A BOUND FOR CANONICAL DIMENSION OF THE SPINOR GROUP

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Abstract. Using the theory of non-negative intersections, duality of the Schubert varieties, and Pieri-type formula for a maximal orthogonal grassmannian, we get an upper bound for the canonical dimension $\operatorname{cd}(\operatorname{Spin}_n)$ of the spinor group Spin_n. A lower bound is given by the canonical 2-dimension $\operatorname{cd}_2(\operatorname{Spin}_n)$, computed in [8]. If n or $n+1$ is a power of 2, no space is left between these two bounds; therefore the precise value of $\operatorname{cd}(\operatorname{Spin}_n)$ is obtained for such n.

In the appendix, we also produce an upper bound for canonical dimension of the semi-spinor group (giving the precise value of the canonical dimension in the case when the rank of the group is a power of 2), compute canonical dimension of the projective orthogonal group, and show that the spinor group represents the unique difficulty when trying to compute the canonical dimension of an arbitrary simple split group, possessing a unique torsion prime.

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1. Introduction

Let F be an arbitrary field (of an arbitrary characteristic). Let X be a smooth algebraic variety over F. A field $L \supset F$ is called a *splitting field* (of X), if $X(L) \neq \emptyset$. A splitting field L is called *generic*, if for any splitting field L' there exists a place $L \to L'$. The *canonical dimension* $\operatorname{cd}(X)$ is defined as the minimum of transcendence degrees of generic splitting fields of X (cf. [8, §2]).

Let G be an algebraic group over F. The *canonical dimension* $\operatorname{cd}(G)$ of G, as introduced in [1], is the maximum of canonical dimensions of G-torsors, defined over field extensions of F (of course, it is not the same as the canonical dimension of the underlying variety of G, which is not an interesting invariant because is always 0). For the spinor group, it is explained in *loc.cit.* that $\operatorname{cd}(\mathrm{Spin}_{2n+1}) = \operatorname{cd}(\mathrm{Spin}_{2n+2})$ (where n is an integer ≥ 1), so that we will discuss only $\operatorname{cd}(\mathrm{Spin}_{2n+1})$ here.

Although the canonical dimension of, say, a smooth projective variety X can be expressed in terms of algebraic cycles on X (see [8, cor. 4.7]), there are no general recipes for computing $\operatorname{cd}(X)$ or $\operatorname{cd}(G)$. A better situation occurs with the *canonical p-dimension* cd_p, a p-relative version of cd, where p is a prime, defined in [8, §3]: a recipe for computing $\operatorname{cd}_p(G)$ of an arbitrary split simple G is obtained in *loc.cit.* In particular, one has

$$\operatorname{cd}_2(\mathrm{Spin}_{2n+1}) = n(n+1)/2 - 2^r + 1 ,$$

where r is the smallest integer such that $2^r > n$ (while $\operatorname{cd}_p(\mathrm{Spin}_{2n+1}) = 0$ for any odd prime p). Since $\operatorname{cd}(G) \geq \operatorname{cd}_p(G)$ for any G and p, we have a lower bound for the canonical dimension of the spinor group, given by its canonical 2-dimension:

$$\operatorname{cd}(\mathrm{Spin}_{2n+1}) \geq \frac{n(n+1)}{2} - 2^r + 1 .$$

In this note we establish the following upper bound for $\operatorname{cd}(\mathrm{Spin}_{2n+1})$:

Theorem 1.1. *For any $n \geq 1$, one has* $\operatorname{cd}(\mathrm{Spin}_{2n+1}) \leq n(n-1)/2$.

The proof is given in section 5. It makes use of the theory of non-negative intersections, of duality between Schubert varieties, and of the Pieri formula for the split maximal orthogonal grassmannian.

Note that the lower bound for $\operatorname{cd}(\mathrm{Spin}_{2n+1})$ coincide with the upper one if (and only if) $n+1$ is a power of 2. Therefore, for such n, we get the precise value:

Corollary 1.2. *If $n+1$ is a power of 2, then* $\operatorname{cd}(\mathrm{Spin}_{2n+1}) = n(n-1)/2$. □

Remark 1.3. For n up to 4, it is easy to see that $\operatorname{cd}(\mathrm{Spin}_{2n+1}) = \operatorname{cd}_2(\mathrm{Spin}_{2n+1})$ (see [1]), but for every $n \geq 5$ the bound of Theorem 1.1 is new.

