From essential to canonical dimension: An overview

(Bielefeld, February 2005)

§1 Tschirnhaus transformations

Let k be a field, K/k be any field extension.
Two polynomials $f, g \in K[X]$ are said to be <u>Tschirnhaus equivalent</u> if there is K-algebra isomorphism $K[X]/\langle f\rangle \xrightarrow[\varphi]{\sim} K[X]/\langle g\rangle$

Such a φ is called a <u>non-degenerate Tschirnhaus transform</u>.
If $f = X^n + a_1 X^{n-1} + \dots + a_n$ and if x denotes the class of X

φ is uniquely determined by the image of x which has the form

$$\varphi(x) = r_0 + r_1 x + \dots + r_{n-1} x^{n-1}$$

and satisfies $f(\varphi(x)) = 0$

Given f of degree n we would like to compute the minimal number of algebraically independent coefficients of f up to Tschirnhaus transformation

<u>examples:</u> <u>$n=2$</u> $\quad f = X^2 + aX + b \underset{\substack{\varphi(x) = x - \frac{a}{2} \\ \text{if } char(k) \neq 2}}{\rightsquigarrow} g = X^2 + c \qquad c = b - \frac{a^2}{4}$

f: 2 parameters; g: 1 parameter

<u>$n=3$</u> $\quad f = X^3 + aX^2 + bX + c \underset{\text{if } char(k) \neq 3}{\overset{\varphi_1}{\rightsquigarrow}} g = X^3 + b'X + c' \underset{\substack{\varphi_2(x) = \frac{c'}{b'}x \\ \text{if } c'b' \neq 0}}{\overset{\varphi_2}{\rightsquigarrow}} h = X^3 + dX + d$

f: 3 parameters; h: 1

<u>$n=4$</u> $\quad f \underset{char k \neq 2}{\rightsquigarrow} X^4 + b'X^2 + c'X + d' \rightsquigarrow X^4 + aX^2 + bX + b$

2 parameters and not 1 in general

All cases are recovered by the generic polynomial of degree n

$$f_{gen}^{(n)} = X^n + a_1 X^{n-1} + \dots + a_n$$

where the a_i are algebraically independent variables over k.

The minimal number of alg coefficients of $f_{gen}^{(n)}$ up to Tschirnhaus Transformation will be denoted by $d_k(n)$

char k = 0 $\qquad d_k(2) = d_k(3) = 1$

$\qquad\qquad d_k(4) = 2$

- Hermite in 1861 shows that $f_{gen}(5)$ can be brought to the form $x^5 + a x^3 + bx + b$

  and thus $d_k(5) \leq 2$

- Klein showed in 1884 that $d_k(5) \neq 1$ which he called the "Kronecker's Theorem" $\Rightarrow d_k(5) = 2$

- Joubert in 1867 shows that $f_{gen}(6)$ can be brought to the form $t^6 + a t^4 + b t^2 + ct + c$ $\Rightarrow d_k(6) \leq 3$

Questions: behavior of $d_k(n)$, bounds?

§2 Buhler and Reichstein's approach (1995) $\boxed{\mathrm{char}(k) = 0}$

Def. Let E/F be a field extension of degree n and $F_0 \subset F$. E/F is said to be defined over F_0, if $\exists$ an extension E_0/F_0 of degree n, contained in E, such that $E_0 F = E$

picture:

$$\begin{tikzcd}[column sep=small] & E \arrow[d, dash] \\ E_0 \arrow[ur, dash] \arrow[d, dash] & F \\ F_0 \arrow[ur, dash] \arrow[d, dash] & \\ k & \end{tikzcd}$$

the essential dimension of E/F, denoted by $ed_k(E/F)$, is the minimal value of $\mathrm{trdeg}(F_0 : k)$ where F_0 ranges over the subfield over which E/F is defined.

Now take $f_{gen}(n) = x^n + a_1 x^{n-1} + \cdots + a_n$ be the generic polynomial of deg n

let $F_n = k(a_1, \ldots, a_n)$ and $E_n = F_n[x] / \langle f_{gen}(n) \rangle$

then $ed(E_n/F_n) = d_k(n)$ (exercise)

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Lemma 1: If E/F is a Galois extension with group G
Then $\exists$ $F_1 \subset F$ and a Galois extension E_1/F_1 with group G
such that $\operatorname{trdeg}(F_1:k) = ed(E/F)$

Lemma 2: Let E/F be an extension of degree n and let $E^{\#}$ be the normal closure of E over F. Then $ed(E/F) = ed(E^{\#}/F)$.

