element $\omega \in K^\times$ such that $\omega^{\sigma-1} = \gamma/f$ and $\omega^{(\sigma-1)^2} = \gamma^{\sigma-1}$. Hence $\langle [\omega]^{(\sigma-1)^2} \rangle = M_\gamma^G$ and we may invoke the statement for ω since $l(\omega) = 3$.

In the latter case, since $N_{K/F}(\gamma^{\sigma-1}) = 1$ and $[\gamma]^{\sigma-1} \in J^G$, from the Exact Sequence Lemma (6) we see that $[\gamma]^{\sigma-1} \in [F^\times] = [K_0^\times]$. Hence $m < 0$ so that $m = -\infty$. Thus there exists an element $\delta \in K^\times$ such that $[N_{K/F}(\delta)]_F \neq [1]_F$ and $[\delta]^{\sigma-1} = [1]$. Again using the Exact Sequence Lemma (6) we see that we may assume that $[N_{K/F}(\delta)]_F = [a]_F$ and $[\delta]^{\sigma-1} = [1]$.

Now let $N_{K/F}(\delta) = ag^p$ for some $g \in F^\times$ and note $N_{K/F}(\gamma g^s/f\delta^s) = 1$. Then as before we have $\omega^{\sigma-1} = \gamma g^s/f\delta^s$ and $[\omega]^{(\sigma-1)^2} = [\gamma]^{(\sigma-1)} \neq [1]$. Hence $\langle [\omega]^{(\sigma-1)^2} \rangle = M_\gamma^G$ and we may invoke the statement for ω since $l(\omega) = 3$. $\square$

Lemma 11 (Fixed Elements of Length 3 Submodules are Norms Lemma)**.** *Suppose that $p = 2$, $n = 2$, $[\gamma] \in J$, $l(\gamma) = 3$, and $[N_{K/F}(\gamma)]_F = [1]_F$. Then there exists $[\alpha] \in J$ such that $M_\gamma^G = \langle N[\alpha] \rangle$.*

Proof. Let $\beta = \gamma^{\sigma-1}$. Then $l(\beta) = 2$ and, since β is in the image of $\sigma - 1$, we have $[N_{K/F}(\beta)]_F = [1]_F$. Because $l(\beta) = 2$ and N_{K/K_1} is equivalent to $1 + \sigma^2 \equiv (\sigma - 1)^2$ on J, we see that $[N_{K/K_1}(\beta)] = [1]$ in J. From the Norm Lemma (8) we conclude that $[N_{K/K_1}(\beta)]_{K_1} = [1]_{K_1}$, and by the Exact Sequence Lemma (6) applied to the $\mathbb{F}_2[H_1]$-module J, we see that $[\beta] \in [K_1^\times]$. Let $b \in K_1^\times$ such that $[b] = [\gamma]^{\sigma-1}$.

Now set $c := N_{K_1/F}(b)$. Observe that $\langle [c] \rangle \subset M_\gamma^G$ and $[c] = [b]^{1+\sigma} = [\gamma]^{\sigma^2-1} = [N_{K/K_1}(\gamma)]$. Hence $N_{K/K_1}(\gamma) = ck^2$ for some $k \in K^\times$, and $k^2 \in K_1^\times \cap K^{\times 2}$. By Kummer theory $k^2 = a_1^s g^2$ for some $s \in \mathbb{Z}$ and $g \in K_1^\times$, whence $N_{K/K_1}(\gamma) = ca_1^s g^2$ and $[N_{K/F}(\gamma)]_F = [a]_F^s$. By hypothesis $s \equiv 0 \bmod 2$. Therefore $N_{K/K_1}(\gamma) = ch^2$ for some $h \in K_1^\times$.

Now $N_{K/F}(\gamma) = N_{K_1/F}(ch^2) = c^2(N_{K_1/F}(h))^2$. Let $\gamma' = bh$. Then $N_{K/F}(\gamma') = c^2(N_{K_1/F}(h))^2$ so that $N_{K/F}(\gamma/\gamma') = 1$. By Hilbert 90 there exists $\alpha \in K^\times$ with $\alpha^{\sigma-1} = \gamma/\gamma'$. Then

$$\begin{aligned}[N_{K/F}(\alpha)] &= [\alpha]^{(\sigma-1)^3} = [\gamma/\gamma']^{(\sigma-1)^2} = [N_{K/K_1}(\gamma\gamma')] \\ &= [ch^2b^2h^2] = [c] = [\gamma]^{(\sigma-1)^2}.\end{aligned}$$

Because $M_\gamma^G = \langle [\gamma]^{(\sigma-1)^2} \rangle$ our statement follows. $\square$

In what follows, let $l_H(\gamma)$ denote the dimension over $\mathbb{F}_p$ of the cyclic $\mathbb{F}_p[H]$-submodule of J generated by $[\gamma]$.

Lemma 12 (Second Fixed Elements are Norms Lemma)**.**

(a) Suppose $p > 2$ and $n \geq 1$. Let $\gamma \in K^\times$ with $[\gamma] \in J \setminus [K_{n-1}^\times]$, and let $H = \mathrm{Gal}(K/K_{n-1})$. Assume that one of the following holds:

- $\xi_p \notin F$
- $\xi_p \in F$ *and* $l_H(\gamma) \geq 3$
- $\xi_p \in F$, $l_H(\gamma) = 2$, *and* $[N_{K/F}(\gamma)]_F = [1]_F$.

Then
$$[\gamma]^{(\sigma-1)^{l(\gamma)-1}} \in [N_{K/F}(K^\times)].$$

(b) Suppose $p = 2$ and $n \geq 2$. Let $\gamma \in K^\times$ and $H = \mathrm{Gal}(K/K_{n-2})$. Assume that one of the following holds:

- $l_H(\gamma) = 4$
- $l_H(\gamma) = 3$ *and* $[N_{K/F}(\gamma)]_F = [1]_F$.

Then
$$[\gamma]^{(\sigma-1)^{l(\gamma)-1}} \in [N_{K/F}(K^\times)].$$

Proof. (a). Since part (a) is true for $n = 1$ by the First Fixed Elements are Norms Lemma (10), let us assume that $n > 1$. The Fixed Submodule Lemma (5) tells us that $l_H(\gamma) \geq 2$, since $[\gamma] \notin [K_{n-1}^\times]$.

Now if $l_H(\gamma) = 2$, we claim that γ is not exceptional for K/K_{n-1}, as follows. Since $l_H(\gamma) = 2 < p$, the Norm Lemma (8) tells us that $[N_{K/K_{n-1}}(\gamma)]_{K_{n-1}} \in \langle [a_{n-1}]_{K_{n-1}} \rangle$. If γ is exceptional for K/K_{n-1}, then $[N_{K/K_{n-1}}(\gamma)]_{K_{n-1}} \neq [1]_{K_{n-1}}$. By the Norm Lemma (8) again, $[N_{K/F}(\gamma)]_F \neq [1]_F$, contradicting our hypothesis. Hence if $l_H(\gamma) = 2$ then γ is not exceptional for K/K_{n-1}, as required.

Let
$$[\beta] = [\gamma]^{(\sigma^{p^{n-1}}-1)^{l_H(\gamma)-1}} = [\gamma]^{(\sigma-1)^{p^{n-1}(l_H(\gamma)-1)}}.$$
We invoke the First Fixed Elements are Norms Lemma (10) and deduce that there exists $[\alpha] \in J$ such that $[\beta] = [N_{K/K_{n-1}}(\alpha)]$. Then
$$[\beta] = [\alpha]^{(\sigma-1)^{p^{n-1}(p-1)}}$$

since $l_H(\alpha) = p$. Set $s = l(\beta)$. Then

$$\begin{aligned}[\alpha]^{(\sigma-1)^{p^n-p^{n-1}}(\sigma-1)^{s-1}} &= [\beta]^{(\sigma-1)^{s-1}} \\ &= [\gamma]^{(\sigma-1)^{p^{n-1}(l_H(\gamma)-1)+s-1}},\end{aligned}$$

and this element is in J^G.

Set $[\lambda] := [\alpha]^{(\sigma-1)^s}$. Then we have

$$[\lambda]^{(\sigma-1)^{p^n-p^{n-1}-1}} = [\alpha]^{(\sigma-1)^{p^n-p^{n-1}+s-1}}.$$

Hence $l(\lambda) = p^n - p^{n-1}$.

Now we claim that $l_H(\lambda) = p - 1$. First, since

$$[\lambda]^{(\sigma^{p^{n-1}}-1)^{p-1}} = [\lambda]^{(\sigma-1)^{p^n-p^{n-1}}} = [1]$$

we see that $l_H(\lambda) \le p - 1$. But since

$$[\lambda]^{(\sigma-1)^{p^{n-1}(p-2)}} = [\lambda]^{(\sigma-1)^{p^n-p^{n-1}-p^{n-1}}}$$

and $p^{n-1} > 1$ (since we assume $n > 1$), we see that

$$[\lambda]^{(\sigma-1)^{p^{n-1}(p-2)}} \ne [1].$$

(Observe that here we use more than we need as $p^{n-1} \ge 1$ is sufficient for the inequality above.) Therefore indeed $l_H(\lambda) = p-1 \ge 2$, since we assume that $p \ge 3$. Observe that since $[\beta] \ne [1]$ we have $s = l(\beta) > 0$. Thus $[\lambda]$ is in the image of $\sigma - 1$ and hence $[N_{K/F}(\lambda)]_F = [1]_F$. Since $l_H(\lambda) = p - 1 < p$, we obtain

$$[N_{K/K_{n-1}}(\lambda)]_{K_{n-1}} \in \langle [a_{n-1}]_{K_{n-1}} \rangle.$$

By the Norm Lemma (8), we deduce that

$$[N_{K/K_{n-1}}(\lambda)]_{K_{n-1}} = [1]_{K_{n-1}}.$$

Hence λ is unexceptional for K/K_{n-1}. Thus we can use the First Fixed Elements are Norms Lemma (10) for λ. We see that there exists $\chi \in K^\times$ such that

$$[\lambda]^{(\sigma^{p^{n-1}}-1)^{l_H(\lambda)-1}} = [\chi]^{(\sigma-1)^{p^n-p^{n-1}}}$$

or equivalently

$$[\lambda]^{(\sigma-1)^{p^n-2p^{n-1}}} = [\chi]^{(\sigma-1)^{p^n-p^{n-1}}}.$$

This means in particular that

$$l(\chi) = l(\lambda) + p^{n-1} = p^n.$$

Putting our calculations together, we obtain

$$\begin{aligned}[N_{K/F}(\chi)] &= [\chi]^{(\sigma-1)^{p^n-1}} = [\lambda]^{(\sigma-1)^{p^n-p^{n-1}-1}} \\ &= [\alpha]^{(\sigma-1)^{p^n-p^{n-1}+s-1}} = [\gamma]^{(\sigma-1)^{p^{n-1}(l_H(\gamma)-1)+s-1}} \\ &= [\gamma]^{(\sigma-1)^{l(\gamma)-1}}\end{aligned}$$

as required.

(b). If $l_H(\gamma) = 3$ then we claim that $[N_{K/K_{n-2}}(\gamma)]_{K_{n-2}} = [1]_{K_{n-2}}$, as follows. Since $l_H(\gamma) < 4$, we have from the Norm Lemma (8) that $[N_{K/K_{n-2}}(\gamma)]_{K_{n-2}} \in \langle [a_{n-2}]_{K_{n-2}} \rangle$. If $[N_{K/K_{n-2}}(\gamma)]_{K_{n-2}} = [a_{n-2}]^s_{K_{n-2}}$ for $s \not\equiv 0 \bmod 2$, then we obtain from the Norm Lemma (8) that $[N_{K/F}(\gamma)]_F = [a]^s_F \neq [1]$, contradicting our hypothesis. Therefore if $l_H(\gamma) = 3$ then $[N_{K/K_{n-2}}(\gamma)]_{K_{n-2}} = [1]_{K_{n-2}}$, as required.

We may then invoke the Fixed Elements of Length 3 Submodules are Norms Lemma (11) and deduce that there exists $\alpha \in K^\times$ such that

$$[\alpha]^{(\sigma^{2^{n-2}}-1)^3} = [\gamma]^{(\sigma^{2^{n-2}}-1)^{l_H(\gamma)-1}}.$$

If instead $l_H(\gamma) = 4$, then by setting $\alpha = \gamma$ we see that α as above exists as well.

In either case, then, we obtain the equation with α above. Hence

$$[\alpha]^{(\sigma-1)^{2^n-2^{n-2}}} = [\gamma]^{(\sigma-1)^{2^{n-2}(l_H(\gamma)-1)}} \neq [1].$$

Set $s := l(\gamma) - 2^{n-2}(l_H(\gamma)-1) > 0$. Then we have

$$[\alpha]^{(\sigma-1)^{2^n-2^{n-2}+s-1}} = [\gamma]^{(\sigma-1)^{2^{n-2}(l_H(\gamma)-1)+s-1}} \neq [1].$$

Furthermore, this element belongs to J^G. Set $[\lambda] := [\alpha]^{(\sigma-1)^s}$. Then

$$[\lambda]^{(\sigma-1)^{2^n-2^{n-2}-1}} = [\alpha]^{(\sigma-1)^{2^n-2^{n-2}+s-1}},$$

whence $l(\lambda) = 2^n - 2^{n-2}$.

