DECOMPOSABLE QUADRATIC FORMS AND INVOLUTIONS

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ABSTRACT. In his book on compositions of quadratic forms, Shapiro asks whether a quadratic form decomposes as a tensor product of quadratic forms when its adjoint involution decomposes as a tensor product of involutions on central simple algebras. We give a positive answer for quadratic forms defined over local or global fields and produce counterexamples over fields of rational fractions in two variables over any formally real field.

1. INTRODUCTION

Every nondegenerate symmetric or skew-symmetric bilinear form b on a finite-dimensional vector space V induces an involution (i.e. an anti-automorphism of period 2) on the endomorphism algebra $\operatorname{End} V$. This involution, known as the *adjoint involution of* b and denoted by ad_b, is characterized by the following property:

$$b(x, f(y)) = b(\operatorname{ad}_b(f)(x), y) \qquad \text{for } x,\, y \in V \text{ and } f \in \operatorname{End} V.$$

If the form b is the tensor product of two nondegenerate symmetric or skew-symmetric bilinear forms, i.e.

$$V = V_1 \otimes V_2 \qquad \text{and} \qquad b = b_1 \otimes b_2,$$

then it is easy to see that the adjoint involution also decomposes,

$$\operatorname{End} V = (\operatorname{End} V_1) \otimes (\operatorname{End} V_2) \qquad \text{and} \qquad \operatorname{ad}_b = \operatorname{ad}_{b_1} \otimes \operatorname{ad}_{b_2},$$

(see [12, Corollary 6.10]), which we denote

$$(\operatorname{End} V, \operatorname{ad}_b) = (\operatorname{End} V_1 \otimes \operatorname{End} V_2, \operatorname{ad}_{b_1} \otimes \operatorname{ad}_{b_2}).$$

In [12, p. 201], Shapiro raises the following question: if the adjoint involution has a nontrivial decomposition of the form

$$(\operatorname{End} V, \operatorname{ad}_b) \simeq (A_1, \sigma_1) \otimes (A_2, \sigma_2)$$

for certain involutions σ_1, σ_2 on central simple algebras A_1, A_2, does it follow that b decomposes nontrivially into a tensor product of bilinear forms? He showed that the answer is positive if A_1 and A_2 are split, see [12, Corollary 6.10] or Proposition 3.1 below.

In this paper, we give an affirmative answer to Shapiro's question over certain fields of characteristic different from 2, including local and global fields, and give

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examples where the answer is negative. Our examples are involution trace forms of central simple algebras of degree $2m$, where m is an odd integer, see Subsection 3.2.

Throughout the paper, the characteristic of the base field F is assumed to be different from 2. Therefore, symmetric bilinear forms correspond bijectively to quadratic forms, and this correspondence is used to define the tensor product of quadratic forms. For any (nondegenerate) quadratic form q, we write ad_q for the adjoint involution of the polar form of q.

In Section 2, we collect results on factorizations of quadratic forms. In particular, we determine the decomposable forms of dimension at most 8. The decomposition of involutions is discussed in Section 3.

2. Decomposable quadratic forms

All the quadratic forms considered in this paper are nondegenerate. A quadratic form q over a field F is called *decomposable* if there is a factorization $q \simeq q_1 q_2$ where q_1 and q_2 are quadratic forms with $\dim q_1, \dim q_2 \geq 2$. In particular, the hyperbolic forms of dimension $n \geq 4$ are decomposable since they are isometric to

$$\langle 1, -1\rangle \otimes \left(\tfrac{n}{2}\langle 1\rangle\right).$$

From the outset, note that the notion of decomposable form is not compatible with Witt equivalence, since every quadratic form q is Witt-equivalent to the decomposable form $q \otimes \langle 1, 1, -1\rangle$.

After reviewing below some general properties of factorizations of quadratic forms, we consider factorizations of forms of small dimension, and determine the indecomposable forms over some special fields.

2.1. General properties.

Lemma 2.1. *If q_1, q_2 are even-dimensional quadratic forms, the discriminant and Clifford invariant of the product q_1q_2 satisfy*

$$\mathrm{disc}(q_1q_2) = 1 \qquad \textit{and} \qquad c(q_1q_2) = (\mathrm{disc}\, q_1, \mathrm{disc}\, q_2)_F.$$

Proof. For $i = 1, 2$, let $d_i \in F^\times$ be such that $\mathrm{disc}\, q_i = d_i F^{\times 2}$. Since q_i is even-dimensional, we have $q_i \equiv \langle 1, -d_i\rangle \bmod I^2F$, hence

$$q_1q_2 \equiv \langle 1 - d_1\rangle\langle 1, -d_2\rangle \bmod I^3F.$$

Therefore,

$$\mathrm{disc}(q_1q_2) = \mathrm{disc}(\langle 1, -d_1\rangle\langle 1, -d_2\rangle) = 1$$

and

$$c(q_1q_2) = c(\langle 1, -d_1\rangle\langle 1, -d_2\rangle) = (d_1, d_2)_F.$$

□

For any quadratic form q over F, let q_a denote the anisotropic kernel of q, and let

$$\dim_a q = \dim q_a.$$

Lemma 2.2. *If a quadratic form q satisfies $\dim q = n_1 n_2$ for some integers n_1, n_2 subject to*

$$n_1 \equiv \dim_a q \bmod 2, \qquad n_2 \equiv 1 \bmod 2, \tag{1}$$

$$n_1 \geq \dim_a q, \qquad n_1, n_2 \geq 2, \tag{2}$$

then q is decomposable.

Proof. Letting $k_1 = \frac{1}{2}(n_1 - \dim_a q)$ and $k_2 = \frac{1}{2}(n_2 - 1)$, we have

$$q = (q_a \perp k_1\langle 1, -1\rangle) \otimes (\langle 1\rangle \perp k_2\langle 1, -1\rangle).$$

□

Besides these (essentially trivial) factorizations, a main source of decompositions is the following well-known result (due to Pfister):

Lemma 2.3. *An anisotropic quadratic form q splits over a quadratic extension $F(\sqrt{d})$ (i.e., becomes hyperbolic over $F(\sqrt{d})$) if and only if $q \simeq \langle 1, -d\rangle \otimes q'$ for some quadratic form q'.*

Proof. See [11, Chapter 2, Theorem 5.2]. □

Corollary 2.4. *Let q be a decomposable quadratic form of dimension $\dim q = 2p$ for some odd prime number p. If $\operatorname{disc} q = 1$, then q is hyperbolic. If $\operatorname{disc} q \neq 1$, then $\dim_a q \equiv 2 \bmod 4$ and q splits over $F(\sqrt{\operatorname{disc} q})$.*

