lecture (45 min):

$\mathbb{R}$: real numbers } $\mathcal{O}$: sheaf of C^∞ functions on $\mathbb{R}$:
$\cup$
p real number :

C^∞

$f \in \mathcal{O}(U)$ $p \in U$ function smooth in neighborhood p : Taylor-series: only well-defined info one can extract from $(f, U) \in \mathcal{O}_p$.

$$f \overset{T(f)}{\rightsquigarrow} \sum_{k \geq 0} \frac{f^{k}(p)}{k!} (t-p)^{k}$$

} Taylor-series of f : is a well-defined function on $\mathcal{O}_p$: local ring of $\mathcal{O}$ at p :

the map is surjective : (E. Borel 1896)

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & \mathcal{O}_p \arrow[r, "T"] & \mathbb{R}[[t-p]] \arrow[r] & 0 \\ & & \mathfrak{m}_p \arrow[u, phantom, sloped, "\subseteq"] & & \end{tikzcd}$$

formal power series in $t-p$

$\mathfrak{m}_p$: maximal ideal of functions vanishing at p : $\mathfrak{m}_p = (t-p)$ generated by a polynomial.

and : $\mathcal{O}_p/\mathfrak{m}_p^{k+1} \cong \mathbb{R}[[t-p]]/(t-p)^{k+1}$ } $\mathcal{O}_p/\mathfrak{m}_p^{k+1}$: is in a natural way : ring of Taylor-polys of order k of function-germs (f, U) C^∞ in neighborhood of p.

$\mathcal{O}_p/\mathfrak{m}_p^{k+1}$: is the fiber at p of a vector bundle $\mathcal{J}^k_{\mathbb{R}}$ } the bundle of principal parts of order k :

Algebraic version : we want to do this purely algebraic :

(Grothendieck Dieudonné) $\mathbb{Z}[t] \subseteq \mathbb{Z}[t,p]$ t,p: alg. indep variables.

$\mathbb{Z}[t] \ni f(t)$ } poly w. integer coeffs. ⇝ $f(t) = f(p+t-p)$

ex: $t^2 = (p+t-p)^2 = p^2 + 2p(t-p) + (t-p)^2$

$t^2 = f(t)$

$= f(p) + \frac{f'(p)}{1!}(t-p) + \frac{f''(p)}{2!}(t-p)^2$ } the algebraic taylor-expansion of $f(t) = t^2$.

this generalize: any $f(t) = f(p+t-p) \overset{\text{binomial+induction theorem}}{=} \sum_{k=0}^{\deg f} \frac{f^{(k)}(p)}{k!}(t-p)^k$ } formal taylor expansion of $f(t)$ in variable p.

We want to truncate: $I = (t-p)^{l+1}$

$$\Rightarrow \quad \underset{A}{\overset{\|}{\mathbb{Z}[t]}} \longrightarrow \mathbb{Z}[t,p]/(t-p)^{l+1} \underset{\substack{t = 1\otimes t \\ p = t\otimes 1}}{\cong} \mathbb{Z}[t]\otimes\mathbb{Z}[t]/I^{l+1}$$

I: ideal of the diagonal.

$$A \xrightarrow{T^l} A\otimes A/I^{l+1}, \quad a \longmapsto 1\otimes a$$

} l'th taylor-expansion of $\underline{\underline{a}}$ T^l: universal differential-operator of order l.

} This works over any ring: (we have not chosen coordinates! = intrinsic definition).

we may: X/S : any separated scheme

$p, q : X \times X \rightrightarrows X$ (projection maps)

$\mathcal{O}_{\Delta^k} = \mathcal{O}_{X\times X}/J^{k+1}$

$\mathcal{E}$: $\mathcal{O}_X$-module

$P^k_{X/S}(\mathcal{E}) := p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{E})$ } module of kth order principal parts of $\mathcal{E}$.

(works for any closed im $Y \hookrightarrow X$ of ringed spaces).
formal scheme = infinite jets.

