TOTARO'S QUESTION FOR G_2, F_4, AND E_6

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Abstract. In a 2004 paper, Totaro asked whether a G-torsor X which has a zero-cycle of degree $d > 0$ will necessarily have a closed étale point of degree dividing d, where G is a connected algebraic group. This question is closely related to several conjectures regarding exceptional algebraic groups. Totaro gave a positive answer to his question in the following cases: G simple, split, and of type G_2, type F_4, or simply connected of type E_6. We extend the list of cases where the answer is "yes" to all groups of type G_2 and some nonsplit groups of type F_4 and E_6. No assumption on the characteristic of the base field is made. The key tool is a lemma regarding linkage of Pfister forms.

For certain linear algebraic groups G over a field k and certain homogeneous G-varieties X, Totaro asked in [To]:

(0.2) *If X has a zero-cycle of degree $d > 0$, does X necessarily have a closed étale point of degree dividing d?*

(This question is closely related to earlier questions raised by Veisfeiler, Serre, and Colliot-Thélène.) This question can be rephrased as:

(0.2′) *If X has a (closed) point over finite extensions $K_1, K_2, \ldots, K_n$ of k, does X necessarily have a point over a separable extension of k of degree dividing every $[K_i : k]$?*

The purpose of this note is to prove the following theorem.

Theorem 0.3. *The answer to* (0.2) *is "yes" when X is a G-torsor and G is*

- *of type G_2,*
- *reduced of type F_4, or*
- *simply connected of type ${}^1E^0_{6,6}$ (split) or of type ${}^1E^{28}_{6,2}$.*

A *G-torsor* is a principal G-bundle over $\operatorname{Spec} k$. Torsors are often called principal homogeneous spaces, as in [Se GC]. It is possible that the answer to (0.2) is "yes" whenever G is semisimple and X is a G-torsor (and, in particular, is affine); no counterexamples are known. In contrast, for X projective, there are examples where the answer is "no" even when $d = 1$, see [Fl] and [Pa].

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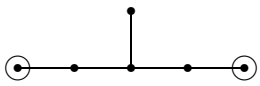
Every group of type F_4 is the group of automorphisms of some uniquely determined Albert k-algebra J. We say that the group is *reduced* if the algebra J is reduced, i.e., if J is not a division algebra. (Recall that Albert algebras are 27-dimensional Jordan algebras. For a survey of the state of the art in 1992, see [PR 94]. Many useful and characteristic-free results about Albert algebras may be found in [Pe].)

The notation "${}^1E_{6,2}^{28}$" is from Tits' classification [Ti 66, p. 58]. Such groups have Tits index

(0.4) 

Totaro proved Theorem 0.3 under the additional assumption that G is split, see [To, 5.1]. The main tool that allows us to extend his result is our Proposition 1.6 and its analogue for cohomology, Cor. 1.7; they assert the existence of field extensions of small dimension that kill a difference of symbols in $H^p(k, \mathbb{Z}/2\mathbb{Z}(p-1))$. The quadratic form theory that underlies Cor. 1.7 occupies half of this paper and concerns mainly the question of so-called linkage of Pfister forms (in all characteristics, including 2).

The last section gives some comments on extending Theorem 0.3 to include other exceptional groups. Except for this final section, we make no assumptions on the base field k.

Notation. Throughout, we write $\boldsymbol{\mu}_n$ for the group scheme of n-th roots of unity, where n is a natural number. We write $H^p(k, \mathbb{Z}/n\mathbb{Z}(p-1))$ for the abelian group defined in [M 03, App. A]; when n is not divisible by the characteristic of k, it is the Galois cohomology group $H^p(k, \boldsymbol{\mu}_n^{\otimes(p-1)})$.

1. Quadratic form preliminaries

For readers who are not very familiar with the algebraic theory of quadratic forms we will first present a few definitions and facts. Other basic results and terminologies that we use without further reference can all be found in Scharlau's book [Sch] for characteristic $\neq 2$, and in Baeza's book [Ba 78] and the article [HL] by Laghribi and the second author for the characteristic 2 case. In particular, we will freely use certain properties of Pfister forms, Pfister neighbors, and function fields of quadratic forms.

If α and β are quadratic forms, then β is called a *subform* of α (denoted $\beta \subset \alpha$), if there exists a form γ such that $\alpha \cong \beta \perp \gamma$, and α is said to *dominate* β (denoted $\beta \prec \alpha$) if β is isometric to the restriction of α to some subspace of the underlying vector space of α.

The Witt index $i_W(\phi)$ of a quadratic form ϕ is the maximal number m such that ϕ contains an orthogonal sum of m hyperbolic planes as a subform. Furthermore, if β is a bilinear form (in characteristic 2), then we denote by β_q the totally singular quadratic form defined by $\beta_q(x) = \beta(x, x)$.

Every nonsingular quadratic form ϕ over k has attached to it an invariant $e_1(\phi)$ living in the Galois cohomology group $H^1(k, \mathbb{Z}/2\mathbb{Z})$. In characteristic not 2, we have $H^1(k, \mathbb{Z}/2\mathbb{Z}) = k^\times/k^{\times 2}$ and $e_1(\phi) = d_\pm(\phi)$ is the signed discriminant. If $\mathrm{char}(k) = 2$, then $H^1(k, \mathbb{Z}/2\mathbb{Z}) = k/\wp(k)$ and $e_1(\phi) = \Delta(\phi)$ is the Arf-invariant. Note that in all characteristics $H^1(k, \mathbb{Z}/2\mathbb{Z})$ is nothing else but the group $H^1(k, \mathbb{Z}/2\mathbb{Z}(0))$ from above.