Let $x_1, \dots, x_n$ be the roots of the generic polynomial $f_{gen}(n)$

then $F_n = k(a_1, \dots, a_n) = k(x_1, \dots, x_n)^{S_n}$

and $$E_n = F_n[x] / \langle f_{gen}(x) \rangle \cong F_n(X_1) = k(x_1, \dots, x_n)^{S_n}(X_1)$$

but $$E_n^{\#} = k(x_1, \dots, x_n)$$

so $$d_k(n) = ed_k(E_n/F_n) = ed_k(E_n^{\#}/F_n)$$

$$= ed_k\left(k(x_1, \dots, x_n) / k(x_1, \dots, x_n)^{S_n}\right)$$

Galois extension with group S_n

(Now comes the geometric approach)

G finite group, X a <u>G-variety</u>, that is an irreducible algebraic variety over k, together with a map of varieties $G \times X \to X$. A G-variety X is called <u>faithful</u> if $G \to \operatorname{Aut}(X)$ is injective

Definition: Let X be a faithful G-variety. The <u>essential dimension of X</u> is the minimal dimension of a faithful G-variety Y such that there exists a G-compression $X \dashrightarrow Y$ that is a dominant G-equivariant rational morphism. This is denoted by $ed_k(X)$

if X is a vector space the variety is called <u>linear</u> $G \to GL(X)$

emphasis on linear faithful representation

Lemma 3: Let X be a faithful G-variety, let $E = k(X)$ = rational functions and $F = E^G$ = G-invariant rational functions

Then $ed_k(X) = ed_k(E/F)$

Thm - Definition: Let X be a faithful G-variety and V be any faithful linear representation of G. Then

$$ed_k(X) \leq ed_k(V)$$

In particular $ed_k(V) = ed_k(V')$ for all faithful linear representation V'

The number $ed_k(V)$ is called the essential dimension of G and denoted by $ed_k(G)$

example $d_k(n) = ed_k\left(k(x_1,\dots,x_n) / k(x_1,\dots,x_n)^{S_n} \right)$

$$= ed_k\left(k(V)/k(V)^{S_n} \right) = ed_k(V) = ed_k(S_n)$$

$V = A^n$ the affine n-space

Remark: if $H \leq G$ then $ed_k(H) \leq ed_k(G)$

Corollary: $d_k(n) \leq d_k(n+1)$

proof. $S_n \leq S_{n+1}$ #

Known facts: if k contains a primitive p^{th}-root of unity, then

$$ed_k(\underbrace{\mathbb{Z}/p \times \dots \times \mathbb{Z}/p}_{r \text{ times}}) = r$$

Corollary: $ed_k(S_n) \geq [\frac{n}{2}]$

proof: $\underbrace{\mathbb{Z}/2 \times \dots \times \mathbb{Z}/2}_{[\frac{n}{2}] \text{ times}} \leq S_n$ #

very little is known on $ed_k(G)$ for finite groups

other result: $ed_k(S_n) \leq n-3 \quad \forall n \geq 5$

$\rightarrow \quad d_k(4) = d_k(5) = 2 \qquad d_k(6) = 3 \qquad d_k(7) = \begin{cases} 3 \\ 4 \end{cases}$ still unknown

— 15 min —

§3 Reichstein's generalisation

k algebraically closed of char 0, G algebraic group over k.

A G-variety X is called generically free if G acts freely (with trivial stabilizers) on a dense open subset of X (here X is not necessarily irreducible)

def: Let X be a generically free G-variety. A G-compression of X is a dominant G-equivariant rational map $X \dashrightarrow X'$ where X' is another generically free G-variety

$$ed_k(X) \overset{def}{:=} \min \{ \dim X' - \dim G \}$$

where the min is taken over all G-compressions $X \dashrightarrow X'$

Thm: same as for G finite but replace faithful by generically free

Known facts: 1) $ed_k(G \times H) \leq ed_k(G) + ed_k(H)$

2) $ed_k(H) + \dim H \leq ed_k(G) + \dim G$ if H closed subgroup of G

3) $ed_k(GL_n) = ed_k(SL_n) = 0$, $ed_k(\mathbb{G}_m^r) = 0$ } more generally: $ed(G) = 0 \Leftrightarrow H^1(K, G) = 0$ $\forall K$

4) $ed_k(O_n) = n$ $\quad ed_k(SO_n) = n-1$ $\forall n \neq 2$ $(ed_k(SO_2) = 0)$