In the last section we explain why the question about canonical dimension of the spinor group is of particular importance, produce an upper bound for canonical dimension of the semi-spinor group (see Theorem 6.2 and Corollary 6.3), and compute canonical dimension of the projective orthogonal group (see Theorem 6.1).

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2. Non-negativity

By *scheme* we mean a separated scheme of finite type over a field. *Variety* is an integral scheme.

Let X be a scheme. Following [4, ch. 12], an algebraic cycle $n_1Y_1+\ldots n_rY_r$ on X (where r is ≥ 0, Y_i are closed subvarieties of X, and n_i are integers) is called *non-negative*, if the coefficients $n_1,\ldots,n_r$ are non-negative. An element α of the integral Chow group $\mathrm{CH}(X)$ is called *non-negative*, if it can be represented by a non-negative cycle.

I thank I. Panin for pointing me out the following fact:

Lemma 2.1. *Let L be a line vector bundle over a smooth variety X. The first Chern class $c_1(L)\in \mathrm{CH}^1(X)$ is non-negative if and only if L has a non-zero global section.*

Proof. Fixing an imbedding of L into the constant $\mathcal{O}_X$-module $F(X)^*$, we get a 1-codimensional cycle C on X, representing $c_1(L)$, such that a function $f\in F(X)^*$ is a global section of L if and only if the cycle $\mathrm{div}(f)+C$ is non-negative, [5]. Since C is rationally equivalent to a non-negative cycle if and only if the cycle $\mathrm{div}(f)+C$ is non-negative for some $f\in F(X)^*$, we are done. □

Corollary 2.2. *Let X be a smooth absolutely irreducible variety over a field F, α an element of $\mathrm{CH}^1(X)$, E/F a field extension. If $\alpha_E\in \mathrm{CH}(X_E)$ is non-negative, then α itself is non-negative.*

Proof. Let L be a line vector bundle over X such that $c_1(X)=\alpha$. Assume that α_E is non-negative. Then the E-vector space $\Gamma(X_E,L_E)$ of global sections of L_E is non-zero by Lemma 2.1. Since $\Gamma(X_E,L_E)=\Gamma(X,L)\otimes_F E$, it follows that $\Gamma(X,L)\neq 0$; therefore, once again by Lemma 2.1, α is non-negative. □

We are going to use the following

Theorem 2.3 ([4, §12.2])**.** *Let X be a smooth variety such that its tangent bundle is generated by the global sections. Then the product of non-negative elements in $\mathrm{CH}(X)$ is non-negative. Moreover, if $\alpha\in\mathrm{CH}(X)$ is represented by a non-negative cycle with support $A\subset X$, while $\beta\in\mathrm{CH}(X)$ is represented by a non-negative cycle with support $B\subset X$, then the product $\alpha\beta\in\mathrm{CH}(X)$ can be represented by a non-negative cycle with support on the intersection $A\cap B$.*

Remark 2.4. If X is a projective homogeneous variety under an action of an algebraic group, then the tangent bundle of X is generated by the global sections. Indeed, there exists a field extension E/F such that the variety X_E is isomorphic to the quotient G/P of a semisimple algebraic group G over E modulo a parabolic subgroup $P\subset G$. Therefore the tangent bundle of the variety X_E is generated by the global sections. Since the property of being generated by global sections is not changed under extension of the base field, the tangent bundle of the variety X is also generated by the global sections.

3. Dual Schubert varieties

Let G be a split semisimple algebraic group, $T\subset B\subset G$ a maximal split torus and a Borel subgroup of G. Let W be the Weyl group of G, and let $S\subset W$ be the set of reflections with respect to the simple roots. We fix a subset $\theta \subset S$, take the subgroup W_θ, generated by θ, and consider the parabolic subgroup $P = P_\theta = BW_\theta B \subset G$.

Using the length function $l\colon W \to \mathbb{Z}_{\geq 0}$, induced by the set S of generators of the group W, we take in each coset of W/W_θ the unique smallest length element and write $W^\theta \subset W$ for the set of representatives thus obtained.