Proof. Let $q = q_1 q_2$ be a nontrivial decomposition of q. We may assume $\dim q_1 = 2$ and $\dim q_2 = p$. Comparing discriminants, we obtain $\operatorname{disc} q = \operatorname{disc} q_1$ since p is odd. Therefore, if $\operatorname{disc} q = 1$, then q_1 is hyperbolic, hence q also is hyperbolic. If $\operatorname{disc} q = dF^{\times 2} \neq 1$, then $F(\sqrt{d})$ splits q_1, hence also q and q_a. By Lemma 2.3, it follows that $q_a \simeq \langle 1, -d\rangle \otimes q'$ for some quadratic form q'. If $\dim q'$ is even, then $\operatorname{disc} q_a = 1$, a contradiction since $\operatorname{disc} q = \operatorname{disc} q_a$. Therefore, $\dim_a q \equiv 2 \bmod 4$. □

The results above yield necessary and sufficient conditions for the decomposability of forms q with $\dim_a q \leq 4$, as we now show.

Proposition 2.5. *A quadratic form q of odd dimension is decomposable whenever its dimension has a nontrivial factorization $\dim q = n_1 n_2$ with $n_1 \geq \dim_a q$. In particular, if $\dim_a q \leq 3$, then q is decomposable if and only if $\dim q$ is neither 1 nor a prime number.*

Proof. If $\dim q$ is odd, then $\dim_a q$ is odd and the conditions (1) hold for any factorization $\dim q = n_1 n_2$. The proposition follows from Lemma 2.2. □

Proposition 2.6. *If $\dim_a q = 2$, then q is indecomposable if and only if $\dim q$ is a power of 2.*

Proof. Note that $\dim q \equiv \dim_a q \bmod 2$, so the hypothesis implies $\dim q$ is even. If it has an odd prime factor p, let $n_1 = (\dim q)/p$ and $n_2 = p$. Then n_1 is even, so q is decomposable by Lemma 2.2. Therefore, q is decomposable if $\dim q$ is not a power of 2.

Conversely, if $\dim q$ is a power of 2, then every nontrivial factorization of q has the form $q = q_1 q_2$ with $\dim q_1$, $\dim q_2$ even. Then $\operatorname{disc} q = 1$ by Lemma 2.1, which is impossible since $\dim_a q = 2$. □

Proposition 2.7. *If $\dim_a q = 4$, then q is indecomposable if and only if one of the following conditions holds:*

(a) $\dim q = 2p$ *for some odd prime p,*
(b) $\dim q$ *is a power of* 2 *and* $\operatorname{disc} q \neq 1$.

Proof. Since $\dim_a q = 4$, it follows that $\dim q$ is even. In case (a), q is indecomposable by Corollary 2.4. If $\dim q$ is a power of 2, then every nontrivial factorization of q is into a product of forms of even dimension. The existence of such a factorization implies $\operatorname{disc} q = 1$ by Lemma 2.1, hence q is indecomposable in case (b). Conversely, Lemma 2.2 shows that q is decomposable whenever $\dim q$ has a factorization $\dim q = n_1 n_2$ with n_1 even, n_2 odd and $n_1 \geq 4$, $n_2 \geq 3$. Since $\dim q$ is even, such a factorization exists unless $\dim q$ is a power of 2 or of the form $2p$ for some odd prime p. To prove that either (a) or (b) holds when q is indecomposable, it only remains to show that q is decomposable if $\dim q$ is a power of 2 and $\operatorname{disc} q = 1$. In this case, q_a has a diagonalization

$$q_a = \langle a, b, c, d \rangle$$

with $abcd \in F^{\times 2}$, so

$$q_a = \langle a, b \rangle \otimes \langle 1, ac \rangle$$

and, for $k = \frac{1}{4}(\dim q) - 1$,

$$q = \langle a, b \rangle \otimes \big(\langle 1, ac \rangle \perp k \langle 1, -1 \rangle \big).$$

□

2.2. **Forms of small dimension.** We may also use Propositions 2.6 and 2.7, together with Corollary 2.4, to discuss the decomposability of forms of dimension at most 8. Of course, quadratic forms whose dimension is 1 or a prime number are indecomposable. Therefore, we need to consider only forms of dimension 4, 6 or 8.

Proposition 2.8. *A quadratic form q of dimension 4 is indecomposable if and only if one of the following conditions holds:*

(a) $\dim_a q = 2$,
(b) $\dim_a q = 4$ *and* $\operatorname{disc} q \neq 1$.

Proof. This readily follows from Propositions 2.6 and 2.7. □

Proposition 2.9. *A quadratic form q of dimension 6 is indecomposable if and only if one of the following conditions holds:*

(a) $\dim_a q = 4$,
(b) $\dim_a q = 6$ *and* $\operatorname{disc} q = 1$,
(c) $\dim_a q = 6$, $\operatorname{disc} q \neq 1$ *and q is not split by* $F(\sqrt{\operatorname{disc} q})$.

Proof. The form q is decomposable if $\dim_a q = 2$, by Proposition 2.6, and indecomposable if $\dim_a q = 4$, by Proposition 2.7. Therefore, it only remains to consider the case where $\dim_a q = 6$, i.e., q is anisotropic. Corollary 2.4 shows that q is indecomposable in cases (b) and (c), and Lemma 2.3 proves that q is decomposable if $\operatorname{disc} q \neq 1$ and q is split by $F(\sqrt{\operatorname{disc} q})$. □

Proposition 2.10. *A quadratic form q of dimension 8 is indecomposable if and only if one of the following conditions holds:*

(a) $\dim_a q = 2$,
(b) $\dim_a q = 4$ *and* $\operatorname{disc} q \neq 1$,
(c) $\dim_a q = 6$,
(d) $\dim_a q = 8$ *and q is not split by any quadratic extension of F.*

Proof. If $\dim_a q = 2$, then q is indecomposable by Proposition 2.6. If $\dim_a q = 4$, then q is indecomposable if and only if $\operatorname{disc} q \neq 1$, by Proposition 2.7. Suppose next $\dim_a q = 6$. If $\operatorname{disc} q \neq 1$, then q is not decomposable by Lemma 2.1. If $\operatorname{disc} q = 1$, then q is Witt-equivalent to an anisotropic Albert form, hence its Clifford invariant has index 4, by [5]. On the other hand, forms which decompose into a product of forms of even dimension have a Clifford invariant of index at most 2 by Lemma 2.1; therefore q is indecomposable. Finally, the case where $\dim_a q = 8$ follows from Lemma 2.3. □

2.3. **Forms over special fields.** Recall from [11, Chapter 2,§16] that the u-invariant of a field F is the supremum of the dimensions of anisotropic quadratic forms over F.

