BIELEFELD - fevrier 05

CLIFFORD ALGEBRA for ANTIAUTOMORPHISMS OF CENTRAL SIMPLE ALGEBRAS

K a field of char $\neq 2$. V a finite dimensional vector space $/K$

A ~~a~~ central simple algebra, τ ~~...~~ anti-aut $/A$.

(sometimes involution)

① The classical case

Let $b: V \times V \to K$ be a non degenerated symmetric bilinear form.

then $C(V,b) = T(V)/\langle v\otimes v - b(v,v),\ v\in V\rangle$, where $T(V)$ is the tensor algebra of V,

is a graded algebra over $\mathbb{Z}/2$.

Define $C_0(V,b) = \big(C(V,b)\big)_0$ the even part of $C(V,b)$.

It can be defined directly as a quotient of a tensor algebra:

$$C_0(V,b) = T(V\otimes V)/I_1 + I_2 \quad \text{where}$$

$$I_1 = \langle v\otimes v - b(v,v),\ v\in V\rangle$$

$$I_2 = \langle u\otimes v\otimes v\otimes w - b(v,v)\, u\otimes w,\ u,v,w\in V\rangle.$$

Result: depending on the parity of $\dim V$, either $C(V,b)$ or $C_0(V,b)$ is a central simple algebra $/K$, and the other one is a "central simple algebra over $K[X]/X^2-d$" where $d = d_\pm(V)$ is the signed discriminant of b.

($\to$ means that it is central over $K[X]/X^2-d$ if this is a field, or a product of 2 c.s alg over K otherwise).

obs: ~~the~~ involution: $V\otimes V \to V\otimes V$, $v\otimes w \to w\otimes v$ induces an involution on $C_0(V,b)$

obs: $C_0(V,b)$, which depends only on the similarity class of (V,b) gives ~~an~~ cohomological invariant of this class (in $Br_2 K$).

② $\boxed{\text{For an involution over a c.s alg}}$. (orthogonal type)

~~If (A, σ) c.s alg with invol~~

Let $A = \mathrm{End}\, V$ and σ_b be the invol on A adjoint to b.
((A,σ) depends up to isomorphism only on the similarity class of (V, b)).

Then $\quad V \otimes V \xrightarrow{\sim} \underline{A}$ (the vectorspace A)
$\quad v \otimes w \longmapsto (u \longmapsto v\, b(w,u))$.

and by this dictionnary, $v \otimes w \to w \otimes v$ becomes $\sigma_b : A \to A$.
$\qquad V\otimes V \to V \otimes V$

$\rightsquigarrow$ we can use this dictionnary to define $C_0(A,\sigma)$ when A is split, and hence in the non-split case.

def: let (A,σ) a c.s alg$/K$ with involution.
(Tits) then $\quad C_0(A,\sigma) = T(\underline{A})/I_1 + I_2 \quad$ where

- $I_1 = < s - \frac{1}{2}\mathrm{Trd}\, s \;,\; s \in \mathrm{Sym}(A,\sigma) >$
  with Trd = reduced trace form $A \to K$
  $\mathrm{Sym}(A,\sigma) = \{ s \in A,\ \sigma(s) = s \}$
- $I_2 = < u - \frac{1}{2}\mu(u) \,,\, u \in \mathrm{Sym}(\underline{A} \otimes \underline{A}, \sigma_2) >$

here σ_2 is defined as follows: let $\mathrm{Sand} : \underline{A} \otimes \underline{A} \xrightarrow{\sim} \mathrm{End}\, \underline{A}$
$\qquad a \otimes b \mapsto (x \mapsto axb)$

~~this~~ if $u \in A \otimes A$, $\quad A \longrightarrow A$
$\qquad x \longmapsto \mathrm{Sand}(u)(\sigma(x))$ is an elt of $\mathrm{End} A$,

and hence commes form an elt $\sigma_2(u) \in \underline{A} \otimes \underline{A}$.

ie σ_2 is defined by: $\forall u \in \underline{A} \otimes \underline{A}\ \forall x \in A \quad (\mathrm{Sand}\, \sigma_2(u))(x) = (\mathrm{Sand}\, u)(\sigma(x))$.