The variety $X = G/P$ is cellular, the cells BwP/P are indexed by $w \in W^\theta$. We write X_w for the closure in G/P of the corresponding cell. The varieties X_w are called (generalized) Schubert varieties; their classes $[X_w] \in \mathrm{CH}(X)$, called (generalized) Schubert classes, form a basis of the group $\mathrm{CH}(X)$. Moreover, $\dim X_w = l(w)$.

Sometimes the upper indexation of the Schubert varieties, which respects their codimension, is more convenient. To define it, let us take the (unique) largest length element $w_0 \in W$ and set $X^w = X_{w'}$, where w' is the smallest length element in the coset $w_0 w W_\theta$ (so that $w' = w_0 w''$, where w'' is the *largest* length element of the coset wW_θ). Now we have $\operatorname{codim} X^w = \dim X_w = l(w)$.

Proposition 3.1 ([10, prop. 1.4])**.** *Let* $\deg\colon \mathrm{CH}(X) \to \mathbb{Z}$ *be the degree homomorphism. Then for any* $w, w' \in W^\theta$ *one has:*

$$\deg([X^w]\cdot[X_{w'}]) = \begin{cases} 1, & \text{if } w' = w;\\ 0, & \text{otherwise.}\end{cases}$$

Because of this property, we refer to the varieties X^w and X_w (as well as to their classes) as *mutually dual*.

Let us now specify the situation: take as G the special orthogonal group $\mathrm{SO}(\phi)$ of a split quadratic form $\phi\colon V \to F$, where V is a $(2n+1)$-dimensional vector space over a field F. Saying *split*, we mean existence in V of an n-dimensional totally isotropic subspace. Let us choose a complete flag

$$\mathcal{F} = (0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n)\ , \quad \dim \mathcal{F}_i = i$$

of totally isotropic subspaces of (V, ϕ) and a subspace D_i in each $\mathcal{F}_i$ such that $\mathcal{F}_i = \mathcal{F}_{i-1} \oplus D_i$. Then we take as $B \subset G$ the stabilizer of $\mathcal{F}$, as P the stabilizer of $\mathcal{F}_n$, and as T the stabilizer of all D_i. The variety $X = G/P$ is therefore the maximal orthogonal grassmannian of ϕ.

Now the Schubert varieties on X are indexed by the strictly decreasing sequences $i_1 > i_2 > \cdots > i_s$ of positive integers, satisfying $n \geq i_1$. The $(i_1 \dots i_s)$-th Schubert variety $X^{i_1\dots i_s}$ is the closed subvariety of the subspaces $W \in X$ such that $\dim(W \cap \mathcal{F}_{n+1-i_t}) \geq t$ for $t = 1, 2, \dots, s$; the variety $X^{i_1\dots i_s}$ has codimension $i_1 + \cdots + i_s$, and we write $e_{i_1\dots i_s}$ for its class in $\mathrm{CH}(X)$. The Schubert classes e_i $(i = 1, 2, \dots, n)$ are called *special*.

As a specific case of Proposition 3.1 we get

Corollary 3.2. *For any* $e_{i_1\dots i_s}$ *there exists* $e_{i'_1\dots i'_{s'}}$ *such that for any* $e_{i''_1\dots i''_{s''}}$

$$\deg(e_{i''_1\dots i''_{s''}} \cdot e_{i'_1\dots i'_{s'}}) = \begin{cases} 1, & \text{if } e_{i''_1\dots i''_{s''}} = e_{i_1\dots i_s};\\ 0, & \text{otherwise.}\end{cases}$$

□

4. Pieri formula

The classical Pieri formula expresses the product of a special Schubert class by an arbitrary Schubert class as a linear combination of Schubert classes in the case of a usual grassmannian.

