A purity theorem for linear algebraic groups

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Abstract

Given a characteristic zero field k and a dominant morphism of affine algebraic k-groups $\mu : G \to C$ one can form a functor from k-algebras to abelian groups $R \mapsto \mathcal{F}(R) := C(R)/\mu(G(R))$. Assuming that C is commutative we prove that this functor satisfies a purity theorem for any regular local k-algebra. Few examples are considered in the very end of the preprint.

1 Introduction

Let $\mathcal{F}$ be a covariant functor from the category of commutative rings to the category of abelian groups. For any domain R consider the sequence

$$\mathcal{F}(R) \to \mathcal{F}(K) \to \bigoplus_{\mathfrak{p}} \mathcal{F}(K)/\mathcal{F}(R_{\mathfrak{p}})$$

where p runs over the height 1 primes of R. We say that $\mathcal{F}$ *satisfies purity for R* if this sequence — which is clearly a complex — is exact. The purity for R is equivalent to the following property

$$\bigcap_{\mathrm{ht}\mathfrak{p}=1} Im[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(K)] = Im[\mathcal{F}(R) \to \mathcal{F}(K)].$$

Let k be a characteristic zero field and $\bar{k}$ be its algebraic closure. By a linear algebraic k-group we mean a reduced affine group k-scheme. In particular, a linear algebraic k-group is always k-smooth. The main results of this paper is the following purity theorem

Theorem 1.0.1 (A). *Let*

$$\mu : G \to C$$

be a dominant morphism of linear algebraic k-groups, with C commutative. Let A be a local ring of a smooth algebraic variety X over k. The functor

$$\mathcal{F} : R \mapsto C(R)/\mu(G(R))$$

satisfies purity for A.

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Theorem 1.0.2 (B). *Let R be a regular local ring containing the field k. Let*

$$\mu : G \to C$$

be a dominant morphism of linear algebraic k-groups, with C commutative. The functor

$$\mathcal{F} : A \mapsto C(A)/\mu(G(A))$$

satisfies purity for R. If K is the fraction field of R this statement can be restated in an explicit way as follows: given an element $c \in C(K)$ suppose that for each height 1 prime ideal $\mathfrak{p}$ in A there exist $a_{\mathfrak{p}} \in C(R_{\mathfrak{p}})$, $g_{\mathfrak{p}} \in G(K)$ with $a = a_{\mathfrak{p}} \cdot \mu(g_{\mathfrak{p}}) \in C(K)$. Then there exist $g_{\mathfrak{m}} \in G(K)$, $a_{\mathfrak{m}} \in C(R)$ such that

$$a = a_{\mathfrak{m}} \cdot \mu(g_{\mathfrak{m}}) \in C(K).$$

After the pioneering articles [C-T/P/S] and [R] on purity theorems for algebraic groups, various versions of purity theorems were proved in [C-T/O], [PS], [Z], [Pa]. The most general result in the so called constant case was given in [Z, Exm.3.3]. This result follows now from our Theorem (A) by taking G to be a k-rational reductive group, $C = \mathbb{G}_{m,k}$ and $\mu : G \to \mathbb{G}_{m,k}$ a dominant k-group morphism. The papers [PS], [Z], [Pa] contain results for the nonconstant case. However they only consider specific examples of algebraic scheme morphisms $\mu : G \to C$.

It seems plausible to expect purity theorem in the following context. Let R be a regular local ring. Let $\mu : G \to T$ be a smooth dominant morphism of R-group schemes with an R-smooth reductive group scheme G and an R-torus T. Let $\mathcal{F}$ be the covariant functor from the category of commutative rings to the category of abelian groups given by $S \mapsto T(S)/\mu(G(S))$. Then $\mathcal{F}$ should satisfies purity for R provided that R contains an infinite field. It might even happen that $\mathcal{F}$ satisfies purity for any regular local ring R.

Another functor for which one should expect purity is the following one. Let R be a regular local ring, G and G' smooth reductive R-group schemes, $\pi : G \to G'$ a smooth dominant R-group scheme morphism. Assume $Z = ker(\pi)$ is finite étale of multiplicative type over R. The boundary operator $\delta_{\pi,R} : G'(R) \to H1_{\text{ét}}(R, Z)$ makes sense and is a group homomorphism [Se, Ch.II, §5.6, Cor.2]. For an R-algebra S set

$$\mathcal{F}(S) = \mathrm{H}^1_{\text{ét}}(S, Z)/\mathrm{Im}(\delta_{\pi,S}). \tag{1}$$

It seems plausible that the functor $\mathcal{F}$ satisfies purity for R provided that R contains an infinite field. It may even happen that, for any regular local ring R, $\mathcal{F}$ satisfies purity. Theorem 2.0.4 states that this is the case if R contains an infinite field k of characteristic zero and G, G' and π are already defined over k.

Note that we use transfers for the functor $R \mapsto C(R)$, but we do not use at all the norm principle for the map $\mu : G \to C$.

The preprint is organized as follows. In Section 2 we construct norm maps following a method from [SV, Sect.6]. In Section 3 we prove the main geometric Lemmas 1.2.3 and 1.2.5. In Section 4 we discuss unramified elements. A key point here is Lemma 1.3.5. In

Section 5 we discuss specialization maps. A key point here is Corollary 1.4.6. In Section 6 we prove Theorem (A). In Section 7 we prove Theorem (B). In Section 8 we consider the functor (1) and prove in 2.0.4 a purity theorem for it. Finally in Section 9 we collect several examples.

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1.1 Norms

Let $k \subset K \subset L$ be field extensions and assume that L is finite separable over K. Let K^{sep} be a separable closure of K and

$$\sigma_i : K \to K^{sep}, \ \ 1 \leq i \leq n$$

the different embeddings of K into L. As in §1, let C be a commutative algebraic group scheme defined over k. We can define a norm map

$$\mathcal{N}_{L/K} : C(L) \to C(K)$$

by

$$\mathcal{N}_{L/K}(\alpha) = \prod_i C(\sigma_i)(\alpha) \in C(K^{sep})^{\mathcal{G}(K)} = C(K) \ .$$

Following Suslin and Voevodsky [SV, Sect.6] we generalize this construction to finite flat ring extensions. Let $p : X \to Y$ be a finite flat morphism of affine schemes. Suppose that its rank is constant, equal to d. Denote by $S^d(X/Y)$ the d-th symmetric power of X over Y.

Lemma 1.1.1. *There is a canonical section*

$$\mathcal{N}_{X/Y} : Y \to S^d(X/Y)$$

which satisfies the following three properties:

(1) *Base change: for any map $f : Y' \to Y$ of affine schemes, putting $X' = X \times_Y Y'$ we have a commutative diagram*

$$\begin{tikzcd} Y' \arrow[r, "\mathcal{N}_{X'/Y'}"] \arrow[d, "f"'] & S^d(X'/Y') \arrow[d, "S^d(Id_X \times f)"] \\ Y \arrow[r, "\mathcal{N}_{X/Y}"] & S^d(X/Y) \end{tikzcd}$$

(2) *Additivity: If $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are finite flat morphisms of degree d_1 and d_2 respectively, then, putting $X = X_1 \coprod X_2$, $f = f_1 \coprod f_2$ and $d = d_1 + d_2$, we have a commutative diagram*

$$\begin{tikzcd}
S^{d_1}(X_1/Y) \times S^{d_2}(X_2/Y) \arrow[rr, "\sigma"] & & S^d(X/Y) \\
& Y \arrow[ul, "\mathcal{N}_{X_1/Y} \times \mathcal{N}_{X_2/Y}"] \arrow[ur, "\mathcal{N}_{X/Y}"'] &
\end{tikzcd}$$

where σ is the canonical imbedding.

(3) *Normalization: If $X = Y$ and p is the identity, then $\mathcal{N}_{X/Y}$ is the identity.*

Proof. We construct a map $\mathcal{N}_{X/Y}$ and check that it has the desired properties. Let $B = k[X]$ and $A = k[Y]$, so that B is a locally free A-module of finite rank d. Let $B^{\otimes d} = B \otimes_A B \otimes_A \cdots \otimes_A B$ be the d-fold tensor product of B over A. The permutation group $\mathfrak{S}_d$ acts on $B^{\otimes d}$ by permuting the factors. Let $S^d(B) \subseteq B^{\otimes d}$ be the A-algebra of all the $\mathfrak{S}_d$-invariant elements of $B^{\otimes d}$. We consider $B^{\otimes d}$ as an $S^d(B)$-module through the inclusion $S^d(B) \subseteq B^{\otimes d}$ of A-algebras. Let I be the kernel of the canonical homomorphism $B^{\otimes d} \to \bigwedge^d(B)$ mapping $b_1 \otimes \cdots \otimes b_d$ to $b_1 \wedge \cdots \wedge b_d$. It is well-known (and easily checked locally on A) that I is generated by all the elements $x \in B^{\otimes d}$ such that $\tau(x) = x$ for some transposition τ. If s is in $S^d(B)$, then $\tau(sx) = \tau(s)\tau(x) = sx$, hence sx is in $S^d(B)$ too. In other words, I is an $S^d(B)$-submodule of $B^{\otimes d}$. The induced $S^d(B)$-module structure on $\bigwedge^d(B)$ defines an A-algebra homomorphism

$$\varphi : S^d(B) \to End_A(\bigwedge^d(B)) .$$

Since B is locally free of rank d over A, $\bigwedge^d(B)$ is an invertible A-module and we can canonically identify $End_A(\bigwedge^d(B)$ with A. Thus we have a map

$$\varphi : S^d(B) \to A$$

and we define

$$Tr_{X/Y} : Y \to S^d(X)$$

as the morphism of Y-schemes induced by φ. The verification of properties (1), (2) and (3) is straightforward. $\square$

Let now C be a commutative k-group scheme and $f : X \to C$ any morphism. We define the norm $N_{X/Y}(f)$ of f as the composite map

$$Y \xrightarrow{Tr_{X/Y}} S^d(X) \xrightarrow{S^d(f)} S^d(C) \xrightarrow{\times} C \tag{2}$$

Here we write " $\times$ " for the group law on C. The norm maps $N_{X/Y}$ satisfy the following conditions

(1) Base change: for any map $f : Y' \to Y$ of affine schemes, putting $X' = X \times_Y Y'$ we have a commutative diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{(id\times f)^*} & C(X') \\ {\scriptstyle N_{X/Y}}\downarrow & & \downarrow{\scriptstyle N_{X'/Y'}} \\ C(Y) & \xrightarrow{f^*} & C(Y') \end{array}$$

(2) multiplicativity: if $X = X_1 \amalg X_2$ then the diagram commutes

$$\begin{array}{ccc} C(X) & \xrightarrow{(id\times f)^*} & C(X_1) \times C(X_2) \\ {\scriptstyle N_{X/Y}}\downarrow & & \downarrow{\scriptstyle N_{X_1/Y}N_{X_2/Y}} \\ C(Y) & \xrightarrow{id} & C(Y) \end{array}$$

(3) normalization: if $X = Y$ and the map $X \to Y$ is the identity then $N_{X/Y} = id_{C(X)}$.