Now consider $l_H(\lambda)$. On the one hand,

$$[\lambda]^{(\sigma^{2^{n-2}}-1)^3} = [\lambda]^{(\sigma-1)^{2^n-2^{n-2}}} = [1],$$

and on the other hand

$$[\lambda]^{(\sigma^{2^{n-2}}-1)^2} \neq [1].$$

We deduce that $l_H(\lambda) = 3$. Observe that since $[\lambda]$ is in the image of $\sigma - 1$ we have $[N_{K/F}(\lambda)]_F = [1]_F$. Since $l_H(\lambda) = 3$, we see that
$$[N_{K/K_{n-2}}(\lambda)]_{K_{n-2}} \in \langle [a_{n-2}]_{K_{n-2}} \rangle.$$

By the Norm Lemma (8), we deduce that
$$[N_{K/K_{n-2}}(\lambda)]_{K_{n-2}} = [1]_{K_{n-2}}.$$
By the Fixed Elements of Length 3 Submodules are Norms Lemma (11), there exists $\chi \in K^\times$ with
$$[\chi]^{(\sigma^{2^{n-2}}-1)^3} = [\lambda]^{(\sigma^{2^{n-2}}-1)^2}.$$
Equivalently,
$$[\chi]^{(\sigma-1)^{2^n-2^{n-2}}} = [\lambda]^{(\sigma-1)^{2^{n-1}}},$$
and therefore $l(\chi) = l(\lambda) + 2^{n-2} = 2^n$.

Summarizing, we have obtained
$$\begin{aligned}[N_{K/F}\chi] &= [\chi]^{(\sigma-1)^{2^n-1}} = [\lambda]^{(\sigma-1)^{2^n-2^{n-2}-1}} \\ &= [\alpha]^{(\sigma-1)^{2^n-2^{n-2}+s-1}} = [\gamma]^{(\sigma-1)^{l(\gamma)-1}}\end{aligned}$$
as required. □

2. Base Cases

Proposition 3. *Theorem 1 holds for $n = 1$.*

Proof. Let $\mathcal{I}$ be an $\mathbb{F}_p$-basis for $[N_{K/F}(K^\times)]$. For each $[x] \in \mathcal{I}$, we construct a free $\mathbb{F}_p[G]$-module $M(x)$, as follows. Choose a representative $x \in F^\times$ for $[x]$ such that $x \in N_{K/F}(K^\times)$. Choose $\gamma \in K^\times$ such that $x = N_{K/F}(\gamma)$. Finally let $M(x)$ be the $\mathbb{F}_p[G]$-submodule of J generated by $[\gamma]$. Since $[N_{K/F}(\gamma)] = [\gamma]^{(\sigma-1)^{p-1}} = [x] \neq [1]$, $\dim_{\mathbb{F}_p} M(x) = p$ and hence $M(x)$ is free. By the Exclusion Lemma (2), the set of modules $M(x)$, $[x] \in \mathcal{I}$, is independent.

Let $Y_1 = \oplus_{\mathcal{I}} M(x)$. Then Y_1 is a free $\mathbb{F}_p[G]$-module with $Y_1^G = [N_{K/F}(K^\times)]$. Let Y_0 be any complement in $[F^\times]$ of Y_1^G. Clearly Y_0 is a trivial $\mathbb{F}_p[G]$-module. Since $Y_0^G \cap Y_1^G = \{0\}$, $Y_0 + Y_1 = Y_0 \oplus Y_1$ by the Exclusion Lemma (2). Moreover, $(Y_0 + Y_1)^G = [F^\times]$.

Now set $\tilde{J} := Y_0 + Y_1$. Then, applying the Inclusion Lemma (1) with $U = \tilde{J}$, $V = J$, and $U + V = J$, we will deduce that $\tilde{J} = J$. Observe first that $(U+V)^G = J^G$ which, by the Fixed Submodule Lemma (5), is $[F^\times]$. Since $\tilde{J}^G = [F^\times]$, we obtain $(U+V)^G \subset U$.

Let $[\gamma] \in J \setminus J^G$. Then $l(\gamma) \geq 2$. If $p=2$ then $[c] = [\gamma]^{(\sigma-1)^{l(\gamma)-1}} = [\gamma]^{(\sigma-1)} = N[\gamma]$. Otherwise, by the First Fixed Elements are Norms Lemma (10), we obtain $[c] = [\gamma]^{(\sigma-1)^{l(\gamma)-1}} = [N_{K/F}(\alpha)]$ for some $\alpha \in K^\times$. In any case, $[c] \in [N_{K/F}(K^\times)]$. Equivalently, switching for the moment to additive notation for convenience, $[c] = \sum_{\mathcal{I}} c_x[x]$ with almost all $c_x = 0$. Now for each $[x]$, $M(x) = M_{\omega(x)}$ for some $\omega(x) \in K^\times$ with $N([\omega(x)]) = [x]$. Hence $[c] = N(\sum c_x[\omega(x)]) \in Y_1 \subset \tilde{J}$. We have shown that for every $[\gamma] \in J \setminus J^G$, $[\gamma]^{(\sigma-1)^{l(\gamma)-1}} = N([\alpha])$ for $[\alpha] \in Y_1 \subset \tilde{J}$. Hence we have satisfied the hypotheses of the Inclusion Lemma (1), and $J \subset \tilde{J}$, as required. $\square$

Proposition 4. *Theorem 2 holds for $n = 1$.*

Proof. Let X be the cyclic submodule of J generated by the given exceptional element $[\delta]$. Since $\delta = \sqrt[p]{a}$ satisfies $[N_{K/F}(\delta)]_F = [a]_F$ and $[\delta]^{(\sigma-1)} = [\xi_p] \in J^G$, we have that $m < 1$.

For the case in which $p = 2$ and $-1 \in N_{K/F}(K^\times)$, let γ satisfy $N_{K/F}(\gamma) = -1$. Then set $\gamma' = \sqrt{a}\gamma$. We have $N_{K/F}(\gamma') = a$ and $[\gamma']^{(\sigma-1)} = [\gamma']^{(1+\sigma)} = [N_{K/F}(\gamma')] = [a] = [1] \in [K^\times_{-\infty}]$. Hence in this case γ' is exceptional and $m = -\infty$. By the definition, then, for any exceptional δ in this case, we have $[\delta]^{(\sigma-1)} = [1]$.

In any case, by the Exact Sequence Lemma (6), we have $[\delta] \notin [F^\times]$. If $m = -\infty$, then X is of dimension 1 and hence $X \cap [F^\times] = \{0\}$. If $m = 0$, then X is of dimension 2 and by the Fixed Submodule Lemma (5), we have that $X^G = X^{(\sigma-1)} = X \cap [F^\times]$.

We proceed to construct Y_1. Let $\mathcal{I}$ be an $\mathbb{F}_p$-basis for $[N_{K/F}(K^\times)]$. For each $[x] \in \mathcal{I}$, we construct a free $\mathbb{F}_p[G]$-module $M(x)$, as follows. Choose a representative $x \in F^\times$ for $[x]$ such that $x \in N_{K/F}(K^\times)$. Choose $\gamma \in K^\times$ such that $x = N_{K/F}(\gamma)$. Finally let $M(x) = M_\gamma$, the $\mathbb{F}_p[G]$-submodule of J generated by $[\gamma]$. Since $[N_{K/F}(\gamma)] = [\gamma]^{(\sigma-1)^{p-1}} = [x] \neq [1]$, $\dim_{\mathbb{F}_p} M(x) = p$ and hence $M(x)$ is free. By the Exclusion Lemma (2), the set of modules $M(x)$, $[x] \in \mathcal{I}$, is independent. Let $Y_1 = \oplus_{\mathcal{I}} M(x)$. Then Y_1 is a free $\mathbb{F}_p[G]$-module with $Y_1^G = [N_{K/F}(K^\times)]$.

Now $X^G \cap Y_1^G = \{0\}$, as follows. Suppose not. Then since $X \cap [F^\times] = \{0\}$ in the case $m = -\infty$, we must have $m = 0$. Let $[f] \in X^G \cap Y_1^G$.

Since Y_1 is free, there exists $[\alpha] \in Y_1$ such that $N[\alpha] = [f]$. Consider $\delta' = \delta/(\alpha)^{(\sigma-1)^{p-2}}$. Then $[N_{K/F}(\delta')]_F \neq [1]_F$ and $[\delta']^{\sigma-1} = [1] \in [K^\times_{-\infty}]$, so that $m = -\infty$, a contradiction. Because $X^G \cap Y_1^G = \{0\}$, by the Exclusion Lemma (2) we have $X + Y_1 = X \oplus Y_1$.

Now let Y_0 be any complement in $[F^\times]$ of the $\mathbb{F}_p$-submodule of J generated by $X \cap [F^\times]$ and Y_1^G. Clearly Y_0 is a trivial $\mathbb{F}_p[G]$-module. Since $Y_0^G \cap (X + Y_1)^G = \{0\}$, we obtain $X + Y_0 + Y_1 = X \oplus Y_0 \oplus Y_1$ from the Exclusion Lemma (2).

If $m = -\infty$ then observe that $[F^\times] = Y_0^G + Y_1^G$, and if $m = 0$ then since $X^G = X^{(\sigma-1)}$, we have $[F^\times] = X^{(\sigma-1)} + Y_0^G + Y_1^G$.

Now set $\tilde{J} = X + Y_0 + Y_1$. We adapt the proof of the Inclusion Lemma (1) to show that $J \subset \tilde{J}$ and hence $J = \tilde{J}$, by induction on the socle series J_i of J.

We first show that if $[\beta] \in J_1 = J^G$ then $[\beta] \in \tilde{J}$. If $[N_{K/F}\beta]_F = [1]_F$, then the Proper Subfield Lemma 9 gives $[\beta] \in [F^\times]$. Since Y_0 is a complement in $[F^\times]$ of the submodule generated by $X \cap [F^\times]$ and Y_1^G, $[\beta] \in \tilde{J}$.

Otherwise $[N_{K/F}\beta]_F \neq [1]_F$. Since $l(\beta) = 1$ we must have $m = -\infty$ and $[\delta] \in J_1$. By the Exact Sequence Lemma (6), both $[N_{K/F}(\beta)]_F$ and $[N_{K/F}(\delta)]_F$ lie in $\langle [a]_F \rangle$, and by the definition of exceptionality, both are generators of $\langle [a]_F \rangle$. Hence $[N_{K/F}(\beta)]_F = [N_{K/F}(\delta)]_F^s$ for some $s \in \mathbb{Z}$, and we set $\beta' = \beta/\delta^s$. Then $[\beta'] \in J^G$ and $[N_{K/F}(\beta')]_F = [1]_F$. By the Exact Sequence Lemma (6), we see that $[\beta'] \in [F^\times]$. As in the preceding paragraph, this gives $[\beta'] \in \tilde{J}$. Then, since $[\delta] \in \tilde{J}$ as well, we obtain $[\beta] \in \tilde{J}$. Hence $J_1 \subset \tilde{J}$.

For the inductive step, assume that $J_i \subset \tilde{J}$ for all $1 \leq i < t \leq p$, and let $[\gamma] \in J_t \setminus J_{t-1}$.

We first claim that in the particular case of $t = 2$, without loss of generality we may assume that γ is unexceptional, as follows. If γ is exceptional and $l(\gamma) = 2$, then by the Fixed Submodule Lemma (5), we see that $m = 0$. We established earlier, however, that if $p = 2$ and $n = 1$ we have $m = -\infty$. Hence $p > 2$. Now since $m = 0$, we have $[\delta]^{(\sigma-1)} \in [F^\times]$, $l(\delta) \leq 2$, and by the definition of exceptionality $l(\delta) \neq 1$, since otherwise $m = -\infty$. Hence $l(\delta) = l(\gamma) = 2$.

Since $p > 2$, we deduce from the Norm Lemma (8) that both of $[N_{K/F}(\gamma)]_F$ and $[N_{K/F}(\delta)]_F$ lie in $\langle [a]_F \rangle$, and by the definition of exceptionality, both are generators. Hence $[N_{K/F}(\gamma)]_F = [N_{K/F}(\delta)]_F^s$ for some $s \in \mathbb{Z}$, and we set $\gamma' = \gamma/\delta^s$. Then $l(\gamma') \leq 2$ and $[N_{K/F}(\gamma')]_F = [1]_F$, so that γ' is unexceptional. Since $[\delta] \in X \subset \tilde{J}$, to show that $[\gamma] \in \tilde{J}$ it is enough to show that $[\gamma'] \in J$. We may therefore assume that γ is unexceptional if $t = 2$.

Now if $p = 2$ then $l(\gamma) = 2$ and

$$[c] = [\gamma]^{(\sigma-1)^{l(\gamma)-1}} = [\gamma]^{(\sigma+1)} = N[\gamma] = [N_{K/F}(\gamma)],$$

and we set $\alpha = \gamma$. Otherwise, $p > 2$ and by the First Fixed Elements are Norms Lemma (10), we have $[c] = [\gamma]^{(\sigma-1)^{l(\gamma)-1}} = [N_{K/F}(\alpha)]$ for some $\alpha \in K^\times$.

In either case, $[c] \in [N_{K/F}(K^\times)]$. Equivalently, switching for the moment to additive notation for convenience, $[c] = \sum_{\mathcal{I}} c_x[x]$ with almost all $c_x = 0$. Now for each $[x]$, $M(x) = M_{\omega(x)}$ for some $\omega(x) \in K^\times$ with $N([\omega(x)]) = [x]$. Hence $[c] = N(\sum c_x[\omega(x)]) \in Y_1 \subset \tilde{J}$. Switching back to multiplicative notation, $[c] = [\alpha]^{(\sigma-1)^{p-1}}$ for some $[\alpha] \in Y_1$. Let $[\gamma'] = [\alpha]^{(\sigma-1)^{p-t}} \in \tilde{J}$. Since $[\gamma/\gamma']^{(\sigma-1)^{t-1}} = [1]$, we find $l(\gamma/\gamma') < t$. By induction, $[\gamma/\gamma'] \in \tilde{J}$, and hence $[\gamma] \in \tilde{J}$ as well. By induction on the socle series, then, $J \subset \tilde{J}$. □

Proposition 5. *Theorem 2 holds in the case $p = 2$, $n = 2$.*

Proof. Let X be the cyclic submodule of J generated by the given exceptional element $[\delta]$. Consider $\theta = \sqrt{a_1}$. Then $[N_{K/F}(\theta)]_F = [a]_F$. Because K/F is Galois we have $a_1^\sigma = a_1 k^2$ for some $k \in K_1^\times$. Therefore $[\theta]^{\sigma-1} = [\pm k] \in K_1^\times$. Hence $m < 2$. Now let δ be any exceptional element in $K^\times$. Because $[N_{K/F}(\delta)]_F \neq [1]_F$ we see that $[\delta] \notin [F^\times]$.