Corollary 2.11. *If $u(F) \leq 4$, a quadratic form q over F is indecomposable if and only if one of the following conditions holds:*

(a) $\dim q$ *is a prime number,*
(b) $\dim_a q = 2$ *and* $\dim q$ *is a power of* 2*,*
(c) $\dim_a q = 4$ *and* $\dim q = 2p$ *for some odd prime* p*,*
(d) $\dim_a q = 4$*,* $\dim q$ *is a power of* 2 *and* $\operatorname{disc} q \neq 1$*.*

In particular, every quadratic form q with $\dim q \equiv 0 \bmod 4$ *and* $\operatorname{disc} q = 1$ *is decomposable.*

Proof. The hypothesis $u(F) \leq 4$ implies $\dim_a q \leq 4$. The corollary therefore readily follows from Propositions 2.5, 2.6 and 2.7. For the last statement, observe that $\dim_a q \neq 2$ if $\operatorname{disc} q = 1$. □

The corollary applies for instance to p-adic fields or to non-formally real number fields. Note that case (d) does not arise for p-adic fields since every anisotropic quadratic form of dimension 4 then has trivial discriminant.

Suppose now F is a real-closed field. Quadratic forms over F are classified by their dimension and signature, which are related by

$$|\operatorname{sgn} q| \leq \dim q \qquad \text{and} \qquad \operatorname{sgn} q \equiv \dim q \bmod 2.$$

Proposition 2.12. *A quadratic form q over a real-closed field F is decomposable if and only if* $\dim q$ *and* $\operatorname{sgn} q$ *have factorizations of the form*

$$\dim q = n_1 n_2, \qquad \operatorname{sgn} q = s_1 s_2$$

with

$$n_i \geq 2, \qquad |s_i| \leq n_i \quad \textit{and} \quad s_i \equiv n_i \bmod 2 \quad \textit{for } i = 1, 2.$$

Proof. If $q = q_1 q_2$ is a nontrivial decomposition, then

$$\dim q = \dim q_1 \dim q_2 \qquad \text{and} \qquad \operatorname{sgn} q = \operatorname{sgn} q_1 \operatorname{sgn} q_2$$

are factorizations of the required form. Conversely, given factorizations as in the statement of the proposition, one may find quadratic forms q_1, q_2 over F with $\dim q_i = n_i$ and $\operatorname{sgn} q_i = s_i$. The forms q and $q_1 q_2$ have the same dimension and signature, hence $q = q_1 q_2$. □

Corollary 2.13. *If a quadratic form q over a real-closed field F satisfies*

$$\dim q \equiv 0 \bmod 4 \qquad \textit{and} \qquad \operatorname{disc} q = 1,$$

then $q = \langle 1, 1\rangle \otimes q'$ for some quadratic form q'. In particular, q is decomposable.

Proof. Since $\dim q$ is even and $\operatorname{disc} q = 1$, it follows that $\operatorname{sgn} q \equiv 0 \bmod 4$. Therefore, we may apply the proposition above with $n_1 = s_1 = 2$. $\square$

The case of quadratic forms of dimension divisible by 4 and trivial discriminant is particularly relevant for Shapiro's question, see Section 3. The following result (of which Corollary 2.13 is a special case) gives further examples where these forms have a nontrivial decomposition.

Proposition 2.14. *Suppose the field F satisfies the following properties:*

- *F is linked, i.e., every tensor product of quaternion F-algebras is Brauer-equivalent to a quaternion algebra,*
- *every quaternion F-algebra is split by a quadratic extension K/F such that $I^3K = 0$.*

Then every quadratic form q over F such that $\dim q \equiv 0 \bmod 4$ and $\operatorname{disc} q = 1$ decomposes as $q = \langle 1, a\rangle \otimes q'$ for some $a \in F^\times$ and some quadratic form q'. If F is formally real, the element a may be chosen to be totally positive, i.e., $a >_P 0$ for every ordering P of F.

Proof. Since F is linked, the Clifford invariant $c(q)$ is Brauer-equivalent to a quaternion F-algebra. Let K/F be a quadratic extension with $I^3K = 0$ which splits $c(q)$. As was shown by Elman and Lam [3, Theorem 3.11], quadratic forms over K are classified by their dimension, discriminant and Clifford invariant, hence q splits over K. If $K = F(\sqrt{d})$, Lemma 2.3 yields

$$q_a = \langle 1, -d\rangle \otimes q_0$$

for some quadratic form q_0. Since $\operatorname{disc} q = \operatorname{disc} q_a = 1$, it follows that $\dim q_0$ is even, hence $\dim q_a \equiv 0 \bmod 4$. Therefore, the Witt index $i_W(q)$ is even. Letting $i_W(q) = 2m$, we have

$$q = q_a \perp 2m\langle 1, -1\rangle = \langle 1, -d\rangle \otimes (q_0 \perp m\langle 1, -1\rangle).$$

We obtain the required factorization with $a = -d$ and $q' = q_0 \perp m\langle 1, -1\rangle$. Note that the condition $I^3K = 0$ implies K is not formally real, hence $d <_P 0$ for every ordering P of F (if any). $\square$

Recall that the virtual cohomological 2-dimension of a field F is the cohomological 2-dimension of $F(\sqrt{-1})$. Fields with virtual cohomological 2-dimension at most 1 satisfy the conditions in Proposition 2.14 since $I^2F(\sqrt{-1}) = 0$.

To obtain further examples, we use the Effective Diagonalization Property (ED), which was characterized in various ways by Prestel and Ware [9]. For instance, a field F is ED if and only if for every valuation v with formally real residue field $\overline{F}$, the group $2v(F^\times)$ has index at most 2 in $v(F^\times)$, and $\overline{F}$ is euclidean when $2v(F^\times) \neq v(F^\times)$. This readily shows that number fields are ED.