1.2 Geometric lemmas

In this Section few geometric lemmas will be proved. Let k be a field of characteristic zero. The main result of this Section are the lemmas 1.2.3 and 1.2.5. Lemma 1.2.3 is a refinement of [OP, Lemma 2] and Lemma 1.2.5 is a refinement of Quillen's trick.

Lemma 1.2.1. *Let F be a field of characteristic zero and let S be an F-smooth algebra which is a domain of dimension 1. Let $\mathfrak{m}_0$ be a maximal ideal with $S/\mathfrak{m}_0 = F$. Let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ be different maximal ideals of S (it might be that $\mathfrak{m}_0 = \mathfrak{m}_i$ for an index i). Then there exists an element $\beta \in S$ such that S is finite over $F[\beta]$ and*

(1) the ideals $\mathfrak{n}_i := \mathfrak{m}_i \cap F[t]$ $(i = 1, 2, \ldots, n)$ are all different from each other and different from $\mathfrak{n}_0 := \mathfrak{m}_0 \cap F[t]$ provided that $\mathfrak{m}_0$ is different from $\mathfrak{m}_i$'s;

(2) the extension $S/F[t]$ is étale at each $\mathfrak{m}_i$'s and at $\mathfrak{m}_0$;

(3) $F[\beta]/\mathfrak{n}_i = S/\mathfrak{m}_i$ for each $i = 1, 2, \ldots, n$;

(4) $\mathfrak{n}_0 = \beta F[\beta]$.

Proof. Consider a closed imbedding $\mathrm{Spec}(S) \hookrightarrow \mathbf{A}^n_F$ and use a generic linear projection to $\mathbf{A}1_F$.

□

Lemma 1.2.2. *Under the hypotheses of Lemma 1.2.1 let $0 \neq f \in S$ be an element that does not belong to a maximal ideal different of $\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Let $N(f) = N_{S/F[\beta]}$ be the norm of f. Then for an element $\beta \in S$ satisfying* (1) *to* (4) *of Lemma 1.2.1 one has*

(a) $N(f) = fg$ for an element $g \in S$;

(b) $fS + gS = S$;

(c) the map $F[\beta]/(N(f)) \to S/(f)$ is an isomorphism.

Proof. It is straightforward.

$\square$

Lemma 1.2.3. *Let A be a local essentially smooth k-algebra with a maximal ideal $\mathfrak{m}$. Let $A[t] \subset R$ be an $A[t]$-algebra which is essentially smooth as a k-algebra and is finite over the polynomial algebra $A[t]$. Assume $R/\mathfrak{m}R$ is a domain and assume there exists an element $h \in R \setminus \mathfrak{m}R$ such that R_h is A-smooth. Let $\epsilon : R \to A$ be an A-augmentation and $I = ker(\epsilon)$. Given an $f \in R$ with $0 \neq \epsilon(f) \in A$ one can find an $s \in R$ satisfying the following conditions*

(1) *R is finite over $A[s]$;*

(2) *$R/sR = R/I \times R/J$ for certain ideal J;*

(3) *$J + fR = R$;*

(4) *$(s-1)R + fR = R$;*

(5) *if $N(f) = N_{R/A[s]}(f)$ then $N(f) = fg \in R$ for certain $g \in R$ ($R/A[t]$ is a projective module since R and $A[t]$ are regular and R is finite over $A[t]$);*

(6) *$fR + gR = R$;*

(7) *the composition map $A[s]/(N(f)) \to R/(N(f)) \to R/(f)$ is an isomorphism.*

Proof. Replacing t by $t - \epsilon(t)$ we may assume that $\epsilon(t) = 0$. Since R is finite over $A[t]$ and R/fR is finite over A we conclude that $A[t]/(N(f))$ is finite over A, hence $A/(tN(f))$ is finite over R too. So setting $u = tN(f)$ we get an integral extension $A[t]/A[u]$. Consider the characteristic polynomial of the operator $R \xrightarrow{tf} R$ as an $A[u]$-module operator. This polynomial vanishes on tf and its free coefficient is $N(tf)$ up to a sing. Thus $N(tf) = tfg$ for an element $g \in R$. Now replacing t by u we may assume that $R/A[t]$ is still integral, $\epsilon(t) = 0$ and $t \in fR$. In fact, $u = N(ft) = ftg$ and whence $\epsilon(u) = \epsilon(t)\epsilon(fg) = 0$ and $u \in fR$.

We use below "bar" to denote the reduction modulo the ideal $\mathfrak{m}R$. Let $F = \bar{A} = A/\mathfrak{m}$. By the assumption of the lemma the F-algebra $\bar{R}$ is an F-smooth domain of dimension 1. Let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ be different maximal ideals of $\bar{R}$ dividing $\bar{f}$ and let $\mathfrak{m}_0 = ker(\bar{\epsilon})$. Let $\beta \in \bar{R}$ be such that the extension $\bar{R}/F[\beta]$ satisfies the conditions (1) to (4) of Lemma 1.2.1.

Let $\alpha \in R$ be a lift of β, that is $\bar{\alpha} = \beta$ in $\bar{R}$. Replacing α by $\alpha - \epsilon(\alpha)$ we may assume that $\epsilon(\alpha) = 0$ and still $\bar{\alpha} = \beta$.

Let $\alpha^n + p_1(t)\alpha^{n-1} + \cdots + p_n(t) = 0$ be an integral dependence equation for α. Let N be an integer large than the $max\{2, deg(p_i(t))\}$, where $i = 1, 2, \ldots, n$. Then for any

$r \in k^\times$ the element $s = \alpha - rt^N$ is such that t is integral over $A[s]$. Thus for any $r \in k^\times$ the ring R is integral over $A[s]$.

On the other hand the element $\bar{s} = \bar{\alpha} - r\bar{t}^N$ still satisfies the conditions (1) to (4) of Lemma 1.2.1 because for any $i = 1, 2, \ldots, n$ one has $\bar{t} \in \mathfrak{m}_i$.

We claim that the element $s \in R$ is the required element (for almost all $r \in k^\times$).

In fact, for almost all $r \in k^\times$ the element s satisfies the conditions (1) to (4) of Lemma [OP, Lemma 2]. It remains to show that the conditions (5) to (7) are satisfied for all $r \in k^\times$.

To prove (5) consider the characteristic polynomial of the operator $R \xrightarrow{f} R$ as an $A[s]$-module operator. This polynomial vanishes on f and its free coefficient is $\pm N(f)$ (the norm of f). Thus $f^n - a_1 f^{n-1} + \cdots \pm N(f) = 0$ and $N(f) = fg$ for an element $g \in R$.

To prove (6) one has to check that this g is a unit modulo the ideal fR. It suffices to check that $\bar{g} \in \bar{R}$ is a unit modulo the ideal $\bar{f}\bar{R}$. The field $F = A/\mathfrak{m}$, the F-algebra $S = \bar{R}$, its maximal ideals $\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_n$ and the element satisfy the hypotheses of Lemma 1.2.2 with β replaced by $\bar{s}$. Now $\bar{g}$ is a unit modulo the ideal $\bar{f}\bar{R}$ by the item (b) of Lemma 1.2.2.

To prove (7) note that R/fR and $A[s]/(N(f))$ are finite A-modules. So it remains to check that the map $\varphi : A[s]/(N(f)) \to R/fR$ is an isomorphism modulo the maximal ideal $\mathfrak{m}$. For that it suffices to verify that the map $\bar{\varphi} : F[\bar{s}]/(N(\bar{f})) \to \bar{R}/\bar{f}\bar{R}$ is an isomorphism where $N(\bar{f}) := N_{\bar{R}/F[\bar{s}]}$. Now $\bar{\varphi}$ is an isomorphism by the item (c) of Lemma 1.2.2. The lemma follows.

$\square$

Corollary 1.2.4. *Under the hypotheses of Lemma 1.2.3 let K be the quotient field of A, $R_K = R \otimes_A K$ and $\epsilon_K = \epsilon \otimes_A id : R_K \to K$. Consider the inclusion $K[s] \subset R_K$. Then the norm $N(f) \in K[s]$ does not vanish at the points* 1 *and* 0 *of the affine line $\mathbf{A}1_K$.*

Proof. The condition (4) of 1.2.3 implies that $N(f)$ does not vanish at the point 1. Since $\epsilon_K(f) \neq 0 \in K$ the conditions (2) and (3) imply that $N(f)$ does not vanish at 0 either.