Recall that in characteristic $\neq 2$ (resp. characteristic 2), a product of p binary quadratic (resp. bilinear) forms $\langle 1, -a_i\rangle$ is called a p-fold Pfister form (resp. p-fold bilinear Pfister form): $\langle\langle a_1, \cdots, a_p\rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_p\rangle$. These forms generate additively the ideal $I^p k$, the p-th power of the fundamental ideal of even-dimensional forms in Wk, the Witt ring of quadratic forms (resp. bilinear forms) over k.

In characteristic 2, a p-fold quadratic Pfister form is a product of a $(p-1)$-fold bilinear Pfister form $\langle\langle a_1, \cdots, a_{p-1}\rangle\rangle$ and a nonsingular quadratic form $[1, a_p] = X^2 + XY + a_pY^2$, $a_p \in k$: $\langle\langle a_1, \cdots, a_{p-1}, a_p]] = \langle\langle a_1, \cdots, a_{p-1}\rangle\rangle \otimes [1, a_p]$. The p-fold quadratic Pfister forms generate the Wk-submodule $I_q^p k = (I^{p-1}k)W_q k$ of the Witt group $W_q k$ of nonsingular quadratic forms over k.

Here, a *p-Pfister form* will always mean either a p-fold Pfister form in characteristic $\neq 2$ or a p-fold quadratic Pfister form if the characteristic is 2, and we will only consider the case $p \geq 1$. Also, we will use in characteristic $\neq 2$ the notation $I_q^p k$ instead of $I^p k$ for uniformity's sake, and we will denote the set of isometry classes of p-Pfister forms over k by $P_p k$.

A quadratic form ϕ is a *Pfister neighbor* of a Pfister form π if $a\phi \subset \pi$ ($\mathrm{char}(k) \neq 2$) resp. $a\phi \prec \pi$ ($\mathrm{char}(k) = 2$) for some $a \in k^\times$, and $2\dim\phi > \dim\pi$.

If ϕ is a Pfister neighbor of π and E/k is any field extension then $i_W(\phi_E) > 0$ if and only if π_E is isotropic if and only if π_E is hyperbolic. If any of these equivalent conditions hold, then $E(\phi)/E$ and $E(\pi)/E$ are purely transcendental, and any form ψ over k which is isotropic over $E(\phi)$ or $E(\pi)$ is so already over E. Applied to $E = k(\phi)$ resp. $E = k(\pi)$, this shows that a form ψ over k is isotropic over $k(\pi)$ if and only if it is isotropic over $k(\phi)$.

In characteristic $\neq 2$, it is well-known that if ϕ and π are Pfister forms with $\pi \subset \phi$, then there exists a Pfister form ρ such that ϕ is isometric to $\rho \otimes \pi$. The analogous result in characteristic 2 is the following.

Lemma 1.1. *Suppose* $\mathrm{char}(k) = 2$. *Let ϕ be a Pfister form over k and let x be in $k^\times$.*

(i) *Let π be a Pfister form with $x\pi \subset \phi$. Then there exists a bilinear Pfister form ρ such that $\phi \cong \rho \otimes \pi$.*
(ii) *Let ρ be a bilinear Pfister form such that $x\rho_q \prec \phi$. Then there exists a Pfister form π such that $\phi \cong \rho \otimes \pi$.*

Proof. We assume that ϕ is anisotropic and leave the (much simpler) case of ϕ being isotropic (and hence hyperbolic) to the reader. We may also assume that $x = 1$ because of the roundness of Pfister forms. (Recall that a form is called *round* if the set of nonzero elements represented by the form is

exactly the group of similarity factors of the form.) Indeed, since π resp. ρ_q represent 1, the hypotheses in (i) resp. (ii) imply that ϕ represents x, hence, the roundness of ϕ implies $\phi \cong x\phi$ and thus $\pi \subset \phi$ resp. $\rho_q \prec \phi$.

(i) Let μ be a quadratic form such that $\phi \cong \mu \perp \pi$. Now $\pi = \langle 1 \rangle \otimes \pi$ and $\langle 1 \rangle$ is a 0-fold bilinear Pfister form. So let $r \geq 0$ be maximal such that there exists an r-fold bilinear Pfister form ρ and a quadratic form η with $\phi \cong \eta \perp (\rho \otimes \pi)$.

Suppose $\dim \eta > 0$, and let $a \in k^\times$ be represented by η. Then $\gamma' = \langle a \rangle_q \perp \rho \otimes \pi$ is dominated by ϕ. In particular, ϕ becomes isotropic and hence hyperbolic over the function field $k(\gamma')$. But γ' is a Pfister neighbor of the Pfister form $\gamma = \langle\langle a \rangle\rangle \otimes \rho \otimes \pi$, hence ϕ becomes isotropic and hence hyperbolic over $k(\gamma)$. Since γ and ϕ represent 1, it follows that there exists a form η' such that $\phi \cong \eta' \perp \gamma$ (see, e.g., [HL, Th. 4.2]). We get a contradiction to the maximality of r as $\gamma \cong \rho' \otimes \pi$ with $\rho' = \langle\langle a \rangle\rangle \otimes \rho$ a bilinear Pfister form of fold $r+1$. Hence $\dim \eta = 0$ and $\phi \cong \rho \otimes \pi$.