If $m = -\infty$, then X is of dimension 1 and therefore $X \cap [F^\times] = \{0\}$. If $m = 0$, then X is of dimension 2, and by the Exact Sequence Lemma (6), observing that $N_{K/F}(\delta^{\sigma-1}) = 1$, we obtain $X^G = X^{(\sigma-1)} = X \cap [F^\times]$. Finally assume that $m = 1$. Observe that then $l(\delta^{\sigma-1}) \neq 1$. Indeed otherwise $N_{K/F}(\delta^{\sigma-1}) = 1$ and the Exact Sequence Lemma (6) implies that $[\delta]^{\sigma-1} \in [F^\times]$, which contradicts our assumption that $m = 1$. Hence $l(\delta) \geq 3$. However since $[\delta]^{\sigma-1} \in [K_1^\times]$ and $(\sigma-1)^2 \equiv \sigma^2 - 1$, we have $l(\delta^{\sigma-1}) \leq 2$, and therefore $l(\delta) \leq 3$. Consequently $l(\delta) = 3$. Since $[N_{K/F}(\delta)]_F \neq [1]_F$ we see that $[\delta] \notin [K_1^\times]$. Therefore $X^{(\sigma-1)} = X \cap [K_1^\times]$.

We proceed to construct Y_2. Let $\mathcal{I}_2$ be an $\mathbb{F}_2$-basis for $[N_{K/F}(K^\times)]$. For each $[x] \in \mathcal{I}_2$, we construct a free $\mathbb{F}_2[G]$-module $M(x)$, as follows. Choose a representative $x \in F^\times$ for $[x]$ such that $x \in N_{K/F}(K^\times)$. Choose $\gamma \in K^\times$ such that $x = N_{K/F}(\gamma)$. Finally let $M(x) = M_\gamma$. Since $[N_{K/F}(\gamma)] = [\gamma]^{(\sigma-1)^3} = [x] \neq [1]$, $\dim_{\mathbb{F}_2} M(x) = 4$ and hence $M(x)$ is free. By the Exclusion Lemma (2), the set of modules $M(x)$, $[x] \in \mathcal{I}_2$, is independent. Let $Y_2 = \oplus_{\mathcal{I}_2} M(x)$. Then Y_2 is a free $\mathbb{F}_2[G]$-module with $Y_2^G = [N_{K/F}(K^\times)]$.

Suppose $X^G \cap Y_2^G \neq \{0\}$. Since $Y_2^G \subset [F^\times]$ and $X \cap [F^\times] = \{0\}$ if $m = -\infty$, we are in the case $m = 0$ or $m = 1$, and $X^G = X^{(\sigma-1)^{m+1}} = X \cap [F^\times]$. In particular, $l(\delta) = m + 2 \leq 3$. Let $f \in F^\times$ satisfy $[f] \in X^G \cap Y_2^G$. Since Y_2 is free, there exists $[\alpha] \in Y_2$ such that $N[\alpha] = [f]$. Let $\delta' = \delta/(\alpha)^{(\sigma-1)^{4-l(\delta)}}$. Then $[N_{K/F}(\delta')]_F = [N_{K/F}(\delta)]_F$ since $\alpha^{(\sigma-1)^{4-l(\delta)}}$ is in the image of $\sigma - 1$. Moreover, $l(\delta') < l(\delta)$. If $m = 0$ then $l(\delta') \leq 1$ and by the definition of exceptionality, $m = -\infty$, a contradiction. If $m = 1$ then $l(\delta') \leq 2$ so that $l((\delta')^{\sigma-1}) \leq 1$ and $[(\delta')^{\sigma-1}] \in J^G$. But since $(\delta')^{\sigma-1}$ is in the image of $\sigma - 1$, we have $[N((\delta')^{\sigma-1})]_F = [1]_F$, and from the Proper Subfield Lemma (9) we obtain $[\delta'] \in [F^\times]$. Then by the definition of exceptionality, $m \leq 0$, again a contradiction. Thus $X^G \cap Y_2^G = \{0\}$.

Because $X^G \cap Y_2^G = \{0\}$, by the Exclusion Lemma (2) we have $X + Y_2 = X \oplus Y_2$.

We proceed to construct Y_1. Let $\mathcal{I}_1$ be an $\mathbb{F}_2$-basis for a complement in $[N_{K_1/F}(K_1^\times)]$ of the $\mathbb{F}_2$-submodule generated by $[N_{K/F}(K^\times)]$ and $X \cap [N_{K_1/F}(K_1^\times)]$. For each $[x] \in \mathcal{I}_1$, we construct an $\mathbb{F}_2[G]$-module $M(x)$ of dimension 2, as follows. Choose a representative $x \in F^\times$ for $[x]$ such that $x \in N_{K_1/F}(K_1^\times)$. Choose $\gamma \in K_1^\times$ such that $x = N_{K_1/F}(\gamma)$. Finally let $M(x) = M_\gamma$. Since $[N_{K_1/F}(\gamma)] = [\gamma]^{(\sigma-1)} = [x] \neq [1]$, $\dim_{\mathbb{F}_2} M(x) = 2$. The $M(x)$, $[x] \in \mathcal{I}_1$, are independent as above. Let $Y_1 = \oplus_{\mathcal{I}_1} M(x)$. Then Y_1 is a direct sum of $\mathbb{F}_2[G]$-modules of dimension 2, and Y_1^G is the $\mathbb{F}_2$-span of $\mathcal{I}_1$. By construction $Y_1^G \cap Y_2^G = \{0\}$ and hence by the Exclusion Lemma (2), we have $Y_1 + Y_2 = Y_1 \oplus Y_2$.

Suppose $X^G \cap (Y_1 + Y_2)^G \neq \{0\}$. Since $(Y_1 + Y_2)^G \subset [F^\times]$ and $X \cap [F^\times] = \{0\}$ if $m = -\infty$, we are in the case $m = 0$ or $m = 1$, and $X^G = X^{(\sigma-1)^{m+1}} = X \cap [F^\times]$. Let $X^G = \langle [x] \rangle$; then $[1] \neq [x] = [y_1] + [y_2]$ for some $[y_1] \in Y_1^G$ and $[y_2] \in Y_2^G$. Since $Y_1^G + Y_2^G \subset [N_{K_1/F}(K_1^\times)]$, we deduce $[x] \in [N_{K_1/F}(K_1^\times)]$. We have already established that $[y_1] \neq [1]$, since $X^G \cap Y_2^G \neq \{0\}$. Hence $[1] \neq [y_1] = [y_2] + [x]$. But then Y_1^G

does not consist of a complement of the $\mathbb{F}_2$-submodule generated by $Y_2^G = [N_{K/F}(K^\times)]$ and $X \cap [N_{K_1/F}(K_1^\times)]$, a contradiction. Hence we have established our equality $X^G \cap (Y_1 + Y_2)^G = \{0\}$.

Because $X^G \cap (Y_1 + Y_2)^G = \{0\}$, by the Exclusion Lemma (2) we have $X + Y_1 + Y_2 = X \oplus Y_1 \oplus Y_2$.

Finally let Y_0 be any complement in $[F^\times]$ of the $\mathbb{F}_2$-submodule of J generated by $X \cap [F^\times]$, Y_1^G, and Y_2^G. Clearly Y_0 is a trivial $\mathbb{F}_2[G]$-module. Since $Y_0^G \cap (X+Y_1+Y_2)^G = \{0\}$, we see that $X+Y_0+Y_1+Y_2 = X \oplus Y_0 \oplus Y_1 \oplus Y_2$ by the Exclusion Lemma (2).

If $m = -\infty$ then observe that $[F^\times] = Y_0^G + Y_1^G + Y_2^G$, and otherwise since $X^G = X^{(\sigma-1)^{m+1}}$, we have $[F^\times] = X^{(\sigma-1)^{m+1}} + Y_0^G + Y_1^G + Y_2^G$. In order to connect this expression with Theorem 2, part (3), in the case $i = 0$, observe that $(\sigma - 1)(\sigma - 1)^{2^m - 1} = (\sigma - 1)^{2^m} = (\sigma - 1)^{m+1}$ for $m = 0$ or 1.

Now let $\tilde{J} = X + Y_0 + Y_1 + Y_2$. We show that $J = \tilde{J}$ by showing that an arbitrary element $[\beta] \in J$ lies in $\tilde{J}$, as follows.

First, if β is exceptional, then since $m \leq 1$ we have $[\beta]^{\sigma-1} \in [K_1^\times]$. Since $1 + \sigma \equiv \sigma - 1$ on J and $[N_{K_1/F}(\gamma)] = [\gamma]^{\sigma+1}$ for $\gamma \in K_1^\times$, we see that $l(\beta) \leq 3$. By the Norm Lemma (8), we have $[N_{K/F}(\beta)]_F = [a]_F^s$ for some $s \not\equiv 0 \bmod p$. Because $p = 2$ and $[N_{K/F}(\beta)]_F \neq [1]_F$ we have $[N_{K/F}(\beta)]_F = [a]_F$. Since δ is exceptional, $[N_{K/F}(\delta)]_F = [a]_F$ as well. Then $\beta' = \beta/\delta$ satisfies $[N_{K/F}(\beta')]_F = [1]_F$ and is therefore unexceptional. Since $[\delta] \in X \subset \tilde{J}$, to show that $[\beta] \in \tilde{J}$ it suffices to show that $[\beta'] \in \tilde{J}$. Therefore we may and do assume that $[\beta]$ is unexceptional.

Observe that the above argument applies not only to elements β that are exceptional, but in fact to all elements β such that $[N_{K/F}(\beta)]_F = [a]_F^s$ for some $s \in \mathbb{Z}$. Therefore we may assume not only that β is unexceptional, but also that $[N_{K/F}(\beta)]_F = [1]_F$.

Suppose that $l(\beta) = 1$ and $[N_{K/F}(\beta)]_F = [1]_F$. From the Exact Sequence Lemma (6) we see that $[\beta] \in [F^\times]$. Since $[F^\times] \subset \tilde{J}$, we obtain $[\beta] \in \tilde{J}$ as well.

Now if $l(\beta) = 2$, then $[\beta]^{(\sigma^2-1)} = [\beta]^{(\sigma-1)^2} = [1]$ and $[\beta] \in J^{H_1}$. Moreover, we assume that $[N_{K/F}(\beta)]_F = [1]_F$. By the Proper Subfield

Lemma (9), we deduce that $[\beta] \in [K_1^\times]$. Hence we may assume that the representative β of $[\beta]$ lies in $K_1^\times$. Then $[\beta]^{(\sigma-1)} = [N_{K_1/F}(\beta)] \subset [N_{K_1/F}(K_1^\times)]$.

If $m = 1$ then

$$[N_{K_1/F}(K_1^\times)] \subset X^{(\sigma-1)^2} + Y_1^G + Y_2^G = X^{(\sigma-1)^2} + Y_1^{(\sigma-1)} + Y_2^{(\sigma-1)^3},$$

since Y_1 is a direct sum of cyclic modules of length 2 and Y_2 a direct sum of cyclic modules of length 4. If $m = 0$ then $X \cap [F^\times] = X^{\sigma-1}$ and therefore

$$[N_{K_1/F}(K_1^\times)] \subset X^{(\sigma-1)} + Y_1^{(\sigma-1)} + Y_2^{(\sigma-1)^3}.$$

If $m = -\infty$ then $X \cap [N_{K_1/F}(K_1^\times)] = \{0\}$ and

$$[N_{K_1/F}(K_1^\times)] \subset Y_1^G + Y_2^G = Y_1^{(\sigma-1)} + Y_2^{(\sigma-1)^3}.$$

In any case, $[\beta]^{(\sigma-1)}$ lies in $\tilde{J}^{\sigma-1}$ and hence there exists $\alpha \in \tilde{J}$ such that $[\alpha]^{(\sigma-1)} = [\beta]^{(\sigma-1)}$. But then $[\alpha/\beta] \in J^G$, which we have already established lies in $\tilde{J}$. Hence $[\beta] \in \tilde{J}$.

Now suppose $l(\beta) \geq 3$ and $[N_{K/F}(\beta)]_F = [1]_F$. By the Fixed Elements of Length 3 Submodules are Norms Lemma (11), we have $[c] = [\beta]^{(\sigma-1)^{l(\beta)-1}} = N[\alpha]$ for some $\alpha \in K^\times$. Equivalently, switching for the moment to additive notation for convenience, $[c] = \sum_{\mathcal{I}_2} c_x[x]$ with almost all $c_x = 0$. As in the proof of the previous theorem, we obtain $[c] = N(\sum c_x[\omega(x)]) \in Y_2 \subset \tilde{J}$. Let $[\beta'] = [\beta] - (\sigma-1)^{4-l(\beta)}(\sum c_x[\omega(x)])$. Then $l(\beta') < l(\beta)$ and we proceed by induction.

Hence $J = \tilde{J}$.

Now we consider the location of $[K_1^\times]$ in J. Since K_1 is the fixed field in K of H_1, we have

$$[K_1^\times] \subset J^{H_1} = X^{H_1} \oplus Y_0 \oplus Y_1 \oplus Y_2^{H_1}.$$

By our construction of Y_0 and Y_1 we see that $Y_0 \oplus Y_1 \subset [K_1^\times]$. By the Submodule-Subfield Lemma (7) we see that $Y_2^{H_1} = Y_2 \cap [K_1^\times]$. Also because $m \leq 1$ we see from the definition of m that $X^{(\sigma-1)} \subset [K_1^\times]$. Hence $X^{(\sigma-1)} + Y_0 + Y_1 + Y_2^{H_1} \subset [K_1^\times]$. It remains to show that this inclusion is an equality.