We are going to use an analogues formula for a split maximal orthogonal grassmanian:

Theorem 4.1 ([6], see also [13]). *Let n be a positive integer, X the maximal orthogonal grassmannian of a split $(2n+1)$-dimensional quadratic form. The following multiplication formula holds for the Schubert classes in $\mathrm{CH}(X)$: for any strictly decreasing sequence of positive integers $x=(x_1,\dots,x_k)$, satisfying $x_1\le n$, and any positive integer $p\le n$, one has*

$$e_p\cdot e_x=\sum_y 2^{m_y^{x,p}}e_y\ ,$$

where the sum runs over all strictly decreasing sequences of integers $y=(y_1,\dots,y_{k+1})$, satisfying

$$n\ge y_1\ge x_1\ge y_2\ge\dots\ge y_k\ge x_k\ge y_{k+1}\ge 0$$

and $y_1+\dots+y_{k+1}=p+x_1+\dots+x_k$ (in the case of $y_{k+1}=0$, we define e_y as $e_{y_1\dots y_k}$). The exponent $m_y^{x,p}$ of the coefficient of e_y is determined as follows:

$$m_y^{x,p}=\begin{cases}\text{the quantity of } i\in[1,\ k] \text{ such that } y_i>x_i>y_{i+1}, & \text{if } y_{k+1}\ne 0;\\ \text{the above quantity minus } 1, & \text{if } y_{k+1}=0.\end{cases}$$

Remark 4.2. Theorem 4.1 is proved in [6] under the assumption that the base field is algebraically closed; it is proved in [13] under the assumption that the base field is $\mathbb{C}$. However, as shown in [3], the multiplication table for the Schubert classes in $\mathrm{CH}(G/B)$, where G is a split semisimple algebraic group and B is its Borel subgroup, depends only on the type of G and does not depend on the base field. Now if $P\subset G$ is a parabolic subgroup, containing B, the pull-back with respect to the projection $G/B\to G/P$ is an injective ring homomorphism, mapping each Schubert class $[X^w]\in\mathrm{CH}(G/P)$ (the notation is introduced in §3) to the "same" Schubert class $[X^w]\in\mathrm{CH}(G/B)$ (see [10, lemma 1.2(b)] and [3, §3.3]).[1] Therefore the multiplication table for the Schubert classes in $\mathrm{CH}(G/P)$ depends only on the type of the pair (G,P) and does not depend on the base field either.

Corollary 4.3. *Under condition of Theorem 4.1, one has $e_1^n=e_n+\dots$, where dots stand for a linear combination of Schubert classes different from e_n.*

Proof. Let x and y be two strictly decreasing sequences of positive integers $\le n$. Let us say that y is a *deformation* of x (and write $x\rightsquigarrow y$), if e_y appears in the formula for $e_1\cdot e_x$, given by Theorem 4.1, in which case we refer to the number $m_y^{x,1}$ as the *exponent* of the deformation.

There is a unique chain of deformations, transforming (1) to (n), namely, the chain $(1)\rightsquigarrow(2)\rightsquigarrow\dots\rightsquigarrow(n)$. Since the exponent of each deformation in the chain is 0, the statement follows. ☐

[1] and, as follows from [10, lemma 1.2(c)], under the push-forward homomorphism, $[X_w]$ is mapped to $[X_w]$ for $w\in W^\theta$, and $[X_w]$ is mapped to 0 for $w\notin W^\theta$.

Remark 4.4. Unfortunately none of the Schubert classes appears with coefficient 1 in the decomposition of e^{n+1}.

5. Proof of Theorem 1.1

Let V be a $(2n+1)$-dimensional vector space over F equipped with a non-degenerate quadratic form $\phi : V \to F$. Let X be the orthogonal grassmannian of n-dimensional (maximal) totally isotropic subspaces in V.

By [8, cor. 4.7], the canonical dimension $\operatorname{cd}(X)$ is the minimum of $\dim Y$, where Y runs over all closed subvarieties of X such that $Y(F(X)) \neq \emptyset$. As explained in [1], $\operatorname{cd}(\mathrm{Spin}_{2n+1})$ is the maximum of $\operatorname{cd}(X)$, taken over all ϕ such that the even Clifford algebra $C_0(\phi)$ is split. Therefore, in order to prove Theorem 1.1, it suffices to prove the following

Proposition 5.1. *Let X be the maximal orthogonal grassmannian of a $(2n+1)$-dimensional quadratic form ϕ such that $[C_0(\phi)] = 0$. Then there exists a closed subvariety $Y \subset X$ of dimension $n(n-1)/2$ such that $Y(F(X)) \neq \emptyset$.*

Proof. Since over the function field $F(X)$ the quadratic form ϕ is split, we may speak about Schubert classes in $\mathrm{CH}(\bar{X})$, where $\bar{X}$ stands for $X_{F(X)}$. We write $\bar{\mathrm{CH}}(X)$ for the image of the restriction homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(\bar{X})$.