(ii) Write $\rho_q \cong \langle 1 \rangle_q \perp \rho'_q$ (with ρ'_q totally singular). Since ρ_q is dominated by ϕ, there exists $b \in k$ such that $\gamma' = [1, b] \perp \rho'_q$ is also dominated by ϕ (see [HL, Cor. 3.3, Lemma 3.5]). Therefore, ϕ becomes isotropic and hence hyperbolic over $k(\gamma')$. But γ' is a Pfister neighbor of the Pfister form $\gamma = \rho \otimes [1, b]$. It follows that ϕ becomes hyperbolic over $k(\gamma)$, and as in the proof of (i), we find a bilinear Pfister form ν such that $\phi \cong \nu \otimes \rho \otimes [1, b]$. The claim follows by putting $\pi = \nu \otimes [1, b]$. □

Remark 1.2. Part (i) in the above lemma is nothing else but [Ba 78, Th. 4.4]. Our proof is quite different from Baeza's original one in that it uses the machinery of function fields of quadratic forms. Part (ii) is basically the analogous result with the roles of bilinear and quadratic Pfister forms reversed.

Lemma 1.3. *Let ϕ and ψ be anisotropic Pfister forms of fold p and q respectively. Then there exists an integer $m \geq 0$ such that $i_W(\phi \perp -\psi) = 2^m$.*

Furthermore, if $m > 0$, then m is maximal for the property such that there exist an m-Pfister form π and Pfister forms (resp. bilinear Pfister forms if $\mathrm{char}(k) = 2$*) ρ and σ of fold $p - m$ and $q - m$, respectively, with $\phi \cong \rho \otimes \pi$ and $\psi \cong \sigma \otimes \pi$.*

Proof. If $\mathrm{char}(k) \neq 2$, this is essentially [EL, Prop.4.4]. In characteristic 2, the result is due to Faivre [Fa], but we include a proof of this case for the reader's convenience.

So let us assume that $\mathrm{char}(k) = 2$. Clearly, ϕ and ψ represent 1, so $i_W(\phi \perp -\psi) \geq 1$. Also, if π is an m-Pfister dividing both ϕ and ψ, then the roundness of Pfister forms implies that π is a subform of both ϕ and ψ, therefore

$$i_W(\phi \perp -\psi) \geq i_W(\pi \perp -\pi) = \dim \pi = 2^m.$$

It thus suffices to consider the case $i_W(\phi \perp -\psi) \geq 2$. We have to show that the Witt index is equal to the maximal dimension of a Pfister form π which

is a subform of both ϕ and ψ. The existence of the bilinear Pfister forms ρ and σ then follows from Lemma 1.1(i).

We first show that there exists a common 1-Pfister form as subform of both ϕ and ψ.

Now $i_W(\phi \perp -\psi) \geq 2$ implies that there is a binary form β (possibly singular) which is dominated both by ϕ and ψ (see [HL, Cor. 3.13]), and we may assume that β represents 1 as ϕ and ψ represent 1. If β is isometric to $[1, b]$ for some $b \in k$, then we are done. Otherwise, β is isometric to $\langle 1, b\rangle_q$ for some $b \in k^\times$. In this case, we necessarily have $p, q \geq 2$ for dimension reasons and since ϕ and ψ are nonsingular. We then apply Lemma 1.1(ii) to find bilinear Pfister forms ρ' and σ', and $c, d \in k$ such that $\phi \cong \langle\langle b\rangle\rangle \otimes \rho' \otimes [1, c]$ and $\psi \cong \langle\langle b\rangle\rangle \otimes \sigma' \otimes [1, d]$. But then there exist $e, f, g \in k$ such that $\langle\langle b, c]] \cong \langle\langle e, g]]$ and $\langle\langle b, d]] \cong \langle\langle f, g]]$ (see, e.g., [Lam] for the equivalent "linkage" result for quaternion algebras). Clearly, $[1, g]$ is the desired form.

Having found a 1-Pfister form as subform of both ϕ and ψ, let now π be an m-Pfister form which is a subform of both ϕ and ψ and with m maximal. Thus, we can write $\phi \cong \phi' \perp \pi$ and $\psi \cong \psi' \perp \pi$, and we have

$$i_W(\phi \perp -\psi) = i_W(\phi' \perp -\psi') + 2^m.$$

If $i_W(\phi \perp -\psi) > 2^m$ then $\phi' \perp -\psi'$ is isotropic, hence ϕ' and ψ' represent a common element $a \in k^\times$, and an argument as in the proof of Lemma 1.1(i) then shows that the $(m+1)$-Pfister $\langle\langle a\rangle\rangle \otimes \pi$ is a subform of both ϕ and ψ, a contradiction to the maximality of m. This completes the proof. □

In the above lemma, we restricted ourselves to the case of anisotropic Pfister forms. It is obvious that if, say, ϕ is isotropic and hence hyperbolic, then $i_W(\phi \perp -\psi) = 2^{p-1}$ or $2^{p-1} + 2^{q-1}$, depending on whether ψ is anisotropic or not. In this case, ϕ divides ψ if ψ is hyperbolic and $q \geq p$, and ψ divides ϕ if $p > q$ or $p = q$ and ψ is hyperbolic. If ψ is anisotropic and $q \geq p$, then any $(p-1)$-Pfister dividing ψ will also divide the hyperbolic form ϕ, but there is obviously no p-Pfister dividing both ϕ and ψ.

Lemma 1.4. *Let ϕ, ψ be p-Pfister forms over k, with $p \geq 2$. Suppose that there is an odd-degree extension L of k such that ϕ and ψ are isometric over a quadratic extension of L. Then there exists a 1-Pfister α dividing both ϕ and ψ. In particular, ϕ and ψ are both hyperbolic over k or some separable quadratic extension of k.*

If $p = 1$, let $c = e_1(\phi \perp -\psi) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. We then have that ϕ and ψ are isometric if and only if $c \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ is trivial. Hence ϕ and ψ are isometric over $k(\sqrt{c})$ ($\mathrm{char}(k) \neq 2$) resp. $k(\wp^{-1}(c))$ ($\mathrm{char}(k) = 2$). But ϕ and ψ need not be divisible by the same 1-Pfister.

