

ZERO CYCLES ON HOMOGENEOUS VARIETIES

DANIEL KRASHEN

ABSTRACT. In this paper we study the group $A_0(X)$ of zero dimensional cycles of degree 0 modulo rational equivalence on a projective homogeneous algebraic variety X . To do this we translate rational equivalence of 0-cycles on a projective variety into R-equivalence on symmetric powers of the variety. For certain homogeneous varieties, we then relate these symmetric powers to moduli spaces of étale subalgebras of central simple algebras which we construct. This allows us to show $A_0(X) = 0$ for certain classes of homogeneous varieties for groups of each of the classical types, extending previous results of Swan / Karpenko, of Merkurjev, and of Panin.

1. INTRODUCTION

The study of algebraic cycles on quadric hypersurfaces has turned out to be unreasonably successful in its applications to quadratic forms. Karpenko, Izhiboldhin, Rost, Merkurjev, Vishik and Voevodsky, to name a few, have used and developed the theory of algebraic cycles in order to solve a number of outstanding conjectures, most notably Voevodsky's recent proof of the Milnor conjecture.

In part inspired by these great successes, there is much interest in studying algebraic cycles on and motives of general projective homogeneous varieties, beyond the quadric hypersurfaces which arise in applications to quadratic forms. Significant progress has been made by various authors in this direction ([Kar00, Bro, CGM05, SZ]).

Despite the progress in understanding general projective homogeneous varieties, the Chow groups of 0-dimensional cycles for such varieties have remained somewhat mysterious. Whereas computations have been performed in various cases (see for example Swan [Swa89] and Merkurjev [Mer95]), the topic has so far resisted general statements or conjectures.

This material is based upon work supported by the National Science Foundation under agreement No. DMS-0111298. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

In this paper, we compute the Chow group of zero cycles on various projective homogeneous varieties by showing that the group $A_0(X)$ of 0 dimensional cycles of degree 0 modulo rational equivalence is trivial in many cases. We give examples of this for certain homogeneous varieties for groups of each of the classical types A_n, B_n, C_n, D_n .

More precisely, in the A_n case (theorem 7.3), we show that $A_0(X) = 0$ for X a Severi-Brauer variety (recovering a result of Panin), and for certain cases when X is a Severi-Brauer flag variety. In all of these examples, we assume that either F is perfect or $\text{char}(F)$ doesn't divide the index of the underlying central simple algebra.

In the B_n and D_n cases (theorem 8.8), we show that $A_0(X) = 0$ for any (orthogonal) involution variety X , assuming that $\text{char}(F) \neq 2$. Involution varieties are twisted forms of quadric hypersurfaces introduced in [Tao94], and are defined in section 8. This generalizes previous results of Swan ([Swa89]) and Karpenko who proved this when X is a quadric hypersurface, and Merkurjev ([Mer95]) who proved this when X has index 2 (see section 2 for the definition of index).

In the C_n case (theorem 8.13), we show that $A_0(X) = 0$ for $X = V_2(A, \sigma)$ a 2'nd generalized involution variety for a central simple algebra A with symplectic involution σ (see section 8) when $\text{ind}(X) = 1$ or 2 and $\text{char}(F) \neq 2$. This gives the first nontrivial computations of this group for such varieties. The case of higher index is still open.

To obtain our results we relate the Chow group of 0-dimensional cycles to the more geometrically naive notion of R-equivalence (i.e. connecting points with rational curves) on symmetric powers of the original variety, along with the slightly weaker notion of H-equivalence which we introduce. This is explained in section 3. Although in some sense, this idea is not new - various aspects of this idea over the complex field appear in [Sam56], and similar ideas were used in Swan's paper ([Swa89]), our formulation of this principle allows us to more fully exploit its uses.

From here, we show that the symmetric powers of certain homogeneous varieties may be related to spaces which parametrize commutative étale subalgebras in a central simple algebra. To make this connection precise, we define moduli spaces of étale subalgebras in section 5. These spaces are very interesting in their own right, as many open questions in the area of central simple algebras concern the existence and structure of certain types of subfields in a division algebra. In sections 6.1, 6.2 and 6.3 we determine show that in certain cases these moduli spaces are R-trivial, and in sections 7 and 8 we apply this to determining the Chow group of zero cycles for certain homogeneous varieties.

There are various known results concerning the group $A_0(X)$ for geometrically rationally connected varieties over certain fields, particularly the finite, local, and global cases (see [KS03], [Kol99], and [CT05]). For example, Colliot-Thélène has conjectured that the torsion in $CH_0(X)$ is finitely generated when F is p -adic, and has obtained positive results in certain cases ([CT05]).

Over an arbitrary ground field, it is clear that the geometrically rationally connected varieties may have very complicated groups of zero cycles, and so it appears difficult to know which classes of varieties have $A_0(X) = 0$. Even restricting to projective homogeneous varieties is not sufficient for this. For example, A. Vishik has pointed out the following example using a result of Karpenko and Merkurjev ([KM90]):

Proposition 1.1. *One may find a field F and a quadratic form q over a vector space V/F such that if we let X be the variety of 2-dimensional totally isotropic subspaces of V , the group of $CH_0(X)$ is infinitely generated (and therefore so is $A_0(X)$).*

Proof. For a given quadratic form q on V/F we may construct the variety X as above. Let Q be the quadric hypersurface in $\mathbb{P}(V)$ defined by the vanishing of q . Thinking of points in Q as isotropic lines in V , we may construct a Chow correspondence from X to Q by setting $Z \in X \times Q$ to be the subvariety described as

$$\{(x, q) \in X \times Q \mid q \subset x\}.$$

This defines a homomorphism $CH_0(X) \rightarrow CH_1(Q)$.

In [KM90], the authors exhibit a quadratic form q on a 7 dimensional vector space such that the associated 5-dimensional quadric Q has an infinite family of independent nontrivial torsion cycles $z_i \in A^4(Q) = CH_1(Q)$. One may check by inspection that these cycles are in the image of the Chow correspondence above, and therefore give infinitely many independent nontrivial elements in $CH_0(X)$. \square

I am grateful to A. Merkurjev who suggested this problem to me while I was a VIGRE assistant professor at UCLA, and whose helpful comments on various drafts of this paper were extremely useful. I would also like to thank D. Saltman who suggested to me the idea of using Pfaffians to prove theorem 6.7, and I. Panin who explained to me how to concretely think of the varieties associated to symplectic involutions. I am also grateful for the comments of an anonymous referee who recommended the use of Hilbert schemes after reading a previous version of this paper. The use of Hilbert schemes of points has considerably cleaned up and shortened the exposition of the paper, as

well as done away with almost all assumptions about the characteristic of the ground field.

After the appearance of this paper in preprint form, Viktor Petrov, Nikita Semenov and Kirill Zainoulline have subsequently applied these methods to compute groups of 0 cycles on homogeneous varieties for various exceptional groups ([PSZ]).

2. PRELIMINARIES AND NOTATION

Let F be a field. All schemes will be assumed to be separated and of finite type over a field (generally F unless specified otherwise). By a variety, we mean an integral scheme. If Z is a closed subscheme of a scheme X , we let $[Z]$ denote the corresponding cycle. Suppose X and Y are schemes over F . For an extension field L/F we denote by X_L the fiber product $X \times_{\text{Spec}(F)} \text{Spec}(L)$. For a morphism $f : X \rightarrow Y$, we write $f(L) : X(L) \rightarrow Y(L)$ for the induced map on the L -points. We denote by $F(X)$ the function field of X . We define the index of a scheme X , as

$$\text{ind}(X) = \text{GCD}\{[L : F] \mid L/F \text{ finite field extension and } X(L) \neq \emptyset\}.$$

If A is a central simple F algebra, we recall that its dimension is a square, and we define the degree of A , $\text{deg}(A) = \sqrt{\dim_F(A)}$. We may write such an $A = M_m(D)$ for some division algebra D unique up to isomorphism, and we define the index of A , $\text{ind}(A) = \text{deg}(D)$. We let $\text{exp}(A)$ denote the order of the class of A in the Brauer group $\text{Br}(F)$. If M is a finite A module, we follow [KMRT98] and define the reduced dimension of M to be $\text{rdim}(M) = \dim_F(M)/\text{deg}(A)$.

We will make frequent use of symmetric powers and Hilbert schemes of points. For this purpose, we will make the following notational shorthands. For a quasiprojective variety X over F , we define the symmetric power $S^n X$ to be the quotient X^n/S_n . We define X_\circ^n to be the configuration space of n distinct points on X - i.e. $X_\circ^n = X^n \setminus \Delta$, where Δ is the big diagonal. We let $X^{(n)}$ be the quotient X_\circ^n/S_n . Note that the quotient morphism $X_\circ^n \rightarrow X^{(n)}$ is étale. For X quasiprojective, we let $X^{[n]}$ denote the Hilbert scheme of n points on X and $\mathcal{U}_X^n \subset X^{[n]} \times X$ denote the universal family over the Hilbert scheme $X^{[n]}$. Note that $X^{(n)}$ is a dense open subscheme of $X^{[n]}$ if $\dim(X) \geq 1$.

In the case that X is given as a subscheme of a Grassmannian $X \subset \text{Gr}(k, m)$, we let $X_*^n \subset X^n$ denote the open subscheme consisting of collections of n subspaces W_1, \dots, W_n which are linearly independent, and we set $X_*^{(n)} = X_*^n/S_n$.

For a scheme X , we define $Z(X)$ to be the set of 0 dimensional cycles on X and $Z_{eff}^n(X)$ to be the subset of degree n effective cycles in $Z(X)$. We have a set map $X^{[n]}(F) \rightarrow Z_{eff}^n(X)$ defined by taking a subscheme $z \subset X$ of degree n to its fundamental class $[z]$. This gives a bijection between the cycles which are a disjoint union of spectrums of separable field extensions of F , and points in $X^{(n)}(F) \subset X^{[n]}(F)$. We will occasionally have to make use of cycles of other dimensions, and we will use the notation $C_i(X)$ to represent the group of i -dimensional cycles on X .

We say that a field L is prime to p closed if every finite algebraic extension E/L has degree a power of p . An algebraic extension L/F is called a prime to p closure if for every finite subextension $F \subset L_0 \subset L$, $[L : F]$ is prime to p , and L is prime to p -closed.

Lemma 2.1. *Suppose X is a scheme over F with $ind(X) = n$, where either $char(F)$ doesn't divide n or F is perfect. Then $X^{(n)}(F) = X^{[n]}(F)$.*

Proof. Given a point $x \in X^{[n]}(F)$, x corresponds to a finite subscheme $Spec(R) \subset X$, where R is a commutative F -algebra of dimension n . By taking a quotient by a maximal ideal of R , we obtain subscheme $Spec(L) \subset X$, L a field of degree at most n . Since $ind(X) = n$, we immediately conclude $Spec(R) = Spec(L)$ and so R is a field. By our hypothesis, R is a separable field extension, and so we see that x corresponds to a point in $X^{(n)}(F)$ as claimed. \square

Lemma 2.2. *Let X be a proper variety such that for any extension field L/F , $X(L) \neq \emptyset$ implies $A_0(X_L) = 0$. If $A_0(X_{F_p}) = 0$ for each prime p dividing $ind(X)$ and every prime to p closure F_p/F then $A_0(X) = 0$.*

Proof. Suppose first that p does not divide $ind(X)$. It then follows that $X(F_p) \neq \emptyset$, and hence by the hypotheses, $A_0(X_{F_p}) = 0$. Therefore, the conditions of the lemma imply $A_0(X_{F_p}) = 0$ for all p .