Let us take the Schubert class $e_1 \in \mathrm{CH}^1(\bar{X})$. By our assumption on $C_0(\phi)$, we have $\bar{\mathrm{CH}}^1(X) = \mathrm{CH}^1(\bar{X})$ (see, e.g., [8, §8.2] or [14, proof of lemma 3.1]; cf. proof of Theorem 6.2); therefore $e_1 \in \bar{\mathrm{CH}}^1(X)$. Moreover, by Corollary 2.2, e_1 is non-negative. It follows by Theorem 2.3 with Remark 2.4 that the n-th power of e_1 is also non-negative, so that we can write

$$e_1^n = n_1[Y_1] + \cdots + n_r[Y_r]$$

with some non-negative integers n_i and some closed subvarieties $Y_i \subset X$. Note that $\dim Y_i = \dim X - n = n(n-1)/2$ for all i.

By Corollary 4.3, we have $e_1^n = e_n + \cdots \in \mathrm{CH}(\bar{X})$, where dots stand for a linear combination of Schubert classes different from e_n. Using Corollary 3.2, we find the Schubert class $e \in \mathrm{CH}(\bar{X})$ dual to e_n, that is, such that $\deg(e_n e) = 1$ while $\deg(e'e) = 0$ for any Schubert class e' different from e_n. For this e we have $\deg(e_1^n e) = 1$. Since the product $e_1^n e$ is non-negative (by Theorem 2.3 with Remark 2.4), it follows that $\deg([\bar{Y}_i] \cdot e) = 1$ (and $n_i = 1$) for some $i \in [1,\ r]$, where $\bar{Y}_i = (Y_i)_{F(X)}$

Let Z be the Schubert variety, representing e. Since the product $[\bar{Y}_i] \cdot [Z]$ is a 0-cycle class of degree 1 and can be represented by a non-negative cycle on the intersection $\bar{Y}_i \cap Z$ (see Theorem 2.3 with Remark 2.4), the scheme $\bar{Y}_i$ has a rational point, that is, $Y_i(F(X)) \neq \emptyset$. □

6. Appendix: Split simple algebraic groups of arbitrary type

In this section we show that the spinor group represents the only difficulty when trying to answer the following question: let G be a split simple algebraic group, having a unique torsion prime p; is it true that $\operatorname{cd}(G) = \operatorname{cd}_p(G)$?

To explain the assumption about the uniqueness of torsion prime, made in this question, let us notice that it is not clear what to expect from $\operatorname{cd}(G)$ in the case when G has more than one torsion prime (recall that $\mathrm{cd}_p(G) > 0$ if and only if p is a torsion prime of G, [8, rem. 6.10]).

Requiring the uniqueness of torsion prime p, we are in the situation where $\mathrm{cd}_p(G)$ is known (see below), so that one may expect to be able to get some interesting information also on $\mathrm{cd}(G)$.

Let us consider all split simple algebraic groups, having a unique torsion prime, type by type. All cited results on canonical dimension are from [1] and on canonical p-dimension from [8]. In each subsection we assume without repeating it, that G is a split simple group of the type under consideration.

6.1. $\mathbf{A}_n$, $n \geq 1$. We have $G \simeq \mathrm{SL}_{n+1}/\mu_l$, where l is a positive integer, dividing $n+1$. The group G has a unique torsion prime p if and only if l is a positive power of p. In this case it is known that $\mathrm{cd}(G) = \mathrm{cd}_p(G) = p^k - 1$, where k is the largest integer such that p^k divides $n+1$.

6.2. $\mathbf{C}_n$, $n \geq 2$. If G is simply connected, then $\mathrm{cd}(G) = 0$.