The statement of the lemma might also be expressed by saying that the two Pfister forms have a common right slot. In other words, there exist $c \in k$ and $a_1, \ldots, a_{p-1}, b_1, \ldots, b_{p-1} \in k^\times$ such that the following holds:

If $\mathrm{char}(k) \neq 2$, we have $\phi \cong \langle\langle a_1, \ldots, a_{p-1}, c\rangle\rangle$, $\psi \cong \langle\langle b_1, \ldots, b_{p-1}, c\rangle\rangle$, and the extension is $k(\sqrt{c})/k$.

If $\mathrm{char}(k) = 2$, we have $\phi \cong \langle\langle a_1, \ldots, a_{p-1}, c]]$, $\psi \cong \langle\langle b_1, \ldots, b_{p-1}, c]]$, and the extension is $k(\wp^{-1}(c))/k$.

Of course, the order of the slots only matters in characteristic 2.

Proof. Suppose first that ϕ is isotropic and hence hyperbolic, and let $b_1, \ldots, b_{p-1} \in k^\times$, $c \in k$ be such that $\psi \cong \langle\langle b_1, \ldots, b_{p-1}, c\rangle\rangle$ ($\mathrm{char}(k) \neq 2$) resp. $\psi \cong \langle\langle b_1, \ldots, b_{p-1}, c]]$ ($\mathrm{char}(k) = 2$). Since ϕ is hyperbolic, we have $\phi \cong \langle\langle -1, \ldots, -1, c\rangle\rangle$ resp. $\phi \cong \langle\langle -1, \ldots, -1, c]]$, and the result is obvious.

So let us assume that both ϕ and ψ are anisotropic. Set q to be the anisotropic part of $\phi \perp -\psi$. Let m be maximal such that ϕ and ψ contain a common m-Pfister as subform. By Lemma 1.3, we have

$$\dim q = 2^{p+1} - 2i_W(\phi \perp -\psi) = 2^{p+1} - 2^{m+1},$$

and it suffices to show that m is not zero; clearly we may assume that $m < p$, i.e., that $\dim q > 0$.

By Springer's Theorem for odd-degree extensions (see, e.g., [Sch, 2.5.3]), q is L-anisotropic. Now q is hyperbolic over a quadratic extension E of L. We consider three cases:

(i) $\mathrm{char}(k) = 2$ and $E = L(\sqrt{b})$ for some $b \in L^\times$ (E/L is inseparable);
(ii) $\mathrm{char}(k) \neq 2$ and $E = L(\sqrt{b})$ for some $b \in L^\times$;
(iii) $\mathrm{char}(k) = 2$ and $E = L(\wp^{-1}(b))$ for some $b \in L^\times$ (E/L is separable);

In cases (i) and (ii), there exists a nonsingular quadratic form η over L such that

$$q_L \cong \langle\langle b\rangle\rangle \otimes \eta$$

(see [Sch, 2.5.2] in case (i) and [Ah] in case (ii)). In case (iii), there exists a bilinear form η over L such that

$$q_L \cong \eta \otimes [1, b]$$

(see [Ba 74]).

In case (i), the nonsingularity of η implies that $\dim \eta \equiv 0 \bmod 2$, hence $\dim q \equiv 0 \bmod 4$ and thus $m \geq 1$.

Suppose we are in case (ii) or (iii), and put $\beta = \langle\langle b\rangle\rangle$ or $\beta = [1, b]$, respectively. Note that β is obviously anisotropic, and that $q_L \in I_q^2 L$ as ϕ and ψ are p-Pfisters with $p \geq 2$. If $\dim \eta$ is odd, then there exists $c \in L^\times$ such that

$$0 \equiv q_L \equiv c\beta \bmod I_q^2 L\ ,$$

but this contradicts the Arason-Pfister Hauptsatz which states that anisotropic forms in $I_q^n L$ are of dimension $\geq 2^n$. Hence, again $\dim \eta \equiv 0 \bmod 2$ and therefore $m \geq 1$. □

The following field-theoretic lemma is folklore but we include a proof for the reader's convenience.

Lemma 1.5. *Let K/k be a finite separable extension of degree $p^n m$ where p is a prime not dividing m. Then there exist finite separable extensions $E/L/k$ such that $K \subset E$, $[E : L] = p^n$, and p does not divide $[L : k]$.*

Proof. Let N/k be any finite Galois extension containing K, and let S' be any p-Sylow of $\mathrm{Gal}(N/K)$. Since $\mathrm{Gal}(N/K)$ is a subgroup of $\mathrm{Gal}(N/k)$, there is a p-Sylow S of $\mathrm{Gal}(N/k)$ containing S'. Let $L = N^S$ and $E = N^{S'}$. Clearly, E contains K, p does not divide $[L : k]$, and by comparing degrees we have $[E : L] = p^n$. □

Proposition 1.6. *Let ϕ and ψ be p-Pfister forms over a field k. Suppose that there exist finite extensions $K_1, \ldots, K_n$ of k such that ϕ and ψ are isometric over K_i for all i. Then there is a Galois extension K of k such that ϕ and ψ are K-isometric and $[K : k]$ divides*

$$g := \gcd\{4, [K_1 : k], [K_2 : k], \ldots, [K_n : k]\}.$$

If $p \geq 2$ and g is even, then K may be chosen to kill both ϕ and ψ. If $g = 1$, then there exists a Galois extension of degree dividing 2 that kills both ϕ and ψ.