We will show that $A_0(X) = 0$ by showing that the degree map $deg : CH_0(X) \rightarrow \mathbb{Z}$ is injective. Let deg_p be the degree map after fibering with F_p . Consider the natural map $\pi_p : X_{F_p} \rightarrow X$, which is a flat morphism. Let $\alpha \in ker(deg)$, and assume that deg_p is injective. In this case, $\pi_p^*(\alpha) \in ker(deg_p) = 0$. This means that we may find irreducible curves $Z_i \subset X_{F_p}$, and a rational function $r_i \in R(Z_i)$, such that $\sum div r_i = \alpha$. But since these subvarieties Z_i , and functions r_i involve only a finite number of coefficients, they are defined over an finite degree intermediate field E , where $F \subset E \subset F_p$. But now we have that if $\pi_E : X_E \rightarrow X$ is the natural map, then $\pi_E^*\alpha = 0$. But $\pi_{E*}\pi_E^*\alpha = [E : F]\alpha$ tells us that $[E : F]\alpha = 0$,

and so $[E : F] \in \text{ann}_{\mathbb{Z}}(\alpha)$. Therefore, since $[E : F]$ is prime to p , $\text{ann}_{\mathbb{Z}}(\alpha) \notin [E : F]$. But because this holds for every prime p , we must have that $\text{ann}_{\mathbb{Z}}(\alpha)$ is not contained in any maximal ideal of \mathbb{Z} and hence $\text{ann}_{\mathbb{Z}}(\alpha) = \mathbb{Z}$. But this implies that $\alpha = 0$. \square

3. CYCLES AND EQUIVALENCE RELATIONS

Let X be a scheme. We say that two points $p_1, p_2 \in X(F)$ are elementarily linked if there exists a rational morphism $\phi : \mathbb{P}^1 \dashrightarrow X$ such that $p_1, p_2 \in \text{im}(\phi(F))$. We define R-equivalence to be the equivalence relation generated by this relation. Let $X(F)/R$ denote the set of equivalence classes of points in $X(F)$ under R-equivalence. We say that X is R-trivial in case $X(F)/R$ is a set of cardinality 1.

If $f : X \rightarrow Y$ is a morphism, we obtain a map of sets $X(F)/R \rightarrow Y(F)/R$ which we denote by f_R . Note that this is well defined, since if $p, q \in X(F)$ are elementarily linked via a rational map $\mathbb{P}^1 \dashrightarrow X$, then the composition $\mathbb{P}^1 \dashrightarrow X \rightarrow Y$ shows that $f(p)$ and $f(q)$ are elementarily linked as well.

Given points $x, y \in X^{[n]}(F)$, we say that x and y are elementarily H-linked if there is a morphism $\phi : \mathbb{P}^1 \rightarrow X^{[n]}$ such that $[\phi(0)] = [x], [\phi(1)] = [y]$. We define H-equivalence, denoted \sim_H , to be the equivalence relation generated by elementary H-linkage. We say that an open subscheme $U \subset X^{[n]}$ is H-trivial if the H-equivalence classes $U(F)/H$ form a set with one element. Note that for $x, y \in X^{(n)}$, $[x] = [y]$ if and only if $x = y$.

We remark that the notions of R and H equivalence carry over in relative versions for any base scheme S by replacing \mathbb{P}_F^1 with \mathbb{P}_S^1 . In particular, if $S \cong \text{Spec}(\oplus E_i)$ where each E_i is a field, it is easy to check that two points are R or H equivalent if and only if the corresponding points are equivalent with respect to each E_i .

The first lemma we prove gives some justification for considering H-equivalence:

Lemma 3.1. *Suppose X is a projective variety, and $\alpha, \beta \in X^{[n]}$. If α and β are H-equivalent, then $[\alpha]$ and $[\beta]$ are rationally equivalent.*

Proof. Without loss of generality, we may assume that α and β are elementarily linked, and choose a morphism $\phi : \mathbb{P}^1 \rightarrow X^{[n]}$ connecting these points (we may assume ϕ is a morphism and not just a rational map since the Hilbert scheme is proper). Pulling back the universal family on $X^{[n]}$ along ϕ , we obtain a flat family $F \subset X \times \mathbb{P}^1$ of 0 dimensional subvarieties of X of degree n on \mathbb{P}^1 . By [Ful98], section 1.6, any two specializations of this to points in \mathbb{P}^1 are rationally equivalent. In particular, α and β are rationally equivalent. \square

Lemma 3.2. *Suppose X is a projective variety over F with $\dim(X) \geq 1$. Then the map $X^{[n]}(F) \rightarrow Z_{\text{eff}}^n(X)$ is surjective.*

Proof. Let $z \subset X$ be an irreducible effective 0 cycle, say $z \cong \text{Spec}(L)$ for L/F a finite field extension. It suffices to show that for any $r > 1$, there is a subscheme $\tilde{z} \subset X$ with $[\tilde{z}] = r[z]$. Without loss of generality, we may assume that $X = \text{Spec}(R)$ is affine. Let $m \subset R$ be the maximal ideal corresponding to z . Since $\dim(R) \geq 1$, we know that $\text{length}(R/m^k)$ is unbounded as k increases. In particular, there exists $k > 0$ such that R/m^k has length $\geq r$ and R/m^{k-1} has length $< r$. Now consider the module m^{k-1}/m^k . We need only show that this module has a submodule M with $\text{length}(M) = r - \text{length}(R/m^{k-1})$. But submodules of m^{k-1}/m^k are the same as L vector spaces with length corresponding to dimension. Since we have subspaces of any desired size, we are done. \square

With this in mind, it is reasonable to extend the definition of elementary H-linkage and H-equivalence to cycles. Namely, if x, y are effective zero cycles of degree n on a regular variety X , we say that they are elementarily H-linked (H-equivalent resp.), if there exist $x', y' \in X^{[n]}(F)$ with $[x'] = x, [y'] = y$ such that x' and y' are elementarily H-linked (H-equivalent resp.).

Corollary 3.3. *There is a natural bijection $X^{[n]}(F)/H = Z_{\text{eff}}^n(X)/H$.*

Proof. This follows immediately from lemma 3.2. \square

It is useful to have a relative version of lemma 3.2 for flat cycles over a curve:

Lemma 3.4. *Suppose X/F is a projective variety, C/F a curve and $\alpha \in C_1(X \times C)$ is an effective cycle such that every component of the support of α is finite and flat over C . Then $\alpha = [Z]$ for some subscheme $Z \subset X \times C$ with $Z \rightarrow C$ flat.*

Note that by the universal property of the Hilbert scheme, this Z must come from a morphism $C \rightarrow X^{[n]}$ by pulling back the universal family.

Proof. Consider the restriction α' of the cycle α to $X_{F(C)}$ (the generic fiber of the family $X \times C$). We have $\alpha' \in Z_{\text{eff}}^n(X_{F(C)})$ for some n , and so we may use lemma 3.2 to find a subscheme $Z' \in X_{F(C)}$ representing it. We may interpret Z' as a point in $X_{F(C)}^{[n]}(F(C)) = X^{[n]}(F(C))$, and therefore obtain a rational morphism $\text{Spec}(F(C)) \rightarrow X^{[n]}$. Since C is a curve and $X^{[n]}$ is proper, we may complete this to a morphism $C \rightarrow X^{[n]}$, and hence obtain a family $Z \subset X \times C$.

By construction it is clear that $[Z]$ and α both have the same restriction to $X_{F(C)}$. We may therefore find an open subset $U \subset C$ such that the cycles α and $[Z]$ are equal. From the fundamental sequence

$$C_1(X \times (C \setminus U)) \rightarrow C_1(X \times C) \rightarrow C_1(X \times U) \rightarrow 0,$$

we see that the difference cycle $[Z] - \alpha$ is supported entirely on $C_1(X \times (C \setminus U))$. But since the support of each cycle is flat over \mathbb{P}^1 , there cannot be any components supported in over $\mathbb{P}^1 \setminus U$, and therefore $[Z] - \alpha = 0$ as claimed. \square

If X/F is a projective variety and L/F a finite field extension of degree n . We define a map of sets

$$\begin{aligned} \mathcal{H} : X(L) &\rightarrow Z_{\text{eff}}^n(X) \\ (\phi : \text{Spec}(L) \rightarrow X) &\mapsto \phi_*[\text{Spec}(L)] \end{aligned}$$

Lemma 3.5. *Let X/F be a projective variety, L/F a finite field extension, and suppose we have $x, y \in X^{[n]}(L)$ with $x \sim_H y$. Then $\mathcal{H}(x) \sim_H \mathcal{H}(y)$. In the case $x, y \in X^{(n)}(L)$ are elementarily linked, so are $\mathcal{H}(x)$ and $\mathcal{H}(y)$.*

Proof. It suffices to consider the case where x and y are elementarily H-linked. Therefore, we may reduce either to the case that $x \sim_R y$ or $[x] = [y]$. If $[x] = [y]$, we may write $[x] = [y] = n[z]$ for $z \cong \text{Spec}(E)$ an irreducible subscheme, and $E \subset L$ a subfield, $n = [L : E]$. Therefore x and y may only differ by an element of $\text{Gal}(L/F)$ and so $\mathcal{H}(x) = \mathcal{H}(y)$.

We may therefore assume that x and y are elementarily linked. Choose $\phi : \mathbb{P}_E^1 \rightarrow X$ with $\phi(0) = x, \phi(\infty) = y$, and let $\rho : \mathbb{P}_E^1 \rightarrow \mathbb{P}^1$ be the natural covering. Since the cycle $(\phi \times \rho)_*[\mathbb{P}_E^1] \in C_1(X \times \mathbb{P}^1)$ satisfies the conditions of lemma 3.4, it follows from lemma 3.4 and the remark just following it that we can find a morphism $\psi : \mathbb{P}^1 \rightarrow X^{[n]}$, where $n = [E : F]$ such that if $C \subset \mathbb{P}^1 \times X$ is the corresponding family, $[C] = (\phi \times \rho)_*[\mathbb{P}_E^1]$.

If we denote by $i_p : \text{Spec}(F) \rightarrow \mathbb{P}^1$, $p = 0, \infty$ the inclusion of points on \mathbb{P}^1 , and consider the pullback diagram:

$$(1) \quad \begin{array}{ccc} \text{Spec}(E) & \longrightarrow & \mathbb{P}_E^1 \\ x \downarrow & & \downarrow \phi \times \rho \\ X & \longrightarrow & X \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \text{Spec}(F) & \xrightarrow{i_0} & \mathbb{P}^1. \end{array}$$

We have $\mathcal{H}(x) = x_*(\text{Spec}(E)) = x_*i_0^![\mathbb{P}_E^1]$ which may be rewritten using [Ful98], theorem 6.2 as $i_0^!(\phi \times \rho)_*[\mathbb{P}_E^1] = i_0^![C]$ which by [Ful98], section 10.1 is the same as $[i_0^{-1}(C)] = [\psi(0)]$, and similarly $\mathcal{H}(y) = [\psi(\infty)]$, showing that these points are elementarily H-linked. \square

Suppose X/F is a projective variety. Given a zero-dimensional subscheme $i : z \hookrightarrow X^{[n]}$, we obtain a family $\mathcal{F} \subset z \times X$. We define the cycle $[n]_*(z) \in Z(X)$ by the formula $[n]_*(z) = \pi_{2*}[\mathcal{F}]$.

Lemma 3.6. *Let X/F be a projective variety. Then the map $X^{[n][m]}(F) \rightarrow Z_{\text{eff}}^{nm}(X)$ defined by mapping a degree m scheme $z \subset X^{[n]}$ to $[n]_*[z]$ passes to H-equivalence.*

Proof. To show this, it suffices to show that if we have $\phi : \mathbb{P}^1 \rightarrow X^{[n][m]}$, $\phi(0) = z$, $\phi(1) = z'$ then $[n]_*[z] \sim_H [n]_*[z']$. To see this, we will construct a morphism $\psi : \mathbb{P}^1 \rightarrow X^{[mn]}$ such that $[\psi(0)] = [n]_*[z]$ and $[\psi(\infty)] = [n]_*[z']$. By the universal property of the Hilbert scheme, this means that we really need to construct a family $\widetilde{W} \subset X \times \mathbb{P}^1$ whose specializations over 0 and ∞ are $[n]_*[z]$ and $[n]_*[z']$ respectively.