$\square$

Lemma 1.2.5. *Let X be a geometrically irreducible k-smooth affine variety of dimension d. Let $x \in X$ be a closed point, $0 \neq f \in k[X]$ and Z be the vanishing locus of f. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\varrho"] \arrow[dr, "q"'] & & \mathbf{A}^d \arrow[dl, "pr"] \\ & \mathbf{A}^{d-1} & \end{tikzcd} \qquad (3)$$

with a linear projection pr such that

1. *there exists a principle open set $X_h \subset X$ containing the fibre $q^{-1}q(x)$ such that the map $q : X_h \to \mathbf{A}^{d-1}$ is smooth;*
2. *the fibre $q^{-1}q(x)$ is geometrically irreducible;*

(3) *the map ϱ is finite;*

(4) *the map $q|_Z : Z \to \mathbf{A}^{d-1}$ is finite;*

(5) *k(x)=k(q(x)).*

Proof. Bertini plus generic projection. Consider a closed imbedding $X \hookrightarrow \mathbf{A}^n_k$, the closure $\bar{X}$ of X in the projective space $\mathbf{P}^n_k$ and the complement X_∞ of X in $\bar{X}$. Let $\bar{Z}$ be the closure of Z in $\mathbf{P}^n_k$ and Z_∞ be the complement of Z in $\bar{Z}$. Since $dimX = d$ one has $dimX_\infty = d-1$. Set $\mathbf{P}^{n-1}_k = \mathbf{P}^n_k - \mathbf{A}^n_k$. Now extend the scalars to an algebraic closure $\bar{k}$ of k. Let $x_1, x_2, \ldots, x_n \in X(\bar{k})$ be the points of $x \otimes_k \bar{k}$. Let $\Pi \subset \mathbf{P}^{n-1}_{\bar{k}}$ be a linear subspace of dimension $n-d$ which is general in the sense that

(i) $\Pi \cap X_{\infty,\, \bar{k}}$ consists of finitely many points;

(ii) $\Pi \cap Z_{\infty,\, \bar{k}} = \emptyset$;

(iii) if $< \Pi, x_i >$ is the space generated by Π and by an x_i then the $< \Pi, x_i > \cap X_{\bar{k}}$ is $\bar{k}$-smooth and irreducible for each $i = 1, 2, \ldots, n$;

(iv) $< \Pi, x_i > \neq < \Pi, x_j >$ for $i \neq j$. The requirement (iii) can be satisfied by a Bertini type theorem for a hyperplane section crossing a given rational point because we are in characteristic zero.

Since the set of linear spaces Π satisfying conditions (i) to (iii) is an open set of a Grassmannian we may choose a Π which is already defined over the ground field k. We may then choose a hyperplaneplane $\pi \subset \Pi$ of dimension $n-d-1$, defined over k and such that

$$\pi \cap X_{\infty,\, \bar{k}} \neq \emptyset. \tag{4}$$

□

Now consider the linear projection $\mathbf{P}^n_k \setminus \pi \to \mathbf{P}^n_d$. Its restriction to $\bar{X}$ defines a finite morphism $\bar{\rho} : \bar{X} \to \mathbf{P}^d_k$. And its restriction to X defines a finite morphism $\rho : X \to \mathbf{A}^d_k$, because it is a base change of $\bar{\rho}$ by means of the open imbedding $\mathbf{A}^d_k \hookrightarrow \mathbf{P}^d_k$.

The linear projection with the center Π defines a morphism $q : X \to \mathbf{A}^{n-1}_k$ which clearly fits in the commutative diagram (3) with a linear projection $pr : \mathbf{A}^d_k \to \mathbf{A}^{d-1}_k$. We claim that these ρ, q and pr are the required morphisms.

To prove this recall that ϱ is finite. The morphism $q|_Z : Z \to \mathbf{A}^{d-1}_k$ is finite too, because $\Pi \cap Z_\infty = \emptyset$. Condition (5) is satisfied because $p \otimes_k \bar{k}$ separates the closed points x_i's of the scheme $x \otimes_k \bar{k}$. The fibre of the morphism $p \otimes_k \bar{k}$ over the point x_i coincides with the intersection $< \Pi, x_i > \cap X_{\bar{k}}$ for each $i = 1, 2, \ldots, n$. After the scalar extension k to $\bar{k}$ the scheme theoretic fibre of q over $q(x)$ becomes a disjoint union of the fibres of $q \otimes_k \bar{k}$ over the points $(q \otimes_k \bar{k})(x_i)$. Since $k(q(x)) = k(x)$ and for each i the intersection $< \Pi, x_i > \cap X_{\bar{k}}$ is irreducible we conclude that the fibre $q^{-1}q(x)$ is irreducible too, and thus condition (2) is verified. Since for each i the intersection $< \Pi, x_i > \cap X_{\bar{k}}$ is smooth we conclude that the scheme theoretic fibre of q over $q(x)$ is smooth too. Thus condition (1) is satisfied. The lemma is proved.

1.3 Unramified elements

Let k be the characteristic zero field. We work in this section with *the category of commutative k-algebras*. Let $\mathcal{F}$ be a covariant functor from the category of commutative k-algebras to the category of abelian groups. Let K be a field containing k and $R \subset K$ be a k-subalgebra whose field of fractions is K. We define the *subgroup of R-unramified elements of K* as

$$\mathcal{F}_{nr,R}(K) = \bigcap_{\mathfrak{p} \in Spec(R)^{(1)}} Im[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(K)],$$

where $Spec(R)^{(1)}$ is the set of hight 1 prime ideals in R. Obviously the image of $\mathcal{F}(R)$ in $\mathcal{F}(K)$ is contained in $\mathcal{F}_{nr,R}(K)$. In most cases $\mathcal{F}(R_{\mathfrak{p}})$ injects into $\mathcal{F}(K)$ and $\mathcal{F}_{nr,R}(K)$ is simply the intersection of all $\mathcal{F}(R_p)$.

A basic functor we are interested in is the following one. Let $\mu : G \to C$ be the morphism of linear algebraic k-groups from Theorem 1.0.1. For a commutative k-algebra R set

$$\mathcal{F}(R) = C(R)/\mu(G(R)). \tag{5}$$

For an element $\alpha \in C(R)$ we will write $\bar{\alpha}$ for its image in $\mathcal{F}(R)$. In this section we will write $\mathcal{F}$ for the functor (5), the only exception being Lemma 1.3.6. We will repeatedly use the following result

Theorem 1.3.1 ([C-T/O]). *Let H be a linear algebraic group over the field k. Let R be a discrete valuation ring with a fraction field K. Then the map of pointed sets*

$$H^1_{ét}(R, H) \to H^1_{ét}(K, H)$$

induced by the inclusion $R \subset K$ has the trivial kernel.

Corollary 1.3.2 ([Ni]). *Let R be a discrete valuation ring with fraction field K. The map $\mathcal{F}(R) \to \mathcal{F}(K)$ is injective.*

Proof. Let H be the kernel of μ. Then the boundary map $\partial : C(R) \to \mathrm{H}^1_{\text{ét}}(R, H)$ fits in a commutative diagram

$$\begin{array}{ccc} C(R)/\mu(G(R)) & \longrightarrow & C(K)/\mu(G(K)) \\ \downarrow & & \downarrow \\ \mathrm{H}^1_{\text{ét}}(R, H) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(K, H). \end{array}$$

in which the vertical arrows have trivial kernels. The bottom arrow has trivial kernel by Theorem 1.3.1. Thus the top arrow has trivial kernel too.

∎

Lemma 1.3.3. *Let $\mu : G \to C$ be the morphism of linear algebraic groups. Let $H =$ $\ker(\mu)$. Then for any field K the boundary map $\partial : C(K)/\mu(G(K)) \to H^1_{ét}(K, H)$ is injective.*

Proof. For a K-rational point $t \in C$ set $H_t = \mu^{-1}(t)$. The action by left multiplication of H on H_t makes H_t into a left principal homogeneous H-space and moreover $\partial(t) \in \mathrm{H}^1_{\text{ét}}(K, H)$ coincides with the isomorphism class of H_t. Now suppose that $s, t \in C(K)$ are such that $\partial(s) = \partial(t)$. This means that H_t and H_s are isomorphic as principal homogeneous H-spaces. We must check that for certain $g \in G(K)$ one has $t = sg$.

Let K^{sep} be a separable closure of K. Let $\psi : H_s \to H_t$ be an isomorphism of left H-spaces. For any $r \in H_s(K^{sep})$ and $h \in H_s(K^{sep})$ one has

$$(hr)^{-1}\psi(hr) = r^{-1}h^{-1}h\psi(r) = r^{-1}\psi(r).$$

Thus for any $\sigma \in Gal(K^{sep}/K)$ and any $r \in H_s(K^{sep})$ one has

$$r^{-1}\psi(r) = (r^\sigma)^{-1}\psi(r^\sigma) = (r^{-1}\psi(r))^\sigma$$

which means that the point $u = r^{-1}\psi(r)$ is a $Gal(K^{sep}/K)$-invariant point of $G(K^{sep})$. So $u \in G(K)$. The following relation shows that the ψ coincides with the right multiplication by u. In fact, for any $r \in H_s(K^{sep})$ one has $\psi(r) = rr^{-1}\psi(r) = ru$. Since ψ is the right multiplication by u one has $t = s\mu(u)$, which proves the lemma.

□

Let K be a field and $x : K^* \to \mathbb{Z}$ be a discrete valuation vanishing on k. Let A_x be the valuation ring of x. Let $\hat{A}_x$ and $\hat{K}_x$ be the completions of A_x and K with respect to x. Let $i : K \hookrightarrow \hat{K}_x$ be the inclusion. By Lemma 1.3.2 the map $\mathcal{F}(\hat{A}_x) \to \mathcal{F}(\hat{K}_x)$ is injective. We will identify $\mathcal{F}(\hat{A}_x)$ with its image under this map. Set

$$\mathcal{F}_x(K) = i_*^{-1}(\mathcal{F}(\hat{A}_x)).$$

The inclusion $A_x \hookrightarrow K$ induces a map $\mathcal{F}(A_x) \to \mathcal{F}(K)$ which is injective by Lemma 1.3.2. So both groups $\mathcal{F}(A)$ and $\mathcal{F}_x(K)$ are subgroups of $\mathcal{F}(K)$. The following lemma shows that $\mathcal{F}_x(K)$ coincides with the subgroup of $\mathcal{F}(K)$ consisting of all elements *unramified* at x.

Lemma 1.3.4. $\mathcal{F}(A_x) = \mathcal{F}_x(K)$.

Proof. We only have to check the inclusion $\mathcal{F}_x(K) \subseteq \mathcal{F}(A_x)$. Let $a_x \in \mathcal{F}_x(K)$ be an element. It determines the elements $a \in \mathcal{F}(K)$ and $\hat{a} \in \mathcal{F}(\hat{A}_x)$ which coincide when regarded as elements of $\mathcal{F}(\hat{K}_x)$. We denote this common element in $\mathcal{F}(\hat{K}_x)$ by $\hat{a}_x$. Let $H = \ker(\mu)$ and let $\partial : C(-) \to \mathrm{H}^1_{\text{ét}}(-, H)$ be the boundary map.