We showed after the definition of exceptional element that $[\delta] \notin [K_{n-1}^\times] = [K_1^\times]$. Therefore $X^{(\sigma-1)} = X \cap [K_1^\times]$, and we have

$$X^{(\sigma-1)} \oplus Y_0 \oplus Y_1 \oplus Y_2^{H_1} \subset [K_1^\times] \subset X^{H_1} \oplus Y_0 \oplus Y_1 \oplus Y_2^{H_1}.$$

Hence each $[k] \in [K_1^\times]$ can be written as

$$[k] = [x] + [y], \text{ where } [x] \in X^{H_1} \text{ and } [y] \in Y_0 \oplus Y_1 \oplus Y_2^{H_1}.$$

Thus

$$[x] = [k] + [y] \in [K_1^\times] + Y_0 + Y_1 + Y_2^{H_1} \subset X \cap [K_1^\times].$$

Therefore we see that $X^{(\sigma-1)} \oplus Y_0 \oplus Y_1 \oplus Y_2^{H_1} = [K_1^\times]$. Observe that if $m = -\infty$ then $X^{(\sigma-1)} = \{0\}$. Since $m \leq 1$ we see that our decomposition of $[K_1^\times]$ is in agreement with Theorem 2, part (3). □

3. Free Submodules and Proof of Theorem 1

For the following proposition, assume Theorems 1 and 2 hold for all extensions of degree p^s, $1 \leq s < n$, and if $p = 2$, then $n > 2$.

Proposition 6. *There exists a submodule*

$$\hat{Y} = \hat{Y}_n \oplus \hat{Y}_{n-1} \oplus \cdots \oplus \hat{Y}_0$$

of J such that

1. $\hat{Y}_i$ *is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i;*
2. $[K_i^\times] = \hat{Y}^{H_i}$ *for $0 \leq i < n$;*
3. $\hat{Y}_n^G = [N_{K/F}(K^\times)]$.

Proof. Let $\mathcal{I}$ be an $\mathbb{F}_p$-base for $[N_{K/F}(K^\times)]$. As usual, for each $[x] \in \mathcal{I}$ construct free independent $\mathbb{F}_p[G]$-modules $M(x)$, $[x] \in \mathcal{I}$, such that $M(x)^G = \langle [x] \rangle$. Set $\hat{Y}_n = \oplus_{[x] \in \mathcal{I}} M(x)$. Hence $\hat{Y}_n$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^n, and $\hat{Y}_n^G = [N_{K/F}(K^\times)]$.

Assume now that $\xi_p \in F^\times$. Since K_{n-1}/F embeds in a cyclic extension K of degree p^n over F, $[a_{n-1}]_{K_{n-1}}^{\bar{\sigma}} = [a_{n-1}]_{K_{n-1}}$ by Kummer theory, where $\bar{\sigma} \in G/H_{n-1}$ is the image of σ under the natural projection $G \to \bar{G} := G/H_{n-1}$. (Indeed since a_{n-1} is a pth power in K, so is $a_{n-1}^{\bar{\sigma}}$; therefore by Kummer theory $[a_{n-1}]_{K_{n-1}}^{\bar{\sigma}} \in \langle [a_{n-1}]_{K_{n-1}} \rangle$. However, viewing $\langle [a_{n-1}]_{K_{n-1}} \rangle$ as $\mathbb{F}_p$, then $\bar{\sigma}$ is an exponent p^{n-1} action on $\mathbb{F}_p$. Since

$$\mathrm{Aut}(\mathbb{F}_p) \cong \mathbb{Z}/(p-1)\mathbb{Z},$$

this action must be the identity. Hence $[a_{n-1}]_{K_{n-1}}^{(\bar{\sigma}-1)} = [1]_{K_{n-1}}$.) Moreover, we have that $[N_{K_{n-1}/F}(a_{n-1})]_F = [a]_F$ by Proposition 1.

Because Theorem 2 holds for $n-1$, we have an $\mathbb{F}_p[\bar{G}]$-module decomposition

$$J(K_{n-1}) = K_{n-1}^{\times}/K_{n-1}^{\times p} = \langle [a_{n-1}]_{K_{n-1}}\rangle \oplus \tilde{Y}_{n-1} \oplus \cdots \oplus \tilde{Y}_0$$

into direct sums $\tilde{Y}_i$ of cyclic $\mathbb{F}_p[\bar{G}]$-modules of dimension p^i and a $\bar{G}$-invariant submodule $\langle [a_{n-1}]_{K_{n-1}}\rangle_{\mathbb{F}_p}$. Indeed we only have to check that a_{n-1} is an exceptional element in K_{n-1}. This follows since we have shown both $[N_{K_{n-1}/F}(a_{n-1})]_F = [a]_F$ and $[a_{n-1}]_{K_{n-1}}^{\bar{\sigma}-1} = [1]_{K_{n-1}}$.

Moreover, by the Submodule-Subfield Lemma (7),

$$\tilde{Y}_{n-1}^{\bar{G}} = [N_{K_{n-1}/F}(K_{n-1}^{\times})]_{K_{n-1}} \cap \tilde{Y}_{n-1}.$$

Because $N_{K_{n-1}/F}$ acts on $J(K_{n-1})$ as $(\bar{\sigma}-1)^{p^{n-1}-1}$ we see that $N_{K_{n-1}/F}$ annihilates the sum $\tilde{Y}_{n-2}\oplus\cdots\oplus\tilde{Y}_0$. Also $[N_{K_{n-1}/F}(a_{n-1})]_{K_{n-1}} = [1]_{K_{n-1}}$. Therefore

$$[N_{K_{n-1}/F}(K_{n-1}^{\times})]_{K_{n-1}} = \tilde{Y}_{n-1}^{\bar{G}}.$$

Assume now that $\xi_p \notin F^{\times}$. Then because Theorem 1 holds for $n-1$, we have an $\mathbb{F}_p[\bar{G}]$-module decomposition

$$J(K_{n-1}) = K_{n-1}^{\times}/K_{n-1}^{\times p} = \tilde{Y}_{n-1} \oplus \cdots \oplus \tilde{Y}_0$$

into direct sums $\tilde{Y}_i$ of cyclic $\mathbb{F}_p[\bar{G}]$-modules of dimension p^i. As before let $\bar{\sigma}$ denote the image of σ under the natural projection $G \to \bar{G}$. Because $N_{K_{n-1}/F}$ acts on $J(K_{n-1})$ as $(\bar{\sigma}-1)^{p^{n-1}-1}$ we see that $N_{K_{n-1}/F}$ annihilates the sum $\tilde{Y}_{n-2} \oplus \cdots \oplus \tilde{Y}_0$. Therefore again

$$[N_{K_{n-1}/F}(K_{n-1}^{\times})]_{K_{n-1}} = [N_{K_{n-1}/F}(\tilde{Y}_{n-1})]_{K_{n-1}} = \tilde{Y}_{n-1}^{\bar{G}}.$$

In both cases $\xi_p \in F^{\times}$, $\xi_p \notin F^{\times}$, consider J as an $\mathbb{F}_p[H_{n-1}]$-module. Then the Exact Sequence Lemma (6) gives us that the images of $\tilde{Y}_0, \ldots, \tilde{Y}_{n-1}$ under the map

$$\epsilon\colon J(K_{n-1}) \to J(K)$$

are direct sums of modules of dimension p^i and are independent. Because the modules $\tilde{Y}_i$ are cyclic as $\mathbb{F}_p[\bar{G}]$-modules, the images $\epsilon(\tilde{Y}_i)$ are cyclic as $\mathbb{F}_p[G]$-modules. Set $\hat{Y}_i = \epsilon(\tilde{Y}_i)$ for $i < n-1$. (Recall that we already defined $\hat{Y}_n$ at the beginning of our proof.)

Set $W := \hat{Y}_n^{H_{n-1}}$. By the Submodule-Subfield Lemma (7),

$$W = \hat{Y}_n^{(\sigma-1)^{p^n-p^{n-1}}} = \hat{Y}_n \cap [K_{n-1}^{\times}].$$

Since $\hat{Y}_n$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^n, W is a direct sum of cyclic modules of dimension p^{n-1} and hence is free as an $\mathbb{F}_p[\bar{G}]$-module. Because $W \subset [K_{n-1}^\times]$, we may consider the image P of the projection map $\mathrm{pr}\colon W \to \epsilon(\tilde{Y}_{n-1})$ from W to the summand $\epsilon(\tilde{Y}_{n-1})$ in the decomposition

$$[K_{n-1}^\times] = \epsilon(J(K_{n-1})) = \epsilon(\tilde{Y}_{n-1}) \oplus \hat{Y}_{n-2} \oplus \cdots \oplus \hat{Y}_0.$$

Observe that $W \cong P$ as $\mathbb{F}_p[G]$-modules. Indeed, since W is a free $\mathbb{F}_p[\bar{G}]$-module, each $[w] \in W \setminus \{0\}$ may be written as $[\tilde{w}]^{(\bar{\sigma}-1)^s}$ for some $0 \le s \le p^{n-1}-1$ and $[\tilde{w}] \in W$ with $l(\tilde{w}) = p^{n-1}$. We have

$$\mathrm{pr}([\tilde{w}])^{(\bar{\sigma}-1)^{p^{n-1}-1}} = [\tilde{w}]^{(\bar{\sigma}-1)^{p^{n-1}-1}} \neq [1],$$

since all other components of $[\tilde{w}]$ are killed by $(\bar{\sigma}-1)^{p^{n-1}-1}$. (Since $n \ge 2$, $p^{n-1}-1 \ge p^{n-2}$.) Therefore $\mathrm{pr}([\tilde{w}])^{(\bar{\sigma}-1)^s} = \mathrm{pr}([w]) \neq [1]$. We conclude that the kernel of the projection map is $[1]$, as required.

Since $M^{\bar{G}} = M^{(\sigma-1)^{p^{n-1}-1}}$ for free $\mathbb{F}_p[\bar{G}]$-modules, we have further obtained that $W^{\bar{G}} = P^{\bar{G}}$; equivalently, $W^G = P^G$. Observe that

$$W^G = W^{\bar{G}} = W^{(\sigma-1)^{p^{n-1}-1}} \subset [N_{K_{n-1}/F} K_{n-1}^\times] = \epsilon(\tilde{Y}_{n-1})^G.$$

By the Free Complement Lemma (3), there exists a free $\mathbb{F}_p[\bar{G}]$-module complement $\hat{Y}_{n-1}$ in $\epsilon(\tilde{Y}_{n-1})$ of P. Since $W = \hat{Y}_n \cap [K_{n-1}^\times]$, we obtain $\hat{Y}_n^G = W^G = P^G$. Now the next idea is to use the fact that $\hat{Y}_n^G = P^G$ to show that $\hat{Y}_n$ and $\hat{Y}_{n-1}$ are independent and $\hat{Y}_n \oplus \hat{Y}_{n-1}$ and $\hat{Y}_{n-2} \oplus \cdots \oplus \hat{Y}_0$ are also independent. Then from the definition of $\hat{Y}_n$ and from our observation on $\hat{Y}_i, i \in \{n-1, \ldots, 0\}$ above it follows immediately that $\hat{Y} = \hat{Y}_n \oplus \cdots \oplus \hat{Y}_0 \subset J$ satisfies conditions (1) and (3) of our proposition. The last part of our proof is then devoted to proving condition (2).

By the Exclusion Lemma (2), $P^G \cap \hat{Y}_{n-1}^G = \{0\}$ implies that $\hat{Y}_{n-1} + \hat{Y}_n = \hat{Y}_{n-1} \oplus \hat{Y}_n$. Then, since $P^G + \hat{Y}_{n-1}^G = \epsilon(\tilde{Y}_{n-1})^G$, we obtain $(\hat{Y}_{n-1} + \hat{Y}_n)^G = \epsilon(\tilde{Y}_{n-1})^G$. Finally, by the Exclusion Lemma (2), $\hat{Y}_{n-1} + \hat{Y}_n$ is independent from $\hat{Y}_{n-2} + \cdots + \hat{Y}_0$. Hence we have a submodule

$$\hat{Y} = \hat{Y}_n \oplus \hat{Y}_{n-1} \oplus \cdots \oplus \hat{Y}_0 \subset J$$

satisfying items (1) and (3).

We turn next to item (2) and prove that $\hat{Y}^{H_{n-1}} = [K_{n-1}^\times]$. Now $\hat{Y}_{n-1} + \cdots + \hat{Y}_0 \subset [K_{n-1}^\times]$ by construction, and $\hat{Y}_n^{H_{n-1}} = W = \hat{Y}_n \cap [K_{n-1}^\times] \subset [K_{n-1}^\times]$ from above. Hence $\hat{Y}^{H_{n-1}} \subset [K_{n-1}^\times]$. We also have the decomposition $[K_{n-1}^\times] = \epsilon(\tilde{Y}_{n-1}) + \hat{Y}_{n-2} + \cdots + \hat{Y}_0$. Therefore it is sufficient to show that $\epsilon(\tilde{Y}_{n-1}) \subset \hat{Y}^{H_{n-1}}$.

Because $\epsilon(\tilde{Y}_{n-1}) = \hat{Y}_{n-1} + P$ it is enough to show that $P \subset \hat{Y}_n^{H_{n-1}} + \hat{Y}_{n-2} + \cdots + \hat{Y}_0 = W + \hat{Y}_{n-2} + \cdots + \hat{Y}_0$. But by the definition of projection, $P \subset W + \hat{Y}_{n-2} + \cdots + \hat{Y}_0$. Hence we conclude that $\hat{Y}^{H_{n-1}} = [K_{n-1}^\times]$, which is item (2) for $i = n-1$.

For $i < n-1$, observe that since Theorems 1 and 2 hold in the case $n-1$, we have

$$(\tilde{Y}_{n-1} + \cdots + \tilde{Y}_0)^{H_i/H_{n-1}} = [K_i^\times]_{K_{n-1}}, \ i < n-1.$$

(If we are in the situation covered by Theorem 1 then this statement is immediate. If we are in the situation covered by Theorem 2 we use the fact that $i(K_{n-1}/F) = -\infty$ and therefore the summand of $[K_i^\times]_{K_{n-1}}$ corresponding to the module generated by an exceptional element is trivial.)