Consider the family corresponding to the map ϕ . This is a subscheme $Z \subset X^{[n]} \times \mathbb{P}^1$ with fibers z and z' over the points 0 and ∞ respectively. Pulling back the universal family on $X^{[n]}$ via the morphism $Z \rightarrow X^{[n]}$, we obtain a family $W \hookrightarrow X \times \mathbb{P}^1 \times Z$, which is degree mn over \mathbb{P}^1 , and such that each component of W is flat over \mathbb{P}^1 . By lemma 3.4, we may find $\widetilde{W} \subset X \times \mathbb{P}^1$ such that $[\widetilde{W}] = \pi_*[W]$, where $\pi : X \times \mathbb{P}^1 \times Z \rightarrow X \times \mathbb{P}^1$ is the projection. It is now routine to check that the fibers over 0 and ∞ of \widetilde{W} give subschemes whose cycles are equal to $[n]_*[z]$ and $[n]_*[z']$ respectively. \square

Definition 3.7. For a scheme X and positive integers n, m , we define $X^{(n,m)}$ to be the fiber product:

$$\begin{array}{ccc} X^{(n,m)} & \longrightarrow & S^m(S^n X) \\ \pi \downarrow & & \downarrow \\ X^{(nm)} & \longrightarrow & S^{nm} X \end{array}$$

Lemma 3.8. *Suppose F is prime to p closed, and X/F a quasiprojective variety. Then the natural morphism $\pi : X^{(n,m)} \rightarrow X^{(nm)}$ is surjective on F -points whenever n, m are powers of p .*

Proof. Since we may identify $S^m S^n X$ with the quotient

$$(X^{nm}) / ((S_n)^m \rtimes S_m),$$

it follows that the degree of the map π is $\frac{nm!}{(n!)^m(m!)}$ which is prime to p (recall that $v_p(p^r!) = \frac{p^r-1}{p-1}$ where v_p is the p -adic valuation). Since π factors through the étale map $X_{\circ}^{m+n} \rightarrow X^{(nm)}$ it is also étale. In particular, if $x \in X^{(nm)}(F)$, the fiber $\pi^{-1}(x)$ is étale over $\text{Spec}(F)$ and hence the spectrum of a direct sum of separable field extensions $\oplus L_i$. Since the total degree of this extension is prime to p , there must be at least one of the field extensions L_i whose degree is not a multiple of p . But since F is prime to p closed, this implies that $L_i = F$, and so the fiber has an F -point as desired. \square

Corollary 3.9. *Let X/F be a projective variety. There is a natural map $X^{[n][m]}(F)/H \rightarrow X^{[nm]}/H$. In the case $\text{ind}(X) = mn$, we have a natural map $X^{(n)(m)}(F)/H \rightarrow X^{(mn)}/H$. If we also have that F is prime to p closed and m, n are p -powers then the map $X^{(n)(m)}(F)/H \rightarrow X^{(mn)}/H$ is surjective.*

Proof. This is immediate from lemmas 2.1, 3.6, 3.8 and corollary 3.3. \square

Lemma 3.10. *Let F be prime to p closed, and suppose X/F is a projective variety. Fix a p -power n . Suppose $X_L^{(n)}$ is H -trivial for every finite field extension L/F of p -power degree m . Then $X^{(nm)}$ is H -trivial. In particular, we show that if $\alpha \in X^{(n)}(F)$, then for all $\beta \in X^{(mn)}(F)$, we have $\beta \sim_H m[\alpha]$.*

Proof. By corollary 3.9, it is sufficient to show that $X^{(n)(m)}$ is H -trivial. Choose $\alpha \in X^{(n)}(F)$, which is nonempty by the hypothesis. We will show that given $\beta \in X^{(n)(m)}(F)$, we can write $[\beta] \sim_H m[\alpha]$. We may write $\beta = \mathcal{H}(\tilde{\beta})$ for some $\tilde{\beta} \in X^{(n)}(E)$ where E/F is a degree m étale extension. Choose $\alpha \in X^{(n)}(F)$, and define $\tilde{\alpha} \in X^{(n)}(E)$ via composing α with the structure morphism $\text{Spec}(E) \rightarrow \text{Spec}(F)$. We then have $\mathcal{H}(\tilde{\alpha}) = n[\alpha]$. Since $X_E^{(n)}$ is H -trivial, $\tilde{\alpha} \sim_H \tilde{\beta}$ and by lemma 3.5, $m[\alpha] \sim_H [\beta]$ as desired. \square

Corollary 3.11. *Suppose X/F is a projective variety with F is prime to p -closed and such that for every finite field extension L/F , $X_L^{(\text{ind}X)}$ is H -trivial. Then for every p -power $n \geq \text{ind}(X)$, $X^{(n)}$ is H -trivial.*

Proof. By lemma 3.10, it suffices to show that $X_L^{\text{ind}(X)}$ is H -trivial, where $[L : F] = n/\text{ind}(X)$. We prove this by induction on $\text{ind}(X)$. If $\text{ind}(X) = 1$, the hypothesis implies that X_E is R -trivial for every extension E/F and the conclusion follows from lemma 3.10 (setting $n = 1$ in the statement of the lemma).

For the general induction case, we either have $\text{ind}(X_L) = \text{ind}(X)$ or $\text{ind}(X_L) < \text{ind}(X)$. In the first case, the hypothesis immediately implies $X_L^{\text{ind}(X)} = X_L^{\text{ind}(X_L)}$ is H-trivial. In the latter case, we have $m(\text{ind}(X_L)) = \text{ind}(X)$ for some p -power m , and by lemma 3.10, to show that $X_L^{\text{ind}(X)}$ is H-trivial, it suffices to show that $X_E^{\text{ind}(X_L)}$ is H-trivial for E/L a degree m extension. Therefore, the result follows from the induction step. \square

Theorem 3.12. *Suppose X/F is a projective variety with F is prime to p closed, $p \neq \text{char}(F)$ or F perfect and such that $(X_L)^{(\text{ind}(X_L))}$ H-trivial for every finite field extension L/F . Then $A_0(X) = 0$.*

Proof. Let $\alpha \in X^{(i)}$, where $i = \text{ind}(X)$. Since by assumption on the characteristic every prime cycle β is represented by a point in $X^{(n)}(F)$ for some p -power n , it follows from corollary 3.11 and lemma 3.10 that $[\beta] \sim_H \frac{n}{i}[\alpha]$. In particular, $CH_0(X) \cong \mathbb{Z}$, generated by $[\alpha]$. \square

Definition 3.13. Suppose $f : X \rightarrow Y$ is a morphism of F -schemes. We say that f has R-trivial fibers if for every field extension L/F and every point $y \in Y(L)$ the fiber X_y is an R-trivial L -scheme. Here X_y is the scheme-theoretic fiber defined as the pullback of f along the morphism $y : \text{Spec}(L) \rightarrow Y$.

Definition 3.14. We define an equivalence relation on projective varieties which we call stable R-isomorphism to be the equivalence relation generated by setting X and Y to be equivalent if there exists $f : X \rightarrow Y$ with R-trivial fibers.

Proposition 3.15. *Suppose $f : X \rightarrow Y$ is an morphism between quasiprojective varieties with R-trivial fibers. Then $\text{ind}(X) = \text{ind}(Y)$. If we let $m = \text{ind}(X) = \text{ind}(Y)$, then there is an induced set map $X^{(m)}(F) \rightarrow Y^{(m)}(F)$ which is surjective on R-equivalence classes and injective in the case X, Y are projective. In particular, if $m = 1$, and X and Y are projective, f is bijective on R-equivalence classes.*

Corollary 3.16. *If X and Y are stably R-isomorphic then $\text{ind}(X) = \text{ind}(Y)$ and there is a bijection $X^{(m)}(F)/R \leftrightarrow Y^{(m)}(F)/R$ where $m = \text{ind}(X) = \text{ind}(Y)$.*

proof of proposition 3.15. It is clear since there is a morphism from X to Y that $\text{ind}(Y)$ divides $\text{ind}(X)$. Since the fibers of f are nonempty, we also have that $\text{ind}(X) | \text{ind}(Y)$.

Let $x \in X^{(m)}(F)$. Considering $x \subset X$ as a finite subscheme, we see as in the proof of lemma 2.1, that $x = \text{Spec}(L)$ for some field extension L/F of degree m . If we consider the image $f(x)$, we find

similarly that $f(x) \cong x$, since otherwise the image would have smaller degree, contradicting $\text{ind}(X) = \text{ind}(Y)$. Therefore, f induces a map $X^{(m)} \rightarrow Y^{(m)}$. We note that this may also be seen as the rational map induced by the morphism $S^m X \rightarrow S^m Y$. Since every $x \in X^{(m)}$ is of the form $\text{Spec}(L)$ for L/F a degree m field extension, we have a commutative diagram such that the vertical arrows are surjective:

$$\begin{array}{ccc} \coprod_{[L:F]} X(L) & \xrightarrow{\coprod f_L} & \coprod_{[L:F]} Y(L) \\ \mathcal{H} \downarrow & & \downarrow \mathcal{H} \\ X^{(m)}(F) & \longrightarrow & Y^{(m)}(F) \end{array}$$

It is clear by tracing the diagram that the map on the bottom must be surjective, and it is also clear that it must preserve R-equivalence classes.

We need only show therefore that the map is injective on R-equivalence classes in the case X and Y are projective. Without loss of generality, we may assume that we have $x, x' \in X^{(m)}(F)$ such that $y = f(x)$ and $y' = f(x')$ are elementarily linked. Choose $\mathbb{P}^1 \rightarrow Y^{(m)}$ linking y and y' . In the case that $y = y' \cong \text{Spec}(L)$, we have that x, x' may be lifted to elements of $X(L)$ which both lie in the same fiber over a point in $Y(L)$. Since by hypothesis, the fibers of f are R-trivial, we therefore have $x \sim_R x'$ due to the fact that \mathcal{H} preserves R-equivalence in this case.