And in text: $\begin{array}{c} X \\ \downarrow \\ \mathrm{Spec}(k) \end{array}$ X : separated scheme of finite type / k

$p \in X$: k-rational point

? $P^k(\mathcal{E})(p) = \mathcal{E}_p / m_p^{k+1}\mathcal{E}_p$

$\rightsquigarrow P^k(p) = \cancel{\;} = \mathcal{O}_p / m_p^{k+1}$; $\mathcal{O}_p$: local ring of $\mathcal{O}_X$ at p.

$m_p \subseteq \mathcal{O}_p$: maximal ideal

hence: $P^k(\mathcal{E})$ has the local properties of $J^k_{\mathbb{R}}$ at k-rational points. (k : base field)

Prop: $\mathcal{E}$ locally free, X/S : all smooth. $\forall k \geq 1$: $0 \to S^k(\Omega^1_{X/S}) \otimes \mathcal{E} \to P^k(\mathcal{E}) \to P^{k-1}(\mathcal{E}) \to 0$ exact

$\Rightarrow$ $\mathcal{E}$: rank e, X/S : relative dim n : $\mathrm{rk}\, P^k(\mathcal{E}) = e\binom{k+n}{n}$ ($P^k(\mathcal{E})$ locally free)

$k=1$: $\mathcal{J} \subseteq \Omega^1_{X/S}$ any $\mathcal{O}_X$-sub-module: $\mathcal{F} := \Omega^1/\mathcal{J}$ $d: \mathcal{O}_X \to \Omega^1 \to \mathcal{F}$ } derivation

⊛ $0 \to \mathcal{F}\otimes\mathcal{E} \to \mathcal{F}\otimes\mathcal{E}\oplus\mathcal{E} \to \mathcal{E} \to 0$ $\left\{ \begin{array}{l} s(x,e) := (sx + ds\otimes e, se) \\ (x,e)s = (xs, es) \end{array} \right\}$

($\mathcal{F}$: foliation)

⊛: exact sequence of $\mathcal{O}_X$-bimodules and:

⊛: is left split by a generalized connection $\left\{ \begin{array}{l} \nabla: \mathcal{E} \to \mathcal{F}\otimes\mathcal{E} \\ \nabla(se) = s\nabla(e) + ds\otimes e \end{array} \right.$

$\mathcal{J}\equiv 0$: ⊛ becomes

$at(\mathcal{E})$ $0 \to \Omega^1\otimes\mathcal{E} \to \boxed{\begin{array}{c}\Omega^1\otimes\mathcal{E}\oplus\mathcal{E} \\ \shortparallel \\ P^1(\mathcal{E})\end{array}} \to \mathcal{E} \to 0$ } 1st order fundamental exact sequence of the principal parts = the Atiyah-sequence.

~~A connection $\nabla: \mathcal{E}\to\Omega^1\otimes\mathcal{E} \rightsquigarrow \Omega^k\otimes\mathcal{E} \to \Omega^{k+1}\otimes\mathcal{E}$: de Rham complex of $(\mathcal{E},\nabla)$~~

~~$ad\nabla$: $End(\mathcal{E}) \to \Omega^1\otimes End(\mathcal{E}) \rightsquigarrow \underset{\overset{\shortparallel}{c_k(\mathcal{E})}}{tr(R^k_\nabla)} \in H^{2k}_{dR}(X)$ } $c_k(\mathcal{E})$: chern-character of $\mathcal{E}$:~~

~~Thm ch: $K_0(X) \to H^*_{DR}(X)$ } the chern-character of X.~~

~~$P^1(\mathcal{E}) \rightleftarrows \mathcal{E}$ } "contains all" info on $ch(\mathcal{E})$~~

~~⊛ Ω^*_X : constructed from P^1_X~~

~~Ⓑ ∇: constructed from $P^1(\mathcal{E})$~~

Generalization:

We get: $\mathrm{at}(\mathcal{E}) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}) \cong$

$\mathcal{E}$: line bundle: $\mathcal{E} \otimes \mathcal{E}^\vee = \mathcal{O}_X$

$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^1_X \otimes \mathcal{E} \otimes \mathcal{E}^\vee) =$

$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^1_X) = H^1(X, \Omega^1_X) \xleftarrow{d\log} \mathrm{Pic}(X)$

ψ

$\mathrm{at}(\mathcal{E}) = c_1(\mathcal{E}) \leftarrow \mathcal{E}$

hence: Atiyah-class calculates the Chern class of a line bundle.