Proof. If $\phi = \langle\langle \cdots, a\rangle\rangle$, $\psi = \langle\langle \cdots, b\rangle\rangle$ ($\mathrm{char}(k) \neq 2$) resp. $\phi = \langle\langle \cdots, a]]$, $\psi = \langle\langle \cdots, b]]$ ($\mathrm{char}(k) = 2$), then $k(\sqrt{a}, \sqrt{b})$ resp. $k(\wp^{-1}(a), \wp^{-1}(b))$ is a Galois extension of degree dividing 4 which obviously kills ϕ and ψ simultaneously.

So we may assume that the gcd g in the statement is 1 or 2. If $g = 1$, then $[K_i : k]$ is odd for some i, hence ϕ and ψ are k-isomorphic by Springer's theorem. In this case, we may choose $a = b$ above and thus get the quadratic Galois extension which kills both forms.

Finally, suppose that $g = 2$. In this case, we have $[K_i : k] \equiv 2 \bmod 4$ for some i which we fix.

Let us first consider the case $\mathrm{char}(k) \neq 2$. We then can write K_i as a separable extension K_i' followed by a purely inseparable one which necessarily is of odd degree. Then $[K_i' : k] \equiv 2 \bmod 4$, and using Springer's theorem, we may then assume that K_i is separable over k by replacing K_i by K_i' if necessary. By Lemma 1.5, there are separable extensions $E/L/k$ such that $K_i \subset E$, $[E : L] = 2$, and $[L : k]$ is odd. Since $K_i \subset E$, we have that ϕ and ψ are isometric over E, and we conclude by applying Lemma 1.4.

Now suppose $\mathrm{char}(k) = 2$. If K_i/k is separable, the same reasoning as above applies. If it is inseparable, then this is only possible for reasons of degree and characteristic if there is an intermediate field L such that L/k is of odd degree (and thus separable) and K_i/L is inseparable of degree 2. Again, Lemma 1.4 yields the desired conclusion. □

The above proposition allows an interpretation in terms of symbols in the group $H^p(k, \mathbb{Z}/2\mathbb{Z}(p-1))$ mentioned in the introduction. Recall that in characteristic $\neq 2$, $H^p(k, \mathbb{Z}/2\mathbb{Z}(p-1))$ is just the usual Galois cohomology group with mod 2 coefficients, $H^p(k, \mathbb{Z}/2\mathbb{Z})$, and that there is a well-defined injective map $P_p k \to H^p(k, \mathbb{Z}/2\mathbb{Z})$ sending $\langle\langle a_1, \cdots, a_p\rangle\rangle$ to the symbol $(a_1) \cup \cdots \cup (a_p)$. This map is well-defined, see Elman-Lam [EL, 3.2] or Arason [Ar, 1.6], and it is injective as a direct consequence of Voevodsky's proof of the Milnor conjecture.

As already mentioned, in characteristic 2, we have $H^1(k, \mathbb{Z}/2\mathbb{Z}(0)) = k/\wp(k)$. One can then show that $H^p(k, \mathbb{Z}/2\mathbb{Z}(p-1))$ is nothing else but the group h_p defined in [AB], namely the quotient of the group $(K_{p-1}k/2K_{p-1}k) \otimes H^1(k, \mathbb{Z}/2\mathbb{Z}(0))$ by the subgroup $\mathcal{R}_p$ generated by elements of type

$$\{a_1, \cdots, a_{p-1}\} \otimes (b) \in (K_{p-1}k/2K_{p-1}k) \otimes H^1(k, \mathbb{Z}/2\mathbb{Z}(0))$$

such that a_i is a norm of the extension $k(\wp^{-1}(b))/k$ for some i (or, equivalently, that a_i is represented by the quadratic form $[1, b]$). Again, we have a well-defined and injective map $P_pk \to h_p$ sending $\langle\langle a_1, \cdots, a_{p-1}, b]]$ to the "symbol" $\{a_1, \cdots, a_{p-1}\} \otimes (b) \bmod \mathcal{R}_p$, see [AB] (or [Ba 01] for a short exposition of these facts).

Corollary 1.7. *Let x and y be symbols in $H^p(k, \mathbb{Z}/2\mathbb{Z}(p-1))$. Suppose that there exist finite extensions $K_1, \ldots, K_n$ of k such that x and y agree over every K_i. Then there exists a Galois extension K of k such that x and y agree over K and $[K : k]$ divides*

$$g := \gcd\{4, [K_1 : k], [K_2 : k], \ldots, [K_n : k]\}.$$

If $p \geq 2$ and g is even, the extension K may be chosen to kill both x and y. If $g = 1$, then there exists a Galois extension of degree dividing 2 that kills both x and y.

2. Proof of Theorem 0.3: type G_2

Every group G of type G_2 and every G-torsor X may be (in a compatible manner) identified with an octonion k-algebra [KMRT, 33.24]. Such algebras are determined by their norm form, which is a 3-Pfister form [SV, §1.7].

Suppose that the G-torsor X is trivial (= has a closed point) over finite extensions $K_1, K_2, \ldots, K_n$ of k. Let ϕ_G and ϕ_X be 3-Pfister forms corresponding to G and X respectively. The quadratic forms are isomorphic over K_i for all i by hypothesis, and Prop. 1.6 gives a separable extension K of k such that ϕ_G and ϕ_X are K-isomorphic and $[K : k]$ divides $[K_i : k]$ for all i. That is, X has a K-point, and the answer to (0.2′)—equivalently, (0.2)—is "yes" for every group G of type G_2 and every G-torsor X.