We are therefore done if we may show that there is a morphism $\mathbb{P}^1 \rightarrow X^{[m]}$ connecting some point in the fiber over y with a point over the fiber of y' . We begin by choosing a morphism $\phi : \mathbb{P}^1 \rightarrow Y^{[m]}$ connecting y and y' , and consider the pullback of the universal family. This gives a curve $C \subset Y \times \mathbb{P}^1$ such that the projection $C \rightarrow \mathbb{P}^1$ is degree m and such that the fibers over 0 and ∞ are equal to y and y' respectively. If we consider the projection morphism $C \rightarrow Y$ restricted to the generic point $\text{Spec}(F(C))$, we obtain a point in $Y(F(C))$. Since f has R-trivial fibers, the fiber over this point in X is nonempty and hence there is a morphism $\text{Spec}(F(C)) \rightarrow X$ such that its composition with f gives the original map $\text{Spec}(F(C)) \rightarrow Y$. Let $\tilde{C} \rightarrow C$ be the normalization of C . Since X is projective, we get a morphism $\tilde{C} \rightarrow X$ such that the diagram

$$\begin{array}{ccccc} \tilde{C} & \longrightarrow & X \times \mathbb{P}^1 & \longrightarrow & Y \times \mathbb{P}^1 \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{P}^1 & & \end{array}$$

is commutative. In particular, if we let $D \subset X \times \mathbb{P}^1$ be the image of \tilde{C} , then D is birational to C and by the universal property of $X[m]$, defines a morphism $\mathbb{P}^1 \rightarrow X[m]$. If we let $U \subset \mathbb{P}^1$ be the open set on which $C \rightarrow \mathbb{P}^1$ is étale, one may check that we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi|_U} & X^{(m)} \\ & \searrow & \vdots \\ & & Y^{(m)} \end{array}$$

Which shows that we may connect points in the fiber over y and y' as desired. \square

Lemma 3.17. *Suppose $f : X \rightarrow \mathbb{P}^1$ is a dominant morphism of quasiprojective varieties such that the fibers are unirational of constant positive dimension. Then we may find $x, y \in X(F)$ such that $f(x) = 0, f(y) = \infty$ and x and y are elementarily linked.*

Proof. Since the generic fiber is unirational, we find a rational map $g : \mathbb{P}^N \times \mathbb{P}^1 \rightarrow X$ commuting with f . By the hypothesis, we may assume that g is defined on an open subset of codimension at least 2, and consequently, at some point in each fiber over \mathbb{P}^1 . Choose points $x', y' \in \mathbb{P}^N(F)$ such that g is defined at $(x', 0)$ and (y', ∞) . Choosing a linear map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^N$ with $\phi(0) = x'$ and $\phi(\infty) = y'$, we find $g \circ \phi$ elementarily links the points $x = g(x', 0)$ and $y = g(y', \infty)$ as desired. \square

Corollary 3.18. *Suppose $f : X \rightarrow Y$ is a morphism with R -trivial fibers which are unirational of constant positive dimension. Then f is bijective on R -equivalence classes.*

Proof. Since f is clearly surjective on R -equivalence classes, we need only to show that it is injective. It suffices to consider the case that $x, x' \in X$ with $f(x)$ and $f(x')$ elementarily linked. But considering a path $\phi : \mathbb{P}^1 \rightarrow Y$ linking the two points, we may pullback the family $f : X \rightarrow Y$ over ϕ to obtain a family over \mathbb{P}^1 . Hence by lemma 3.17, we can reduce to the case that $f(x) = f(x')$. But in this case we are done since the fibers of f are R -trivial by hypothesis. \square

4. PRELIMINARIES ON SEVERI-BRAUER FLAG VARIETIES

Definition 4.1. Let A be an central simple algebra of degree n . Choose positive integers $n_1 < \dots < n_k < n$. The Severi-Brauer flag variety of

type (n_1, \dots, n_k) , denoted $V_{n_1, \dots, n_k}(A)$, is the variety whose points correspond to flags of ideals $I_{n_1} \subset I_{n_2} \subset \dots \subset I_{n_k}$, where I_{n_i} has reduced dimension n_i . More precisely $V_{n_1, \dots, n_k}(A)$ represents the following functor:

$$V_{n_1, \dots, n_k}(A)(R) = \left\{ (I_1, \dots, I_k) \left| \begin{array}{l} I_i \in \text{Gr}(n_i n, A)(R) \\ \text{is a right ideal of } A_R \text{ and } I_i \subset I_{i+1} \end{array} \right. \right\}$$

In particular, in the case $k = 1$, the variety $V_i(A)$ is the i 'th generalized Severi-Brauer variety of A ([Bla91]), which parametrizes right ideals of A which are locally direct summands of reduced rank i . The same definition generalizes easily to sheaves of Azumaya algebras of constant degree over a base scheme S .

Theorem 4.2. *Suppose A is a central simple F -algebra. Then the Severi-Brauer flag variety $V_{n_1, \dots, n_k}(A)$ is stably R -isomorphic to $V_d(D)$, where D is any central simple algebra Brauer equivalent to A and*

$$d = \text{GCD}\{n_1, \dots, n_k, \text{ind}(A)\}.$$

This result relies on a number of intermediate results:

Proposition 4.3. *Suppose A is a central simple F -algebra, and we have positive integers $n_1 < \dots < n_k$ such that $\text{ind}(A) | n_i$, each i . Then any two points on the Severi-Brauer flag variety $V_{n_1, \dots, n_k}(A)$ are elementarily linked. In particular, $V_{n_1, \dots, n_k}(A)$ is R -trivial.*

Corollary 4.4. *Suppose A is a central simple F -algebra, and we have positive integers n such that $\text{ind}(A) | n$. Then any two F -points in the generalized Severi-Brauer variety $V_n(A)$ are elementarily linked. In particular, $V_n(A)$ is R -trivial.*

Lemma 4.5. *Suppose $A = \text{End}_{r,D}(V)$ for some F -central division algebra D , where V is a right D -space. Let $i = \text{deg}(D)$, and let $I \subset A$ be a right ideal of reduced dimension ri (note that every ideal has reduced dimension a multiple of i). Then there exists a D -subspace $W \subset V$ of dimension r such that $I = \text{Hom}_{r,D}(V, W) \subset \text{End}_{r,D}(V)$.*

Equivalently, writing $A = M_m(D)$, we may consider I to be the set of matrices such that each column is a vector in W .

Proof. Choose a right ideal $I \subset A$, and let $W = \text{im}(I)$. It is enough to show that $I = \text{Hom}_{r,D}(V, W)$. The claim concerning reduced dimension will follow immediately from a dimension count. Since it is clear by definition that $I \subset \text{Hom}_{r,D}(V, W)$, it remains to show that the reverse inclusion holds. We do this by showing that I contains a basis for $\text{Hom}_{r,D}(V, W)$. Let e_1, \dots, e_m be a basis for V , and f_1, \dots, f_r be a basis for W . We must show that the transformation $T_{i,j} \in I$ where

$T_{i,j}(e_k) = f_j \delta_{i,k}$. Since $f_i \in \text{im}(i)$ for some $i \in I$, we know that there exists $v \in V$ such that $i(v) = f_j$. Now define $a \in \text{End}_{r,D}(V)$ to be given by $a(e_k) = v \delta_{i,k}$. Now, $ia(e_k) = f_j \delta_{i,k}$ and $ia \in I$ as desired. \square

proof of 4.3. Let $A = M_m(D)$ for some division algebra D with $\text{deg}(D) = i = \text{ind}(A)$, $mi = n = \text{deg}(A)$. We may therefore write $A = \text{End}_{r,D}(V)$ for some right D -space V of dimension m . Choose flags of ideals

$$(I_1, \dots, I_k), (I'_1, \dots, I'_k) \in V_{n_1, \dots, n_k}(A)(F).$$

We will show that there is a rational morphism $f : \mathbb{A}^1 \dashrightarrow V_{n_1, \dots, n_k}(A)$, such that $f(0) = (I_1, \dots, I_k)$ and $f(1) = (I'_1, \dots, I'_k)$.

By lemma 4.5, we may write $I_j = \text{Hom}(V, W_j)$, $I'_j = \text{Hom}(V, W'_j)$. Choose bases $w_{j,1}, \dots, w_{j,l_j}$ for W_j and $w'_{j,1}, \dots, w'_{j,l_j}$ for W'_j , where $l_j = n_j/i$. Define morphisms $f_{j,l} : \mathbb{A}^1 \rightarrow V$ by $f_{j,l}(t) = w_{j,l}t + w'_{j,l}(1-t)$. We may combine these to get rational morphisms $\mathbb{A}^1 \dashrightarrow \text{Gr}(n_j n, A)$ by taking t to the $n_j n$ -dimensional space of matrices in $M_m(D)$ whose columns are right D -linear combinations of the vectors $w_{j,1}t + w'_{j,1}(1-t), \dots, w_{j,l_j}t + w'_{j,l_j}(1-t)$. By 4.5, this corresponds to a rational morphism $f_j : \mathbb{A}^1 \rightarrow V_{n_j}(A)$. One may check that $f_j(0) = I_j$ and $f_j(1) = I'_j$. Further, for any t , $f_j(t) \subset f_{j+1}(t)$. Therefore, we may put these together to yield a rational morphism $f : \mathbb{A}^1 \dashrightarrow V_{n_1, \dots, n_k}(A)$ with $f(0) = (I_1, \dots, I_k)$ and $f(1) = (I'_1, \dots, I'_k)$. \square

Remark 4.6. *In fact the proof above shows that if we are given $n < m$ with $\text{ind}(A)|n, m$, and we fix $I \in V_n(A)$ $\pi^{-1}(I)$ is R -trivial where $\pi : V_{n,m}(A) \rightarrow V_n(A)$ is the natural projection. In fact, given $\alpha, \beta \in \pi^{-1}(I)$, the path constructed in the proof above to connect α and β as points in $V_{n,m}(A)$ lies entirely in the fiber $\pi^{-1}(I)$ showing they are R -equivalent there as well.*

proof of theorem 4.2. Let $X = V_{n_1, \dots, n_k}(A)$ and $Y = V_d(D)$. Consider the product variety $X \times Y$ together with its natural projections π_1, π_2 onto X and Y respectively. I claim that both projections have R -trivial fibers, which would prove the theorem.

Suppose we have $x : \text{Spec}(L) \rightarrow X$ or $x : \text{Spec}(L) \rightarrow Y$. This would imply that $X(L) \neq \emptyset$ or $Y(L) \neq \emptyset$, and in either case this in turn says that $\text{ind}(A)|d$. Since the scheme theoretic fiber over x is isomorphic to either X_L or Y_L respectively, we know that since $\text{ind}(A_L) = \text{ind}(D_L)|d$ that the fibers are R -trivial by proposition 4.3. \square

Definition 4.7. Suppose A is a central simple algebra and $I \subset A$ is a right ideal of reduced dimension l . Given integers $n_1, \dots, n_k < l$, we define the variety $V_{n_1, \dots, n_k}(I)$ to be the subvariety of $V_{n_1, \dots, n_k}(A)$ consisting of flags of ideals all of which are contained within I .

For these varieties, we have a theorem which generalizes a result from [Art82] on Severi-Brauer varieties:

Theorem 4.8. *Suppose A is a central simple algebra. Let $I \subset A$ be a right ideal of reduced dimension l . Then there exists a degree l algebra D which is Brauer equivalent to A such that for any $n_1, \dots, n_k < k$,*

$$V_{n_1, \dots, n_k}(I) = V_{n_1, \dots, n_k}(D)$$

In order to prove this theorem, we will use the following lemma:

Lemma 4.9. *Let A be an Azumaya algebra with center R , a Noetherian commutative ring, and suppose that I is a right ideal of A such that A/I is a projective R -module. Then there exists an idempotent element $e \in I$ such that $I = eA$.*

Proof. Since A/I is projective as an R -module, by [DI71] it is also a projective (right) A -module. This implies that the short exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$$

splits as a sequence of right A -modules, and therefore, there exists a right ideal $J \subset A$ such that $A = I \oplus J$. We may therefore uniquely write $1 = e + f$, with $e \in I$ and $f \in J$. Now,

$$e = (e + f)e = e^2 + fe.$$

Since $e^2 \in I$, $fe \in J$ this gives $fe \in I \cap J = 0$ and so $e^2 = e$. Finally, $I = (e + f)I = eI + fI$, and this gives $fI \in J \cap I = 0$. Consequently, we have $eA \subset I = eI \subset eA$ so $I = eA$ as desired. \square

proof of theorem 4.8. By 4.9, we know that $I = eA$ for some idempotent $e \in A$. Set $D = eAe$.