Again using Theorem 1 and Theorem 2 as well as the equality $\hat{Y}^{H_{n-1}} = [K_{n-1}^\times]$ and the fact that

$$\epsilon \colon [K_{n-1}^\times]_{K_{n-1}} \to J \text{ with } \epsilon([K_{n-1}^\times]_{K_{n-1}}) = [K_{n-1}^\times]$$

is an $\mathbb{F}_p[G]$-homomorphism, we obtain for each $i \in \{0, 1, \ldots, n-2\}$ that

$$\begin{aligned}
\hat{Y}^{H_i} &= (\hat{Y}^{H_{n-1}})^{H_i/H_{n-1}} = [K_{n-1}^\times]^{H_i/H_{n-1}} \\
&= (\epsilon(\tilde{Y}_{n-1} + \cdots + \tilde{Y}_0))^{H_i/H_{n-1}} \\
&= \epsilon([K_i^\times]_{K_{n-1}}) = [K_i^\times],
\end{aligned}$$

as required. □

Proof of Theorem 1. The case $p = 2$, $n = 1$ was treated in Proposition 3. For the remaining case of $\xi_p \notin F$ and $p > 2$, we proceed by induction. The base case of $n = 1$ is Proposition 3. Assume then that $n > 1$ and the Theorem holds for $n-1$. By Proposition 6 above, there exists an $\mathbb{F}_p[G]$-submodule $\hat{Y} = \oplus \hat{Y}_i \subset J$, where each $\hat{Y}_i$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i, $[K_i^\times] = \hat{Y}^{H_i}$, $0 \le i < n$, and $\hat{Y}_n^G = [N_{K/F}(K^\times)]$. Set $Y_i = \hat{Y}_i$ and $Y = \oplus \hat{Y}_i$. All that remains is to show that $J \subset Y$.

We adapt the proof of the Inclusion Lemma (1) to show that $J \subset Y$, by induction on the socle series J_i of J. We first show that $J_{p^{n-1}} \subset Y$, as follows. Consider Y and J as $\mathbb{F}_p[H_{n-1}]$-modules. By the Fixed Submodule Lemma (5), $J^{H_{n-1}} = [K_{n-1}^\times]$, and we have already shown that $[K_{n-1}^\times] = Y^{H_{n-1}} \subset Y$, so $J^{H_{n-1}} \subset Y$. Since $J^{H_{n-1}}$ is the kernel of $\sigma^{p^{n-1}} - 1 \equiv (\sigma-1)^{p^{n-1}}$, $J^{H_{n-1}} = J_{p^{n-1}}$. Hence $J_{p^{n-1}} = [K_{n-1}^\times] \subset Y$.

For the inductive step, assume that $J_i \subset Y$ for all $i < t$ for some $p^{n-1} < t \le p^n$, and let $[\gamma] \in J_t \setminus J_{t-1}$. Hence $l(\gamma) = t$. Therefore $[\gamma] \notin [K_{n-1}^\times]$, and by the Second Fixed Elements are Norms Lemma (12), part (a), there exists $[\chi] \in J$ such that $[\gamma]^{(\sigma-1)^{t-1}} = [N_{K/F}(\chi)] \in Y_n^G$. Since Y_n is a free $\mathbb{F}_p[G]$-module, there exists $[\chi'] \in Y_n$ such that $[N_{K/F}(\chi')] = [\chi']^{(\sigma-1)^{p^n-1}} = [\chi]^{(\sigma-1)^{l(\chi)-1}}$. Set $[\gamma'] = [\chi']^{(\sigma-1)^{p^n-t}} \in Y_n \subset Y$. Then $l(\gamma/\gamma') < t$. By induction $[\gamma/\gamma'] \in Y$, and since $[\gamma'] \in Y$, we obtain $[\gamma] \in Y$ as well. □

4. Exceptional Elements

Assume that $\xi_p \in F$ and, if $p = 2$, then either $n > 1$ or $-1 \in N_{K/F}(K^\times)$. Recall that in Proposition 4 in section 2 we proved that Theorem 2 holds for extensions of degree p and in Proposition 5 we proved that Theorem 2 holds in the case $p = 2$ and $n = 2$. Assume then that Theorem 2 holds for extensions of degree p^s for $1 \le s < n$.

In the next lemma we assume that $n \ge 2$ and, if $p = 2$, that $n \ge 3$ as well. These conditions allow us to use Proposition 6, by which we assume that we have a fixed submodule $\hat{Y} = \hat{Y}_n \oplus \hat{Y}_{n-1} \oplus \cdots \oplus \hat{Y}_0$ of J with properties (1), (2) and (3) listed in Proposition 6.

Lemma 13. *Suppose $\delta \in K^\times$ satisfies $[N_{K/F}(\delta)]_F \neq [1]_F$ and $p^t + 2 \le l(\delta) \le p^{t+1}$, for some $t \in \{0, 1, \dots, n-2\}$. Then there exists $\delta' \in K^\times$ with $[N_{K/F}(\delta')]_F \neq [1]_F$ and $l(\delta') < l(\delta)$.*

Proof. Let $[\beta] = [\delta]^{(\sigma-1)}$ and $[\gamma] = [\delta]^{(\sigma-1)^{l(\delta)-1}}$. Since $l(\beta) < p^{t+1}$, $[\beta] \in J^{H_{t+1}}$, and since $[\beta] \in J^{\sigma-1}$, $[N_{K/F}(\beta)]_F = [1]_F$. By the Proper Subfield Lemma (9), we have $[\beta] \in [K_{t+1}^\times]$.

By Proposition 6, $[\beta] \in \hat{Y}^{H_{t+1}}$. Moreover, $p^t + 1 \le l(\beta) < p^{t+1}$. Let

$$\begin{aligned} W &= \hat{Y}_n^{H_{t+1}} \oplus \hat{Y}_{n-1}^{H_{t+1}} \oplus \cdots \oplus \hat{Y}_{t+1}^{H_{t+1}} \\ &= \hat{Y}_n^{H_{t+1}} \oplus \hat{Y}_{n-1}^{H_{t+1}} \oplus \cdots \oplus \hat{Y}_{t+2}^{H_{t+1}} \oplus \hat{Y}_{t+1}. \end{aligned}$$

By Proposition 6 and the Submodule-Subfield Lemma (7), W is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of length p^{t+1}.

Let β' be the component of β in W. Because $p^t + 1 \leq l(\beta)$ and $(\sigma - 1)^{p^t}$ is trivial on $\hat{Y}_t \oplus \cdots \oplus \hat{Y}_0$, we see that $l(\beta) = l(\beta')$ and also

$$[\gamma] = [\delta]^{(\sigma-1)^{l(\delta)-1}} = [\beta]^{(\sigma-1)^{l(\beta)-1}} = [\beta']^{(\sigma-1)^{l(\beta')-1}}.$$

Since W is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of length p^{t+1} and contains $[\beta']$, of length strictly less than p^{t+1}, $[\beta']$ lies in the image of $(\sigma - 1)$ on W. Hence there exists $[\alpha'] \in W$ such that $[\alpha']^{(\sigma-1)} = [\beta']$. Therefore $l(\alpha') = l(\delta)$ and

$$[\gamma] = [\delta]^{(\sigma-1)^{l(\delta)-1}} = [\alpha']^{(\sigma-1)^{l(\delta)-1}}.$$

Moreover, by Proposition 6, $W \subset \hat{Y}^{H_{t+1}} = [K_{t+1}^\times] \subset [K_{n-1}^\times]$ and therefore $[N_{K/F}(\alpha')]_F = [1]_F$. Now set $\delta' = \delta/\alpha'$. Then $[\delta']^{(\sigma-1)^{l(\delta)-1}} = [1]$ so that $l(\delta') < l(\delta)$, and $[N_{K/F}(\delta')]_F = [N_{K/F}(\delta)]_F \neq [1]$. □

Proposition 7. *Suppose that $\xi_p \in F$ and, if $p = 2$, that $n > 1$ or $-1 \in N_{K/F}(K^\times)$. Then $m < n$ and, for any exceptional element δ, $l(\delta) = p^m + 1$. Moreover, this length is the minimal $l(z)$ for all z with $[N_{K/F}(z)]_F \neq [1]_F$.*

Observe that the proposition implies that for any exceptional element δ, $l(\delta) < p^n$. (Indeed $p^m + 1 \leq p^{n-1} + 1$ and $p^{n-1} + 1 \leq p^n$ unless $p = 2$ and $n = 1$. If $p = 2$, $n = 1$, and $-1 \in N_{K/F}(K^\times)$, then let $-1 = N_{K/F}(\theta)$, where $\theta \in K^\times$. Observe that $\delta = \sqrt{a}\theta$ satisfies $[N_{K/F}(\delta)]_F = [a]_F$ and $[\delta]^{(\sigma-1)} = [1]$. Hence $l(\delta) < 2$.) By the Norm Lemma (8), then $[N_{K/F}(\delta)]_F = [a]_F^s$, and by definition of exceptional element, $s \not\equiv 0 \bmod p$. By choosing an appropriate power of δ, we have that there exists an exceptional element δ with $[N_{K/F}(\delta)]_F = [a]_F$.

Proof. We first prove that $m < n$. Assume first that $p > 2$ or $p = 2$ and $n > 1$. Consider $\delta = \sqrt[p]{a_{n-1}}$. We observed in the proof of Proposition 2 that $N_{K/F}(\delta) = a_0 = a$. Now $\delta^\sigma = \sqrt[p]{a_{n-1}^\sigma}$ for a suitable pth root of unity. Because K/K_{n-1} is Galois we see from Kummer theory that $a_{n-1}^\sigma = a_{n-1}k_{n-1}^p$ for some $k_{n-1} \in K_{n-1}^\times$. Hence $\delta^{\sigma-1} \in K_{n-1}^\times$, and therefore $m \leq n - 1$, as required.

Now consider the case when $p = 2$, $n = 1$, and $-1 = N_{K/F}(\theta)$ for some $\theta \in K^\times$. Then set $\delta = \sqrt{a}\theta$. As we observed above, $[N_{K/F}(\delta)]_F = [a]_F$ and $[\delta]^{\sigma-1} = [1]$, showing that $m = -\infty < 1$.

Now let δ be an arbitrary exceptional element. Clearly $[\delta] \neq [1]$ since $[N_{K/F}(\delta)]_F \neq [1]_F$; hence $l(\delta) \geq 1$. If $m = -\infty$, then $[\delta]^{\sigma-1} = [1]$ so that $l(\delta) \leq 1$ and because of our convention $p^{-\infty} = 0$ we are done.

Hence assume that $m \geq 0$. Then set $[\beta] := [\delta]^{\sigma-1} \in [K_m^\times]$. Also

$$[N_{K_m/F}(\beta)] = [\beta]^{(\sigma-1)^{p^m-1}} \in [F^\times].$$

Therefore $l(\delta) \leq 1 + (p^m - 1) + 1 = p^m + 1$.

Now suppose $[z] \in J$ satisfies $[N_{K/F}(z)]_F \neq [1]_F$ and $l(z)$ is minimal among all such z. Since $[N_{K/F}(\delta)]_F \neq [1]_F$ and δ above has $l(\delta) \leq p^m + 1$, we see that $l(z) \leq p^m + 1$. Now suppose, contrary to our statement, that $l(z) < p^m + 1$. If $m = 0$ then $l(z) = 1$ and hence $[z]^{(\sigma-1)} \in [K_{-\infty}^\times]$, contradicting the minimality of m. Otherwise $m \geq 1$ and repeated application of Lemma 13 yields $\delta' \in K^\times$ such that $[N_{K/F}(\delta')]_F \neq [1]_F$ and $l(\delta') \leq p^{m-1} + 1$. (Observe that we can indeed apply Lemma 13, since $l(\delta') \leq l(z) \leq p^m \leq p^{(n-2)+1}$, where the last inequality holds since $m < n$.)

Let $[\beta'] = [\delta']^{(\sigma-1)}$. Then $l(\beta') \leq p^{m-1}$ so that $[\beta'] \in J^{H_{m-1}}$, and since $[\beta']$ is in the image of $(\sigma-1)$, $[N_{K/F}(\beta')]_F = [1]_F$. By the Proper Subfield Lemma (9), we see that $[\beta'] \in [K_{m-1}^\times]$. Hence $[N_{K/F}(\delta')]_F \neq [1]_F$, and $[\delta']^{\sigma-1} \in [K_{m-1}^\times]$, contradicting the minimality of m. Therefore $l(\delta) = p^m + 1$. □

Now assume that $\xi_p \in F$ and, if $p = 2$, then $n \geq 2$.

Proposition 8. *If δ is an exceptional element of K/F, then δ is an exceptional element of K/K_i for $0 \leq i < n$ if $p > 2$ and for $0 \leq i < n-1$ if $p = 2$.*

Proof. Since $K_0 = F$, the proposition is clear for $i = 0$. We therefore assume that $i > 0$.

If δ is an exceptional element of K/F, then Proposition 7 tells us that $l(\delta) = p^m+1$ for $m < n$. If $p > 2$, then for each $i \in \{0, 1, \ldots, n-1\}$ we have

$$l(\delta) = p^m + 1 \leq p^{n-1} + 1 \leq p^n - p^{n-1} \leq p^n - p^i.$$

If $p = 2$ and $n \geq 2$, then similarly for each $i \in \{0, 1, \ldots, n-2\}$ we have

$$l(\delta) = 2^m + 1 \leq 2^{n-1} + 1 \leq 2^n - 2^{n-2} \leq 2^n - 2^i.$$

Since the Norm Lemma (8) gives $[N_{K/F}(\delta)]_F \neq [1]_F$ if and only if $[N_{K/K_i}(\delta)]_{K_i} \neq [1]_{K_i}$, it follows $[N_{K/K_i}(\delta)]_{K_i} \neq [1]_{K_i}$.

Let $\tau = \sigma^{p^i}$. Then $(\tau - 1) \equiv (\sigma - 1)^{p^i}$ on J, and so $[\delta]^{(\sigma-1)} \in [K_m^\times]$ implies that $[\delta]^{(\tau-1)} \in [K_m^\times]$. Now we define intermediate fields $\{K'_{-\infty}, K'_0, \dots, K'_{n-i}\}$ of K/K_i by $K'_j := K_{j+i}$.