$ed_k(PGL_n) \leq n^2 - 2n$ $\quad \forall n \geq 4$

$ed_k(PGL_{n^r}) \geq 2^r$ for any $n \geq 2$ $r \geq 1$

$ed(PGL_n) = 2$ if $n = 2, 3, 6$

if n_1, n_2 are relatively primes then

- $ed(PGL_{n_i}) \leq ed(PGL_{n_1 n_2})$
- $ed(PGL_{n_1 n_2}) \leq ed(PGL_{n_1}) + ed(PGL_{n_2})$
- $ed(G_2) = 3$
- $ed(F_4) = 5$
- $9 \leq ed(E_8)$

$ed(PO_n) \geq n-1$

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§4 Merkurjev's functional point of view

Let k be any field, denote by $\mathcal{C}_k$ the category of all field extensions of k

Definition: Let $F: \mathcal{C}_k \to$ Sets be a covariant functor

For K/k and $a \in F(K)$ we write $ed(a) \leq n$ if there exists $E \subseteq K$ such that

i) $a \in \operatorname{im}(F(E) \to F(K))$

ii) $\operatorname{trdeg}(E:k) = n$

Of course $ed(a) = n$ if $ed(a) \leq n$ and $ed(a) \not\leq n-1$

We put $ed_k(F) = \sup_{K/k} \{ ed(a) \mid a \in F(K) \} \in \mathbb{N} \cup \{\infty\}$

example of functors

1) Let X be a k-scheme. It defines a functor $X: \mathcal{C}_k \to$ Sets

by setting $X(K) = \operatorname{Mor}(\operatorname{Spec} K, X) =$ K-rational points of X

2) More generally let X be a k-scheme and G be a k-group-scheme acting on X. We have an orbit functor denoted by $X/G: \mathcal{C}_k \to$ Sets

$$K \mapsto X(K)/G(K)$$

3) Let G be a group-scheme of finite type, then we have the Galois cohomology functor $K \mapsto H^1(K, G)$

$$ed_k(H^1(-,G)) \overset{\text{not}}{=} ed_k(G)$$

advantages:
- k any field
- easier statements
- compare functors
- $F \simeq G \Rightarrow ed(F) = ed(G)$

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- $ed(F \times G) \leq ed(F) + ed(G)$

Known facts : - $ed_k(X) = \dim X$

- if $F \twoheadrightarrow G$ is a surjective morphism (ie $F(K) \twoheadrightarrow G(K)$ $\forall K/k$)

then $ed_k(G) \leq ed_k(F)$

- cohomological invariants

Corollary : $ed_k(O_n) \leq n$

proof : $H^1(K, O_n) \longleftrightarrow$ isometry classes of non-degenerate quadratic forms of rank n over K $\overset{not}{=} Q_n(K)$

$$\mathbb{G}_m \times \dots \times \mathbb{G}_m \twoheadrightarrow Q_n$$

variety of dimension n

$$(a_1, \dots, a_n) \longmapsto \langle a_1, \dots, a_n \rangle$$

$$ed_k(\mathbb{Z}/4) = \begin{cases} 1 & \text{if } -1 = \square \text{ and char } k \neq 2 \\ 2 & \text{otherwise} \end{cases}$$

$$ed_k(S^1) = \begin{cases} 0 & \text{if } -1 = \square \text{ and char } k \neq 2 \\ 1 & \text{otherwise} \end{cases}$$

$ed_k(G) = ed_{k(t)}(G)$ for any infinite field k

Special case of $ed(X/G)$ ← hard in general

$X_{d,n}$ = projective space of homogenous polynomial of degree d in n variables $\simeq \mathbb{P}^{m-1}$

$G = PGL_n$ $\qquad F_{d,n} = X_{d,n}/PGL_n$ $\qquad$ where $m = \binom{d+n-1}{n-1}$

$F_{3,n} = Cub_n$

Thm : if char $k \neq 2,3$ then
(2003)

$ed_k(Cub_3) = 3$ $\qquad$ 10 coefficients

+ other results

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§ 5 Canonical dimension (geometric)

We look closer at $ed_k(X/G)$

Let $x \in X(L)$ a L-rational point $x : \operatorname{Spec} L \to X$

and take Y a k-model of L (k-scheme such that $k(Y) \simeq L$)

x induces a rational morphism $\varphi_x : Y \dashrightarrow X$ (and vice versa)

In the same way $g \in G(L)$ induces $\varphi_g : Y \dashrightarrow G$

and $g \cdot x \in X(L)$ is given by

$$F_{g,x} : Y \dashrightarrow G \times X \xrightarrow{\cdot} X$$
$$y \longmapsto (\varphi_g(y), \varphi_x(y))$$