Let $\xi = \partial(a) \in \mathrm{H}^1_{\text{ét}}(K, H)$, $\hat{\xi} = \partial(\hat{a}) \in \mathrm{H}^1_{\text{ét}}(\hat{A}_x, H)$ and $\hat{\xi}_x = \partial(\hat{a}_x) \in \mathrm{H}^1_{\text{ét}}(\hat{K}_x, H)$. Clearly, $\hat{\xi}$ and ξ both coincide with $\hat{\xi}_x$ when regarded as elements of $\mathrm{H}^1_{\text{ét}}(\hat{K}_x, H)$. Thus one can glue ξ and $\hat{\xi}$ to get a $\xi_x \in \mathrm{H}^1_{\text{ét}}(A_x, H)$ which maps to ξ under the map induced by the inclusion $A_x \hookrightarrow K$ and maps to $\hat{\xi}$ under the map induced by the inclusion $A_x \hookrightarrow \hat{A}_x$.

We now show that ξ_x has the form $\partial(a'_x)$ for a certain $a'_x \in \mathcal{F}(A_x)$. In fact, observe that the image ζ of ξ in $\mathrm{H}^1_{\text{ét}}(K, G)$ is trivial. By Theorem 1.3.1 the map

$$\mathrm{H}^1_{\text{ét}}(A_x, G) \to \mathrm{H}^1_{\text{ét}}(K, G)$$

has trivial kernel. Therefore the image ζ_x of ξ_x in $\mathrm{H}^1_{\text{ét}}(A_x, G)$ is trivial. Thus there exists an element $a'_x \in \mathcal{F}(A_x)$ with $\partial(a'_x) = \xi_x \in \mathrm{H}^1_{\text{ét}}(A_x, H)$.

We now prove that a'_x coincides with a_x in $\mathcal{F}_x(K)$. Since $\mathcal{F}(A_x)$ and $\mathcal{F}_x(K)$ are both subgroups of $\mathcal{F}(K)$, it suffices to show that a'_x coincides with the element a in $\mathcal{F}(K)$. By Lemma 1.3.3 the map

$$\mathcal{F}(K) \xrightarrow{\partial} \mathrm{H}^1_{\text{ét}}(K, H)$$

is injective. Thus it suffices to check that $\partial(a'_x) = \partial(a)$ in $\mathrm{H}^1_{\text{ét}}(K, H)$. This is indeed the case because $\partial(a'_x) = \xi_x$ and $\partial(a) = \xi$, and ξ_x coincides with ξ when regarded over K. We have proved that $a'_x \in \mathcal{F}(A_x)$ coincides with a_x in $\mathcal{F}_x(K)$. Thus the inclusion $\mathcal{F}_x(K) \subseteq \mathcal{F}(A_x)$ is proved, whence the lemma.

□

Lemma 1.3.5. *Let $A \subset B$ be a finite extension of Dedekind k-algebras. Let $0 \neq f \in B$ be such that B/fB is reduced.*

Suppose $N_{B/A}(f) = fg \in B$ for a certain $g \in B$ coprime with f. Suppose the composite map $A/N(f)A \to B/N(f)B \to B/fB$ is an isomorphism. Let F and E be the quotient fields of A and B respectively. Let $\beta \in C(B_f)$ be such that $\bar{\beta} \in \mathcal{F}(E)$ is B-unramified. Then, for $\alpha = N_{E/F}(\beta)$, the class $\bar{\alpha} \in \mathcal{F}(F)$ is A-unramified.

Proof. The only primes at which $\bar{\alpha}$ could be ramified are those which divide $N(f)$. Let $\mathfrak{p}$ be one of them. Check that $\bar{\alpha}$ is unramified at $\mathfrak{p}$.

To do this we consider all primes $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ in B lying over $\mathfrak{p}$. Let $\mathfrak{q}_1$ be the unique prime dividing f and lying over $\mathfrak{p}$. Then

$$\hat{B}_{\mathfrak{p}} = \hat{B}_{\mathfrak{q}_1} \times \prod_{i \neq 1} \hat{B}_{\mathfrak{q}_i}$$

with $\hat{B}_{\mathfrak{q}_1} = \hat{A}_{\mathfrak{p}}$. If F, E are the fields of fractions of A and B then

$$E \otimes \hat{F}_{\mathfrak{p}} = \hat{E}_{\mathfrak{q}_1} \times \cdots \times \hat{E}_{\mathfrak{q}_n}$$

and $\hat{E}_{\mathfrak{q}_1} = \hat{F}_{\mathfrak{p}}$. We will write $\hat{E}_i$ for $\hat{E}_{\mathfrak{q}_i}$ and $\hat{B}_i$ for $\hat{B}_{\mathfrak{q}_i}$. Let $\beta \otimes 1 = (\beta_1, \ldots, \beta_n) \in C(\hat{E}_1) \times \cdots \times C(\hat{E}_n)$. Clearly for $i \geq 2$ $\beta_i \in C(\hat{B}_i)$ and $\beta_1 = \mu(\gamma_1)\beta'_1$ with $\beta'_1 \in C(\hat{B}_1) = C(\hat{A}_{\mathfrak{p}})$ and $\gamma_1 \in G(\hat{E}_1) = G(\hat{F}_{\mathfrak{p}})$. Now $\alpha \otimes 1 \in C(\hat{F}_{\mathfrak{p}})$ coincides with the product

$$\beta_1 N_{\hat{L}_2/\hat{K}_{\mathfrak{p}}}(\beta_2) \cdots N_{\hat{L}_n/\hat{K}_{\mathfrak{p}}}(\beta_n) = \mu(\gamma_1)[\beta'_1 N_{\hat{L}_2/\hat{K}_{\mathfrak{p}}}(\beta_2) \cdots N_{\hat{L}_n/\hat{K}_{\mathfrak{p}}}(\beta_n)].$$

Thus $\overline{\alpha \otimes 1} = \bar{\beta}'_1 \overline{N_{\hat{L}_2/\hat{K}_{\mathfrak{p}}}(\beta_2)} \cdots \overline{N_{\hat{L}_n/\hat{K}_{\mathfrak{p}}}(\beta_n)} \in \mathcal{F}(\hat{A}_{\mathfrak{p}})$. Let $i : F \hookrightarrow \hat{F}_{\mathfrak{p}}$ be the inclusion and $i_* : \mathcal{F}(F) \to \mathcal{F}(\hat{F}_{\mathfrak{p}})$ be the induced map. Clearly $i_*(\bar{\alpha}) = \overline{\alpha \otimes 1}$ in $\mathcal{F}(\hat{F}_{\mathfrak{p}})$. Now Lemma 1.3.4 shows that the element $\bar{\alpha} \in \mathcal{F}(F)$ belongs to $\mathcal{F}(A_{\mathfrak{p}})$. Hence $\bar{\alpha}$ is A-unramified.

□

Lemma 1.3.6 (Unramifiedness Lemma). *Let $\mathcal{F}$ be a covariant functor from the category of commutative k-algebras to the category of abelian groups. Let S and R be noetherian domains with quotient fields K and L respectively. Let $S \xrightarrow{i} R$ be an injective flat* *homomorphism of finite type and let $j : K \to L$ be the induced inclusion of the quotient fields. Then for each localization $R' \supset R$ of R the map*

$$j_* : \mathcal{F}(K) \to \mathcal{F}(L)$$

takes S-unramified elements to R'-unramified elements.

Proof. Let $v \in K^*$ and let $\mathfrak{r}$ be height 1 primes of R'. Then $\mathfrak{q} = R \cap \mathfrak{r}$ is a height 1 prime of R. Let $\mathfrak{p} = S \cap \mathfrak{q}$. Since the S-algebra R is flat of finite type one has $\mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}(\mathfrak{p})$. Thus $\mathrm{ht}(\mathfrak{p})$ is 1 or 0. The commutative diagram

$$\begin{array}{ccc} \mathcal{F}(K) & \longrightarrow & \mathcal{F}(L) \\ \uparrow & & \uparrow \\ \mathcal{F}(S_{\mathfrak{p}}) & \longrightarrow & \mathcal{F}(R'_{\mathfrak{r}}) \end{array}$$

shows that the class $j_*(v)$ is in the image of $\mathcal{F}(R'_{\mathfrak{r}})$ and hence the class $j_*(v) \in \mathcal{F}(L)$ is R'-unramified.

□

1.4 Specialization maps

In this Section we consider the functor $R \mapsto \mathcal{F}(R)$ defined by (5). For a regular ring R containing k and each height one prime $\mathfrak{p}$ in R we construct specialization maps $s_{\mathfrak{p}} : \mathcal{F}_{nr,R}(K) \to \mathcal{F}(k(\mathfrak{p}))$, where K is the quotient field of R and $k(\mathfrak{p})$ is the residue field of R at the prime $\mathfrak{p}$.

Definition 1.4.1. *Let $Ev_{\mathfrak{p}} : C(R_{\mathfrak{p}}) \to C(k(\mathfrak{p}))$ and $ev_{\mathfrak{p}} : \mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(k(\mathfrak{p}))$ be the maps induced by $R_{\mathfrak{p}} \to k(\mathfrak{p})$. Define a homomorphism $s_{\mathfrak{p}} : \mathcal{F}_{nr,R}(K) \to \mathcal{F}(k(\mathfrak{p}))$ by $s_{\mathfrak{p}}(\alpha) = ev_{\mathfrak{p}}(\tilde{\alpha})$ where $\tilde{\alpha}$ is a lift of α in $\mathcal{F}(R_{\mathfrak{p}})$. Corollary 1.3.2 shows that the map $s_{\mathfrak{p}}$ is well-defined. It is called the specialization map. The map $ev_{\mathfrak{p}}$ is called the evaluation map at the prime $\mathfrak{p}$.*

Obviously for $\alpha \in C(R_{\mathfrak{p}})$ one has $s_{\mathfrak{p}}(\bar{\alpha}) = \overline{Ev_{\mathfrak{p}}(\alpha)} \in \mathcal{F}(k(\mathfrak{p}))$.

Lemma 1.4.2 ([H]). *Let H be a linear algebraic group over the field k. Let R be a k-algebra which is a Dedekind domain with quotient field K. If $\xi \in H^1_{ét}(K, H)$ is an R-unramified element for the functor $H^1_{ét}(-, H)$ then ξ can be lifted to an element of $H^1_{ét}(R, H)$.*

Proof. Patching.