Let $X_I = V_{n_1, \dots, n_k}(I)$, and $X_D = V_{n_1, \dots, n_k}(D)$. To prove the theorem, we will construct mutually inverse maps (natural transformations of functors) $\phi : X_I \rightarrow X_D$ and $\psi : X_D \rightarrow X_I$. For a commutative Noetherian F -algebra R , and for $\mathcal{J} = (J_1, \dots, J_k) \in X_I(R)$, we define $\phi(\mathcal{J}) = (J_1e, \dots, J_ke) = (eJ_1e, \dots, eJ_ke)$. For $\mathcal{K} = (K_1, \dots, K_k) \in X_D(R)$, define $\psi(\mathcal{K}) = (K_1A_R, \dots, K_kA_R)$. To see that these are mutually inverse, we need to show that for each $i = 1, \dots, k$, we have $J_i e A = J_i$ and that $K_i A_R e = K_i$. For the second we have

$$K_i A_R e = K_i e = K_i$$

since $K_i \subset eA_R e$. For the first, we note that by the lemma, we have $J_i = hA_i$ for some idempotent h . But then

$$J_i \supset J_i e A_R = J_i I \supset J_i^2 = hA_R hA_R = hA_R = J_i$$

and so $J_i = J_i e A_R$ and we are done. \square

5. MODULI SPACES OF ÉTALE SUBALGEBRAS

Let S be a Noetherian scheme, and let A be a sheaf of Azumaya algebras over S . Our goal in this section is to study the functor $\text{ét}(A)$, which associates to every S -scheme X , the set of sheaves of commutative étale subalgebras of A_X . We will show that this functor is representable by a scheme which may be described in terms of the generalized Severi-Brauer variety of A .

Unless said otherwise, all products are fiber products over S . If X is an S -scheme with structure morphism $f : X \rightarrow S$, then we write A_X for the sheaf of \mathcal{O}_X -algebras $f^*(A)$. For a S -scheme Y , we occasionally write Y_X for $Y \times X$, thought of as an X -scheme.

Every sheaf of étale subalgebras may be assigned a discrete invariant, which we call its type, and therefore our moduli scheme is actually a disjoint union of other moduli spaces.

To begin, let us define the notion of type.

Definition 5.1. Let R be a local ring, and B/R an Azumaya algebra. If $e \in B$ is an idempotent, we define the rank of e , denoted $r(e)$ to be the reduced rank of the right ideal eB .

Let E be a sheaf of étale subalgebras of A/S , and let $p \in S$. Let R be the local ring of p in the étale topology (so that R is a strictly Henselian local ring). Then taking étale stalks, we see that E_p is an étale subalgebra of A_p/R , and it follows that

$$E_p = \bigoplus_{i=1}^k R e_i,$$

for a uniquely defined collection of idempotents e_i , which are each minimal idempotents in S_p .

Definition 5.2. The type of E at the point p is the unordered collection of positive integers $[r(e_1), \dots, r(e_m)]$.

Definition 5.3. We say that E has type $[n_1, \dots, n_m]$ if it has this type for each point $p \in S$.

Remark 5.4. Since $1 = \sum e_i$, the ideals $I_i = e_i A$ span A . Further it is easy to see that the ideals I_i are linearly independent since $e_i a = e_j b$ implies $e_i a = e_i e_i a = e_i e_j b = 0$. We therefore know that the numbers making up the type of E give a partition of $\text{deg}(A_p)$.

Some additional notation for partitions will be useful. Let $\rho = [n_1, \dots, n_m]$. For a positive integer i , let $\rho(i)$ be the number of occurrences of i in ρ . Let $S(\rho)$ be the set of distinct integers n_i occurring

in ρ , and let $N(\rho) = |S(\rho)|$. Let

$$\ell(\rho) = \sum_{i \in S(\rho)} \rho(i) = m$$

be the length of the partition.

Suppose A/S is an sheaf of Azumaya algebras, and suppose S is a connected, Noetherian scheme. Let $\rho = [n_1, \dots, n_m]$ be a partition of $n = \deg(A)$. Let $\acute{e}t_\rho(A)$ be the functor which associates to every S scheme X the set of étale subalgebras of A_X of type ρ . That is, if X has structure map $f : X \rightarrow S$,

$$\acute{e}t_\rho(A)(X) = \left\{ \begin{array}{l} \text{sub-}\mathcal{O}_X\text{-modules} \\ E \subset f^*A \end{array} \middle| \begin{array}{l} E \text{ is a sheaf of commutative étale} \\ \text{subalgebras of } f^*A \text{ of type } \rho \end{array} \right\}$$

Our first goal will be to describe the scheme which represents this functor. We use the following notation:

$$V(A)^\rho = \prod_{i \in S(\rho)} V_i(A)^{\rho(i)}.$$

We define $V(A)_*^\rho$ to be the open subscheme parametrizing ideals which are linearly independent. That is to say, for a S -scheme X , if $I_1, \dots, I_{\ell_\rho}$ is a collection of sheaves of ideals in A_X , representing a point in $V(A)^\rho(X)$, then by definition, this point lies in $V(A)_*^\rho$ if and only if $\bigoplus I_i = A$.

Let S_ρ be the subgroup $\prod_{i \in S(\rho)} S_{\rho(i)}$ of the symmetric group S_n . For each i , we have an action of $S_{\rho(i)}$ on $V_i(A)^{\rho(i)}$ by permuting the factors. This induces an action of S_ρ on $V(A)^\rho$, and on $V(A)_*^\rho$. Denote the quotients of these actions by $S^\rho V(A)$ and $V(A)_*^{(\rho)}$ respectively. We note that since the action on $V(A)_*^\rho$ is free, the quotient morphism $V(A)_*^\rho \rightarrow V(A)_*^{(\rho)}$ is a Galois covering with group S_ρ .

Theorem 5.5. *Let $\rho = [n_1, \dots, n_m]$ be a partition of n . Then the functor $\acute{e}t(A)_\rho$ is represented by the scheme $V(A)_*^{(\rho)}$.*

Proof. To begin, we first note that both $\acute{e}t(A)_\rho$ and the functor represented by $V(A)_*^{(\rho)}$ are sheaves in the étale topology. Therefore, to show that these functors are naturally isomorphic, it suffices to construct a natural transformation $\psi : V(A)_*^{(\rho)} \rightarrow \acute{e}t(A)_\rho$, and then show that this morphism induces isomorphisms on the level of stalks.

Let X be an S -scheme, and let $p : X \rightarrow V(A)_*^{(\rho)}$. To define $\psi(X)(p)$, since both functors are étale sheaves, it suffices to define it on an étale

cover of X . Let \tilde{X} be the pullback in the diagram

$$(2) \quad \begin{array}{ccc} \tilde{X} & \longrightarrow & V(A)_*^\rho \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{p} & V(A)_*^{(\rho)} \end{array}$$

Since the quotient morphism π is étale, so is the morphism $\tilde{X} \rightarrow X$. Therefore we see that after passing to an étale cover, and replacing X by \tilde{X} , we may assume that $p = \pi(q)$ for some $q \in V(A)_*^\rho(X)$. Passing to another cover, we may also assume that $X = \text{Spec}(R)$.

Since $p = \pi(q)$, we may find right ideals $I_1, \dots, I_{\ell(\rho)}$ of A_R such that $\oplus I_i = A_R$, which represent q . Writing

$$1 = \sum e_i, \quad e_i \in I_i,$$

we define $E_p = \oplus e_i R$. This is a split étale extension of R , which is a subalgebra of A , and we set $\psi(p) = E_p$. One may check that this defines a morphism of sheaves. Note that this definition with respect to an étale cover gives a general definition since the association $(I_1, \dots, I_\ell) \mapsto E_p$ is S_ρ invariant.

To see that ψ is an isomorphism, it suffices to check that it is an isomorphism on étale stalks. In other words, we may restrict to the case that $X = \text{Spec}(R)$, where R is a strictly Henselian local ring.

We first show that ψ is injective. Suppose E is an étale subalgebra of A_R of type ρ . Since R is strictly Henselian, we have

$$E = \bigoplus_{i \in S(\rho)} \bigoplus_{j=1}^{\rho(i)} e_{i,j} R.$$

By definition, since the type of E is ρ , if we let $I_{i,j} = e_{i,j} A$, then we the tuple of ideals $(I_{i,j})$ defines a point $q \in V(A)_*^\rho(R)$. Further, since $\sum e_{i,j} = 1$, we actually have $q \in V(A)_*^\rho(R)$. If we let $p = \pi(q)$, then tracing through the above map yields $\psi(R)(p) = E$. Therefore ψ is surjective.

To see that it is injective, we suppose that we have a pair of points $p, p' \in V(A)_*^{(\rho)}(R)$. By forming the pullbacks as in equation 2, since R is strictly Henselian, we immediately find that in each case, because \tilde{X} is an étale cover of X , it is a split étale extension, and hence we have sections. This means we may write

$$p = \pi(I_1, \dots, I_{\ell(\rho)}), \quad p' = \pi(I'_1, \dots, I'_{\ell(\rho)}).$$

Note that in order to show that $p = p'$ it suffices to prove that the ideals are equal after reordering. Now, if $E_p = E_{p'}$, then both rings have the

same minimal idempotents. However, by remark 5.4, the ideals are generated by these idempotents. Therefore, the ideals coincide after reordering, and we are done. \square

Since we now know that the functor $\acute{e}t_\rho(A)$ is representable, we will abuse notation slightly and refer to it and the representing variety by the same name.

Definition 5.6. $\acute{e}t(A)$ is the disjoint union of the schemes $\acute{e}t_\rho(A)$ as ρ ranges over all the partitions of $n = \deg(A)$.

Corollary 5.7. *The functor which associates to any S -scheme X the set of étale subalgebras of A_X is representable by $\acute{e}t(A)$.*

Remark 5.8. *By associating to an étale subalgebra $E \subset A_X$ its underlying module, we obtain a natural transformation to the Grassmannian functor, $\acute{e}t_\rho(A) \rightarrow \text{Gr}(\ell(\rho), A)$.*

6. SUBFIELDS OF CENTRAL SIMPLE ALGEBRAS

In this section and for the remainder of the paper, we specialize back to the case where $S = \text{Spec}(F)$, and A is a central simple F -algebra. If E is an étale subalgebra of A , then taking the étale stalk at $\text{Spec}(F)$ amounts to extending scalars to the separable closure F^{sep} of F . Let G be the absolute Galois group of F^{sep} over F . Writing

$$E \otimes F^{\text{sep}} \cong \bigoplus_{i \in S(\rho)} \bigoplus_{j=1}^{\rho(i)} e_{i,j} F^{\text{sep}},$$

we have an action of G on the idempotents $e_{i,j}$. One may check that the idempotents $e_{i,j}$ are permuted by G , and there is a correspondence between the orbits of this action and the idempotents of E . In particular we have

Lemma 6.1. *In the notation above, if E is a subfield of A , then $|S(\rho)| = 1$.*

Proof. E is a field if and only if G acts transitively on the set of idempotents. On the other hand, this action must also preserve the rank of an idempotent, which implies that all the idempotents have the same rank. \square

Therefore, if we are interested in studying the subfields of a central simple algebra, we may restrict attention to partitions of the above type. If $m|n = \deg(A)$, we write

$$\acute{e}t_m(A) = \acute{e}t_{[\frac{n}{m}, \frac{n}{m}, \dots, \frac{n}{m}]}$$

Note that every separable subfield of dimension m is represented by a F -point of $\text{ét}_m(A)$, and in the case that A is a division algebra, this gives a 1-1 correspondence. In particular, elements of $\text{ét}_n(A)(F)$ are in natural bijection with the maximal separable subfields of A .

Proposition 6.2. *Suppose A is a central simple F algebra of degree $md = n$, and suppose $\text{ét}_m(A)(F) \neq \emptyset$. Then $\text{ét}_m(A)$ is unirational.*

Proof. Note that if $F = \overline{F}$, any two étale subalgebras of type $[d, \dots, d]$ are conjugate under the action of $GL_1(A) = A^*$. Therefore, if $L \subset A$ is an étale subalgebra of the appropriate type, the morphism $GL_1(A) \rightarrow \text{ét}_m(A)$ defined by $g \rightarrow [gLg^{-1}]$ is dominant. Since $GL_1(A)$ is rational, $\text{ét}_m(A)$ is unirational. \square

6.1. Maximal subfields. Note that if $a \in A$ is an element whose characteristic polynomial has distinct roots, then the field $F(a)$ is a maximal étale subalgebra of A .