Lemma 2.15. *Every quaternion division algebra over an ED field F is split by a non-formally real quadratic extension of F.*

Proof. Let $Q = (a, b)_F$ be a quaternion division F-algebra. Since F is ED, Theorem 2 of [9] shows that the form $\langle 1, a, b, -ab\rangle$ is almost isotropic, which means that for every diagonalization $\langle 1, a, b, -ab\rangle \simeq \langle c_1, c_2, c_3, c_4\rangle$, there is an integer n such that $n\langle c_1\rangle \perp \langle c_2, c_3, c_4\rangle$ is isotropic. In particular, there is an integer n such that $n\langle 1\rangle \perp \langle a, b, -ab\rangle$ is isotropic, hence $\langle a, b, -ab\rangle$ represents an element $u \in F^\times$ such that $-u$ is a sum of squares. The algebra Q is then split by $F(\sqrt{u})$, which is a non-formally real quadratic extension of F. $\square$

Corollary 2.16. *If a quadratic form q over a global field F satisfies*

$$\dim q \equiv 0 \bmod 4 \qquad \textit{and} \qquad \operatorname{disc} q = 1,$$

then $q = \langle 1, a\rangle \otimes q'$ for some totally positive $a \in F^\times$ and some quadratic form q'. In particular, q is decomposable.

Proof. Global fields are linked and ED, and non-formally real global fields K satisfy $I^3K = 0$. Therefore, the corollary follows from Proposition 2.14 and Lemma 2.15. $\square$

3. Decomposable adjoint involutions

Let q be a quadratic form on an F-vector space V and let ad_q be the adjoint involution on $\operatorname{End}_F V$. In this section, we consider decompositions of algebras with involution

$$(\operatorname{End}_F V, \mathrm{ad}_q) \simeq (A_1, \sigma_1) \otimes_F (A_2, \sigma_2) \tag{3}$$

where A_1, A_2 are central simple F-algebras and σ_1, σ_2 are involutions on A_1 and A_2 respectively. Since $\operatorname{End}_F V$ represents the trivial element in the Brauer group of F, the Brauer classes of A_1 and A_2 are opposite. However, σ_1 yields an isomorphism between A_1 and its opposite algebra, hence A_1 and A_2 are also Brauer-equivalent. Since ad_q is an orthogonal involution, the involutions σ_1 and σ_2 must be both orthogonal or both symplectic, by [12, Proposition 6.9] or [5, (2.23)].

In [12, p. 201], Shapiro asks whether a nontrivial decomposition as in (3) implies that q is decomposable.

3.1. **Positive results on Shapiro's question.** The case where A_1 and A_2 are split and σ_1, σ_2 are orthogonal is discussed in Shapiro's book [12]:

Proposition 3.1. *Let q, q_1, q_2 be quadratic forms on F-vector spaces V, V_1, V_2. The following statements are equivalent:*

(a) $(\operatorname{End}_F V, \mathrm{ad}_q) \simeq (\operatorname{End}_F V_1, \mathrm{ad}_{q_1}) \otimes_F (\operatorname{End}_F V_2, \mathrm{ad}_{q_2})$;
(b) *there exists $\lambda \in F^\times$ such that $q \simeq \langle\lambda\rangle q_1 q_2$.*

Proof. See [12, Corollary 6.10, p. 112]. $\square$

Corollary 3.2. *Suppose that one of the algebras A_1, A_2 in* (3) *is split. Then A_1 and A_2 are both split and the quadratic form q is decomposable if* $\deg A_1$, $\deg A_2 > 1$. *Moreover, if σ_1 and σ_2 are symplectic, then q is hyperbolic.*

Proof. The first part is clear, since A_1 and A_2 are Brauer-equivalent. Let $A_1 = \operatorname{End}_F V_1$ and $A_2 = \operatorname{End}_F V_2$ for some F-vector spaces V_1, V_2. If σ_1 and σ_2 are orthogonal, then there exist quadratic forms q_1 on V_1 and q_2 on V_2 such that $\sigma_1 = \mathrm{ad}_{q_1}$ and $\sigma_2 = \mathrm{ad}_{q_2}$. By Proposition 3.1, it follows from (3) that $q = \langle\lambda\rangle q_1 q_2$ for some $\lambda \in F^\times$, hence q is decomposable if $\dim V_1$, $\dim V_2 > 1$, i.e. if $\deg A_1$, $\deg A_2 > 1$.

If σ_1 and σ_2 are symplectic, then they are adjoint to some skew-symmetric bilinear forms b_1 on V_1 and b_2 on V_2. By [12, Corollary 6.10], the polar form b_q of q satisfies $b_q \simeq \lambda b_1 \otimes b_2$ for some $\lambda \in F^\times$, hence q is hyperbolic. Since hyperbolic forms of dimension at least 4 are decomposable, the proof is complete. $\square$

We now use results from Section 2 to derive from (3) the decomposability of q over special fields.

Proposition 3.3. *Suppose the adjoint involution* ad_q *decomposes as in* (3) *with* $\deg A_1$, $\deg A_2 > 1$. *Then the quadratic form q is decomposable if the field F satisfies any of the following conditions:*

- *the Brauer group of F is trivial;*
- $u(F) \leq 4$;
- *F is real-closed;*
- *F is a global field.*

Proof. Whenever A_1 and A_2 are split—which is certainly the case if the Brauer group of F is trivial—the decomposability of q follows from Corollary 3.2. We may therefore restrict our discussion to the case where A_1 and A_2 are not split. The Brauer class of A_1 and A_2 then has order 2, hence $\deg A_1$ and $\deg A_2$ are even, and therefore $\dim q \equiv 0 \bmod 4$. Moreover, by [12, Lemma 10.25] or [5, (7.3)], $\mathrm{disc}\, q = 1$. The decomposability of q follows from Corollary 2.11 if $u(F) \leq 4$, from Corollary 2.13 if F is real-closed, and from Corollary 2.16 if F is global. □

The decomposability of q can also be derived from certain types of decompositions of ad_q:

Proposition 3.4. *Suppose the adjoint involution* ad_q *decomposes as in* (3) *with* $\deg A_1$, $\deg A_2 > 1$. *Then the quadratic form q is decomposable in each of the following cases:*

(a) *A_1 and A_2 are Brauer-equivalent to a quaternion F-algebra Q and σ_1, σ_2 are symplectic;*
(b) $\deg A_1 = 2$ *and σ_1, σ_2 are orthogonal;*
(c) $\deg A_1 = 4$ *and σ_1, σ_2 are symplectic;*
(d) $\deg A_1 = 4$, *σ_1, σ_2 are orthogonal and* $\mathrm{disc}\, \sigma_1 = 1$;
(e) $\deg A_1 = 8$, *σ_1, σ_2 are symplectic and A_1 is not a division algebra;*
(f) $\deg A_1 = 8$, *σ_1, σ_2 are orthogonal,* $\mathrm{disc}\, \sigma_1 = 1$ *and one of the factors of the Clifford algebra $C(A_1, \sigma_1)$ is split.*

Moreover, in case (a) *the form q is divisible by the norm form n_Q of Q, and in case* (b) *it is divisible by the form* $\langle 1, -\mathrm{disc}\, \sigma_1 \rangle$.