□

Theorem 1.4.3. *Let K be a field of characteristic zero and let H be a linear algebraic group over K. Then the canonical map $H^1_{ét}(K, H) \to H^1_{ét}(K[t], H)$ is bijective.*

Proof. Let $i : \{0\} \hookrightarrow \mathbf{A}1_K$ be the origin of the affine line. Let $i^* : \mathrm{H}^1_{\text{ét}}(K[t], H) \to \mathrm{H}^1_{\text{ét}}(K, H)$ be the pull-back map. To prove the theorem we need the following

Claim 1.4.4. *The following two statements are equivalent.*

(1) For any linear algebraic group H over K the map i^* is injective.

(2) For any linear algebraic group H over K the map i^* has the trivial kernel.

Clearly the first statement implies the second one. Now suppose the second statement holds and prove the first one. For that consider $\xi, \xi' \in \mathrm{H}^1_{\text{ét}}(K[t], H)$ and set $\xi_0 = i^*(\xi), \xi'_0 = i^*(\xi')$. Let H_0 be the inner form of the group H with respect to the H-torsor $\mathcal{H}_0$. For any K-scheme S there is a well-known bijection $\phi_S : \mathrm{H}^1_{\text{ét}}(S, H) \to \mathrm{H}^1_{\text{ét}}(S, H_0)$ between non-pointed sets. It takes the H-torsor $\mathcal{H}_0 \times_K S$ to the trivial H_0-torsor $H_0 \times_K S$, where $\mathcal{H}_0$ is an H-torsor over K representing the class ξ_0. These bijections respect K-scheme morphisms. Suppose $\xi_0 = \xi'_0$, then

$$i^*(\phi_{\mathbf{A}1}(\xi)) = \phi_K(\xi_0) = * = \phi_K(\xi'_0) = i^*(\phi_{\mathbf{A}1}(\xi')) \in \mathrm{H}^1_{\text{ét}}(K, H_0).$$

So $\phi_{\mathbf{A}1}(\xi) = * = \phi_{\mathbf{A}1}(\xi') \in \mathrm{H}^1_{\text{ét}}(K[t], H_0)$. Whence $\xi = \xi'$ and the injectivity follows. With this Claim in hand prove the theorem as follows.

The surjectivity of the map i^* is obvious. To prove its injectivity just use the Claim and mimic the proof of [C-T/O, Thm.2.1, Property P2]. Let $p : \mathbf{A}1_K \to \mathrm{Spec}(K)$ be the structural morphism. Since $p \circ i = id$ and i^* is bijective p^* is bijective.

□

We need the following theorem.

Theorem 1.4.5 (Homotopy invariance). *Let k be the field and $\mu : G \to C$ be the algebraic group morphism from Theorem 1.0.1. Let $k \subset K$ be a field extension and $K(t)$ be the rational function field in one variable. Let $R \mapsto \mathcal{F}(R)$ be the functor defined in 5. Then one has*

$$\mathcal{F}(K) = \mathcal{F}_{nr,K[t]}(K(t)).$$

Proof. Injectivity is clear, because the composition

$$\mathcal{F}(K) \to \mathcal{F}_{un}(K(t)) \xrightarrow{s_0} \mathcal{F}(K)$$

coincides with the identity (here s_0 is the specialization map at the point zero defined in 4.6).

It remains to check the surjectivity. Let $a \in \mathcal{F}_{un}(K(t))$ and let $H = ker(\mu)$. Then by Lemma 1.3.2 the element $\partial(a) \in H^1_{et}(K(t), H)$ is a class which for every $x \in \mathbb{A}_K 1$ belongs to the image of $H^1_{et}(\mathcal{O}_x, H)$. Thus by Lemma 1.4.2, $\partial(a)$ can be represented by an element $\xi \in H^1_{et}(K[t], H)$, where $K[t]$ is the polynomial ring. By Theorem 1.4.3, the map

$$H^1_{et}(K, H) \to H^1_{et}(K[t], H)$$

is an isomorphism. Then $\xi = \rho(\xi_0)$ for an element $\xi_0 \in H^1_{et}(K, H)$. Consider the diagram

$$\begin{tikzcd}
& a \arrow[r, mapsto] & \xi \arrow[r, mapsto] & * \\
1 \arrow[r] & \mathcal{F}(K(t)) \arrow[r, "\partial"] & H^1_{et}(K(t), H) \arrow[r] & H^1_{et}(K(t), G) \\
1 \arrow[r] & \mathcal{F}(K) \arrow[r, "\partial"] \arrow[u, "\epsilon"] & H^1_{et}(K, H) \arrow[r] \arrow[u, "\rho"] & H^1_{et}(K, G) \arrow[u, "\eta"] \\
& a_0 \arrow[r, mapsto] & \xi_0 ,
\end{tikzcd}$$

in which all the maps are canonical and all the vertical arrows have trivial kernels. Since ξ goes to the trivial element in $H^1_{et}(K(t), G)$, one concludes that ξ_0 goes to the trivial element in $H^1_{et}(K, G)$. Thus there exists an element $a_0 \in \mathcal{F}(K)$ such that $\partial(a_0) = \xi_0$. The map $\mathcal{F}(K(t)) \to H^1_{et}(K(t), H)$ is injective by Lemma 1.3.3. Thus $\epsilon(a_0) = a$. □

Corollary 1.4.6. *Let $R \mapsto \mathcal{F}(R)$ be the functor defined in (5). Let*

$$s_0, s_1 : \mathcal{F}_{un}(K(t)) \to \mathcal{F}(K)$$

be the specialization maps at zero and at one (at the primes (t) and (t-1)). Then $s_0 = s_1$.

Proof. It is an obvious consequence of Theorem 1.4.5. □

1.5 Proof of Theorem (A)

Proof. By assumption there exists a smooth d-dimensional k-algebra $S = k[t_1, t_2, \dots, t_n]$ and a prime ideal $\mathfrak{p}$ in R such that $A = S_{\mathfrak{p}}$. We first reduce the proof to the case in which $\mathfrak{p}$ is maximal. To do this, choose a maximal ideal m containing p. Since k is of characteristic zero it is infinite. By a standard general position argument we can find d algebraically independent elements $X_1, \dots, X_d$ such that S is finite over $k[X_1, \dots, X_d]$ and étale at m. After a linear change of coordinates we may assume that S/p is finite over $B = k[X_1, \dots, X_m]$, where m is the dimension of S/p. Clearly S is smooth over B at m and thus, for some $h \in S \setminus m$, the localization S_h is smooth over B. Let $\mathcal{S}$ be the set of nonzero elements of B, $k' = \mathcal{S}^{-1}B$ the field of fractions of B and $S' = \mathcal{S}^{-1}S_h$. The prime ideal $p' = \mathcal{S}^{-1}p_h$ is maximal in S', the k'-algebra S' is smooth and $S = S'_{p'}$.

From now on and till the end of this proof we assume that $A = \mathcal{O}_{X,x}$ is the local ring of a closed point x of a smooth d-dimensional irreducible affine variety X over k. Replacing k by its algebraic closure in $k[X]$ we may assume furthermore that this X is a k-smooth geometrically irreducible affine variety over k.

Let $\xi \in C(K)$ be such that the $\bar{\xi} \in \mathcal{F}(K)$ is A-unramified. We may assume that $\xi \in C(k[X]_{\mathrm{f}})$ for an appropriative non-zero $\mathrm{f} \in k[X]$ with a reduced k-algebra $k[X]/(\mathrm{f})$. Shrinking X we may assume that $\bar{\xi} \in \mathcal{F}(k[X]_{\mathrm{f}})$ is $k[X]$-unramified. By Lemma 1.2.5 we can find a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\varrho"] \arrow[dr, "q"'] & & \mathbf{A}^d \arrow[dl, "pr"] \\ & \mathbf{A}^{d-1} & \end{tikzcd} \tag{6}$$

with a linear projection pr and a function $\mathrm{h} \in k[X]$ such that the conditions (1) to (5) of are satisfied. Let $U = \mathrm{Spec}(A)$ the point $x \in X$ is the closed point of U. Consider the Cartesian square of schemes

$$\begin{tikzcd} & \mathcal{X} \arrow[r, "p_X"] \arrow[d, "p"] & X \arrow[d, "q"] \\ U \arrow[ur, "\delta"] \arrow[r, "id"] & U \arrow[r, "r"] & \mathbf{A}^{d-1} \end{tikzcd}$$

where $r = q|_U$, $\mathcal{X} = U \times_{\mathbf{A}^{d-1}} X$, p is the first projection and δ is the diagonal map.

Applying base change by means of r to the diagram (6) we get a commutative triangle

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\lambda"] \arrow[dr, "p"'] & & U \times \mathbf{A}1 \arrow[dl, "pr"] \\ & U & \end{tikzcd} \tag{7}$$

with a finite surjective λ. Set $f = p_X^*(\mathrm{f})$ and $h = p_X^*(\mathrm{h})$, where h is from the condition (1) of Lemma 1.2.5.

Now check that the field k, the k-algebra A with the maximal ideal $\mathfrak{m} := \mathfrak{m}_x$, the ring R of regular functions on the affine scheme $\mathcal{X}$, the inclusion $\lambda^* : A[t] \hookrightarrow R$ induced by λ, the function h, the A-augmentation $\epsilon := \Delta^* : R \to A$ and the function f satisfy the assumption of Lemma 1.2.3.

As a local ring of a k-smooth variety A is essentially k-smooth. Since r is essentially smooth and X is k-smooth, $\mathcal{X}$ is essentially k-smooth and so is R. Since ϱ is finite R is finite over $A[t]$ too. Since $k(q(x)) = k(x)$ and the fibre of q over $q(x)$ is geometrically irreducible so is the fibre of p over $x \in U$. This implies that $R/\mathfrak{m}$ is a domain. By base change, the open subscheme $\mathcal{X}_h$ contains the closed fibre of p and the restriction of p to $\mathcal{X}_h$ is smooth. The assumptions are satisfied.