Theorem 6.3. *Let $U \subset A$ be the Zariski open subset of elements of A whose characteristic polynomials have distinct roots. Then there is a dominant rational map $U \rightarrow \text{ét}_n(A)$ which is surjective on F -points.*

Proof. This argument is a geometric analog of one in [KS04]. Consider the morphism $U \rightarrow Gr(n, A)$ defined by taking an element a to the n -plane spanned by the elements $1, a, a^2, \dots, a^{n-1}$. Since the characteristic polynomial of a has distinct roots, this n -plane is a maximal étale subalgebra. Therefore by the remark at the end of section 5, we obtain a morphism $U \rightarrow \text{ét}_n(A)$. This morphism can be described as that which takes an element of A to the étale subalgebra which it generates. Since every étale subalgebra of A can be generated by a single element, this morphism is surjective on F -points. Since this also holds after fibering with the algebraic closure, it follows also that this morphism is surjective at the algebraic closure and hence dominant. \square

Theorem 6.4. *Suppose A has degree n . Then $\text{ét}_n(A)$ is R -trivial.*

Proof. Since any two points on A as an affine space are elementarily linked, the open subscheme $U \subset A$ from the previous theorem is R -trivial. Therefore since U is R -trivial and there is a map $U \rightarrow \text{ét}_n(A)$ which is surjective on F -points, it follows that $\text{ét}_n(A)$ is R -trivial as well. \square

6.2. Degree 4 algebras. In this section we assume that $\text{char}(F) \neq 2$.

Lemma 6.5. *Let A be a degree 4 central simple F -algebra. Then $V_2(A)$ is isomorphic to an involution variety $V(B, \sigma)$ of a degree 6 algebra with*

orthogonal involution σ (see section 8 for the definition of involution varieties).

Proof. Consider the map

$$Gr(2, 4) \rightarrow \mathbb{P}^5,$$

given by the Plücker embedding. Fixing V a 4 dimensional vector space, we may consider this as the map which takes a 2 dimensional subspace $W \subset V$ to the 1 dimensional subspace $\wedge^2 W \subset \wedge^2 V$. This morphism gives an isomorphism of $Gr(2, 4)$ with a quadric hypersurface. This quadric hypersurface may be thought of as the quadric associated to the bilinear form on $\wedge^2 V$ defined by $\langle \omega_1, \omega_2 \rangle = \omega_1 \wedge \omega_2 \in \wedge^4 V \cong F$. Note that one must choose an isomorphism $\wedge^4 V \cong F$ to obtain a bilinear form, and so it is only defined up to similarity. Nevertheless, the quadric hypersurface and associated adjoint (orthogonal) involution depend only on the similarity class and are hence canonically defined.

Since the Plücker embedding defined above is clearly $PGL(V)$ invariant, using [Art82], for any degree 4 algebra A given by a cocycle $\alpha \in H^1(F, PGL_4)$, we obtain a morphism:

$$V_2(A) \rightarrow V(B),$$

where B is given by composition of α with the standard representation $PGL(V) \rightarrow PGL(V \wedge V)$. By [Art82] this implies that B is similar to $A^{\otimes 2}$ in $Br(F)$. Also, it is easy to see that the quadric hypersurface and hence the involution is PGL_4 invariant, and hence descends to an involution σ on B . We therefore obtain an isomorphism $V_2(A) \cong V(B, \sigma)$ as claimed. \square

Theorem 6.6. *Suppose A is a degree 4 central simple F algebra. Then $\acute{e}t_2(A)$ is R -trivial.*

Proof. By lemma 8.7 (although located near the end of this paper, it does not require the present result), it follows that any two points in $V(B, \sigma)^{(2)}(F)$ are elementarily linked. In particular, since $\acute{e}t_2(A)(F) = V_2(A)_*^{(2)}(F) \subset V_2(A)^{(2)}(F) = V(B, \sigma)^{(2)}(F)$ by lemma 6.5, we may conclude that any two points on $\acute{e}t_2(A)(F)$ are also elementarily linked. \square

6.3. Exponent 2 algebras. We assume again in this section that $\text{char}(F) \neq 2$. We will show in this section that for an algebra of exponent 2, the variety $\acute{e}t_{\deg(A)/2}(A)$ is R -trivial. It will be useful, however, to prove the slightly more general fact below:

Theorem 6.7. *Suppose A is an algebra of exponent 2 and degree $n = 2m$. Then every nonempty open subvariety $U \subset \acute{e}t_m(A)$ is R -trivial.*

The idea of the argument presented in the proof of this theorem is due to D. Saltman.

For the remainder of the section, fix A as in the hypotheses of the theorem above. For an involution τ (symplectic or orthogonal), let $Sym(A, \tau)$ denote the subspace of elements of A fixed by τ .

Lemma 6.8. *Suppose $a \in A$ generates an étale subalgebra $F(a) \in \text{ét}_m(A)$. Then there is a symplectic involution σ on A such that $F(a) \subset A^\sigma$.*

Proof. Since A has exponent 2, we may choose τ to be an arbitrary symplectic involution on A . Set $L = F(a)$, and let $\phi = \tau|_L : L \rightarrow A$. By the Noether Skolem theorem ([DI71], Cor 6.3), there is an element $u \in A$ such that conjugation by u restricted to $F(a)$ gives ϕ .

I claim we may choose u so that $\tau(u) + u$ is invertible. If this is the case, set $v = \tau(u) + u$. Since $ul = \tau(l)u$ for all $l \in L$, we may take τ of both sides to obtain $\tau(l)\tau(u) = \tau(u)l$. Adding opposite sides of these two equations yields $vl = \tau(l)v$, or in other words $\text{inn}_{v^{-1}} \circ \tau|_L = \text{id}_L$. But since v is τ -symmetric, $\sigma = \text{inn}_{v^{-1}} \circ \tau$ is a symplectic involution, and by construction $\sigma(l) = l$ for $l \in L$, proving the lemma. Hence we need only prove the claim. This is done as follows:

Suppose w is any element of A^* such that $\text{inn}_w|_L = \tau|_L$. Let $Q = C_A(L)$ be the centralizer of L in A . For any $q \in Q^*$, it is easy to check that $\text{inn}_{uq}|_L = \text{inn}_u|_L$. Define a linear map $f : Q \rightarrow A$ by $f(q) = uq + \tau(uq)$. The condition that $f(q) \in A^*$ is an open condition, defining an open subvariety $U \in Q$. I claim that U is not the empty subvariety. Note that since Q is an affine space and F is infinite, then F -points are dense on Q and this would imply that U contains an F -point.

To check that U is not the empty subvariety, it suffices to check that $U(\overline{F}) \neq \emptyset$. In other words, we must exhibit an element q in $Q_{\overline{F}} = Q \otimes_F \overline{F}$ such that $uq + \tau(uq)$ is invertible in $A_{\overline{F}} = A \otimes_F \overline{F}$ (we have abused notation here by writing u in place of $u \otimes 1$). To do this, first choose a symplectic involution γ on $A_{\overline{F}}$ such that $\gamma|_{\tau(L_{\overline{F}})} = \text{id}_{\tau(L_{\overline{F}})}$. By [KMRT98] we may find an element $r \in \text{Sym}(A_{\overline{F}}, \tau)$ such that $\gamma \circ \tau = \text{inn}_r$. We therefore have

$$\text{inn}_r|_{L_{\overline{F}}} = \gamma \circ \tau|_{L_{\overline{F}}} = \tau|_{L_{\overline{F}}} = \text{inn}_w|_{L_{\overline{F}}}.$$

This in turn implies that $\text{inn}_{w^{-1}r}|_{L_{\overline{F}}} = \text{id}_{L_{\overline{F}}}$, or in other words $w^{-1}r \in C_{A_{\overline{F}}}(L_{\overline{F}}) = Q_{\overline{F}}$. Since r is τ -symmetric, we also have $r + \tau(r) = 2r \in A_{\overline{F}}^*$ (since $\text{char}(F) \neq 2$). Now setting $q = w^{-1}r$, we have $wq = r$, and so q satisfies the required hypotheses - i.e. $q \in U(\overline{F})$. \square

proof of theorem 6.7. Let L_1, L_2 be subfields of A of degree m , represented by points $[L_1], [L_2] \in U(F) \subset \text{ét}_m(A)(F)$. By the previous lemma, we may find a symplectic involutions σ_1, σ_2 such that $L_i \subset \text{Sym}(A, \sigma_i)$. By [KMRT98], proposition 2.7, there is an element $u \in \text{Sym}(A, \sigma)$ such that $\sigma_2 = \text{inn}_u \circ \sigma_1$, where inn_u denotes conjugation by u . Define a morphism $\mathbb{A}^1 \rightarrow \text{Sym}(A, \sigma)$ by mapping t to $v_t = tu + (1-t)$. Note that since $v_t \in \text{Sym}(A, \sigma)$, for $t \in U$ we have that $\gamma_t = \text{inn}_{v_t} \circ \sigma_1$ is a symplectic involution by [KMRT98], proposition 2.7. Let Prp_{σ_i} be the Pfaffian characteristic polynomial on $\text{Sym}(A, \sigma_i)$ (see [KMRT98], page 19), which is a degree m polynomial. Every element in $\text{Sym}(A, \sigma_i)$ satisfies the degree m polynomial Prp_{σ_i} , and further, there are dense open sets of elements in $\text{Sym}(A, \sigma_i)$, $i = 1, 2$ whose Pfaffians have distinct roots (for example we may choose generators of L_1 and L_2 respectively). Since $\text{Sym}(A, \sigma_i)$ is a rational variety and F is infinite, the F -points in $\text{Sym}(A, \sigma_i)$ are dense. Note also that since $[L_i] \in U(F)$, the restriction $[F(a)] \in U$ gives a nonempty open condition on $\text{Sym}(A, \sigma_i)$. Therefore, there is an element α_1 in $\text{Sym}(A, \sigma)$ such that the Pfaffians of both α_1 and $u\alpha_1 \in \text{Sym}(A, \sigma_2)$ have distinct roots, and such that $[F(\alpha_i)] \subset U(F)$.

Let $\alpha_2 = u\alpha_1$, and set $E_i = F(\alpha_i)$.

We will now show that the points $[L_1]$ and $[L_2]$ are R-equivalent by first showing $[L_i]$ may be connected to $[E_i]$ by a rational curve, and then showing $[E_1]$ and $[E_2]$ may also be connected by a rational curve.

To connect $[E_1]$ and $[E_2]$, we define $\phi : \mathbb{A}^1 \rightarrow A$ via $\phi(t) = v_t\alpha_1$. By construction, $E_1 = F(\phi(0))$ and $E_2 = F(\phi(1))$. Note also that $\phi(t) \in \text{Sym}(A, \gamma_t)$, and so it satisfies the Pfaffian characteristic polynomial Prp_{γ_t} . The condition that $\text{Prp}_{\gamma_t}(\phi(t))$ has distinct roots gives an open condition on t which is nontrivial (e.g. $t = 0, 1$), and the condition that $[F(\phi(t))] \in U$ also gives a nontrivial open condition, which together define a Zariski dense open set of \mathbb{A}^1 . As in the proof of theorem 6.3, we may therefore obtain a rational morphism $\mathbb{A}^1 \dashrightarrow U \subset \text{ét}_m(A)$ via $t \mapsto [F(\phi(t))]$. It is easy to check that this morphism sends 0 to $[E_1]$ and 1 to $[E_2]$.