$\mathrm{at}(\mathcal{E}) = 0$.

---

and: $0 \to \Omega^1_X \otimes \mathcal{E} \to P^1(\mathcal{E}) \overset{\nabla}{\rightleftarrows} \mathcal{E} \to 0$ left split: by a connection

$\nabla : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E} \rightsquigarrow$ get the de Rham-complex of $(\mathcal{E}, \nabla)$

and $(\mathrm{End}(\mathcal{E}), \mathrm{ad}(\nabla))$ $\quad R_\nabla \in \mathcal{E} \to \Omega^2 \otimes \mathcal{E} \Rightarrow$

$R_\nabla \in \Omega^2 \otimes \mathrm{End}(\mathcal{E}) \ldots$

$$R^k_\nabla \in \Omega^{2k} \otimes \mathrm{End}(\mathcal{E}) \quad \Big| \quad \overline{\mathrm{tr}(R^k_\nabla)} \in H^{2k}_{dR}(X)$$

$\downarrow \mathrm{tr}$

Ω^{2k}

Theorem: $\mathrm{ch} : K_0(A) \to H^*_{dR}(A)$) is a ring-homomorphism ⑧ the Chern-character of A ⑧ with values in algebraic de Rham cohomology.

$$(\mathcal{E}, \nabla) \longmapsto \sum_{n \geq 0} \frac{\mathrm{tr}(R_\nabla^n)}{n!} = \exp(R_\nabla)$$

Q: what when $0 \to \Omega^1_X \otimes \mathcal{G} \to P^1(\mathcal{G}) \to \mathcal{G} \to 0$ non-split ?

We dualize: wtz: an $\nabla : \mathcal{E} \to \Omega^1 \otimes \mathcal{G}$ ⇝ gives a covariant derivation

$$\bar{\nabla} : T_X = \mathrm{Der}(A) \to \mathrm{End}_k(E) \qquad \bar{\nabla}(s)(ae) = a\bar{\nabla}(s)(e) + s(a)e$$

(Leibniz-property.

∇ flat $\Rightarrow$ $\bar{\nabla}$: map of Lie-algebras.

We may consider instead of $\bar{\nabla}$ globally defined on $\mathrm{Der}(A)$:

pick a sub-Lie-algebra. $\mathfrak{g} \subseteq \mathrm{Der}_k(A)$

Conf.: we may: for any $\nabla : \mathfrak{g} \to \mathrm{End}_k(W)$: partial connection.

$\rightsquigarrow$ $R_\nabla^n \in C^{2n}(\mathfrak{g}, \mathrm{End}(W))$ } (shuffle-product)

$$\begin{tikzcd} R_\nabla^n \arrow[d, "\mathrm{trace}"] \\ \mathrm{trace}(R_\nabla^n)\ C^{2n}(\mathfrak{g}|_U, \mathcal{O}_U) \end{tikzcd}$$

$U \subseteq \mathrm{Spec}(A)$: open subset of $\mathrm{Spec}(A)$ when W is locally free

[illegible]

~~[illegible]~~ $\mathrm{tr}\exp(R_\nabla) \in C^*(\mathfrak{g}|_U, \mathcal{O}_U) \rightsquigarrow$

$\rightsquigarrow$ thm ① $\mathrm{ch}(W, \nabla) \in H^*(\mathfrak{g}|_U, \mathcal{O}_U)$

$K_0(\mathfrak{g})$: Grothendieck ring of locally free A-mod W with $\mathfrak{g}$-connect.

Theorem ② $\boxed{\mathrm{ch} : K_0(\mathfrak{g}) \to H^*(\mathfrak{g}, A)}$ is a ring-homomorphism.

Corollary: X a smooth k-algebra of finite type: $\mathfrak{g} = \mathrm{Der}(A)$ If X, $K_0(\mathfrak{g}) = K(\mathrm{Der}(A)) = K_0(A) \to H^*(\mathfrak{g}, A) = H^*_{dR}(A)$ } is the classical case.

So: We do get a new chern-class theory in this generalized setting and there is a chern-character.

Q: which other concepts in the classical case generalize?

- Index-theorems } unknown
- Cartier-iso morphism } [illegible].