Now we can find an $s \in R$ satisfying the conditions (1) to (7) of Lemma 1.2.3. The function s defines a finite morphism $\pi : \mathcal{X} \to U \times \mathbf{A}1$. Since $\mathcal{X}$ and $\mathcal{Y} := U \times \mathbf{A}1$ are regular schemes and π is finite surjective it is finite and flat by a theorem of Grothendieck [E, Cor.18.17]. So for any map $f : \mathcal{Y}' \to \mathcal{Y}$ of affine schemes, putting $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ we have the norm map given by (2)

$$C(\mathcal{X}') \to C(\mathcal{Y}').$$

Set $\mathcal{D} = \mathrm{Spec}(R/J)$ and $\mathcal{D}_1 = \mathrm{Spec}(R/(s-1)R)$. Clearly $\mathcal{D}_1$ is the scheme theoretic pre-image of $U \times \{1\}$ and the disjoint union $\mathcal{D}_0 := \mathcal{D} \coprod \delta(U)$ is the scheme theoretic pre-image of $U \times \{0\}$ under the morphism π. In particular $\mathcal{D}$ is finite flat over U. We will write Δ for $\delta(U)$. Recall that $\xi \in C(k[X]_{\mathrm{f}})$ and set $\zeta = p_X^*(\xi) \in C(R_f)$. Since the function $f \in R$ is co-prime to the ideals J and $(s-1)R$ one can form the following element

$$\xi' = N_{\mathcal{D}_1/U}(\zeta|_{\mathcal{D}_1}) N_{\mathcal{D}/U}(\zeta|_{\mathcal{D}})^{-1} \in C(U) := C(A). \tag{8}$$

Claim 1.5.1 (Main). *One has $\bar{\xi}'_K = \bar{\xi}_K$ in $\mathcal{F}(K)$, where K is the fraction field of both $k[X]$ and $k[U] = A$.*

To complete the proof, it remains to prove the claim. To do this it is convenient to fix some notations. For an A-module M set $M_K = M \otimes_A K$, for an $\mathcal{Y}$-scheme $\mathcal{Z}$ set $\mathcal{Z}_K = \mathrm{Spec}(K) \times_{\mathcal{Y}} \mathcal{Z}$, for a $\mathcal{Y}$-morphism $\varphi : \mathcal{Z} \to \mathcal{W}$ set $\varphi_K = \mathrm{Spec}(K) \times_{\mathcal{Y}} \varphi$. Clearly one has $A_K = K$, $U_K = \mathrm{Spec}(K)$, $\mathcal{X}_K = \mathrm{Spec}(R_K)$ and $(\mathbf{A}1 \times U_K)_K$ is just the affine line $\mathbf{A}1_K$. Its closed subschemes $U_K \times \{1\}$ and $U_K \times \{0\}$ coincide with the points 1 and 0 of $\mathbf{A}1_K$. The morphism $p_K : \mathcal{X}_K \to \mathrm{Spec}(K)$ is smooth by the condition (1) of Lemma 1.2.3. The morphism π_K is finite flat and fits in the commutative triangle

$$\begin{tikzcd} \mathcal{X}_K \arrow[rr, "\pi_K"] \arrow[dr, "p_K"'] & & \mathbf{A}1_K \arrow[dl] \\ & \mathrm{Spec}(K) & \end{tikzcd} \tag{9}$$

One has $\mathcal{D}_K = \mathrm{Spec}(R_K/J_K)$, $\mathcal{D}_{1,K} = \mathrm{Spec}(R_K/(s-1)R_K)$. Clearly $\mathcal{D}_{1,K}$ is the scheme theoretic pre-image of the point $\{1\} \in \mathbf{A}1_K$ and the disjoint union $\mathcal{D}_{0,K} := \mathcal{D}_K \coprod \Delta_K$ is the scheme theoretic pre-image of the point $\{0\} \in \mathbf{A}1_K$ under the morphism π_K. Let

$$\zeta_s := N_{K(\mathcal{X}_K)/K(\mathbf{A}1)}(\zeta) \in C(K(\mathbf{A}1)) = C(K(s)).$$

To prove Claim 1.5.1 we need the following one:

Claim 1.5.2. *The class $\bar{\zeta}_s \in \mathcal{F}(K(s))$ is $K[s]$-unramified.*

Assuming this claim we complete the proof of Claim 1.5.1. By Claim 1.5.2 we can apply the specialization maps to $\bar{\zeta}_s$. By Corollary 1.4.6 the specializations at 0 and 1 of the element $\bar{\zeta}_s$ coincide, that is

$$s_1(\bar{\zeta}_s) = s_0(\bar{\zeta}_s) \in \mathcal{F}(K). \tag{10}$$

By Corollary 1.2.4 the function $N_{R_K/K[s]}(f) \in K[s]$ does not vanishes as at 1, so at 0 and by the condition (5) of Lemma 1.2.3 one has $\zeta_s \in C(K[s]_{N(f)})$. Using the relation between specialization and evaluation maps described in Definition 1.4.1 one has a chain of relations

$$\overline{Ev_1(\zeta_s)} = s_1(\bar{\zeta}_s) = s_0(\bar{\zeta}_s) = \overline{Ev_0(\zeta_s)} \in \mathcal{F}(K). \tag{11}$$

Base change and the multiplicativity properties of the norm map (2) imply relations

$$Ev_1(\zeta_s) = N_{\mathcal{D}_{1,K}/U_K}(\zeta|_{\mathcal{D}_{1,K}}) \tag{12}$$

and

$$Ev_0(\zeta_s) = N_{\mathcal{D}_K \coprod \Delta_K/U_K}(\zeta|_{\mathcal{D}_K \coprod \Delta_K}) = N_{\mathcal{D}_K/U_K}(\zeta|_{\mathcal{D}_K}) N_{\Delta_K/U_K}(\zeta|_{\Delta_K}) \tag{13}$$

So we have a chain of relations in $\mathcal{F}(K)$

$$\overline{N_{\mathcal{D}_{1,K}/U_K}(\zeta|_{\mathcal{D}_{1,K}})} = \overline{Ev_1(\zeta_s)} = \overline{Ev_0(\zeta_s)} = \overline{N_{\mathcal{D}_K/U_K}(\zeta|_{\mathcal{D}_K})} \cdot \overline{N_{\Delta_K/U_K}(\zeta|_{\Delta_K})}$$

By the normalization property of the norm map (2) one has $N_{\Delta_K/U_K}(\zeta|_{\Delta_K}) = \delta_K^*(\zeta)$. Since $\delta_K^*(\zeta) = \xi_K \in C(K)$ we have a chain of relations in $\mathcal{F}(K)$

$$\bar{\xi}_K = \overline{\delta_K^*(\zeta)} = \overline{N_{\Delta_K/U_K}(\zeta|_{\Delta_K})} = \overline{N_{\mathcal{D}_{1,K}/U_K}(\zeta|_{\mathcal{D}_{1,K}})} \cdot (\overline{N_{\mathcal{D}_K/U_K}(\zeta|_{\mathcal{D}_K})})^{-1}$$

By the base change property of the norm map (2) one has

$$\xi'_K = N_{\mathcal{D}_{1,K}/U_K}(\zeta|_{\mathcal{D}_{1,K}}) N_{\mathcal{D}_K/U_K}(\zeta|_{\mathcal{D}_K})^{-1} \in C(U_K) := C(K). \tag{14}$$

So $\bar{\xi}'_K = \bar{\xi}_K$. Claim 1.5.1 follows.

It remains to prove Claim 1.5.2. Recall that the class $\bar{\xi} \in \mathcal{F}(k[X]_{\mathrm{f}})$ is $k[X]$-unramified. Since $q|_{X_{\mathrm{h}}} : X_{\mathrm{h}} \to \mathbf{A}^{d-1}$ is flat (it is smooth) so is $r : U \hookrightarrow X_{\mathrm{h}} \to \mathbf{A}^{d-1}$. Since the $p_X : \mathcal{X} \to X$ is the base change of r it coincides with a composition map $\mathcal{X} \hookrightarrow X_{\mathrm{h}} \times_{\mathbf{A}^{d-1}} X \xrightarrow{q \times id} X$ where $q \times id$ is a flat morphism of finite type. By Lemma 1.3.6 the class $\bar{\zeta} \in \mathcal{F}(R_f)$ is R-unramified. Thus the same class $\bar{\zeta}$ when regarded in $\mathcal{F}(K[\mathcal{X}_K]_f)$ is $K[\mathcal{X}_K]$-unramified.

Now check that the inclusion of K-algebras $K[s] \subset R_K = K[\mathcal{X}_K]$ and the function $f \in R$ satisfy the hypotheses of Lemma 1.3.5. We first check that the K-algebra R_K/fR_K is reduced. Recall that the ring $k[X]/(\mathrm{f})$ is reduced. As we mentioned above, the morphism $p_X : \mathcal{X} \to X$ is essentially smooth, hence the ring $R/(f)$ is reduced too and thus its localization R_K/fR_K is reduced. Since the extension $K[s] \subset R_K = K[\mathcal{X}_K]$ and the functions $f \in R$ satisfy the conditions (5) to (7) of Lemma 1.2.3 they also satisfy the hypotheses of Lemma 1.3.5. Thus by Lemma 1.3.5 the class $\bar{\zeta}_s$ is $K[s]$-unramified. This implies Claim 1.5.2. The proof of the theorem is completed.

□

1.6 Proof of Theorem (B)

Proof. To prove Theorem (B) we now recall a celebrated result of Dorin Popescu (see [P] or, for a self-contained proof, [Sw]).

Let k be a field and R a local k-algebra. We say that R is *geometrically regular* if $k' \otimes_k R$ is regular for any finite extension k' of k. A ring homomorphism $A \to R$ is called *geometrically regular* if it is flat and for each prime ideal $\mathfrak{q}$ of R lying over $\mathfrak{p}$, $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} = k(\mathfrak{p}) \otimes_A R_{\mathfrak{q}}$ is geometrically regular over $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$.

Observe that any regular local ring containing a field k is geometrically regular over the prime field of k.

Theorem 1.6.1 (Popescu's theorem). *A homomorphism $A \to R$ of noetherian rings is geometrically regular if and only if R is a filtered direct limit of smooth A-algebras.*

Proof of Theorem B. Let R be a regular local ring containing the field k. Since k is of characteristic zero one can apply Popescu's theorem. So R can be presented as a filtered direct limit of smooth k-algebras A_α, We first observe that we may replace the direct system of the A_α's by a system of essentially smooth local k-algebras. In fact, if $\mathfrak{m}$ is the maximal ideal of R, we can replace each A_α by $(A_\alpha)_{\mathfrak{p}_\alpha}$, where $\mathfrak{p}_\alpha = \mathfrak{m} \cap A_\alpha$. Note that in this case the canonical morphisms $\varphi_\alpha : A_\alpha \to R$ are local and every A_α is a regular local ring, in particular a factorial ring.