Finally, to connect $[L_i]$ and $[E_i]$, choose a generator β_i of L_i . Since both β_i and α_i are σ_i -symmetric, and since the σ_i -symmetric elements of A form a linear space, we may obtain a morphism $\mathbb{A}^1 \rightarrow \text{Sym}(A, \sigma_i)$ by $t \mapsto t\beta_i + (1-t)\alpha_i$. Since the general element in the image of this morphism has distinct roots for its Pfaffian, and generates an étale algebra in U , we obtain, as in the proof of theorem 6.3 a rational morphism $\mathbb{A}^1 \dashrightarrow \text{ét}_m(A)$ with $0 \mapsto [F(\alpha_i)] = [E_i]$ and $1 \mapsto [F(\beta_i)] = [L_i]$. \square

7. SEVERI-BRAUER FLAG VARIETIES

Suppose A/F is a central simple algebra with $\text{char}(F)$ not dividing $\text{ind}(A)$. Recall $V_d(A)_*^{(m)}$ denotes the open set in $V_d(A)^{(m)}$ consisting of ideals which are linearly independent as subspaces of A .

Lemma 7.1. *Suppose A is a central simple algebra of index i and $md = i$. Then $\text{ind}(V_d(A)) = m$.*

Proof. Without loss of generality, we may assume F is prime to p closed, $p \neq \text{char}(F)$. By [Bla91], $V_d(A)(L) \neq \emptyset$ if and only if $\text{ind}(A_L)|d$. Therefore, it suffices to consider the case that A is a division algebra and that d is a power of p . Let $E \subset A$ be a maximal separable subfield. Since F is prime to p closed and $p \neq \text{char}(F)$, E has a Galois closure which is a p -group and so it has subextensions of every size dividing $i = [E : F] = \text{deg}(A)$. In particular, $V_d(A)_*^{(m)}(F) = \text{ét}_m(A)(F) \neq \emptyset$, and so $\text{ind}(V_d(A))|m$.

On the other hand, suppose $V_d(A)(L) \neq \emptyset$ for some field L . Since $\text{ind}(A_L)|d$, we may choose a maximal subfield $E \subset A_L$ with $[E : L]|d$. Since E splits A , it must contain a maximal subfield of A , and so $\text{deg}(A) = i|[E : F] = [E : L][L : F]d[L : F]$. Therefore, we have $m|[L : F]$, and in particular, $m|\text{ind}(V_d(A))$, completing the proof. \square

Lemma 7.2. *Suppose A is an F -central simple algebra of degree n with index i , and suppose $i = md$. If either i is prime to $\text{char}(F)$ or F is perfect, then $V_d(A)^{(m)}(F) = V_d(A)_*^{(m)}(F)$.*

Proof. Suppose $x \in V_d(A)^{(m)}(F)$. Write $x \cong \text{Spec}(L)$ for L/F a degree m étale extension. Choose a Galois extension E/F with group G such that $L \otimes E \cong \bigoplus^m E$. Then x gives a collection of m ideals $I_1, \dots, I_m \subset A_E$ each of reduced dimension d . Setting $I = \sum I_i$, note that the natural G action on A_E restricts to an action on I , and hence by descent, I corresponds to an ideal $\bar{I} \subset A$ with $\text{rdim}(\bar{I}) = \text{rdim}(I)$. In particular, since $\text{ind}(A) = i$, we have $i|\text{rdim}(I)$. But this means the ideals I_i are all linearly independent and therefore x corresponds to a point of $V_d(A)_*^{(m)}(F)$ as claimed. \square

Theorem 7.3. *Let $X = V_{n_1, \dots, n_k}(A)$. Let*

$$d = \text{gcd}\{n_1, \dots, n_k, \text{ind}(A)\}.$$

Then $X^{(\text{ind}(X))}$ is R -trivial if any of the following conditions hold:

- (1) $d = 1$,
- (2) $d = 2$ and either $\text{ind}(A)|4$ or $\text{exp}(A)|2$,
- (3) d and $\frac{\text{deg}(A)}{\text{gcd}\{\text{deg}(A), d\}}$ are relatively prime.

In particular, in each of these situations we have $A_0(X) = 0$.

Proof. Let D be the underlying division algebra of A , and let $Y = V_d(D)$. By theorem 4.2, Y and X are stably R-isomorphic. Therefore by proposition 3.16, it suffices to show $Y^{(m)}$ is R-trivial. If we let $i = \text{ind}(A) = \text{deg}(D)$, then we have $m = \text{ind}(Y) = i/d$ by lemma 7.1. By lemma 7.2, the inclusion $\acute{e}t_m(D) \subset Y^{(m)}$ is surjective on F -points. Therefore it suffices to show that $\acute{e}t_m(D)$ is R-trivial. But this follows from theorems 6.4, 6.6 and 6.7 respectively. \square

8. INVOLUTION VARIETIES

We assume in this section that the field F has characteristic not 2.

Definition 8.1. Let (A, σ) be an algebra with an involution (always assumed to be of the first kind, either orthogonal or symplectic). We define the radical of a right ideal $I \subset A$ to be $I \cap I^\perp$, where $I^\perp = r.\text{ann}(\sigma(I))$. We say that a right ideal I is regular with respect to σ if $\text{rad}(I) = 0$, or equivalently, $A = I \oplus I^\perp$.

We let $V_i(A)_{\text{reg}}$ be the subscheme of $V_i(A)$ consisting of regular ideals. It is not hard to show that $V_k(A)_{\text{reg}}$ forms an open subvariety of the generalized Severi-Brauer variety $V_k(A)$. Hence, (A, σ) has a regular ideal of reduced dimension k if and only if $\text{ind}(A) | k$.

We define the generalized involution variety $V_k(A, \sigma)$ to be the subvariety of the Grassmannian representing the following functor of points:

$$(3) \quad V_k(A, \sigma)(R) = \left\{ I \in \text{Gr}(n^2 - nk, A)(R) \mid \begin{array}{l} I \text{ is a left ideal of } A_R \\ \text{and } \sigma(I)I = 0 \end{array} \right\}$$

When $k = 1$, we write $V(A, \sigma)$ for $V_1(A, \sigma)$ and call this the involution variety associated to (A, σ) .

Definition 8.2. Let I be a right ideal of (A, σ) , and choose $l < \text{rdim}(I)$. Define the subinvolution variety $V_l(I, \sigma)$ as the variety representing the functor:

$$V_l(I, \sigma)(R) = \{ J \in V_l(A, \sigma)(R) \mid J \subset I \}$$

The behavior of this variety depends on the ideal I - in particular on whether it is regular, isotropic or neither.

Theorem 8.3. *Suppose (A, σ) is an algebra with involution. Let $I \subset A$ be a regular right ideal of reduced dimension k . Then there exists a degree k algebra with involution of the same type (D, τ) which is Brauer equivalent to A and such that for any $l \leq k$, we have:*

$$V_l(I, \sigma) = V_l(D, \tau)$$

proof of theorem 8.3. By the fact that I is regular, we may write $A = I \oplus I^\perp$, and as in lemma 4.9, $I = eA$ where $1 = e + f$, with $e \in I, f \in I^\perp$. We set $D = eAe$. By [Pie82], D is Brauer equivalent to A . By descent, one sees that $\sigma(e) = e$. This implies that the involution σ restricts to an involution of D , and we denote this restriction by τ .

To prove the theorem, we will construct mutually inverse maps (natural transformations of functors) $\phi : V_l(I, \sigma) \rightarrow V_l(D, \tau)$ and $\psi : V_l(D, \tau) \rightarrow V_l(I, \sigma)$. For a commutative Noetherian F -algebra R , and for $J \in V_l(I, \sigma)(R)$, we define $\phi(J) = Je = eJe \subset D$. For $K \in V_l(D, \tau)$, define $\psi(K) = KA$. It follows from an argument identical the one in the proof of theorem 4.8 that these are mutually inverse. \square

For an isotropic ideal, we have the following:

Lemma 8.4. *Suppose (A, σ) is an algebra with involution. Let $I \subset A$ be an isotropic ideal of reduced dimension k . Then there exists a degree k algebra D which is Brauer equivalent to A such that for any $l \leq k$,*

$$V_l(I, \sigma) = V_l(D)$$

Proof. This follows immediately from the fact that any ideal J contained in I is automatically isotropic. Therefore, $V_l(I, \sigma) = V_l(I)$. By theorem 4.8, we have $V_l(I) = V_l(D)$ as claimed. \square

8.1. Orthogonal involution varieties.

Lemma 8.5. *Suppose V is a vector space space, and q is an isotropic quadratic form on V . Then the quadric hypersurface $C(q)$ is a rational variety and any two F -points on $C(q) \subset \mathbb{P}(V)$ are elementarily linked.*

Proof. Since q is isotropic, choose $p \in C(q)$. Consider the variety of lines in $\mathbb{P}(V)$ passing through p . This is isomorphic to $\mathbb{P}^{\dim(V)-2}$, and hence is \mathbb{R} -trivial. It is easy to see that the rational morphism $\mathbb{P}^{\dim(V)-2} \dashrightarrow C(q)$ given by taking a line through p to its other intersection point with $C(q)$ is a birational isomorphism, well defined off of the intersection of the tangent space to $T_p C(q) \subset \mathbb{P}(V)$ with $C(q)$. In particular, the F -points on $C(q)$ are infinite and dense. Now choose points $p_1, p_2 \in C(q)(F)$. We may choose p such that the rational morphism defined above has both p_i in its image by choosing p in the open complement of $T_{p_i} C(q)$. Using this map we may connect p_1 and p_2 by a single rational curve by connecting their preimages in $\mathbb{P}^{\dim(V)-2}$. \square

Lemma 8.6. *Suppose (A, σ) is an algebra with orthogonal involution, and let $X = V(A, \sigma)$. Then either $\text{ind}(X) = 1$ or $\text{ind}(X) = \max\{\text{ind}(A), 2\}$.*

Proof. Consider the case where $\text{ind}(A) \leq 2$. In this case, we may choose an ideal $I \subset V(A, \sigma)_{\text{reg}}$, and we have $V(I, \sigma) \subset X$ is a subscheme which by descent is isomorphic to the spectrum of a degree 2 étale extension E/F . This means $\text{ind}(X)$ is 1 or 2. In particular, if $\text{ind}(A) = 2$, then $\text{ind}(X) = 2$ since $V(I, \sigma) \subset X \subset V(A)$, and $\text{ind}(V(A)) = 2$, which verifies the theorem in this case.

In the case $\text{ind}(A) > 2$, we may reduce to the case that F is prime to 2 closed. In particular, since $\text{ind}(V_2(A)) = \text{ind}(A)/2$ by lemma 7.1, we may find a field extension E/F of degree $\text{ind}(A)/2$ such that $\text{ind}(A_E) = 2$ (note that A_E is not split since $[E : F] < \text{ind}(A)$). By the first case, $\text{ind}(X_E) = 2$, and so there is a quadratic extension L/E such that $X(L) \neq \emptyset$. Therefore $\text{ind}(X) | \text{ind}(A)$. But since $X \subset V(A)$ and $\text{ind}(V(A)) = \text{ind}(A)$, the reverse holds as well, and we have $\text{ind}(X) = \text{ind}(A)$. \square

Theorem 8.7. *Suppose (A, σ) is a central simple F -algebra with orthogonal involution and let $X = V(A, \sigma)$. If F is prime to 2-closed, then $X^{(\text{ind}(X))}$ is R -trivial.*

Proof. The case $\text{ind}(X) = 1$ follows immediately from lemma 8.5.

If $\text{ind}(X) \geq 2$, we consider the morphism $f : X_*^{(2)} \rightarrow V_2(A)$ defined by taking a pair of ideals to their sum. We note that since a pair of 1 dimensional subspaces are either equal or independent, it follows by descent that $X_*^{(2)} = X^{(2)}$. Since $\text{ind}(X) \neq 1$, every ideal $I \in V_2(A)(F)$ is either regular or isotropic, since otherwise $\text{rad}(I)$ would be a point of $X(F)$. Therefore it follows that the fiber over an ideal $I \in V_2(A)(F)$ is $V(I, \sigma)^{(2)}$, which is either $\text{Spec}(E)^{(2)} = \text{Spec}(F)$ for some quadratic étale E/F (if I is regular) or $V(Q)^{(2)} = \text{ét}_2(Q)$ for some quaternion algebra Q (if I is isotropic). In either case the fiber is nonempty (and R -trivial by theorem 6.4). Let $P = f^{-1}(V_2(A)_{\text{reg}})$. Since $f|_P$ is an isomorphism, we may regard P as an open subvariety of $V_2(A)$.