If G is not simply connected, then 2 is its unique torsion prime and $\mathrm{cd}(G) = \mathrm{cd}_2(G) = 2^{k+1} - 1$, where k is the largest integer such that 2^k divides n.

6.3. $\mathbf{B}_n$, $n \geq 3$. The prime 2 is the unique torsion prime here.

If G is simply connected, then $G \simeq \mathrm{Spin}_{2n+1}$.

If G is not simply connected, then $G \simeq \mathrm{SO}_{2n+1}$. In this case it is known that $\mathrm{cd}(G) = \mathrm{cd}_2(G) = n(n+1)/2$ (originally proved in [7]).

6.4. $\mathbf{D}_n$, $n \geq 4$. The prime 2 is the unique torsion prime here.

If G is simply connected, then $G \simeq \mathrm{Spin}_{2n}$.

If G is not simply connected, then either $G \simeq \mathrm{SO}_{2n}$ or $G \simeq \mathrm{PGO}^+_{2n}$. If n is odd, there are no other possibilities. If n is even, then one more possibility is added: $G \simeq \mathrm{Spin}^\sim_{2n}$ (the semi-spinor group).

For $G = \mathrm{SO}_{2n}$, we know that $\mathrm{cd}(G) = \mathrm{cd}_2(G) = n(n-1)/2$.

In the case when $G = \mathrm{PGO}^+_{2n}$, we know that $\mathrm{cd}_2(G) = n(n-1)/2 + 2^k - 1$, where k is the largest integer such that 2^k divides n. We compute $\mathrm{cd}(G)$ now:

Theorem 6.1. *For $G = \mathrm{PGO}^+_{2n}$, one has $\mathrm{cd}(G) = \mathrm{cd}_2(G)$.*

Proof. We only need to show that $\mathrm{cd}(G) \leq \mathrm{cd}_2(G)$.

The set of isomorphism classes of G-torsors is mapped surjectively and with trivial kernel onto the set of isomorphism classes of central simple algebras of degree $2n$ with an orthogonal pair of trivial discriminant, [9, §29F].

Let A be such an algebra. The scheme of rank n isotropic ideals in A has two (possibly non-isomorphic) components; let X be a component of this scheme. Note that $\dim X = n(n-1)/2$. Furthermore, let Y be the Severi-Brauer variety of A. The algebra with orthogonal pair A is split (i.e., its class is the distinguished point of the set of isomorphism classes) if and only if the variety $Y \times X$ has a rational point. Therefore, to get the required bound for $\mathrm{cd}(G)$, it suffices to get the same bound for $\mathrm{cd}(Y \times X)$ (see [8, rem. 3.7]).

The index of the algebra $A_{F(X)}$ divides n and the index of A itself is a power of 2. Therefore, $\operatorname{ind} A_{F(X)}$ divides 2^k, and we may write A as the tensor product of a degree 2^k central simple algebra D and a matrix algebra M. Let Z^o be the Severi-Brauer variety of D; the Severi-Brauer variety of M is a projective space $\mathbb{P}$. Tensor product of ideals gives rise to a closed imbedding $Z^o \times \mathbb{P} \hookrightarrow Y_{F(X)}$ (which becomes a Segre imbedding over $F(Y \times X)$). Choosing a closed rational point on $\mathbb{P}$, we identify Z^o with a closed subvariety of $Y_{F(X)}$. Let Z be the closure of Z^o in $Y \times X$.

The constructed closed subvariety $Z \subset Y \times X$ has the dimension

$$\dim X + \dim Z^o = \frac{n(n-1)}{2} + 2^k - 1\ ,$$

which coincides with the bound we want for $\mathrm{cd}(Y \times X)$. To show that this number is really a bound for $\mathrm{cd}(Y \times X)$, it suffices to show that $Z(F(Y \times X)) \neq \emptyset$, [8, cor. 4.7].