Proof. In case (a), we may assume Q is a division algebra, since the proposition follows from Corollary 3.2 if A_1 and A_2 are split. Let γ be the conjugation involution on Q. Since σ_1 and σ_2 are symplectic, it follows from [12, Proposition, p. 202] or [1, Proposition 3.4] that there exist quadratic forms q_1', q_2' on F-vector spaces V_1', V_2' such that

$$(A_1, \sigma_1) \simeq (Q, \gamma) \otimes_F (\mathrm{End}_F V_1', \mathrm{ad}_{q_1'}), \qquad (A_2, \sigma_2) \simeq (Q, \gamma) \otimes_F (\mathrm{End}_F V_2', \mathrm{ad}_{q_2'}).$$

On the other hand, by [5, (11.1)], we have

$$(Q, \gamma) \otimes_F (Q, \gamma) \simeq (\mathrm{End}_F Q, \mathrm{ad}_{n_Q}).$$

Therefore, (3) implies

$$(\mathrm{End}_F V, \mathrm{ad}_q) \simeq (\mathrm{End}_F Q, \mathrm{ad}_{n_Q}) \otimes_F (\mathrm{End}_F V_1', \mathrm{ad}_{q_1'}) \otimes_F (\mathrm{End}_F V_2', \mathrm{ad}_{q_2'}).$$

It follows by [12, Corollary 6.10] that $q \simeq \langle \lambda \rangle n_Q q_1' q_2'$ for some $\lambda \in F^\times$, hence q is decomposable if $\dim V_1'$ or $\dim V_2'$ is at least 2. If $\dim V_1' = \dim V_2' = 1$, then q is a multiple of n_Q hence it is decomposable by Proposition 2.8.

In case (b), the quaternion algebra with involution (A_1, σ_1) becomes hyperbolic over $F(\sqrt{\operatorname{disc}\sigma_1})$, hence q also becomes hyperbolic over this extension. By Lemma 2.3, there exists a quadratic form q' such that

$$q_a \simeq \langle 1, -\operatorname{disc}\sigma_1\rangle \otimes q'.$$

Since $\operatorname{disc} q = 1$ by [12, Lemma 10.25] or [5, (7.3)], the dimension of q' is even. Therefore, the Witt index $i_W(q)$ is even. Letting $i_W(q) = 2m$, we have

$$q = q_a \perp 2m\langle 1, -1\rangle \simeq \langle 1, -\operatorname{disc}\sigma_1\rangle \otimes (q' \perp m\langle 1, -1\rangle).$$

Cases (c)—(f) reduce to (a) or (b) since in each case there is a decomposition of the form

$$(A_1, \sigma_1) \simeq (Q, \sigma) \otimes_F (A', \sigma')$$

for some quaternion F-algebra Q: see [12, Proposition 10.21] for case (c), [12, Proposition 10.26] for case (d), [2, Theorem 7] for case (e) and [5, (42.11)] for case (f). □

3.2. **Negative results on Shapiro's question.** For any central simple algebra with orthogonal or symplectic involution (A, σ) over a field F, the involution trace form $T_\sigma\colon A \to F$ is defined by

$$T_\sigma = \operatorname{Trd}_A(\sigma(x)x),$$

where Trd_A is the reduced trace. By [5, (11.1)], the map $\sigma_*\colon A \otimes_F A \to \operatorname{End}_F A$ defined by $\sigma_*(a \otimes b)(x) = ax\sigma(b)$ for a, b, $x \in A$ carries the involution $\sigma \otimes \sigma$ on $A \otimes_F A$ to the adjoint involution of T_σ, so

$$(\operatorname{End}_F A, \operatorname{ad}_{T_\sigma}) \simeq (A, \sigma) \otimes (A, \sigma).$$

Therefore, $\operatorname{ad}_{T_\sigma}$ is decomposable if $\deg A > 1$. However, we show in this subsection that the quadratic form T_σ is indecomposable for certain orthogonal involutions σ on central simple algebras A of degree $\deg A \equiv 2 \bmod 4$, providing a negative solution to Shapiro's question.

Remark. If the base field F is a global field (or a real-closed field, or satisfies $u(F) \le 4$), Proposition 3.3 shows that the form T_σ is decomposable when $\deg A > 1$, since its adjoint involution is decomposable.

We start our construction of examples where T_σ is indecomposable with a lemma from elementary number theory.

Lemma 3.5. *Suppose m, k, l, r, $s \ge 1$ are integers such that*

$$m^2 = kl, \qquad (m-1)^2 = rs, \tag{4}$$

$$k \ge r \ge 1, \qquad l \ge s \ge 1. \tag{5}$$

If $m \equiv s \equiv 1 \bmod 2$, then $l = 1$.

Proof. The hypotheses show that l is odd. Let $k - r = a$ and $l - s = 2b$. By (5), we have $k > a \ge 0$ and $l > 2b \ge 0$, hence, by the first equation in (4),

$$m^2 > 2ab \ge 0. \tag{6}$$

By definition of a and b,

$$2ab = (k-r)(l-s) = (k-r)l + k(l-s) + rs - kl,$$

hence, by (4),

$$2ab = al + 2bk + (m-1)^2 - m^2.$$

It follows that

$$al + 2bk = 2m - 1 + 2ab,$$

hence, using the first equation in (4),

$$(al - 2bk)^2 = (al + 2bk)^2 - 8abkl = (2m - 1 + 2ab)^2 - 8abm^2.$$

This equation shows that

$$(2m - 1 + 2ab)^2 \geq 8abm^2,$$

hence

$$(2ab - 1)(2ab - 1 + 4m - 4m^2) \geq 0. \tag{7}$$

By (6) we have

$$2ab - 1 + 4m - 4m^2 < 4m - 1 - 3m^2,$$

and the right side is easily seen to be nonpositive for all $m \geq 1$. Therefore, (7) yields $2ab \leq 1$, hence $a = 0$ or $b = 0$ since a and b are nonnegative integers. If $a = 0$, then (4) yields

$$m^2 = kl \qquad \text{and} \qquad (m-1)^2 = ks.$$

Since m^2 and $(m-1)^2$ are relatively prime, it follows that $k = 1$, hence $s = (m-1)^2 \equiv 0 \bmod 2$, a contradiction. Therefore, $b = 0$ and (4) implies $l = 1$ since m^2 and $(m-1)^2$ are relatively prime. □

Theorem 3.6. *Let Q be a quaternion algebra over a field F and let m be an odd integer. If an orthogonal involution σ on the matrix algebra $M_m(Q)$ satisfies the following conditions:*

(a) *the biquaternion algebra $Q \otimes_F (\operatorname{disc}\sigma, -1)_F$ is division,*
(b) *there is an ordering P on F such that $\operatorname{sgn}_P T_\sigma = 4(m-1)^2$,*

then T_σ is an indecomposable form.