3. Proof of Theorem 0.3: type F_4

An Albert k-algebra J has p-Pfister form invariants $f_p(J)$ for $p = 3, 5$. Every Albert algebra also has a "mod 3" invariant $g_3(J)$ living in the group $H^3(k, \mathbb{Z}/3\mathbb{Z}(2))$. (The correspondence between 3-Pfister forms and elements of $H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$ identifies the pair $(f_3(J), g_3(J))$ with an element of $H^3(k, \mathbb{Z}/6\mathbb{Z}(2))$. This element is the Rost invariant of J with respect to the split group of type F_4, up to sign.) The algebra J is reduced if and only if $g_3(J)$ is zero [Ro 91], [PR 96]. There is a unique Albert algebra — the *split* Albert algebra — with f_3 hyperbolic and g_3 equal to zero.

Let G be a reduced group of F_4, i.e., G is of the form $\mathrm{Aut}(J)$ for a reduced Albert k-algebra J. We are given a G-torsor X and finite extensions

$K_1, K_2, \ldots, K_n$ of k such that X is trivial over K_i for all i. The torsor X is the collection $\mathrm{Iso}(J', J)$ of isomorphisms $J' \xrightarrow{\sim} J$ for some Albert k-algebra J', and by hypothesis J' and J are K_i-isomorphic for all i. We will construct a separable extension K of k such that J' and J are K-isomorphic and the degree $[K : k]$ divides the gcd g of the degrees $[K_i : k]$. For $\ell = 2, 3$, we first construct an ℓ-primary separable extension E_ℓ of k which makes the mod ℓ invariants of J and J' equal.

$\underline{\ell = 2:}$ If g is odd, then $[K_i : k]$ is odd for some i. Hence f_3 and f_5 agree for J and J', and we take $E_2 = k$. If g is even, let E_2 be the Galois extension provided by Prop. 1.6 that kills $f_3(J)$ and $f_3(J')$. Since f_3 divides f_5, the extension E_2 kills $f_5(J)$ and $f_5(J')$ also.

$\underline{\ell = 3:}$ If $g_3(J')$ is zero, we may take $E_3 = k$. If g is not divisible by 3, then $[K_i : k]$ is not divisible by 3 for some i and $g_3(J')$ is zero. Otherwise, J' is a division algebra and so contains a separable cubic subfield, which we take to be E_3. Then $J' \otimes_k E_3$ has zero divisors because $E_3 \otimes_k E_3$ does, whence $g_3(J')$ is killed by E_3.

Set K to be the compositum of E_2 and E_3 in some separable closure of k. Since E_2 is Galois over k, the degree $[K : k]$ divides the product $[E_2 : k][E_3 : k]$, hence K has degree dividing g. By construction, K kills $g_3(J')$, so J' is reduced over K. Since $f_i(J)$ agrees with $f_i(J')$ over K for $i = 3, 5$, the algebras J and J' are isomorphic over K by [SV, 5.8.1], [Pe, 4.1].

4. The connecting homomorphism for E_6

The purpose of this section is to prove a technical result we will need to prove Theorem 0.3 for groups of type E_6.

We say that a group G *has trivial Tits algebras* if the k-algebra $\mathrm{End}_G(V)$ is a field for every irreducible representation V of G. From here until the end of the paper, H^1 denotes flat cohomology as in [DG] or [Wa]. For smooth groups—in particular, for all the groups we consider here except for central subgroups of semisimple groups in bad characteristic—it agrees with the Galois cohomology defined in [Se GC].

Lemma 4.1. *Let C be a central subgroup of an algebraic group G of type E_6. If G is isotropic and has trivial Tits algebras, then the natural map*

$$\partial\colon (G/C)(k) \to H^1(k, C)$$

is surjective.

This lemma is an E_6-analogue of a well-known fact about classical groups. Specifically, let G be the spin group of a nonsingular quadratic from q on a vector space V. Let C be the kernel of the "vector representation" $G \to GL(V)$; it is isomorphic to $\boldsymbol{\mu}_2$. The group $H^1(k, C)$ is identified with $k^\times/k^{\times 2}$ by the Kummer exact sequence, and the connecting homomorphism ∂ is the spinor norm. When q is isotropic, ∂ is surjective, as is well-known in

quadratic form theory [Ba 78, p. 78]. The proof of Lemma 4.1 below is easily adapted to give an alternative proof that the spinor norm is surjective.

The crux case of Lemma 4.1 is where G is of type ${}^1E_{6,2}^{28}$, precisely the case we need in the rest of the paper. When k has good characteristic (i.e., not 2 or 3), the result in the crux case is an easy consequence of the theory of Albert algebras. We give an algebraic-group-theoretic proof that is valid in all characteristics.

We will use without comment the fact that ∂ fits naturally into an exact sequence

$$(G/C)(k) \xrightarrow{\partial} H^1(k, C) \xrightarrow{\iota} H^1(k, G).$$

In particular, ∂ is surjective if and only if ι is the zero map.

Proof of Lemma 4.1. Clearly we may assume that C is not the trivial group, hence that G is not adjoint, i.e., G is simply connected. Since the center of G is a twisted form of $\boldsymbol{\mu}_3$, the subgroup C must be the entire center.

The case where G is split is standard. Indeed, G contains a maximal k-torus T that is k-split. The center C is contained in every maximal torus, hence the map ι above factors through $H^1(k, T)$. The group $H^1(k, T)$ is zero by Hilbert 90, since T is split.