First consider the case $m < i$. We have that $l(\delta) = p^m + 1$, and so then $l(\delta) \leq p^i$. Hence $[\delta]^{\tau-1} = [\delta]^{(\sigma-1)^{p^i}} = [1]$. Since we have shown that δ satisfies $[N_{K/K_i}(\delta)]_{K_i} \neq [1]_{K_i}$, δ is an exceptional element of K/K_i with $i(K/K_i) = -\infty$.

Now consider the case $m \geq i$. In this case we have shown that δ satisfies $[N_{K/K_i}(\delta)]_{K_i} \neq [1]_{K_i}$ and $[\delta]^{\tau-1} \in [K'^{\times}_{m-i}]$. All that remains is to show that no $\delta' \in K^\times$ exists with $[N_{K/K_i}(\delta')]_{K_i} \neq [1]_{K_i}$ and $[\delta']^{\tau-1} \in [K'^{\times}_j]$ for $j < m - i$. Suppose such a δ' exists. We may assume that this δ' has a minimal length among all elements z with $[N_{K/K_i}(z)]_{K_i} \neq [1]_{K_i}$. By the remark made after Proposition 7 we see that we may further assume that $[N_{K/K_i}(\delta')]_{K_i} = [a_i]_{K_i}$. Therefore $[N_{K/F}(\delta')]_F = [N_{K_i/F}(a_i)]_F = [a]_F \neq [1]_F$.

If $j = -\infty$, then since $(\tau - 1) \equiv (\sigma - 1)^{p^i}$, we obtain $l(\delta') \leq p^i \leq p^m$. On the other hand, if $j \geq 0$ then $m > i$. Moreover, since $(\tau - 1) \equiv (\sigma - 1)^{p^i}$ and $[N_{K'_j/F}(\gamma)] = [\gamma]^{(\sigma-1)^{p^{i+j}-1}}$ for $[\gamma] \in [K'^{\times}_j]$, we have

$$l(\delta') \leq p^i + (p^{i+j} - 1) + 1 \leq p^m.$$

In either case, this violates the condition of Proposition 7, since then $l(\delta)$ is not minimal among lengths $l(\delta')$ for $[N_{K/F}(\delta')]_F \neq [1]_F$. □

5. Proof of Theorem 2

We first adapt the proof of Theorem 1 to prove the following analogue for the case of Theorem 2. We assume here that Theorem 2 holds for $n - 1$ and, if $p = 2$, then $n > 2$.

Proposition 9. *Let $\xi_p \in F$, $n \geq 2$, and $\delta \in K^\times$ be any exceptional element of K/F. Then the $\mathbb{F}_p[G]$-module J decomposes as*

$$J = X + \hat{Y}, \quad \hat{Y} = \hat{Y}_n \oplus \hat{Y}_{n-1} \oplus \cdots \oplus \hat{Y}_0,$$

where

(1) *X is the cyclic $\mathbb{F}_p[G]$-module generated by $[\delta]$;*
(2) *$\hat{Y}_i$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i with*
$$[K_i^\times] = \hat{Y}^{H_i}, \quad 0 \le i < n;$$
(3) $\hat{Y}_n^G = [N_{K/F}(K^\times)]$.

Proof. By Proposition 6, there exists an $\mathbb{F}_p[G]$-submodule $\hat{Y} = \oplus \hat{Y}_i \subset J$, where each $\hat{Y}_i$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i, $[K_i^\times] = \hat{Y}^{H_i}$, $0 \le i < n$, and $\hat{Y}_n^G = [N_{K/F}(K^\times)]$. Let X be defined as in the statement of the Theorem and set $\hat{J} = X + \hat{Y}$. We have that $\hat{J}$ is an $\mathbb{F}_p[G]$-submodule of J.

Assume first that $p > 2$. Consider $\hat{J}$ and J as $\mathbb{F}_p[H_{n-1}]$-modules. By Proposition 8, δ is exceptional for K/K_{n-1} and so by Proposition 4 (which is just Theorem 2 in case $n = 1$), J decomposes as $\bar{X} \oplus \bar{Y}_1 \oplus \bar{Y}_0$, where $\bar{X} \subset X$ is the $\mathbb{F}_p[H_{n-1}]$-submodule generated by $[\delta]$, $\bar{Y}_0 \subset J^{H_{n-1}}$, and $\bar{Y}_1^{H_{n-1}} + \bar{Y}_0 \subset [K_{n-1}^\times]$ (by Theorem 2, case $n = 1$, part (3)). Hence $J^{H_{n-1}} \subset X + [K_{n-1}^\times] \subset \hat{J}$. (Here we use the fact that $\bar{X} \subset X$ and $\bar{Y}_0^{H_{n-1}} = \bar{Y}_0$.)

Now suppose that $[\Gamma] \in J \setminus (X + [K_{n-1}^\times])$. Our goal is to show that $[\Gamma] = [\theta] + [\gamma]$ with $[\theta] \in X$ and $[\gamma]^{(\sigma-1)^{l(\gamma)-1}} \in \hat{Y}_n^G$. Then, with this result in hand, we will adapt the proof of the Inclusion Lemma (1) to show that $J \subset \hat{J}$.

Write $l_H(\Gamma)$ for the length of the cyclic $\mathbb{F}_p[H_{n-1}]$-submodule of J generated by Γ. Since $[\Gamma] \notin J^{H_{n-1}}$, we find $l_H(\Gamma) \ge 2$.

If $l_H(\Gamma) = 2$ and Γ is exceptional, we find γ and θ as follows. By Proposition 8, δ and Γ are exceptional elements for K/K_{n-1}. Since $[N_{K/K_{n-1}}(\Gamma)]_{K_{n-1}} \neq [1]_{K_{n-1}}$, we see that for a suitable power $s \in \mathbb{Z}$, $[N_{K/K_{n-1}}(\Gamma)]_{K_{n-1}} = [N_{K/K_{n-1}}(\delta)]_{K_{n-1}}^s$. Set $\theta = \delta^s$ and $\gamma = \Gamma/\theta$. Then $[N_{K/K_{n-1}}(\gamma)]_{K_{n-1}} = [1]_{K_{n-1}}$ and so $[N_{K/F}(\gamma)]_F = [1]_F$. Moreover, $l(\gamma) > p^{n-1}$ since otherwise $[\gamma] \in J^{H_{n-1}}$ and by the Exact Sequence Lemma (6) we would have $[\gamma] \in [K_{n-1}^\times]$, contradicting our assumption on Γ. Thus we have $l(\gamma), l(\Gamma) > p^{n-1}$. Also since the maximum length of the elements in X is at most $p^{n-1} + 1$ by Proposition 7, we have $l(\theta) \le p^{n-1} + 1$. Now if $l(\Gamma) > l(\theta)$ then $l(\gamma) = l(\Gamma/\theta) = l(\Gamma)$, and if $l(\Gamma) = l(\theta)$ then $l(\Gamma) = l(\theta) = p^{n-1} + 1$ and $p^{n-1} < l(\gamma) \le p^{n-1} + 1$, showing that in this case as well $l(\Gamma) = l(\gamma)$. Thus in all cases $l(\Gamma) = l(\gamma)$ and therefore also $l_H(\gamma) = l_H(\Gamma)$.

Otherwise, let $\theta = 1$ and $\gamma = \Gamma$. Clearly $l(\gamma) = l(\Gamma)$ and $l_H(\gamma) = l_H(\Gamma)$.

In either case, our choice of γ is made in order to make sure that we have either $l_H(\gamma) \geq 3$ or both $l_H(\gamma) = 2$ and $[N_{K/F}(\gamma)]_F = [1]_F$. These are the necessary hypotheses to apply the Second Fixed Elements are Norms Lemma (12), part (a), by which we obtain that there exists $[\chi] \in J$ such that $[\gamma]^{(\sigma-1)^{l(\gamma)-1}} = [N_{K/F}(\chi)] \in \hat{Y}_n$. Hence we have shown that for all $[\Gamma] \in J \setminus (X + [K_{n-1}^\times])$, we have that $[\Gamma] = [\theta] + [\gamma]$ with $[\theta] \in X$, $[\gamma]^{(\sigma-1)^{l(\gamma)-1}} \in \hat{Y}_n^G$.

Now we adapt the proof of the Inclusion Lemma (1) to show that $J \subset \hat{J}$, by induction on the socle series J_i of J. Since $\sigma^{p^{n-1}} - 1 \equiv (\sigma-1)^{p^{n-1}}$, $J^{H_{n-1}} = J_{p^{n-1}}$. Hence $J_{p^{n-1}} \subset \hat{J}$ and our base case for the induction is $J_{p^{n-1}}$.

For the inductive step, assume that $J_i \subset \hat{J}$ for all $i < t$ for some $p^{n-1} < t \leq p^n$, and let $[\Gamma] \in J_t \backslash J_{t-1}$. Then $l(\Gamma) = t$. If $[\Gamma] \in X + [K_{n-1}^\times]$, we have already shown that $[\Gamma] \in \hat{J}$. Therefore we assume that this is not the case.

By our result above, we may write $[\Gamma] = [\theta] + [\gamma]$ with $[\theta] \in X$ and

$$[\gamma]^{(\sigma-1)^{l(\gamma)-1}} = [N_{K/F}(\chi)] \in \hat{Y}_n^G$$

for some $[\chi] \in J$. Moreover, as we have shown, we may assume that $l(\gamma) = l(\Gamma)$. To show that $[\Gamma] \in \hat{J}$, it is enough to show that $[\gamma] \in \hat{J}$. Since $\hat{Y}_n$ is a free $\mathbb{F}_p[G]$-module, there exists $[\chi'] \in \hat{Y}_n$ such that $[N_{K/F}(\chi')] = [\chi']^{(\sigma-1)^{p^n-1}} = [\chi]^{(\sigma-1)^{l(\chi)-1}}$. Set $[\gamma'] = [\chi']^{(\sigma-1)^{p^n-t}} \in \hat{Y}_n \subset \hat{J}$. Then $l(\gamma/\gamma') < t$. By induction $[\gamma/\gamma'] \in \hat{J}$, and since $[\gamma'] \in \hat{J}$, $[\gamma] \in \hat{J}$ as well. Hence our induction is complete.

The case $p = 2$ follows similarly with the following modifications. Replace $H := H_{n-1}$ with $H := H_{n-2}$ and K_{n-1} with K_{n-2}. Thus we consider $\hat{J}$ and J as $\mathbb{F}_2[H]$-modules. By Proposition 5 (our theorem in the base case $p = 2$ and $n = 2$) and by Proposition 8 we may write

$$J = \bar{X} \oplus \bar{Y}_0 \oplus \bar{Y}_1 \oplus \bar{Y}_2,$$

where $\bar{X} \subset X$ is the cyclic $\mathbb{F}_2[H]$-module generated by $[\delta]$ and for $i = 0, 1, 2$, the summand $\bar{Y}_i$ is a direct sum of cyclic $\mathbb{F}_2[H]$-modules of dimension 2^i. By Proposition 5 we also have

$$\begin{aligned} J^{H_{n-1}} &\subset \bar{X}^{H_{n-1}} \oplus (\bar{Y}_0 \oplus \bar{Y}_1 \oplus \bar{Y}_2)^{H_{n-1}} \\ &\subset X \oplus [K_{n-1}^\times] \subset \hat{J}. \end{aligned}$$

Now suppose that $[\Gamma] \in J \setminus (X + [K_{n-1}^\times])$. Again we want to show that $[\Gamma] = [\theta] + [\gamma]$ with $[\theta] \in X$ and $[\gamma]^{(\sigma-1)^{l(\gamma)-1}} \in \hat{Y}_n^G$. We have $[\Gamma] \notin J^{H_{n-1}}$ and so $l_H(\Gamma) \geq 3$.

If $l_H(\Gamma) = 3$ then

$$[N_{K/K_{n-2}}(\Gamma)]_{K_{n-2}} = [a_{n-2}]_{K_{n-2}}^s$$

for some $s \in \mathbb{Z}$. Set $\theta = \delta^s$ and $\gamma = \Gamma/\theta$. Then

$$[N_{K/K_{n-2}}(\gamma)]_{K_{n-2}} = [1]_{K_{n-2}},$$

whence $[N_{K/F}(\gamma)]_F = [1]_F$. Also $[\gamma] \notin X + [K_{n-1}^\times]$ and therefore $l(\gamma) > 2^{n-1}$. On the other hand, $l(\theta) \leq 2^{n-1} + 1$ by Proposition 7. Hence we see again that $l(\gamma) = l(\Gamma)$ and in particular $l_H(\gamma) \geq 3$.

Otherwise, if $l_H(\Gamma) = 4$ then let $\theta = 1$ and $\gamma = \Gamma$. Clearly $l(\gamma) = l(\Gamma)$ and $l_H(\gamma) = l_H(\Gamma)$.

In either case, we see from the Second Fixed Elements are Norms Lemma (12), part (b), that there exists $[\chi] \in J$ such that $[N_{K/F}(\chi)] = [\gamma]^{(\sigma-1)^{l(\gamma)-1}}$.

From now on the proof that $\hat{J} = J$ in the case $p = 2$ is identical with the proof above for the case $p > 2$. □

Proof of Theorem 2. The case $n = 1$ is Proposition 4. The cases $p = 2$ and $n = 2$ were established in Proposition 5. We proceed by induction on n. Assume that the Theorem holds for $n - 1$. By Proposition 9, we write $J = X + \hat{Y}$, $\hat{Y} = \oplus \hat{Y}_i$, where $\hat{Y}_i$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i and for $i < n$, $[K_i^\times] = \hat{Y}^{H_i}$.