Proof. Suppose $T_\sigma = q_1 q_2$ for some quadratic forms q_1, q_2. We use condition (a) to show that $\dim q_1$ and $\dim q_2$ cannot be both even, and condition (b) to show that the decomposition is trivial if one of $\dim q_1$, $\dim q_2$ is odd.

Suppose first $\dim q_1 \equiv \dim q_2 \equiv 0 \bmod 2$. By Lemma 2.1, the Clifford invariant $c(q_1 q_2)$ has index at most 2. On the other hand, the Hasse invariant of T_σ was computed by Lewis in [6] and Quéguiner in [10]. From their computation, and from the relations between the Hasse and the Clifford invariant (see [11, p. 81]), it follows that $c(T_\sigma)$ is represented by the biquaternion algebra $Q \otimes_F (\operatorname{disc}\sigma, -1)_F$, which has index 4 if condition (a) holds. Therefore, this case leads to a contradiction.

Suppose next that $\dim q_2$ is odd. Since $\dim T_\sigma = 4m^2$, the dimension of q_1 is a multiple of 4. Let $\dim q_1 = 4k$ and $\dim q_2 = l$, so

$$m^2 = kl, \qquad k, l, m \geq 1 \text{ and } m \equiv 1 \bmod 2.$$

Comparing signatures at P, we also have

$$\operatorname{sgn}_P T_\sigma = 4(m-1)^2 = \operatorname{sgn}_P q_1 \operatorname{sgn}_P q_2,$$

with $\operatorname{sgn}_P q_1 \equiv \dim q_1 \equiv 0 \bmod 2$ and $\operatorname{sgn}_P q_2 \equiv \dim q_2 \equiv 1 \bmod 2$. Therefore, $\operatorname{sgn}_P q_1$ is a multiple of 4. Letting $|\operatorname{sgn}_P q_1| = 4r$ and $|\operatorname{sgn}_P q_2| = s$, we have

$$(m-1)^2 = rs, \qquad r, s \geq 1 \text{ and } s \equiv 1 \bmod 2.$$

Lemma 3.5 shows that $l = 1$, so the decomposition $T_\sigma = q_1 q_2$ is trivial. □

Let $\overline{}$ be the conjugation involution on a quaternion division F-algebra Q. For $x \in M_m(Q)$, we define $\overline{x}$ by letting $\overline{}$ act on each entry of x, so the map $x \mapsto \overline{x}^t$ is a symplectic involution on $M_m(Q)$. To obtain orthogonal involutions, consider diagonal matrices of pure quaternions

$$\Delta = \operatorname{diag}(\alpha_1, \dots, \alpha_m)$$

with $\overline{\alpha_i} = -\alpha_i \neq 0$ for $i = 1, \dots, m$, and define $\sigma_\Delta \colon M_m(Q) \to M_m(Q)$ by

$$\sigma_\Delta(x) = \Delta^{-1} \cdot \overline{x}^t \cdot \Delta \qquad \text{for } x \in M_m(Q). \tag{8}$$

Since $\overline{\Delta}^t = -\Delta$, the map σ_Δ is an orthogonal involution, see [5, (2.7)]. Moreover, since every orthogonal involution on the endomorphism algebra of a finite-dimensional Q-vector space is adjoint to some skew-hermitian form with respect to $\overline{}$ (see [5, (4.2)]), every orthogonal involution on $M_m(Q)$ is conjugate to an involution σ_Δ for some Δ as above.

Proposition 3.7. *The discriminant of σ_Δ and the signature of T_{σ_Δ} at an ordering P of F are determined as follows:*

(a) $\operatorname{disc} \sigma_\Delta = \alpha_1^2 \dots \alpha_m^2 F^{\times 2} \in F^\times / F^{\times 2}$.

(b) *Let F_P be a real closure of F at P. If Q is not split by F_P, then*

$$\operatorname{sgn}_P T_{\sigma_\Delta} = 0.$$

If Q is split by F_P, choose a pure quaternion $\beta \in Q^\times$ such that $\beta^2 <_P 0$, let m_1 be the number of indices i such that $\alpha_i^2 <_P 0$ and $\operatorname{Trd}_Q(\alpha_i\beta) >_P 0$, and let m_2 be the number of indices i such that $\alpha_i^2 <_P 0$ and $\operatorname{Trd}_Q(\alpha_i\beta) <_P 0$; then

$$\operatorname{sgn}_P T_{\sigma_\Delta} = 4(m_1 - m_2)^2.$$

Proof. The arguments[1] in the proof of [5, (7.3)(2)] show that

$$\operatorname{disc} \sigma_\Delta = (-1)^m \operatorname{Nrd}_{M_m(Q)}(\Delta) F^{\times 2} = (-1)^m \operatorname{Nrd}_Q(\alpha_1) \dots \operatorname{Nrd}_Q(\alpha_m) F^{\times 2}.$$

Part (a) follows, since $\alpha_i^2 = -\operatorname{Nrd}_Q(\alpha_i)$.