Suppose now that G is of type ${}^1E_{6,2}^{28}$, i.e., has Tits index (0.4). Fix a maximal k-torus T in G containing a maximal k-split torus. Fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_6\}$ of G with respect to T, numbered as in [Bo]. Let Λ (resp. Λ_r) denote the weight (resp. root) lattice. Let ω_i be the fundamental weight corresponding to α_i. The absolute Galois group Γ of k acts naturally on Λ and the fixed sublattice Λ^Γ consists of the weights ω such that $(\alpha_i, \omega) = 0$ for all $i \neq 1, 6$ by [BT, p. 108, Cor. 6.9]. In particular, the weight ω_1 lies in Λ^Γ.

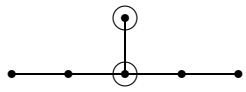
The weight (resp. root) lattice is the group of cocharacters $\mathbb{G}_m \to T/C$ (resp. $\mathbb{G}_m \to T$), and the weight ω_1 corresponds to a k-defined cocharacter $\overline{\ell}$. Consulting the tables in the back of [Bo], we find that ω_1 is not in the root lattice, but $3\omega_1$ is. Let $\ell\colon \mathbb{G}_m \to T$ be the k-cocharacter corresponding to $3\omega_1$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_3 & \longrightarrow & \mathbb{G}_m & \xrightarrow{3} & \mathbb{G}_m & \longrightarrow & 1 \\
 & & \downarrow & & \ell\downarrow & & \downarrow\overline{\ell} & & \\
1 & \longrightarrow & C & \longrightarrow & T & \longrightarrow & T/C & \longrightarrow & 1
\end{array}$$

Here, the arrow $\boldsymbol{\mu}_3 \to C$ arises because the composition $\boldsymbol{\mu}_3 \to \mathbb{G}_m \to T/C$ is the zero map. Since ω_1 is not in the root lattice, ℓ is an injection, ergo the map $\boldsymbol{\mu}_3 \to C$ is an injection. That is, the center C of G is contained in a rank 1 k-split torus, namely the image of ℓ. The map ι is zero by the same argument as in the case where G is split. (This paragraph and the previous one are an adaptation of the arguments behind the applications of

the Gille-Merkurjev Norm Principle in the last section of [M 96]. The map $\boldsymbol{\mu}_3 \to C$ constructed here is $h(\overline{\ell})$ in the notation of [M 96, 1.3].)

If G is of type ${}^1E_6$, the remaining possibility is that it has Tits index



We claim that this is impossible. Indeed, the semisimple anisotropic kernel G_{an} of G is of type ${}^1A_2 \times {}^1A_2$. Since the Tits algebras of G are trivial, so are the Tits algebras of G_{an} by [Ti 71, 5.5.5], hence G_{an} is isotropic. This is a contradiction.

Now suppose that G is of type ${}^2E_6$ and let K be the separable quadratic extension of k over which G is of type ${}^1E_6$. The hypothesis on the Tits algebras ensures that G is split or of type ${}^1E_{6,2}^{28}$ over K, hence that C is contained in a K-defined rank 1 split torus S in G. Not only is the map

$$\partial_K \colon (G/C)(K) \to H^1(K, C)$$

surjective, it is even surjective when one restricts to the rational subgroup S/C of G/C. The Gille-Merkurjev Norm Principle from [Gi], [M 96] shows that the image of ∂ contains the corestriction $\mathrm{cor}_{K/k} H^1(K, C)$. Since K/k is quadratic and C is 3-torsion, a restriction-corestriction argument shows that the corestriction $H^1(K, C) \to H^1(k, C)$ is surjective. □

We remark that the Gille-Merkurjev Norm Principle is proved for Galois cohomology and here we are using flat cohomology, and the two cohomology theories may give different values for $H^1(k, C)$ in characteristic 3. However, the proof of the norm principle in [M 96] goes through with no changes since flat and Galois cohomology agree for reductive groups. The only point requiring checking is Merkurjev's Lemma 3.11, which is easy to translate.

Remark 4.2. The same sort of proof gives an E_7 version of Lemma 4.1: Let C be a central subgroup of an algebraic group G of type E_7. *If G is isotropic, has trivial Tits algebras, and is not of type $E_{7,1}^{66}$, then the natural map $(G/C)(k) \to H^1(k, C)$ is surjective.*

5. Proof of Theorem 0.3: type E_6

In this entire section, we assume that G is a simply connected group of type E_6 that is split or of type ${}^1E_{6,2}^{28}$. We write Z for the center of G; it is isomorphic to the group scheme $\boldsymbol{\mu}_3$ of cube roots of unity.

Lemma 5.1. *The group G contains a subgroup H that is reduced of type F_4, and the natural map*

$$H^1(k, H) \times H^1(k, Z) \to H^1(k, G)$$

is surjective.

Proof. Write G_{sp} for the split simply connected group of type E_6. The split group H_{sp} of type F_4 is contained in G_{sp}, and the induced map

$$H^1(k, H_{\mathrm{sp}}) \times H^1(k, Z) \to H^1(k, G_{\mathrm{sp}}) \tag{5.2}$$

is surjective. This has been shown in [Ga, 3.4] under the assumption that k has characteristic different from 2, 3, but this restriction is unnecessary by the following reasoning. In all characteristics G_{sp} is the group of isometries of a cubic form—a norm for an Albert algebra J—and the elements of norm 1 in J form an open orbit in $\mathbb{P}(J)$ [As, 3.16(3)]. The stabilizer of the identity element in J is of type F_4 in all characteristics by [Sp, 4.6], hence the stabilizer of the identity element in $\mathbb{P}(J)$ is $F_4 \times \boldsymbol{\mu}_3$. The conclusion now follows by [Ga, Lemma 3.1], where the last paragraph of the proof is replaced with an appeal to [DG, p. 373, Prop. III.4.4.6b], a flat cohomology analogue of [Se GC, I.5.4, Prop. 37]. This completes the proof when G is split.