---

anything we can do will simultaneously generalize 3 different theories + we introduce Lie-algebra cohomology into the picture: seem fun.

$L \subseteq \mathrm{Der}_k(A) \subseteq \mathrm{Diff}(A/k)$ } ring of diff-operators on A.

∃ generalized universal enveloping algebra $U(L,A)$: { 1-1 cor: left $U(L,A)$-mods / A-modules w. flat L-connec

(sheafified)

∃ generalized PBW-theorem for $U(L,A)$ when L is locally free finite rank A-module } powerfull tool!

and (Ekedahl...) : $U(L,A)^{*} \overset{s}{\underset{t}{\leftleftarrows}} A$ commutative A-algebroid ⇝ it may rise to an algebraic (= formal) equivalence-relation in schemes R/R_0 $(R \rightrightarrows R_0)$ and $\mathcal{U}(L,A)$: is (in the case A carries) in a natural way the (non-commutative) ring of functions on the stack-quotient $[R/R_0]$

Unifying concept: the jetbundles / Principal parts.

(A Lie-algebra $L \subseteq Der(A)$ $\overset{cor}{\rightsquigarrow}$ quotient P/J of first order jets.)

Q: higher order jets: What can be derived from them? What do we know about them? need to clarify ...

$\mathbb{P}^1_u, \mathbb{P}^1_{\mathbb{Z}} \rightsquigarrow P^1(\mathcal{O}(d)) = ?$ ⇝ given by matrices

---

(sheaf of $\mathcal{O}$-bimodules) Left : $\begin{bmatrix} t^d & 0 \\ dt^{d-1} & -t^{d-2} \end{bmatrix}$ char $F \mid d$: $\mathcal{O}(d) \oplus \mathcal{O}(d-2)$ as left

Right $\begin{bmatrix} u^d & 0 \\ 0 & u^{d-2} \end{bmatrix} \rightsquigarrow \mathcal{O}(d) \oplus \mathcal{O}(d-2)$ as right

<u>hence</u>: if char$(F) \mid d$: $0 \to \underset{\mathcal{O}(d-2)}{\underset{\wr}{\Omega^1 \otimes \mathcal{O}(d)}} \to P^1(\mathcal{O}(d)) \to \mathcal{O}(d) \to 0$ is left split.

in positive char: $\mathcal{O}(d)$ has a connection, but <u>not</u> in char 0.

$$\begin{bmatrix} 1 & -\frac{1}{d}t \\ 0 & 1/d \end{bmatrix} \begin{bmatrix} t^d & 0 \\ dt^{d-1} & -t^{d-2} \end{bmatrix} \begin{bmatrix} 1/t & 1 \\ d & 0 \end{bmatrix} = \begin{bmatrix} t^{d-1} & 0 \\ 0 & t^{d-1} \end{bmatrix}$$

$\Rightarrow$ char 0. $P^1(\mathcal{O}(d)) = \mathcal{O}(d-1) \oplus \mathcal{O}(d-1)$.

need: involutive subbl.

$V \subseteq T_{P^1}$, $\nabla : V \to \mathrm{End}(\mathcal{O}(d))$

deduce chern cls in char 0.

on $\mathbb{P}^1_h$: all bundles split:

Thm: $P^l(\mathcal{O}(d)) = \overset{l+1}{\oplus} \mathcal{O}(d-l)$ $1 \le l$ $\begin{cases} l \le d \\ d < 0 \end{cases}$

$P^l(\mathcal{O}(d)) = \mathcal{O}^{d+1} \oplus \mathcal{O}(-l-1)^{l-d}$ $0 \le d < h$

as left $\mathcal{O}$-modules

$P^l(\mathcal{O}(d)) = \mathcal{O}(d) \oplus \mathcal{O}(d-l-1)^l$ as right $\mathcal{O}$-modules

---

Alternative: $\mathbb{P}^1_h = SL(V)/P$ $V = \{e_0, e_1\}$ $P = \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}$ $V^* = \{x_0, x_1\}$ $L = \{e_0\}$ $L^* = \{e_1\}$ $S^d(L^*) = \mathcal{O}(d)$