If Q is not split by F_P, then $\operatorname{sgn}_P T_{\sigma_\Delta} = 0$ by [7, Theorem 1] (or [5, (11.11)]). For the rest of the proof, suppose Q is split by F_P. For $x = (x_{ij})_{1 \le i,j \le m} \in M_m(Q)$ we have $\sigma_\Delta(x) = (x'_{ij})_{1 \le i,j \le m}$ where

$$x'_{ij} = \alpha_i^{-1} \overline{x_{ji}} \alpha_j.$$

Therefore,

$$T_{\sigma_\Delta}(x) = \sum_{i,j=1}^m \operatorname{Trd}_Q(\alpha_i^{-1} \overline{x_{ji}} \alpha_j x_{ji}).$$

Letting $T_{ij} \colon Q \to F$ denote the quadratic form

$$T_{ij}(x) = \operatorname{Trd}_Q(\alpha_i^{-1} \overline{x} \alpha_j x),$$

we thus have

$$T_{\sigma_\Delta} = \overset{\perp}{\bigoplus_{1 \le i,j \le m}} T_{ij},$$

[1] There is a misprint in the statement of [5, Proposition (7.3)(2)]: the formula given there yields $\det(\operatorname{Int}(u) \circ \sigma)$ instead of $\operatorname{disc}(\operatorname{Int}(u) \circ \sigma)$.

hence

$$\operatorname{sgn}_P T_{\sigma_\Delta} = \sum_{i,j=1}^{m} \operatorname{sgn}_P T_{ij}.$$

If α_i and α_j anticommute, then 1 and α_i span a totally isotropic subspace of Q for T_{ij}, so T_{ij} is hyperbolic and $\operatorname{sgn}_P T_{ij} = 0$. Note that in this case we have either $\alpha_i^2 >_P 0$ or $\alpha_j^2 >_P 0$ since Q splits over F_P. If α_i and α_j do not anticommute, pick a nonzero quaternion α' which anticommutes with α_i. Computation shows that $(1, \alpha_i, \alpha_j\alpha', \alpha_j\alpha_i\alpha')$ is an orthogonal basis of Q for T_{ij}, and that T_{ij} has the following diagonalization in this basis:

$$T_{ij} = \langle T(\alpha_i^{-1}\alpha_j)\rangle\langle 1, -\alpha_i^2, -\alpha_j^2\alpha'^2, \alpha_i^2\alpha_j^2\alpha'^2\rangle.$$

Note that $\alpha'^2 >_P 0$ if $\alpha_i^2 <_P 0$ since Q splits over F_P. It follows that $\operatorname{sgn}_P T_{ij} = 0$ unless $\alpha_i^2 <_P 0$ and $\alpha_j^2 <_P 0$; in this case, $\operatorname{sgn}_P T_{ij} = \pm 4$. Therefore, letting

$$t_{ij} = \begin{cases} 0 & \text{if } \alpha_i^2 >_P 0 \text{ or } \alpha_j^2 >_P 0, \\ 1 & \text{if } \alpha_i^2 <_P 0,\ \alpha_j^2 <_P 0 \text{ and } \operatorname{Trd}_Q(\alpha_i^{-1}\alpha_j) >_P 0, \\ -1 & \text{if } \alpha_i^2 <_P 0,\ \alpha_j^2 <_P 0 \text{ and } \operatorname{Trd}_Q(\alpha_i^{-1}\alpha_j) <_P 0, \end{cases}$$

we have

$$\operatorname{sgn}_P T_{\sigma_\Delta} = 4\sum_{i,j=1}^{m} t_{ij}.$$

Now, let $\beta \in Q^\times$ be a pure quaternion such that $\beta^2 <_P 0$. If a pure quaternion $\alpha \in Q$ satisfies $\operatorname{Trd}_Q(\alpha\beta) = 0$, then β anticommutes with α, hence $\alpha^2 >_P 0$ since Q splits over F_P. For $i = 1, \dots, m$, let

$$s_i = \begin{cases} 0 & \text{if } \alpha_i^2 >_P 0, \\ 1 & \text{if } \alpha_i^2 <_P 0 \text{ and } \operatorname{Trd}_Q(\alpha_i\beta) >_P 0, \\ -1 & \text{if } \alpha_i^2 <_P 0 \text{ and } \operatorname{Trd}_Q(\alpha_i\beta) <_P 0, \end{cases}$$

so that

$$\sum_{i=1}^{m} s_i = m_1 - m_2.$$

To complete the proof, we have to show

$$\sum_{i,j=1}^{m} t_{ij} = \Big(\sum_{i=1}^{m} s_i\Big)^2.$$

It clearly suffices to prove $t_{ij} = s_i s_j$ for $i, j = 1, \dots, m$, which amounts to showing that if $\alpha_i^2 <_P 0$ and $\alpha_j^2 <_P 0$, then $\operatorname{Trd}_Q(\alpha_i^{-1}\alpha_j) >_P 0$ if and only if $\operatorname{Trd}_Q(\alpha_i\beta)$ and $\operatorname{Trd}_Q(\alpha_j\beta)$ have the same sign. Since $\alpha_i^2 = -\operatorname{Nrd}_Q(\alpha_i)$ and $\operatorname{Trd}_Q(\alpha_i\alpha_j) = \alpha_i^2 \operatorname{Trd}_Q(\alpha_i^{-1}\alpha_j)$, it is equivalent to show that if $\operatorname{Nrd}_Q(\alpha_i) >_P 0$ and $\operatorname{Nrd}_Q(\alpha_j) >_P 0$, then

$$\operatorname{Trd}_Q(\alpha_i\beta)\operatorname{Trd}_Q(\alpha_j\beta)\operatorname{Trd}_Q(\alpha_i\alpha_j) <_P 0.$$

To check this statement, we may extend scalars to a real closure of F at P. We may therefore substitute for Q a matrix algebra over a real-closed field. The following lemma completes the proof:

Lemma 3.8. *Let K be an ordered field and let α, α', $\beta \in M_2(K)$ be matrices with trace zero. If $\det\alpha$, $\det\alpha'$, $\det\beta > 0$, then $\operatorname{tr}(\alpha\alpha')\operatorname{tr}(\alpha\beta)\operatorname{tr}(\alpha'\beta) < 0$.*

Proof. Since $\operatorname{tr}(\alpha) = \operatorname{tr}(\beta) = 0$,

$$\operatorname{tr}(\alpha\beta) = \alpha\beta + \beta\alpha.$$

Therefore, letting $\alpha_0 = \alpha - \beta\alpha\beta^{-1}$, we have

$$2\alpha = \operatorname{tr}(\alpha\beta)\beta^{-1} + \alpha_0 \qquad \text{and} \qquad \operatorname{tr}(\alpha_0\beta^{-1}) = 0. \tag{9}$$

The condition $\det\alpha > 0$ then yields

$$\operatorname{tr}(\alpha\beta)^2 \det\beta^{-1} + \det\alpha_0 > 0. \tag{10}$$

Similarly, letting $\alpha'_0 = \alpha' - \beta\alpha'\beta^{-1}$,

$$2\alpha' = \operatorname{tr}(\alpha'\beta)\beta^{-1} + \alpha'_0, \qquad \operatorname{tr}(\alpha'_0\beta^{-1}) = 0, \tag{11}$$

and

$$\operatorname{tr}(\alpha'\beta)^2 \det\beta^{-1} + \det\alpha'_0 > 0. \tag{12}$$