Now suppose that G is of type ${}^1E_{6,2}^{28}$. The semisimple anisotropic kernel G_{an} is simply connected of type D_4. It has trivial Tits algebras [Ti 71, 5.5.5], so is obtained by twisting the split simply connected group Spin_8 of type D_4 by a 1-cocycle α with values in Spin_8. The group Spin_8 is even a subgroup of H_{sp} in G_{sp}. Twisting the split group G_{sp} by α produces a group of type ${}^1E_6$ with the same semisimple anisotropic kernel as G, hence the group is isomorphic to G by Tits' Witt-type theorem. Twisting H_{sp} by α, we find a group H of type F_4 that is reduced. Twisting everything in (5.2) by α gives the lemma. □

Let r_G denote the Rost invariant of G as defined in [M 03]. The composition

$$H^1(k, H) \longrightarrow H^1(k, G) \xrightarrow{r_G} H^3(k, \mathbb{Q}/\mathbb{Z}(2)), \tag{5.3}$$

is equal to an integer multiple nr_H of the Rost invariant $r_H \colon H^1(k, H) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$. Over a separable closure of k, the subgroup H from Lemma 5.1 is similar to the standard inclusion of the split F_4 into the split E_6, and the integer multiple for this inclusion is 1 by [Ga, 2.4]. Since this multiplying factor does not change under scalar extensions [M 03, Prop. 7.9(4)], the composition (5.3) is equal to r_H.

By Lemma 4.1, the image of the map $H^1(k, Z) \to H^1(k, G)$ is zero. Since the Rost invariant is compatible with twisting, for $(\alpha, \lambda) \in H^1(k, H) \times H^1(k, Z)$ we have:

$$r_G(\alpha, \lambda) = r_H(\alpha). \tag{5.4}$$

Proposition 5.5. *The Rost invariant $H^1(k, G) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ has kernel zero.*

Proof. Suppose β is in the kernel. It is the image of some pair (α, λ) in $H^1(k, H) \times H^1(k, Z)$ such that $r_H(\alpha)$ is zero by 5.1 and (5.4).

The group H is the automorphism group of a reduced Albert algebra J. Let J' be the Albert algebra corresponding to α. Since $r_H(\alpha)$ is zero, the

Rost invariants of J and J' relative to the split group of type F_4 agree, i.e., $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$. In particular, J' is also reduced. An explicit formula for the cubic norm form on a reduced Albert algebra is well-known, and since the 3-Pfister forms $f_3(J)$ and $f_3(J')$ are the same, it is easy to cook up an isometry between the norms on J and J'. Since G may be viewed as the group of isometries of the cubic norm on J, the image of α in $H^1(k, G)$ is zero by descent.

The group $H^1(k, Z)$ acts on $H^1(k, G)$, and β is $\lambda \cdot (\operatorname{im} \alpha)$. Hence β is in the image of the map $H^1(k, Z) \to H^1(k, G)$. But this map has image zero by Lemma 4.1, hence β is zero. □

Now we may prove Theorem 0.3 for our G. By hypothesis, we are given a G-torsor X and finite extensions $K_1, K_2, \ldots, K_n$ of k such that X is trivial over K_i for all i. Fix a pair (α, λ) in $H^1(k, H) \times H^1(k, Z)$ mapping to X. Then $r_H(\alpha)$ equals $r_G(X)$, and it is killed by K_i for all i. Let K be the extension constructed from H, α, and the K_i as in §3. The extension K kills $r_G(X)$, hence trivializes X by the proposition. By construction, K is separable over k of degree dividing $[K_i : k]$ for all i. This completes the proof of the theorem.

6. Final remarks

In the special case where $d = 1$ and X is a G-torsor, Totaro's question overlaps an earlier question by Serre [Se 95, p. 233]. But even this simpler case is wide open for general groups of type F_4. *For k of characteristic $\neq 2, 3$, the following are equivalent:*

(6.1) *For every k-group G of type F_4, every G-torsor X, and $d = 1$, the answer to* (0.2) *is "yes".*

(6.2) *Albert k-algebras are classified by their invariants f_3, f_5, and g_3.*

Indeed, the equivalence of (6.1) and (6.2) is an easy consequence of a theorem of Rost from [Ro 02], which says: If J, J' are Albert k-algebras such that f_3, f_5, and g_3 agree for J and J', then there exist extensions K_1, K_2 such that J, J' are isomorphic over K_1 and K_2 and such that the degrees $[K_1 : k]$ and $[K_2 : k]$ are coprime. But proving or disproving (6.2) is Question #1 on Petersson-Racine's list of open problems in [PR 94]; it is viewed by many as the principal outstanding problem concerning Albert algebras.

Suppose for the moment that (6.1) is true. To extend (6.1) to the case of general d, one would want an analogue of Corollary 1.7 for the prime 3.

If the answer to Totaro's question is yes for G simply connected of type ${}^1E_6$ with trivial Tits algebras, then one can deduce (in a manner very similar to the above) that two Albert algebras are isotopic if and only if they have the same f_3 and g_3 invariants. Whether or not that is true is Question #4 in Petersson-Racine's list.

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