In the case $\text{ind}(X) = 2$, we have by corollary 4.4 that any two points in P are elementarily linked. Therefore we may conclude that $X^{(2)}$ is R -trivial if we can show that any point in $X^{(2)}(F)$ is R -equivalent to one in $P(F)$. Let $\alpha \in X^{(2)}(F)$ be arbitrary, let $J = f(\alpha)$, and choose a right ideal $I \in V_2(A)_{\text{reg}}(F)$. By corollary 4.4, we may find a morphism $\phi : \mathbb{P}^1 \rightarrow V_2(A)$ with $\phi(0) = J, \phi(\infty) = I$. Since the generic point of \mathbb{P}^1 maps into $V_2(A)_{\text{reg}} \cong P \subset X^{(2)} \subset X^{[2]}$, we may lift ϕ to a morphism $\psi : \mathbb{P}^1 \rightarrow X^{[2]}$ such that $\psi(0) \in V(J, \sigma)^{[2]}$ and $f(\psi(\infty)) = I$. But since $\text{ind}(X) = 2$, it follows from lemma 2.1 that $V(J, \sigma)^{[2]}(F) = V(I, \sigma)^{(2)}(F)$. In particular, since this is an R -trivial variety, we find

that $f^{-1}(I) = \psi(\infty) \sim_R \psi(0) \sim_R \alpha$. Since $f^{-1}(I) \in P(F)$, $X^{(2)}$ is R -trivial.

Suppose now that $i > 2$. By lemma 3.8 it suffices to show that $X^{(2)(i/2)}$ is R -trivial. Choose $\beta, \beta' \in X^{(2)(i/2)}(F)$. In the case that $\beta, \beta' \in P^{(i/2)}(F)$, the conclusion follows theorem 6.7 since $P^{(i/2)} = V_2(A)_{reg}^{(i/2)}$ and $(V_2(A)_{reg})_*^{(i/2)}(F) = (V_2(A)_{reg})^{(i/2)}(F)$ and the fact that $(V_2(A)_{reg})_*^{(i/2)}$ is an open subvariety of $\acute{e}t_2(A)$.

Therefore we are done if we can show that for every $\beta \in X^{(2)(i/2)}(F)$, there is a $\beta' \in P^{(i/2)}(F)$ with $\beta \sim \beta'$. Given such a β , we may write $\beta = \mathcal{H}(\tilde{\beta})$ for some $\tilde{\beta} \in X^{(2)}(L)$, for L/F a degree $i/2$ field extension. By changing our focus to proving the same thing for $\tilde{\beta}$, we may assume by lemma 3.5, that $L = F$, $i = 2$, in which case we are done by the argument in the $i = 2$ case. \square

Theorem 8.8. *Suppose (A, σ) is a central simple algebra with orthogonal involution. Then $A_0(V(A, \sigma)) = 0$.*

Proof. This follows directly from theorems 3.12 and 8.7. \square

8.2. Symplectic involution varieties. Let (A, σ) be an algebra with symplectic involution and index at most 4. Note that since every reduced dimension 1 right ideal is isotropic, the variety $V(A, \sigma)$ is the same as $V(A)$. We therefore focus our attention on the first nontrivial case $V_2(A, \sigma)$.

Lemma 8.9. *Let $X = V_2(A, \sigma)$ as above. Then $ind(X)$ is 1 or 2.*

Proof. Without loss of generality, we may assume that F is prime to 2-closed. Suppose $X(F) = \emptyset$. We must show that X has a point in a quadratic extension of F . Choose $I \in V_4(A)_{reg}$, and consider the generalized subinvolution variety $V_2(I, \sigma)$ which is a closed subscheme of X . By theorem 8.3, $V_2(I, \sigma) \cong V_2(D, \tau)$ for some degree 4 algebra with symplectic involution τ . It therefore suffices to consider the case that $deg(A) = 4$, and this follows from the following proposition. \square

Proposition 8.10. *Suppose A is a degree 4 algebra with symplectic involution σ . Then $V_2(A, \sigma)$ is isomorphic to a quadric hypersurface in \mathbb{P}^4 .*

Proof. By 6.5, recall that the Plücker embedding descends to show $V_2(A)$ as a quadric hypersurface in $V(B)$ where B is a degree 6 central simple algebra similar to $A^{\otimes 2}$. In particular, since $exp(A)|2$, we have $V(B) \cong \mathbb{P}^5$.

At the separable closure, if we write $A = End(W)$, $B = End(\wedge^2 W)$, this corresponds to the Plücker embedding $Gr(2, W) \hookrightarrow \mathbb{P}(\wedge^2 W)$. The

symplectic involution σ is adjoint to a form ω on W which defines an element of $W^* \wedge W^* = \mathcal{O}_{\mathbb{P}(\wedge^2)}(2)$, and the zeros off this element in $Gr(2, W)$ are exactly the isotropic subspaces. By descent, this corresponds to a hyperplane in $\mathbb{P}^5 = V(B)$, whose intersection with the embedded $V_2(A)$ is $V_2(A, \sigma)$. Hence, by intersecting our quadric $V_2(A)$ with an additional hyperplane, we obtain a quadric $V_2(A, \sigma)$ in \mathbb{P}^4 as claimed. \square

Corollary 8.11. *Suppose $X = V_2(A, \sigma)$ for an algebra A of degree 4. Then $X^{(ind(X))}$ is R -trivial.*

Proof. This follows from proposition 8.10 and theorem 8.7. \square

Theorem 8.12. *Let $X = V_2(A, \sigma)$, and assume F is prime to 2-closed. Then $X^{(ind(X))}$ is R -trivial.*

Proof. We first consider the case $ind(X) = 1$. In this case, if we have two points $I_1, I_2 \in X(F)$, note that the ideals $J \in X(F)$ such that J is linearly disjoint from the I_i 's form a dense open subvariety $U \subset X(F)$. Since the group $Sp(A, \sigma)$ acts on X with dense orbits and is unirational, we may find an element $a \in Sp(A, \sigma)(F)$ such that $a(I_1) \in U(F)$. In particular, $U(F)$ is nonempty. Choose $J \in U(F)$, and let $V \subset X$ be the dense open subscheme of right ideals of reduced dimension 2 which are linearly disjoint from J . We have a morphism $f : V \rightarrow V_4(A)$ via $f(I) = J + I$. The image Y of f is open in the subvariety of ideals $K \in V_4(A)$ with $J \subset K$. It follows from remark 4.6 that Y is R -trivial. Note that if $K \in Y(F)$, the fiber $f^{-1}(K)$ is open in $V_2(K, \sigma)$ which is R -trivial and rational by lemma 8.5. Therefore by corollary 3.18, V is R -trivial, which implies $I_1 \sim_R I_2$.

Now consider the case $ind(X) = 2$. Let $f : X_*^{(2)} \rightarrow V_4(A)$ as before. Given $\alpha \in X^{(2)}(F)$, I claim that $\alpha \sim_R \alpha'$ for some $\alpha' \in X_*^{(2)}$ such that $f(\alpha')$ is a regular ideal. To see this, we write $\alpha = \mathcal{H}(\beta)$ for $\beta \in X(L)$, L/F a degree 2 field extension. Since $Sp(A_L, \sigma_L)$ is unirational ([Bor91], thm 18.2) and acts on X_L with dense orbits, we may choose $\psi : \mathbb{P}_L^1 \rightarrow Sp(A, \sigma)$ such that $\psi(0) = id$, and $\alpha' = \psi(\infty)(\alpha)$ is in the open set of elements such that $\alpha' \in X_*^{(2)}$ and $f(\alpha')$ is regular. The path $\phi : \mathbb{P}^1 \rightarrow X(L)$ via $\phi(t) = \psi(t)(\alpha)$ shows that $\alpha \sim_R \alpha'$ as claimed.

Let $P = f^{-1}(V_4(A)_{reg})$. We have reduced to showing that P is R -trivial. But by theorem 8.3, the fibers are of the form $V_2(D, \tau)^{(2)}$ for D an index 4 algebra with symplectic involution τ . Since $V_2(D, \tau)$ is isomorphic to a quadric hypersurface in \mathbb{P}^4 , by proposition 8.10, $V_2(D, \tau)^{(2)}$ is birational to the Grassmannian of projective lines in \mathbb{P}^4 , and hence they are unirational of constant positive dimension. Since

they are also R-trivial by 8.7, we conclude from corollary 3.18 that P is R-trivial as desired. \square

Theorem 8.13. *Let A be a central simple algebra with symplectic involution σ and index at most 4 and let $X = V_2(A, \sigma)$. Then $A_0(X) = 0$.*

Proof. This follows from theorems 8.12 and 3.12. \square

REFERENCES

- [Art82] M. Artin. Brauer-Severi varieties. In *Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981)*, pages 194–210. Springer, Berlin, 1982.
- [Bla91] Altha Blanchet. Function fields of generalized Brauer-Severi varieties. *Comm. Algebra*, 19(1):97–118, 1991.
- [Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [Bro] Patrick Brosnan. On motivic decompositions arising from the method of Bialynicki-Birula. arXiv:math.AG/0407305.
- [CGM05] Vladimir Chernousov, Stefan Gille, and Alexander Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J.*, 126(1):137–159, 2005.
- [CT05] J.-L. Colliot-Thélène. Un théorème de finitude pour le groupe de Chow des zéro-cycles d’un groupe algébrique linéaire sur un corps p -adique. *Invent. Math.*, 159(3):589–606, 2005.
- [DI71] Frank DeMeyer and Edward Ingraham. *Separable algebras over commutative rings*. Springer-Verlag, Berlin, 1971.
- [Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.
- [Kar00] N. A. Karpenko. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz*, 12(1):3–69, 2000.
- [KM90] N. A. Karpenko and A. S. Merkur’ev. Chow groups of projective quadrics. *Algebra i Analiz*, 2(3):218–235, 1990.
- [KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
- [Kol99] János Kollár. Rationally connected varieties over local fields. *Ann. of Math. (2)*, 150(1):357–367, 1999.
- [KS03] János Kollár and Endre Szabó. Rationally connected varieties over finite fields. *Duke Math. J.*, 120(2):251–267, 2003.
- [KS04] Daniel Krashen and David J. Saltman. Severi-Brauer varieties and symmetric powers. In *Algebraic Transformation Groups and Algebraic Varieties*, volume 132 of *Encyclopaedia Math. Sci.*, pages 59–70. Springer, Berlin, 2004.

- [Mer95] A. S. Merkur'ev. Zero-dimensional cycles on some involutive varieties. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 227(Voprosy Teor. Predstav. Algebr i Grupp. 4):93–105, 158, 1995.
- [Pie82] Richard S. Pierce. *Associative algebras*. Springer-Verlag, New York, 1982. Studies in the History of Modern Science, 9.
- [PSZ] V. Petrov, N. Semenov, and K. Zainoulline. Zero-cycles on a twisted Cayley plane. arXiv:math.AG/0508200.
- [Sam56] Pierre Samuel. Rational equivalence of arbitrary cycles. *Amer. J. Math.*, 78:383–400, 1956.
- [Swa89] Richard G. Swan. Zero cycles on quadric hypersurfaces. *Proc. Amer. Math. Soc.*, 107(1):43–46, 1989.
- [SZ] Nikita Semenov and Kirill Zainoulline. On motivic decompositions arising from the method of Bialynicki-Birula. preprint at <http://www.mathematik.uni-bielefeld.de/LAG/man/189.html>.
- [Tao94] David Tao. A variety associated to an algebra with involution. *J. Algebra*, 168(2):479–520, 1994.