Let us write $\bar{Z}$, $\bar{X}$, and $\bar{Y}$ for Z, X, and Y over the field $F(Y \times X)$. The variety $\bar{Y}$ is isomorphic to a projective space; let $H \subset \bar{Y}$ be a hyperplane. Let $x \in \bar{X}$ be a rational point. The degree of the product $[\bar{Z}] \cdot ([H]^{2^k-1} \times [x])$ is 1. Moreover, this product can be represented by a non-negative cycle with support on $\bar{Z}$ (Theorem 2.3 with Remark 2.4). Therefore, the scheme $\bar{Z}$ has indeed a rational point. □

For the semi-spinor group $G = \mathrm{Spin}^{\sim}_{2n}$, we know that $\mathrm{cd}_2(G) = n(n-1)/2 + 2^k - 2^r$, where k is the same as above, while r is the smallest integer such that $2^r \geq n$. We show now, how the bound for $\mathrm{cd}(\mathrm{Spin}_{2n}) = \mathrm{cd}(\mathrm{Spin}_{2n-1})$, established in this paper (Theorem 1.1), produces a bound for $\mathrm{cd}(\mathrm{Spin}^{\sim}_{2n})$.

Theorem 6.2. *For k being the largest integer such that 2^k divides n, one has*

$$\mathrm{cd}(\mathrm{Spin}^{\sim}_{2n}) \leq \frac{(n-1)(n-2)}{2} + 2^k - 1\ .$$

Proof. The set of isomorphism classes of G-torsors for $G = \mathrm{Spin}^{\sim}_{2n+2}$ is mapped surjectively and with trivial kernel onto the set of isomorphism classes of degree $2n$ central simple algebras with an orthogonal pair of trivial discriminant and trivial component of the Clifford algebra.

Let A be such an algebra. For a component X of the scheme of rank n isotropic ideals of A, let us consider the homomorphism $\mathrm{CH}^1(X) \to \mathrm{CH}^1(X_{\bar{F}})$, where $\bar{F}$ is an algebraic closure of F. Since the absolute Galois group of F acts trivially on $\mathrm{CH}^1(X_{\bar{F}}) = \mathbb{Z} \cdot e_1$, the cokernel of this homomorphism is identified with the relative Brauer group

$$\mathrm{Br}(F(X)/F) = \mathrm{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}(F(X))\big)$$

of the function field of X (see, e.g., [2, proof of th. 3.1]), which is trivial, if the corresponding to X component of the Clifford algebra of A is trivial, [11, appendix] (see [12, rem 9.2] for the characteristic 2 case). Therefore the homomorphism $\mathrm{CH}^1(X) \to \mathrm{CH}^1(X_{\bar{F}})$ is surjective for an appropriate choice of X. Note that if the second component of the Clifford algebra is not trivial (what happens if and only if the algebra A is not trivial [9, (9.14)]), the component of the scheme of the ideals is uniquely determined by this condition; in particular, X can not be chosen arbitrary in this proof.

Let Y be the Severi-Brauer variety of A. The algebra A is split if and only if the variety $Y \times X$ has a rational point. Therefore, to get the required bound for $\mathrm{cd}(G)$, it suffices to get the same bound for $\mathrm{cd}(Y \times X)$.

Let us agree to write $\bar{T}$ for the scheme $T_{F(Y \times X)}$, if T is a scheme over F. The variety $\bar{X}$ is isomorphic to a component of the maximal orthogonal grassmannian of a hyperbolic

$2n$-dimensional quadratic form and therefore is isomorphic to the maximal orthogonal grassmannian of a split $(2n-1)$-dimensional quadratic form, considered in the main part of the paper. In particular, $\dim X = n(n-1)/2$. Moreover, $\bar{\mathrm{CH}}^1(X) = \mathrm{CH}^1(\bar{X})$ (where $\bar{\mathrm{CH}}(X)$ stands for the image of $\mathrm{CH}(X) \to \mathrm{CH}(\bar{X})$), so that the first special Schubert class $e_1 \in \mathrm{CH}(\bar{X})$ lies in $\bar{\mathrm{CH}}(X)$. Acting as in the proof of Theorem 1.1, we find a component T of the support of a non-negative cycle, representing e_1^{n-1}, such that $\deg([\bar{T}] \cdot e) = 1$, where $e \in \mathrm{CH}(\bar{X})$ is the Schubert class dual to e_{n-1}. Note that $\dim T = (n-1)(n-2)/2$.