Let now L be the field of fractions of R and, for each α, let K_α be the field of fractions of A_α. For each index α let $\mathfrak{a}_\alpha$ be the kernel of the map $\varphi_\alpha : A_\alpha \to R$ and $B_\alpha = (A_\alpha)_{\mathfrak{a}_\alpha}$. Clearly, for each α, K_α is the field of fractions of B_α. The composition map $A_\alpha \to R \to L$ factors through B_α and hence it also factors through the residue field k_α of B_α. Since R is a filtering direct limit of the A_α's we see that L is a filtering direct limit of the B_α's.

Let $\xi \in C(L)$ be such that the class $\bar{\xi} \in \mathcal{F}(L)$ is R-unramified. We need the following two lemmas.

Lemma 1.6.2. *Let $B = \mathcal{O}_{X,x}$ be the local ring of a k-smooth variety at a point x. Let $k(x)$ be its residue field and K be its field of fractions. Let $\eta, \rho \in \mathcal{F}(B)$ be such that $\eta_K = \rho_K \in \mathcal{F}(K)$. Then $\eta(x) = \rho(x) \in \mathcal{F}(k(x))$.*

Lemma 1.6.3. *There exists an index α and an element $\xi_\alpha \in C(B_\alpha)$ such that the class $\bar{\xi}_\alpha \in \mathcal{F}(K_\alpha)$ is A_α-unramified.*

Assuming these two claims we complete the proof as follows. Consider a commutative diagram

$$\begin{tikzcd}
A_\alpha \arrow[rr, "\varphi_\alpha"] \arrow[d] & & R \arrow[d] \\
B_\alpha \arrow[r] \arrow[d] & k_\alpha \arrow[r] & L \\
K_\alpha. & &
\end{tikzcd}$$

By Lemma 1.6.3 the class $\bar{\xi}_\alpha \in \mathcal{F}(K_\alpha)$ is A_α-unramified. Hence by Theorem A there exists an element $\eta \in C(A_\alpha)$ such that $\bar{\xi}_\alpha = \bar{\eta} \in \mathcal{F}(K_\alpha)$. By Lemma 1.6.2 the elements $\bar{\xi}_\alpha$ and $\bar{\eta}$ have the same image in $\mathcal{F}(k_\alpha)$. Hence $\bar{\xi} \in \mathcal{F}(L)$ coincides with the image of the element $\varphi_\alpha(\bar{\eta})$ in $\mathcal{F}(L)$. It remains to prove the two Lemmas.

Proof of Lemma 1.6.2. Induction on $dim(B)$. The case of dimension 1 follows fromTheorem 1.3.1. To prove the general case choose an $f \in B$ such that $\eta = \rho \in \mathcal{F}(B_f)$. Let π be a regular parameter in B without common factors with f and let $\bar{B} = B/\pi B$. Then for the image $\overline{\eta - \rho}$ of $\eta - \rho$ in $\mathcal{F}(\bar{B})$ we have $(\overline{\eta - \rho})_f = \overline{(\eta - \rho)_f} = 0 \in \mathcal{F}(\bar{B}_{\bar{f}})$. By the inductive hypotheses one has $\overline{\eta - \rho} = 0 \in \mathcal{F}(\bar{B})$. Thus $\eta(x) = \rho(x) \in \mathcal{F}(k(x))$.

Proof of Lemma 1.6.3. Choose an $f \in R$ such that ξ is defined over R_f. Then ξ is ramified at most at those hight one primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ which contains f. Since the class $\bar{\xi} \in \mathcal{F}(L)$ is R-unramified there exists, for any $\mathfrak{p}_i$, an element $\sigma_i \in G(L)$ and an element $\xi_i \in C(R_{\mathfrak{p}_i})$ such that $\xi = \mu(\sigma_i)\xi_i \in C(L)$. We may assume that ξ_i is defined over R_{h_i} for some $h_i \in R - \mathfrak{p}_i$ and that σ_i is defined over R_{g_i} for some $g_i \in R$.

We can find an index α such that A_α contains lifts $f_\alpha, h_{1,\alpha}, \dots, h_{r,\alpha}, g_{1\alpha}, \dots, g_{r,\alpha}$ and moreover

(1) $C(A_{\alpha,f_\alpha})$ contains a lift ξ_α of ξ,

(2) $C(A_{\alpha,h_{i,\alpha}})$ contains a lift of $\xi_{i,\alpha}$ of ξ_i,

(3) $G(A_{\alpha,g_{i,\alpha}})$ contains a lift of $\sigma_{i,\alpha}$ of σ_i.

Since none of the $f_\alpha, h_{1,\alpha}, \dots, h_{r,\alpha}, g_{1\alpha}, \dots, g_{r,\alpha}$ vanishes in R, the elements $\xi_\alpha, \xi_{1,\alpha}, \dots, \xi_{ir,\alpha}$ and $\sigma_{1,\alpha}, \dots, \sigma_{r,\alpha}$ may be regarded as elements of $C(B_\alpha)$ and $G(B_\alpha)$ respectively.

We know that $\xi_{i,\alpha}\mu(\sigma_{i,\alpha})$ and ξ_α map to the same element in $C(L)$. Hence replacing α by a larger index, we may assume that $\xi_\alpha = \xi_{i,\alpha}\mu(\sigma_{i,\alpha}) \in C(B_\alpha)$. We claim that the class $\bar{\xi}_\alpha \in \mathcal{F}(K_\alpha)$ is A_α-unramified. To prove this note that the only primes at which $\bar{\xi}_\alpha$ could be ramified are those which divide f_α. Let $\mathfrak{q}_\alpha$ be one of them. Check that $\bar{\xi}_\alpha$ is unramified at $\mathfrak{q}_\alpha$. Let $q_\alpha \in A_\alpha$ be a prime element such that $q_\alpha A_\alpha = \mathfrak{q}_\alpha$. Then $q_\alpha r_\alpha = f_\alpha$ for an element r_α. Thus $qr = f \in R$ for the images of q_α and r_α in R. Since the homomorphism $\varphi_\alpha : A_\alpha \to R$ is local, $q \in \mathfrak{m}_R$. The relation $qr = f$ shows that $q \in \mathfrak{p}_i$ for some index i. Thus $q_\alpha \in \varphi_\alpha^{-1}(\mathfrak{p}_i)$ and $\mathfrak{q}_\alpha \subset \varphi_\alpha^{-1}(\mathfrak{p}_i)$. On the other hand $h_{i,\alpha} \in A_\alpha - \varphi_\alpha^{-1}(\mathfrak{p}_i)$, because $h_i \in R - \mathfrak{p}_i$. Thus $h_{i,\alpha} \in A_\alpha - \mathfrak{q}_\alpha$. Now the relation $\xi_\alpha = \xi_{i,\alpha}\mu(\sigma_{i,\alpha}) \in C(B_\alpha)$ with $\xi_{i,\alpha} \in C(A_{\alpha,h_{i,\alpha}})$ shows that $\bar{\xi}_\alpha$ is unramified at $\mathfrak{q}_\alpha$. Thus $\bar{\xi}_\alpha$ is unramified at each hight one prime in A_α containing f_α. Since $\xi_\alpha \in C(A_{\alpha,f_\alpha})$ we conclude that $\bar{\xi}_\alpha$ is A_α-unramified. The lemma follows. The theorem is proved.

□

2 Applications

In this Section we prove a purity theorem for reductive groups. Let G be a reductive group over the characteristic zero field k and $Z \xrightarrow{i} G$ a closed central subgroup of G. Let $G' = G/Z$ be the factor group, $\pi : G \to G'$ be the projection. For any k-algebra R consider the boundary operator $\delta_{\pi,R} : G'(R) \to \mathrm{H}^1_{\text{ét}}(R, Z)$. It is a group homomorphism [Se, Ch.II, §5.6, Cor.2]. Set

$$\mathcal{F}(R) = \mathrm{H}^1_{\text{ét}}(R, Z)/Im(\delta_{\pi,R}).$$

Theorem 2.0.4. *Let R be a regular local ring containing the field k. The functor $\mathcal{F}$ satisfies purity for R. If K is the fraction field of R this statement can be restated in an explicit way as follows:*

given an element $\xi \in H^1_{ét}(K, Z)$ suppose that for each height 1 prime ideal $\mathfrak{p}$ in R there exist $\xi_\mathfrak{p} \in H^1_{ét}(R_\mathfrak{p}, Z)$, $g_\mathfrak{p} \in G'(K)$ with $\xi = \xi_\mathfrak{p} + \delta_\pi(g_\mathfrak{p}) \in H^1_{ét}(K, Z)$. Then there exist $\xi_\mathfrak{m} \in H^1_{ét}(R, Z)$, $g_\mathfrak{m} \in G'(K)$, such that

$$\xi = \xi_\mathfrak{m} + \delta_\pi(g_\mathfrak{m}) \in H^1_{ét}(K, Z).$$

Proof. Since G is reductive the group Z is of multiplicative type. So we can find a commutative separable k algebra l and a closed embedding $Z \hookrightarrow R_{l/k}(\mathbb{G}_{m,\ l})$ into the permutation torus $T^+ = R_{l/k}(\mathbb{G}_{m,\ l})$. Let $G^+ = (G \times T^+)/Z$ and $T = T^+/Z$, where Z is embedded in $G \times T^+$ diagonally. Clearly $G^+/G = T$. Consider a commutative diagram

$$\begin{tikzcd}
 & \{1\} & \{1\} & & \\
 & G' \arrow[u] \arrow[r, "id"] & G' \arrow[u] & & \\
\{1\} \arrow[r] & G \arrow[u, "\pi"] \arrow[r, "j^+"] & G^+ \arrow[u, "\pi^+"] \arrow[r, "\mu^+"] & T \arrow[r] & \{1\} \\
\{1\} \arrow[r] & Z \arrow[u, "i"] \arrow[r, "j"] & T^+ \arrow[u, "i^+"] \arrow[r, "\mu"] & T \arrow[u, "id"'] \arrow[r] & \{1\} \\
 & \{1\} \arrow[u] & \{1\} \arrow[u] & &
\end{tikzcd}$$

with exact rows and columns. For a local k-algebra A one has $\mathrm{H}^1_{\text{ét}}(A, T^+) = \{*\}$ by Hilbert 90 and this diagram gives rise to a commutative diagram of pointed sets

$$\begin{tikzcd}
 & & \mathrm{H}^1_{\text{ét}}(A, G') \arrow[r, "id"] & \mathrm{H}^1_{\text{ét}}(A, G') \\
G^+(A) \arrow[r, "\mu^+_A"] & T(A) \arrow[r, "\delta^+_A"] & \mathrm{H}^1_{\text{ét}}(A, G) \arrow[u, "\pi_*"] \arrow[r, "j^+_*"] & \mathrm{H}^1_{\text{ét}}(A, G^+) \arrow[u, "\pi^+_*"] \\
T^+(A) \arrow[u, "i^+_*"] \arrow[r, "\mu_A"] & T(A) \arrow[u, "id"] \arrow[r, "\delta_A"] & \mathrm{H}^1_{\text{ét}}(A, Z) \arrow[u, "i_*"] \arrow[r, "\mu"] & \{*\} \arrow[u, "i^+_*"] \\
 & & G'(A) \arrow[u, "\delta_\pi"] &
\end{tikzcd}$$

with exact rows and columns. It follows that π^+_* has trivial kernel and one has a chain of group isomorphisms

$$\mathrm{H}^1_{\text{ét}}(A, Z)/Im(\delta_{\pi,A}) = ker(\pi_*) = ker(j^+_*) = T(A)/\mu^+(G^+(A)).$$