So to find $g \in G(L)$ such that $g\,x$ is defined over a minimal extension (for trdeg) is the same as to find $\varphi_g : Y \dashrightarrow G$ such that $\dim(F_{g,x}(Y))$ is minimal (exercise)

$$ed([x]) = \min_{g \in G(L)} \{ \dim F_{g,x}(Y) \}$$

If X is irreducible, take $x = \eta$ the generic point of X

$Y = X$ $\quad \varphi_x = id_X : X \dashrightarrow X$

$f = \varphi_g : X \dashrightarrow G$

$$F : X \dashrightarrow X \qquad \leftarrow \text{canonical form map}$$
$$x \longmapsto f(x)\, x$$

$$ed([\eta]) = \min_{f : X \dashrightarrow G} \{ \dim F(X) \}$$

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Berhuy-Reichstein definition (2004) char $k = 0$ $k = \bar{k}$

$$cd(X,G) = \min_{\substack{F \text{ canonical} \\ \text{form map}}} \{ \dim F(X) - \dim X + \dim G \}$$

remember $F : X \dashrightarrow X$ is canonical if it is of the form $x \mapsto f(x).x$ where $f : X \dashrightarrow G$

When X is generically free $F : X \dashrightarrow X$ is canonical iff it commutes with the rational quotient map $X \overset{\pi}{\dashrightarrow} X/G$

Prop: if V is a generically free linear representation of G and X is any generically free G-variety then $cd(X,G) \leq cd(V,G)$

def: $cd(G) = cd(V,G)$ for some V generically free $= \max_{\substack{X \text{ gen free} \\ G\text{-variety}}} cd(X,G)$

Back to $ed(F_{d,n}) = ed(\mathbb{P}^{m-1}/PGL_n)$:

$$ed(F_{d,n}) \geq ed([n]) = cd(\mathbb{P}^{m-1}, PGL_n) + \dim \mathbb{P}^{m-1} - \dim PGL_n$$

$$= cd(\mathbb{P}^{m-1}, PGL_n) + m - 1 - n^2 + 1$$

$$= cd(\mathbb{P}^{m-1}, PGL_n) + m - n^2$$ $ed(F_{d,n}) \geq ed([n])$

$$\underset{\text{lemma}}{=} cd(\mathbb{A}^m, \mathbb{G}_m \times GL_n) + m - n^2$$ but don't know how to prove the equality

for good (n,d) $\rightsquigarrow$ $$\underset{\text{def}}{=} cd(\mathbb{G}_m \times GL_n) + m - n^2$$

$$\underset{\text{lemma}}{=} cd(GL_n/\mu_d) + m - n^2$$

special iff $\gcd(n,d) = 1$

$\boxed{\text{rajouter}}$

Known:

$cd(G) = cd(G^0)$, $cd(G) = 0$ iff G^0 is special

–40 min

§6 Canonical dimension (functorial)

If X is a generically free G-variety the rational quotient map may be viewed as a G-torsor or principal homogeneous space over G

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d, dashed] \\ \operatorname{Spec} k(X/G) \arrow[r] & X/G \end{tikzcd} \qquad k(X/G) = k(X)^G = K$$

generic fiber

X defines an element $\alpha \in H^1(K, G)$

Lemma: For any canonical form map $F: X \dashrightarrow X$ the following are equivalent

i) $\alpha_E = 1_E \in H^1(E, G) \qquad (K \subseteq E)$

ii) $F(X)$ has a E-rational point

in particular $k(F(X)) = E \Rightarrow \alpha_{k(F(X))}$ is a splitting field

One can show that $k(F(X))$ is a <u>generic splitting field for α</u>

So that $cd(X, G) = \min \operatorname{trdeg}(L : K) \qquad K = k(X)^G$

where L runs through the generic splitting fields for α

This leads to the following generalisation:

– 45 min –

Let $F : \mathcal{C}_k \to Sets^*$ be a covariant functor with values in the category of pointed sets.
Let K/k and $a \in F(K)$. A splitting field for a is an extension L/K such that $a_L = * \in F(L)$.
A splitting field E/K is called generic for a if $\forall\ L/K$ splitting field there exists a K-place $\varphi : E \rightsquigarrow L$.
The canonical dimension of a, $cd(a)$, is the min of $trdeg(E:K)$ for all generic splitting fields E/K of a.

$$cd(F) = \sup_{K/k} \{cd(a) \mid a \in F(K)\}$$

$$cd(G) = cd(H^1(-,G))$$ by our considerations

This is helpful for computations and it generalizes to k arbitrary.
Aim: Study more carefully what happens for morphisms of functors $F \to G$

Recently Karpenko and Merkurjev gave another functorial approach which generalises this one. They compute $cd(G)$ for a large class of algebraic groups using Chow theory.

– 50 min –