Since the variety $X_{F(T)}$ has a rational point, the index of the algebra $A_{F(T)}$ divides n. Since the index of A itself is a power of 2, $\operatorname{ind} A_{F(T)}$ divides 2^k, and we may write A as the tensor product of a degree 2^k central simple algebra D and a matrix algebra M. Let Z^o be the Severi-Brauer variety of D. Acting as in the proof of Theorem 6.1, we identify Z^o with a closed subvariety of $Y_{F(T)}$. Let Z be the closure of Z^o in $Y \times T \subset Y \times X$.

The constructed closed subvariety $Z \subset Y \times X$ has the dimension

$$\dim T + \dim Z^o = \frac{(n-1)(n-2)}{2} + 2^k - 1 ,$$

which coincides with the bound we want for $\mathrm{cd}(Y \times X)$. To show that this number is really a bound for $\mathrm{cd}(Y \times X)$, it suffices to show that $Z(F(Y \times X)) \neq \emptyset$, i.e., that $\bar{Z}$ has a rational point.

The variety $\bar{Y}$ is isomorphic to a projective space; let $H \subset \bar{Y}$ be a hyperplane. The degree of the product $[\bar{Z}] \cdot ([H^{2^k-1}] \times e)$ is 1. Moreover, this product can be represented by a non-negative cycle with support on $\bar{Z}$. Therefore, the scheme $\bar{Z}$ has indeed a rational point. □

Note that the upper bound of Theorem 6.2 coincides with the lower bound, given by cd_2, if (and only if) n is a power of 2. Therefore, in this case, we get the precise value:

Corollary 6.3. *If n is a power of 2, then* $\mathrm{cd}(\mathrm{Spin}^{\sim}_{2n}) = n(n-1)/2$. □

6.5. $\mathbf{G}_2$**.** The prime 2 is the unique torsion prime here, and one knows that

$$\mathrm{cd}(G) = \mathrm{cd}_2(G) = 3 .$$

6.6. $\mathbf{F}_4$ **and** $\mathbf{E}_n$**,** $n = 6, 7, 8$**.** We have multiple torsion primes here.

References

[1] G. Berhuy and Z. Reichstein. *On the notion of canonical dimension for algebraic groups.* Adv. Math., to appear.

[2] J.-L. Colliot-Thélène, D. Madore. *Surfaces de Del Pezzo sans point rationnel sur un corps de dimension cohomologique un.* J. Inst. Math. Jussieu **3** (2004), no. 1, 1–16.

[3] M. Demazure. *Désingularisation des variétés de Schubert généralisées.* Ann. Sci. Ecole Norm. Sup. (4) **7** (1974), 53–88.

[4] W. Fulton. *Intersection Theory.* Springer-Verlag; Berlin, Heidelberg, New York, Tokyo, 1984.

[5] R. Hartshorne. *Algebraic Geometry.* Springer-Verlag; New York, Heodelberg, Berlin, 1977.

[6] H. Hiller, B. Boe. *Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n.* Adv. Math. **62** (1986), 49–67.

[7] N. A. Karpenko. *Canonical dimension of orthogonal groups.* Transform. Groups, to appear.

[8] N. A. Karpenko and A. S. Merkurjev. *Canonical p-dimension of algebraic groups.* Linear Algebraic Groups and Related Structures (Preprint Server), December 2004, 22 p.

[9] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol. *The Book of Involultions.* American Mathematical Society Colloquium publications, v. **44**, 1998.

[10] B. Köck. *Chow motif and higher Chow theory of G/P.* Manuscripta Math. **70** (1991), 363–372.

[11] A. S. Merkurjev, I. A. Panin, A. R. Wadsworth. *Index reduction formulas for twisted flag varieties, I.* K-Theory **10** (1996), no. 6, 517–596.

[12] A. S. Merkurjev, I. A. Panin, A. R. Wadsworth. *Index reduction formulas for twisted flag varieties, II.* K-Theory **14** (1998), no. 2, 101–196.

[13] P. Pragacz, J. Ratajski. *Pieri formula for isotropic Grassmannians; the operator approach.* Manuscripta Math. **79** (1993), no. 2, 127–151.

[14] B. Totaro. *The torsion index of the spin group.* Duke Math. J., to appear.

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