We define the Y_i and Y as follows. When $m = -\infty$, set $Y_i = \hat{Y}_i$ and $Y = \sum Y_i$. Now suppose that $m \geq 0$, and let $[\beta] \in [K_m^\times]$ satisfy $[\beta] = [\delta]^{(\sigma-1)}$. By Proposition 7, we see that $l(\beta) = p^m$ and so the cyclic $\mathbb{F}_p[G]$-submodule M_β generated by $[\beta]$ is a free $\mathbb{F}_p[G/H_m]$-submodule. Moreover, we have already established that $[K_m^\times] \subset \hat{Y}$, so $M_\beta \subset \hat{Y}$.

Now let $[\gamma] = [\beta]^{(\sigma-1)^{p^m-1}} \in \hat{Y}^G$. Suppose that $[\gamma] \in W := \hat{Y}_{m+1} + \cdots + \hat{Y}_n$. Then since W^G is in the image of $(\sigma-1)^{p^m+1}$ on W, there exists $[\alpha] \in W$ such that $[\alpha]^{(\sigma-1)^{p^m+1}} = [\gamma]$. Hence $[\beta'] = [\alpha]^{(\sigma-1)}$, being in the image of $(\sigma-1)$, satisfies $[N_{K/F}(\beta')]_F = [1]_F$, while $l(\beta') = p^m + 1 = l(\delta)$ and $[\beta']^{(\sigma-1)^{p^m}} = [\delta]^{(\sigma-1)^{p^m}}$. Hence $[N_{K/F}(\delta/\beta')]_F \neq [1]_F$ and $l(\delta/\beta') \leq p^m$. But this contradicts the minimality of $l(\delta)$ among lengths $l(z)$ with $[N_{K/F}(z)]_F \neq [1]_F$, a contradiction. Hence $[\gamma] \notin W$.

However, since $[\gamma]$ is in the image of $(\sigma-1)^{p^m-1}$ on $\hat{Y}$, $[\gamma] \in \hat{Y}_m \oplus \cdots \oplus \hat{Y}_n$. Let $[\gamma']$ be the component of $[\gamma]$ in $\hat{Y}_m$. By the previous paragraph, $[\gamma'] \neq [1]$. Now since $[\gamma']$ lies in $\hat{Y}_m^G$, we have that $[\gamma']$ is in the image of $(\sigma-1)^{p^m-1}$ on $\hat{Y}_m$. Now let $[\beta]_{(m)} \in \hat{Y}_m$ be a projection of $[\beta]$ into Y_m. (Since $M_\beta \subset \hat{Y}$ this projection is well defined.) Moreover since $[\gamma'] = [\beta]_{(m)}^{(\sigma-1)^{p^m-1}} \neq [1]$ we see that $[\beta]_{(m)}$ generates a cyclic $\mathbb{F}_p[G]$-submodule $M_{[\beta]_{(m)}}$ of $\hat{Y}_m$ which is a free $\mathbb{F}_p[G/H_m]$-submodule of $\hat{Y}_m$. By the Free Complement Lemma (3), there exists a free $\mathbb{F}_p[G/H_m]$-complement Y_m of $M_{[\beta]_{(m)}}$ in $\hat{Y}_m$. Having defined Y_m, we set all other $Y_i = \hat{Y}_i$, $i \neq m$, and $Y = \sum Y_i$.

Since the $\hat{Y}_i$ are all independent, the Y_i are independent. Assume now that $m > 0$. Then $X + \sum Y_i = X + \sum \hat{Y}_i$, because clearly $X + \sum Y_i \subset X + \sum \hat{Y}_i$, and $\hat{Y}_m \subset X + \sum Y_i$ follows from our construction of Y_m. Hence we have $X + Y = X + \hat{Y} = J$. Because in the case $m = -\infty$ we set $Y_i = \hat{Y}_i$ for all $i \in \{0, 1, \ldots, n\}$ we see that $J = X + \hat{Y} = X + Y$ as well. To show the sum is direct, consider first the case $m = -\infty$. Here $X^G = X$, and by the Fixed Submodule Lemma (5) $X^G \cap Y^G$ is trivial. Hence the Exclusion Lemma (2) gives X and Y are independent. When $m \geq 0$, X^G is generated by $[\gamma]$, which by construction satisfies $[\gamma] \notin Y^G$. Again using the Exclusion Lemma (2), we have X and Y are independent.

We now show that $X^{(\sigma-1)} \oplus Y^{H_i} = \hat{Y}^{H_i}$ for $i \geq m$. (Here $X^{(\sigma-1)}$ means the image of X under $(\sigma-1)$.) First observe that since X and Y are independent we indeed have $X^{(\sigma-1)} + Y^{H_i} = X^{(\sigma-1)} \oplus Y^{H_i}$. If $m = -\infty$ then $X^{(\sigma-1)} = \{0\}$ and the equality $X^{(\sigma-1)} \oplus Y^{H_i} = \hat{Y}^{H_i}$ is a trivial statement. Assume now that $m \geq 0$. We have

$$X^{(\sigma-1)} \subset [K_m^\times] \subset [K_i^\times] \subset \hat{Y}^{H_i}.$$

Hence

$$X^{(\sigma-1)} \oplus Y^{H_i} \subset \hat{Y}^{H_i}.$$

To obtain the reverse inclusion, observe that $Y_i = \hat{Y}_i$, $i \neq m$, and $\hat{Y}_m \subset X^{(\sigma-1)} + Y_m$ by our construction of Y_m. Finally since $\hat{Y}_m^{H_i} = \hat{Y}_m$ and $Y_m^{H_i} = Y_m$ we see that also $\hat{Y}^{H_i} \subset X^{(\sigma-1)} \oplus Y^{H_i}$. Thus we indeed have the desired equality

$$X^{(\sigma-1)} \oplus Y^{H_i} = \hat{Y}^{H_i} = [K_i^\times],$$

for each $i \in \{m, m+1, \dots, n-1\}$ if $m \geq 0$ and for each $i \in \{0, 1, \dots, n-1\}$ if $m = -\infty$. For $i < m$, observe that since X is cyclic of length $p^m + 1$ and $(\sigma^{p^i} - 1) \equiv (\sigma - 1)^{p^i}$ on J,

$$X^{H_i} = X^{(\sigma-1)(\sigma^{p^i}-1)^{p^{m-i}-1}}.$$

Then, since $[K_i^\times] = \hat{Y}^{H_i} = (X^{(\sigma-1)})^{H_i} \oplus Y^{H_i}$ for all $i \leq m$, we are done. $\square$

6. Proofs of Corollaries

Proof of Corollary 1. Recall that if M is a cyclic $\mathbb{F}_p[G]$-module of dimension l, then the $l+1$ submodules of M are cyclic, given by $(\sigma-1)^i M$, $i = 0, 1, \dots, l$, and have annihilators $\langle(\sigma-1)^{l-i}\rangle \subset \mathbb{F}_p[G]$, respectively. By Theorem 1, $[K_i^\times] = J^{H_i} = \oplus Y_i^{H_i}$, and Y_j is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^j.

Now $H_i = \langle \sigma^{p^i} \rangle$ and $(\sigma^{p_i} - 1) \equiv (\sigma-1)^{p^i}$ on J. When $j < i$, observe that Y_j is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension $p^j < p^i$, and so $Y_j = Y_j^{H_i}$. When $j \geq i$, the submodule $Y_j^{H_i}$ is given by $Y_j^{(\sigma-1)^{p^j-p^i}}$.

On $[K_i^\times]$, $N_{K_i/F} \equiv (\sigma-1)^{p^i-1}$. For $j < i$, since $Y_j^{H_i}$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension $p^j < p^i$, $N_{K_i/F}$ annihilates $Y_j^{H_i}$. For $j \geq i$, $Y_j^{H_i}$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i and so applying $N_{K_i/F}$ to $Y_j^{H_i}$ yields $Y_j^{(\sigma-1)^{p^j-1}} = Y_j^G$. Hence we have the first statement.

Now a cyclic $\mathbb{F}_p[G]$-module of dimension p^i is a free $\mathbb{F}_p[G/H_i]$-module on one generator, and for direct sums M of such modules,

$$\operatorname{rank}_{\mathbb{F}_p[G/H_i]} M = \dim_{\mathbb{F}_p} M^G.$$

Observe that Y_j, $j < n$, is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension $p^j < p^n$. Applying $N_{K/F}$ to J, then, we see that $Y_n^G = [N_{K/F}(K^\times)]$. Moreover, with a descending induction we see that Y_i^G is a complement of $[N_{K_{i+1}/F}(K_{i+1}^\times)]$ in $[N_{K_i/F}(K_i^\times)]$. Hence we have the second statement. $\square$

Proof of Corollary 2. We begin as in the previous proof. If $m = -\infty$, in fact, then the previous proof carries over without modification. Hence we assume that $m \geq 0$.

By Theorem 2, $[K_i^\times] = X' \oplus Y_i^{H_i}$, where X' is a cyclic $\mathbb{F}_p[G]$-module of dimension p^i if $i \leq m$ and of dimension p^m if $i \geq m$. As in the previous proof, $Y_j^{H_i} = Y_j$ for $j < i$ and $Y_j^{H_i}$ for $j \geq i$ is a direct sum of cyclic $\mathbb{F}_p[G]$-modules of dimension p^i. Similarly, $N_{K_i/F}$ annihilates $Y_j^{H_i}$, $j < i$, and yields Y_j^G when $j \geq i$. Applying $N_{K_i/F}$ annihilates X' when $m < i$ and otherwise yields $(X')^G = X^G$. Hence we have the statements locating $[N_{K_i/F}(K_i^\times)]$.

For the statements establishing ranks, we proceed as in the previous proof. Observe that since $X \cap [K_m^\times] = X^{\sigma-1}$ is a cyclic $\mathbb{F}_p[G]$-submodule of dimension p^m, we obtain $X^G \subset [N_{K_m/F}(K_m^\times)]$. If $m \neq n-1$, then since $X \cap [K_{m+1}^\times] = X^{\sigma-1}$ is a cyclic $\mathbb{F}_p[G]$-submodule of dimension $p^m < p^{m+1}$, we see that $X^G \cap [N_{K_{m+1}/F}(K_{m+1}^\times)] = \{0\}$. If $m = n-1$ then $X \cap [K_{m+1}^\times] = X$ is a cyclic $\mathbb{F}_p[G]$-submodule of dimension $p^m + 1$, which is annihilated by $N_{K_{m+1}/F}$ unless $p^m + 1 = p^{m+1} = p^n$—that is, $p = 2$, $m = 0$, $n = 1$. But this latter case violates the hypothesis of Theorem 2. Hence $X^G \subset [N_{K_m/F}(K_m^\times)] \setminus [N_{K_{m+1}/F}(K_{m+1}^\times)]$ under our hypotheses.

Again, since Y_j, $j < n$, and X are direct sums of cyclic $\mathbb{F}_p[G]$-submodules of dimension less than p^n, applying $N_{K/F}$ to J yields $Y_n^G = [N_{K/F}(K^\times)]$. A descending induction yields that Y_i^G, $m < i < n$, is a complement of $[N_{K_{i+1}/F}(K_{i+1}^\times)]$ in $[N_{K_i/F}(K_i^\times)]$. But Y_m^G is a complement of $[N_{K_{m+1}/F}(K_{m+1}^\times)] + X^G$ in $[N_{K_m/F}(K_m^\times)]$. For $i < m$, then as before Y_i^G is a complement of $[N_{K_{i+1}/F}(K_{i+1}^\times)] = (X + Y_{i+1} + \cdots + Y_n)^{H_i}$ in $[N_{K_i/F}(K_i^\times)]$. Hence we have the statements establishing $\operatorname{rank}_{\mathbb{F}_p[G/H_i]} Y_i$. $\square$

7. Proof of Theorem 3

We shall first prove the first equality in Theorem 3 which says that

$$m = i(K/F) = \min\{s \mid \xi_p \in N_{K/F}(K^\times) N_{K_{n-1}F}(K_s^\times)\}.$$

In order to do so we shall calculate $(\sqrt[p]{N_{K/F}(\alpha)})^{\sigma-1}$, with a suitable $\alpha \in K^\times$, in two ways. Then comparing our results we shall see that we are indeed dealing with the equation

$$E_i : \xi_p = N_{K/F}(\beta)N_{K_{n-1}/F}(\gamma), \qquad \beta \in K^\times,\ \gamma \in K_i^\times,\ 0 \le i < n$$

and that our number $m = i(K/F)$ depends upon the smallest $i \in \{-\infty, 0, 1, \ldots, n-1\}$ such that E_i is solvable for a suitable $\beta \in K^\times$ and $\gamma \in K_i^\times$. The following lemma contains the key expression for $\left(\sqrt[p]{N_{K/F}(\alpha)}\right)^{\sigma-1}$.

Lemma 14. *Suppose that $\alpha^{\sigma-1} = \gamma k^p$ with $\gamma \in K_i^\times$, $0 \le i < n$, and $k \in K^\times$. Suppose additionally that if $p = 2$ then $n > 1$.*

Then

$$\left(\sqrt[p]{N_{K/F}(\alpha)}\right)^{\sigma-1} = N_{K/F}(k)\sqrt[p]{N_{K/F}(\gamma)},$$

where

$$\sqrt[p]{N_{K/F}(\gamma)} = \left(N_{K_i/F}(\gamma)\right)^{p^{n-i-1}}.$$

Proof. First we claim that

$$N_{K/F}(\alpha) = (k^p)^S \alpha^{p^n} \gamma^S,$$

where

$$S := (p^n - 1) + (p^n - 2)\sigma + \cdots + \sigma^{p^n-2} \in \mathbb{Z}[G].$$

Observe that

$$\begin{aligned}
\alpha &= \alpha \\
\alpha^\sigma &= k^p \alpha \gamma \\
\alpha^{\sigma^2} &= ((k^p)^\sigma k^p)\, \alpha\, (\gamma\gamma^\sigma) \\
\alpha^{\sigma^3} &= \left((k^p)^{\sigma^2}(k^p)^\sigma k^p\right) \alpha \left(\gamma\gamma^\sigma\gamma^{\sigma^2}\right) \\
&\cdots \\
\alpha^{\sigma^{p^n-1}} &= \left(\prod_{j=0}^{p^n-2} (k^p)^{\sigma^j}\right) \alpha \left(\prod_{j=0}^{p^n-2} \gamma^{\sigma^j}\right).
\end{aligned}$$

Our result is then the product of the equations.