$\exists$: equivalence of categories: <u>P-modules</u> $\cong$ <u>$SL(V)$-homogeneous vectorbundles on $\mathbb{P}^1$</u>

Problem: classify $P^l(\mathcal{O}(d))$ as left & right P-modules.

$$P^\ell(\mathcal{O}(d)) = \begin{cases} \mathrm{Sym}^{\ell-d}(L^*) \otimes \mathrm{Sym}^\ell(V^*) & 1 \le k \le d \\ \mathrm{Sym}^{\ell-d}(L) \otimes \mathrm{Sym}^\ell(V) & d \le 0,\ k \ge 1 \\ \mathrm{Sym}^d(L^*) \otimes \mathrm{Sym}^{\ell+1}(L) \otimes \mathrm{Sym}^{\ell-d-1}(V) & 0 \le d \lneq k \end{cases}$$

left P-mod

$$P^\ell(\mathcal{O}(d)) = \begin{cases} \mathrm{Sym}^d(L) \otimes \mathrm{Sym}^{\ell+d+1}(L) \otimes \mathrm{Sym}^{\ell-d-1}(V) & 0 \le d < k \\ \mathrm{Sym}^d(L^*) \otimes \mathrm{Sym}^{d-\ell-1}(L^*) \otimes \mathrm{Sym}^{\ell-1}(V) & d \geqslant 0 \end{cases}$$

on right P-mod

Anything works on $G(m,n)$ ...

By the work of Reznikov/Mazur/Deninger...: (Arithmetic topology)
primes in rings of integers of number fields
behave like knots in 3-manifolds; maybe these new
invariants can shed light on this mysterious behaviour?
(explain)

Generalization: the sheaf of differential oper. D_X: filtered by order

$$\begin{tikzcd}
 & D_X \arrow[ddd] \\
\mathcal{O} \arrow[ur, dashed] & \\
D_X^{\le 1} \arrow[u, phantom, "\supseteq"] \arrow[ddr] & \\
\mathrm{Der}_X \arrow[u, phantom, "\subseteq"] \arrow[r, "\nabla"'] & \mathrm{End}(\mathcal{E})
\end{tikzcd}$$

a map $\nabla : D_X \to \mathrm{End}(\mathcal{E})$:
$\nabla(fe) = f\nabla(e) + \partial(f)e$? a stratification on $\mathcal{E}$.

thm: (Grothendieck): k: alg closed $\mathcal{E}$: bundle on $\mathbb{P}^n_k$ w. stratification: a trivial.

in char p: D_X: filtered by sub-rings $D_X^{p^r}$ for
all powers of the prime p:
a: $\nabla : \mathrm{Der}_X^{\le 1} \to \mathrm{End}(\mathcal{E})$ ... are
$U(L,A) \subseteq D_X^{p}$:

we only get action of subalg. how do we repair
$U(L,X) \subseteq D_X^{p}$ this? Answer: the
Jets.

$$\begin{tikzcd}
P^k(E) \arrow[d] & \\
P^{k-1}(E) & \\
P^1(E) \arrow[r] & E \arrow[l, bend right, "s_1"'] \arrow[ul, bend right, "s_{k-1}"'] \arrow[uul, bend right, "s_k"']
\end{tikzcd}$$

$$\begin{tikzcd}
E\otimes P^m\otimes P^n \arrow[r] & E\otimes P^n\otimes P^m \\
E \arrow[u] \arrow[r] \arrow[ur, "K^{m,n}"] & E\otimes P^n\otimes P^m \arrow[u]
\end{tikzcd}$$

$\{K^{m,n}; S_i\}$ gives rise the action

$$\mathrm{Diff}^k \longrightarrow \mathrm{End}(E)$$

$K^{1,1}: E \to E\otimes P^1\otimes P^1$: curvature of a connection.

$K^{m,n}$: generalized curvature = functorial

Question: ① Can we generalize the de Rham complex to give new cohomology-theories in which these [illegible] cocycle-condition $K^{n,n}$ gives rise to invariants generalizing the chern-classes of E?

② why? for a bundle E on P^n_k by Grothendieck's theorem: such invariants (if they exist) will measure how a bundle deviates from being free, and we know by Mumford/Horrocks example such bundles do exist.