This is again a linear invol, and hence $\mathrm{Sym}(\underline{A}\otimes\underline{A}, \sigma_2)$ makes sens.

Finaly, $\mu(a \otimes b) = ab \quad \mu : \underline{A} \otimes \underline{A} \to \underline{A}$.

Involution: σ induces an involution over $C_0(A,\sigma)$.

Results: • $\mathcal{C}_0(A,\sigma) = \mathcal{C}_0(V_L, b_L)^{Gal(L/k)}$ if $A\otimes_k L \xrightarrow{\sim} \mathrm{End}\, V_L$, $\sigma_L \longleftrightarrow b_L$.

which was the original definition by Jacobson (1964)

• $\mathcal{C}_0(A,\sigma)$ depends only on the isom. class of (A,σ)

• if $\deg A$ is even $d = \mathrm{Nrd}\, x$ is indep^t of the elt $x \in A^\times \cap \mathrm{Skew}(\sigma) = \{y \in A^\times, \sigma(y) = -y\} \neq \emptyset$.

d = discriminant of (A,σ).

$\longrightarrow$ $\mathcal{C}_0(A,\sigma)$ is "c.s" over $K[X]/X^2-d$

it gives hence a cohomological invariant of the isom. class of (A,σ). lives in fact in $Br_2 K/\langle A\rangle$

③ $\boxed{\text{For antiautomorphisms}}$

Let σ be an antiautomorphism over A cs/K, not nec. involutive

known invariants (Cortella - Tignol) ⊛ $a_\sigma \in A^\times$ the asymetry of (A,σ), γ_σ the linear isom. attached to (A,σ):

in the split case $A = \mathrm{End}\, V$ $\sigma \longleftrightarrow \sigma_b$ where b bilinear, not sym, bilinear form.

then a_σ et γ_σ are def by: $\forall x,y \in V$ $\forall f \in \mathrm{End}\, V$

$$b(x,y) = b(y, a_\sigma(x)) \text{ and } b(y, f(x)) = b(x, \gamma_\sigma(f)(y)).$$

(hence $\gamma_\sigma(f) = \sigma(f)\, a_\sigma$).

rmk: a_σ satisfies $\sigma^2 = \mathrm{Int}\, a_\sigma$ and $\sigma(a_\sigma) = a_\sigma^{-1}$.

(which defines it up to a sign).

$\longrightarrow$ was entirely defined in the non-split case.

Advantage: γ_σ is an involutive linear application $\underline{A} \to \underline{A}$, hence is better than σ even if its not multiplicative.

In fact $\gamma_\sigma(xyz) = \sigma^{-1}(z)\,\gamma_\sigma(y)\,\sigma(x)$.

⊛ discriminant: in even degree, same as before but replacing σ by γ_σ: $d = \mathrm{Nrd}\, x$ where $x \in A^\times \cap \mathrm{Skew}\,\gamma_\sigma \neq \emptyset$.

<u>Clifford algebra</u> $\to$ it seems that we only have to replace everywhere in the preceding def σ by γ_σ.

<u>pb</u>: in the split case, then $C_0(A,\sigma)$ is a quotient of $T(V\otimes V)$, but not the even part of a quotient of $T(V)$

$\Longrightarrow$ <u>Correction</u> in I_1, just replace σ by γ_σ.

in I_2, σ is replaced by $x \mapsto a_\sigma \gamma_\sigma(x) a_\sigma$

<u>Result</u>: $C_0(A,\sigma)$ is an inv^t of the isomorphism class of (A,σ), that behave well under scalar extension, and such that in the split case $C_0(A,\sigma) = \left(\frac{T(V)}{\langle v\otimes a_\sigma v - b(v,v),\, v\in V\rangle}\right)_0$

<u>Questions</u>: dim ? "Simple" (in which way ?) ? Centre ? cohomological invariant ?

$\to$ <u>computations</u>: <u>in degree 2</u>

$$C_0(A,\sigma) = K[X]/X^2 - d\,\mathrm{Nrd}(a_\sigma+1)$$

(<u>rmk</u>: if $a_\sigma - 1 \in A^\times$, then $d = \mathrm{Nrd}(a_\sigma - 1)$)