Now $[\alpha]^{(\sigma-1)} = [\gamma]$, and because $[\gamma] \in [K_i^\times]$ and $[N_{K_i/F}(\beta)] = [\beta]^{(\sigma-1)^{p^i-1}}$ for $\beta \in K_i^\times$, we obtain $[\gamma]^{(\sigma-1)^{p^i}} = [1]$. Hence $[\alpha]^{(\sigma-1)^{p^i+1}} = [1]$. Now $p^i + 1 < p^n$ unless $p = 2$ and $n = 1$, a case we have excluded. Hence $l(\alpha) < p^n$, whence $[N_{K/F}(\alpha)] = [1]$, and so $N_{K/F}(\alpha) \in K^{\times p}$.

Therefore $\gamma^S \in K^{\times p}$ as well, and we may choose a pth root $\sqrt[p]{\gamma^S} \in K^\times$. We then choose

$$\sqrt[p]{N_{K/F}(\alpha)} = k^S \alpha^{p^{n-1}} \sqrt[p]{\gamma^S}.$$

(Because $(\sqrt[p]{N_{K/F}(\alpha)})^{\sigma-1}$ does not depend upon the choice of a pth root of $N_{K/F}(\alpha)$ we see that we are free to make this choice.)

Our next claim is that

$$\left(\sqrt[p]{\gamma^S}\right)^{\sigma-1} = \frac{(N_{K_i/F}(\gamma))^{p^{n-i-1}}}{\gamma^{p^{n-1}}}.$$

Let L be the Galois closure of $K(\sqrt[p]{\gamma})$ over F. Since $[\gamma]$ lies in the image of $\sigma - 1$ on J, we have $[N_{K/F}(\gamma)]_F = [1]_F$. Let $\hat{\sigma}$ be any pullback of σ to $\mathrm{Gal}(L/F)$. Then

$$\sqrt[p]{\gamma}^{(\hat{\sigma}^{p^n}-1)} = \sqrt[p]{\gamma}^{(1+\hat{\sigma}+\cdots+\hat{\sigma}^{p^n-1})(\hat{\sigma}-1)} = \left(\sqrt[p]{N_{K/F}(\gamma)}\right)^{(\hat{\sigma}-1)} = 1.$$

(Observe that the equation is independent of the choice of the pth root of $N_{K/F}(\gamma)$.) Hence $\hat{\sigma}^{p^n}$ leaves $\sqrt[p]{\gamma}$ fixed. Now the field L is generated over K by all elements $\sqrt[p]{\tilde{\gamma}}$, where $\tilde{\gamma}$ runs through all conjugate elements γ^τ for $\tau \in G = \mathrm{Gal}(K/F)$. Therefore $[N_{K/F}(\tilde{\gamma})]_F = [1]_F$ for each such $\tilde{\gamma}$ and the same argument as above shows that $\hat{\sigma}^{p^n}$ leaves each $\sqrt[p]{\tilde{\gamma}}$ fixed. Since $\hat{\sigma}$ restricted to K is σ we see that $\hat{\sigma}^{p^n}$ leaves every element of K fixed. Hence $\hat{\sigma}^{p^n}$ leaves every element of L fixed as well. Therefore $\hat{\sigma}^{p^n} = 1 \in \mathrm{Gal}(L/F)$.

Set $\hat{S} = (p^n - 1) + (p^n - 2)\hat{\sigma} + \cdots + \hat{\sigma}^{p^n-2}$, $\hat{N} = 1 + \hat{\sigma} + \cdots + \hat{\sigma}^{p^n-1}$, and note that $\hat{N} = \hat{N}_1\hat{N}_2$, where $\hat{N}_1 = 1 + \hat{\sigma}^{p^i} + \hat{\sigma}^{2p^i} + \cdots + \hat{\sigma}^{(p^{n-i}-1)p^i}$ and $\hat{N}_2 = 1 + \hat{\sigma} + \cdots + \hat{\sigma}^{p^i-1}$. Further observe that $\hat{N}_1 \equiv N_{K/K_i}$ on $K^\times$, $\hat{N}_2 \equiv N_{K_i/F}$ on K_i, and $(\hat{\sigma} - 1)\hat{S} = \hat{N} - p^n$.

We calculate $(\sqrt[p]{\gamma^S})^{\sigma-1}$ in two cases. First assume that $\gamma \in K_0^\times$. Then $\gamma^S = \gamma^{p^n(p^n-1)/2}$ and since in the case $p = 2$ we assume that $n \geq 2$ we see that γ^S is a pth power of an element in $K_0^\times$ and therefore $(\sqrt[p]{\gamma^S})^{\sigma-1} = 1$ confirming our claim in this case. Next assume that

$i > 0$. Then we have

$$\left(\sqrt[p]{\gamma^S}\right)^{\sigma-1} = \left(\sqrt[p]{\gamma}^{\hat{S}}\right)^{\sigma-1} = \frac{(\sqrt[p]{\gamma})^{\hat{N}}}{(\sqrt[p]{\gamma})^{p^n}} = \frac{\left(\sqrt[p]{\gamma}^{\hat{N}_1}\right)^{\hat{N}_2}}{\gamma^{p^{n-1}}}$$
$$= \frac{\left(\xi_p^c \gamma^{p^{n-i-1}}\right)^{\hat{N}_2}}{\gamma^{p^{n-1}}} = \frac{(N_{K_i/F}(\gamma))^{p^{n-i-1}}}{\gamma^{p^{n-1}}},$$

where ξ_p^c is a suitable pth root of 1.

Returning to $\sqrt[p]{N_{K/F}(\alpha)}$, we may write

$$\begin{aligned}\left(\sqrt[p]{N_{K/F}(\alpha)}\right)^{\sigma-1} &= k^{S(\sigma-1)}(\alpha^{p^{n-1}})^{\sigma-1}\left(\sqrt[p]{\gamma^S}\right)^{\sigma-1}\\ &= k^{N-p^n}(\alpha^{\sigma-1})^{p^{n-1}}\left(\sqrt[p]{\gamma^S}\right)^{\sigma-1}\\ &= \frac{N_{K/F}(k)}{k^{p^n}}(\gamma k^p)^{p^{n-1}}\frac{(N_{K_i/F}(\gamma))^{p^{n-i-1}}}{\gamma^{p^{n-1}}}\\ &= N_{K/F}(k)(N_{K_i/F}(\gamma))^{p^{n-i-1}}.\end{aligned}$$

□

Proof of Theorem 3. We have three equalities to establish, and we begin by showing $m = \min\left\{s \mid \xi_p \in N_{K/F}(K^\times)N_{K_{n-1}/F}(K_s^\times)\right\}$.

If $m = -\infty$ then $l(\delta) = 1$ for δ an exceptional element. Hence $[\delta] \in J^G$ and $[N_{K/F}(\delta)]_F \neq [1]_F$. However, for all $f \in F^\times$, we have $[N_{K/F}(f)]_F = [1]_F$. Hence $[\delta] \in J^G \setminus [F^\times]$. Therefore, by the Fixed Submodule Lemma (5), $\xi_p \in N_{K/F}(K^\times)$. Going the other way, if $\xi_p \in N_{K/F}(K^\times)$, the Fixed Submodule Lemma (5) tells us that there exists an exceptional element in J^G and so $m = -\infty$. Hence the Theorem holds when $m = -\infty$.

Assume then that $m \geq 0$. Consider $\alpha \in K^\times$ with $l(\alpha) < p^n$. By the Norm Lemma (8), $[N_{K/F}(\alpha)]_F \in \langle [a]_F \rangle$. It follows that $[N_{K/F}(\alpha)]_F \neq [1]_F$ if and only if $\sqrt[p]{N_{K/F}(\alpha)}^{\sigma-1}$ is a nontrivial pth root of unity, say ξ^t.

Now assume that $[N_{K/F}(\alpha)]_F \neq [1]_F$ and $[\alpha]^{(\sigma-1)} = [\gamma]$, $\gamma \in K_i^\times$, $i < n$. Then by Lemma 14, $\xi_p^t = N_{K/F}(k)(N_{K_i/F}(\gamma))^{p^{n-i-1}}$, and, by taking an appropriate power of ξ_p^t, we have that

$$\xi_p \in N_{K/F}(K^\times)(N_{K_i/F}(K_i^\times))^{p^{n-i-1}}.$$

Since there exists an exceptional element α with $[\alpha]^{(\sigma-1)} \in [K_m^\times]$, observe that $N_{K_{n-1}/F}(\gamma) = (N_{K_i/F}(\gamma))^{p^{n-i-1}}$ to conclude that $\xi_p \in N_{K/F}(K^\times)N_{K_{n-1}/F}(K_m^\times)$. Hence the minimum s is less than or equal to m.

Going the other way, assume that $\xi_p = N_{K/F}(k)N_{K_s/F}(\gamma)^{p^{n-s-1}}$ for $k \in K$ and $\gamma \in K_s^\times$, $s < n$. Then $1 = N_{K/F}(k^p\gamma)$ and so by Hilbert 90 there exists $\delta \in K^\times$ with $\delta^{\sigma-1} = \gamma k^p$. Since $[\gamma] \in [K_s^\times]$ and $\sigma^{p^s} - 1 \equiv (\sigma-1)^{p^s}$ annihilates $[K_s^\times]$, we have $l(\delta) \leq p^s + 1 < p^n$. By Lemma 14, $\sqrt[p]{N_{K/F}(\delta)}^{(\sigma-1)} = N_{K/F}(k)(N_{K_s/F}(\gamma))^{p^{n-s-1}} = \xi_p$. Since $l(\delta) < p^n$ and $\sqrt[p]{N_{K/F}(\delta)}^{(\sigma-1)}$ is a nontrivial pth root of unity, we use the equations above to deduce that $[N_{K/F}(\delta)]_F \neq [1]_F$. Therefore by the definition of exceptionality, $m \leq s$.

We now establish the remaining two equalities. For convenience, we set

$$T := \left\{t \mid \exists[\delta] \in J^{H_{t+1}}, [N_{K/K_{t+1}}(\delta)]_{K_{t+1}} \neq [1]_{K_{t+1}}\right\}$$

and

$$S := \left\{s \mid \xi_p \in N_{K/K_{s+1}}(K^\times)\right\}.$$

Observe that $n-1 \in T$ because $\{0\} \neq X \subset J$ by Theorem 2 and $N_{K/K_n}(k) = k$ for each $k \in K^\times$, and $n-1 \in S$ since $\xi_p \in F^\times \subset K^\times = N_{K/K_n}(K^\times)$. Hence the minima are well-defined. It remains to show that $m = \min T = \min S$.

To see that $m = \min T$, consider $t \in T$ with $t \leq n-2$ such that there exists $[z] \in J^{H_{t+1}}$ with $[N_{K/K_{t+1}}(z)]_{K_{t+1}} \neq [1]_{K_{t+1}}$. The Norm Lemma (8) gives that $[N_{K/F}(z)]_F \neq [1]_F$ if and only if $[N_{K/K_{t+1}}(z)]_{K_{t+1}} \neq [1]_{K_{t+1}}$, except possibly if $l(z) > p^n - p^{t+1}$. But this latter case occurs only if $t = n-1$, since otherwise $[z] \in J^{H_{n-1}}$ and so $l(z) \leq p^{n-1} \leq p^n - p^{t+1}$. Now for δ an exceptional element of K/F, we have $l(\delta) = p^m + 1 \leq l(z)$, by Proposition 7, and hence $[z] \in J^{H_{t+1}}$ implies $l(\delta) = p^m + 1 \leq l(z) \leq p^{t+1}$. Hence $m \leq t$. In the case $t = n-1$, again Proposition 7 gives $m < n$ and hence $m \leq t$. We conclude that $m \leq \min T$.

For the other direction, observe that for δ an exceptional element of K/F, then $l(\delta) = p^m + 1$ and therefore we have $[\delta] \in J^{H_{m+1}}$. Further, by the Norm Lemma (8), $[N_{K/K_{m+1}}(\delta)]_{K_{m+1}} \neq [1]_{K_{m+1}}$, except possibly if $m = n-1$. (Here we have used

$$p^m + 1 \leq p^{n-1} \leq p^n - p^{n-1} \leq p^n - p^{m+1}$$

whenever $m \leq n-2$.) When $m = n-1$, then clearly $m \geq \min T$. Otherwise $m \in T$ and we deduce that $m \geq \min T$. We conclude that $m = \min T$.

Finally, we establish that $\min T = \min S$. Let $t \in T$ with $t \leq n-2$, and let z satisfy $[z] \in J^{H_{t+1}}$ and $[N_{K/K_{t+1}}(z)]_{K_{t+1}} \neq [1]_{K_{t+1}}$. From the Fixed Submodule Lemma (5), part (2), we obtain $z^{(\sigma^{p^{t+1}}-1)} = \lambda^p$ with $N_{K/K_{t+1}}(\lambda) = \xi_p^\nu$ for some $\nu \in \mathbb{Z}$ not divisible by p. Choosing an appropriate power of z, we may assume $\nu = 1$. Hence $t \in S$. Since $n-1 \in T \cap S$, we have $T \subset S$.

Conversely, suppose $s \in S$ and $s \leq n-2$ satisfies $\xi_p \in N_{K/K_{s+1}}(\lambda)$ for $\lambda \in K^\times$. We have $1 = N_{K/K_{s+1}}(\lambda^p)$, and by Hilbert 90 we see that there exists $\delta \in K^\times$ such that $\delta^{\sigma^{p^{s+1}}-1} = \lambda^p$. Hence $[\delta] \in J^{H_{s+1}}$, and again using the Fixed Submodule Lemma (5) and its proof we see that $[N_{K/K_{s+1}}(\delta)]_{K_{s+1}} \neq [1]_{K_{s+1}}$. Hence $s \in T$. Since $n-1 \in T \cap S$, we have $S \subset T$. $\square$

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