By (9) and (11),

$$4\operatorname{tr}(\alpha\alpha') = \operatorname{tr}(\alpha_0\alpha'_0) + \operatorname{tr}(\alpha\beta)\operatorname{tr}(\alpha'\beta)\operatorname{tr}(\beta^{-2}).$$

Since $\operatorname{tr}(\beta) = 0$, we have $\beta^2 = -\det\beta$, hence the preceding equation, together with (10) and (12), yields

$$\begin{aligned} 4\operatorname{tr}(\alpha\alpha')\operatorname{tr}(\alpha\beta)\operatorname{tr}(\alpha'\beta) &= \operatorname{tr}(\alpha_0\alpha'_0)\operatorname{tr}(\alpha\beta)\operatorname{tr}(\alpha'\beta) - 2\operatorname{tr}(\alpha\beta)^2\operatorname{tr}(\alpha'\beta)^2\det\beta^{-1} \\ &< \operatorname{tr}(\alpha_0\alpha'_0)\operatorname{tr}(\alpha\beta)\operatorname{tr}(\alpha'\beta) + \operatorname{tr}(\alpha\beta)^2\det\alpha'_0 \\ &\quad + \operatorname{tr}(\alpha'\beta)^2\det\alpha_0. \end{aligned}$$

Since $\operatorname{tr}(\alpha_0) = \operatorname{tr}(\alpha'_0) = 0$, the right side of the last inequality is

$$\det(\operatorname{tr}(\alpha\beta)\alpha'_0 - \operatorname{tr}(\alpha'\beta)\alpha_0).$$

To conclude the proof, it suffices to observe that the determinant of any matrix which anticommutes with β is negative, otherwise $M_2(K)$ would be generated by two anticommuting matrices with negative square, i.e. $(-1,-1)_K \simeq M_2(K)$. $\square$

Remark. It follows from [7, Theorem 1] that the signature of the involution trace form T_σ is a square for any orthogonal or symplectic involution σ on a central simple algebra A. In the terminology of [7] (see also [5, (11.10)]), Proposition 3.7(b) states that if Q is split by F_P, then

$$\operatorname{sgn}_P \sigma_\Delta = |m_1 - m_2|.$$

We proceed to give examples satisfying the hypotheses of Theorem 3.6, for any odd integer $m \geq 3$.

Let F_0 be an arbitrary ordered field and let $F = F_0(x,y)$, the field of rational fractions in two indeterminates over F_0. The following classical construction (see [8, p. 75]) extends the ordering on F_0 to an ordering P on F such that $x <_P 0$ and $0 <_P y <_P 1$: consider the (x,y)-adic valuation on F defined on $F_0[x,y] \setminus \{0\}$ by

$$v\Big(\sum_{i,j} a_{ij}x^iy^j\Big) = \min\{(i,j) \mid a_{ij} \neq 0\},$$

where the minimum is for the lexicographic order on $\mathbb{Z}^2$. If $f \in F^\times$ satisfies $v(f) = (m,n)$, then $v((-x)^{-m}y^{-n}f) = 0$ and we may consider the residue $\overline{(-x)^{-m}y^{-n}f} \in F_0$. Set

$$f >_P 0 \qquad \text{if and only if} \qquad \overline{(-x)^{-m}y^{-n}f} > 0 \text{ in } F_0. \tag{13}$$

Then $-x >_P 0$, $y >_P 0$ and $1-y >_P 0$, so $x <_P 0$ and $0 <_P y <_P 1$.

Consider the quaternion F-algebra $Q = (x,y)_F$. For any odd integer $m \geq 3$, consider the following pure quaternions in Q:

$$\alpha_1 = i+k, \quad \alpha_2 = j, \quad \alpha_3 = \cdots = \alpha_m = i,$$

and let σ_Δ be the involution on $M_m(Q)$ defined as in (8). Since $y >_P 0$, the algebra Q is split by any real closure of F at P. To compute $\operatorname{sgn}_P T_{\sigma_\Delta}$, we use Proposition 3.7 with $\beta = i$. Since

$$\alpha_1^2 = x(1-y) <_P 0, \quad \alpha_2^2 = y >_P 0 \quad \text{and} \quad \operatorname{Trd}_Q(i(i+k)) = \operatorname{Trd}_Q(i^2) = 2x <_P 0,$$

we have, in the notation of Proposition 3.7, $m_1 = 0$ and $m_2 = m-1$, so

$$\operatorname{sgn}_P T_{\sigma_\Delta} = 4(m-1)^2.$$

By Proposition 3.7, we also have

$$\operatorname{disc} \sigma_\Delta = \alpha_1^2 \ldots \alpha_m^2 F^{\times 2} = (x-xy)yx^{m-2}F^{\times 2} = (1-y)yF^{\times 2}.$$

To see that the hypotheses of Theorem 3.6 hold, it remains to prove that the tensor product

$$(x,y)_F \otimes_F (y(1-y), -1)_F$$

is a division algebra, or, equivalently by a theorem of Albert (see [5, (16.5)]), that an associated Albert form such as

$$q = \langle x, y, -xy, y(y-1), y(y-1), 1\rangle$$

is anisotropic. Since

$$q = \langle 1, y, y(y-1), y(y-1)\rangle \perp \langle x\rangle\langle 1, -y\rangle,$$

a degree argument (see for instance [4, Lemma 1.4(i)]) shows that q is isotropic over F if and only if one of the forms $\langle 1, y, y(y-1), y(y-1)\rangle$, $\langle 1, -y\rangle$ is isotropic over $F_0(y)$. The latter is clearly anisotropic since $y \notin F_0(y)^{\times 2}$. Using the $(y-1)$-adic valuation on $F_0(y)$, one may construct as above (see (13)) an ordering P_0 on $F_0(y)$ such that $y >_{P_0} 1$. Then

$$\operatorname{sgn}_{P_0}\langle 1, y, y(y-1), y(y-1)\rangle = 4,$$

so $\langle 1, y, y(y-1), y(y-1)\rangle$ is anisotropic over $F_0(y)$.

Thus, Theorem 3.6 shows that the quadratic form T_{σ_Δ} is indecomposable, even though its adjoint involution is decomposable, as it is isomorphic to $\sigma_\Delta \otimes \sigma_\Delta$.

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