Clearly these isomorphisms respect k-homomorphisms of local k-algebras. The functor $A \mapsto T(A)/\mu^+(G^+(A))$ satisfies purity for the regular local k-algebra R by Theorem (B). Hence the functor $A \mapsto \mathrm{H}^1_{\text{ét}}(A, Z)/Im(\delta_{\pi,A})$ satisfies purity for R. $\square$

3 Examples

We follow here the notation of The Book of Involutions [KMRT]. The field k is a characteristic zero field. The functors (15) to (29) satisfy purity for regular local rings containing k as follows either from Theorem 2.0.4 or from Theorem (B).

(1) Let G be a simple algebraic group over the field k, Z a central subgroup, $G' = G/Z$, $\pi : G \to G'$ the canonical morphism. For any k-algebra A let $\delta_{\pi,R} : G'(R) \to \mathrm{H}^1_{\text{ét}}(R, Z)$ be the boundary operator. One has a functor

$$R \mapsto \mathrm{H}^1_{\text{ét}}(R, Z)/\mathrm{Im}(\delta_{\pi,R}). \tag{15}$$

(2) Let (A, σ) be a finite separable k-algebra with an orthogonal involution. Let $\pi : \mathrm{Spin}(A, \sigma) \to \mathrm{PGO}^+(A, \sigma)$ be the canonical morphism of the spinor k-group scheme to the projective orthogonal k-group scheme. Let $Z = ker(\pi)$. For a k-algebra R let $\delta_R : \mathrm{PGO}^+(A, \sigma)(R) \to \mathrm{H}^1_{\text{ét}}(R, Z)$ be the boundary operator. One has a functor

$$R \mapsto \mathrm{H}^1_{\text{ét}}(R, Z)/Im(\delta_R). \tag{16}$$

In $(2a)$ and $(2b)$ below we describe this functor somewhat more explicitly following [KMRT].

(2a) Let $C(A, \sigma)$ be the Clifford algebra. Its center l is an étale quadratic k-algebra. Assume that $deg(A)$ is divisible by 4. Let $\Omega(A, \sigma)$ be the extended Clifford group [KMRT, Definition given just below (13.19)]. Let $\underline{\sigma}$ be the canonical involution of $C(A, \sigma)$ as it is described in [KMRT, just above (8.11)]. Then $\underline{\sigma}$ is either orthogonal or symplectic by [KMRT, Prop.8.12]. Let $\underline{\mu} : \Omega(A, \sigma) \to R_{l/k}(\mathbb{G}_{m,l})$ be the multiplier map defined in [KMRT, just above (13.25)] by $\underline{\mu}(\omega) = \underline{\sigma}(\omega) \cdot \omega$. Set $R_l = R \otimes_k l$. For a field or a local ring R one has $\mathrm{H}^1_{\text{ét}}(R, Z)/Im(\delta_R) = R_l^\times / \underline{\mu}(\Omega(A, \sigma)(R))$ by [KMRT, the diagram in (13.32)]. Consider the functor

$$R \mapsto R_l^\times / \underline{\mu}(\Omega(A, \sigma)(R)). \tag{17}$$

It coincides with the functor $R \mapsto \mathrm{H}^1_{\text{ét}}(R, Z)/Im(\delta_R)$ on local rings containing k.

(2b) Now let $deg(A) = 2m$ with odd m. Let $\tau : l \to l$ be the involution of l/k. The kernel of the morphism $R_{l/k}(\mathbb{G}_{m,l}) \xrightarrow{id-\tau} R_{l/k}(\mathbb{G}_{m,l})$ coincides with $\mathbb{G}_{m,k}$. Thus $id - \tau$ induces a k-group scheme morphism which we denote $\overline{id - \tau} : R_{l/k}(\mathbb{G}_{m,l})/\mathbb{G}_{m,k} \hookrightarrow R_{l/k}(\mathbb{G}_{m,l})$. Let $\underline{\mu} : \Omega(A, \sigma) \to R_{l/k}(\mathbb{G}_{m,l})$ be the multiplier map defined in [KMRT, just above (13.25)] by $\underline{\mu}(\omega) = \underline{\sigma}(\omega) \cdot \omega$. Let $\kappa : \Omega(A, \sigma) \to R_{l/k}(\mathbb{G}_{m,l})/\mathbb{G}_{m,k}$ be the k-group scheme morphism described in [KMRT, Prop.13.21]. The composition $\overline{(id - \tau)} \circ \kappa$ lands in $R_{l/k}(\mathbb{G}_{m,l})$. Let $U \subset \mathbb{G}_{m,k} \times R_{l/k}(\mathbb{G}_{m,l})$ be a closed k-subgroup consisting of all (α, z) such that $\alpha^4 = N_{l/k}(z)$.

Set $\mu_* = (\underline{\mu}, [\overline{(id - \tau)} \circ \kappa] \cdot \underline{\mu}2) : \Omega(A, \sigma) \to \mathbb{G}_{m,k} \times R_{l/k}(\mathbb{G}_{m,l})$. This k-group scheme morphism lands in U. So we get a k-group scheme morphism $\mu_* : \Omega(A, \sigma) \to \mathrm{U}$. On the level of k-rational points it coincides with the one described in [KMRT, just above (13.35)]. For a field or a local ring one has

$$\mathrm{H}^1_{\text{ét}}(R, Z)/Im(\delta_R) = \mathrm{U}(R)/[\{(N_{l/k}(\alpha), \alpha^4) | \alpha \in R_l^\times\} \cdot \mu_*(\Omega(A, \sigma)(R))].$$

Consider the the functor

$$R \mapsto \mathrm{U}(R)/[\{(N_{l/k}(\alpha), \alpha^4) | \alpha \in R_l^\times\} \cdot \mu_*(\Omega(A, \sigma)(R))]. \tag{18}$$

It coincides with the functor $R \mapsto \mathrm{H}^1_{\text{ét}}(R, Z)/Im(\delta_R)$ on local rings containing k.

(3) Let $\Gamma(A,\sigma)$ be the Clifford group k-scheme of (A,σ). Let $\mathrm{Sn} : \Gamma(A,\sigma) \to \mathbb{G}_{m,k}$ be the spinor norm map. It is dominant. Consider the functor

$$R \mapsto R^{\times}/\mathrm{Sn}(\Gamma(A,\sigma)(R)). \tag{19}$$

Purity for this functor was originally proved in [Z, Thm.3.1]. In fact, $\Gamma(A,\sigma)$ is k-rational.

(4) We follow here the Book of Involutions [KMRT, §23]. Let A be a separable finite dimensional k-algebra with center l and k-involution σ such that k coincides with all σ-invariant elements of l, that is $k = l^{\sigma}$. Consider the k-group schemes of similitudes of (A,σ):

$$\mathrm{Sim}(A,\sigma)(R) = \{a \in A_R^{\times} \mid a \cdot \sigma_R(a) \in l_K^{\times}\}.$$

We have a k-group scheme morphism $\mu : \mathrm{Sim}(A,\sigma) \to \mathbb{G}_{m,k},\ a \mapsto a \cdot \sigma(a)$. It gives an exact sequence of algebraic k-groups

$$\{1\} \to \mathrm{Iso}(A,\sigma) \to \mathrm{Sim}(A,\sigma) \to \mathbb{G}_{m,k} \to \{1\}.$$

One has a the functor

$$R \mapsto R^{\times}/\mu(\mathrm{Sim}(A,\sigma)(R)). \tag{20}$$

Purity for this functor was originally proved in [Pa, Thm.1.2]. Various particular cases are obtained considering unitary, symplectic and orthogonal involutions.

(4a) In the case of an orthogonal involution σ the connected component $\mathrm{GO}^{+}(A,\sigma)$ [KMRT, (12.24)] of the similitude k-group scheme $\mathrm{GO}(A,\sigma) := \mathrm{Sim}(A,\sigma)$ has the index two in $\mathrm{GO}(A,\sigma)$. The restriction of μ to $\mathrm{GO}^{+}(A,\sigma)$ is still a dominant morphism to $\mathbb{G}_{m,k}$. One has a functor

$$R \mapsto R^{\times}/\mu(\mathrm{GO}^{+}(A,\sigma)(R)) \tag{21}$$

It seems that its purity does not follow from [Pa, Thm.1.2]. In fact we do not know whether the norm principle holds for $\mu : \mathrm{GO}^{+}(A,\sigma) \to \mathbb{G}_{m,k}$ or not.

(5) Let A be a central simple algebra (csa) over k and $\mathrm{Nrd} : GL_{1,A} \to \mathbb{G}_{m,k}$ the reduced norm morphism. One has a functor

$$R \mapsto R^{\times}/\mathrm{Nrd}(\mathrm{GL}_{1,A}(R)). \tag{22}$$

Purity for this functor was originally proved in [C-T/O, Thm.5.2].

(6) Let (A,σ) be a finite separable k-algebra with a unitary involution such that its center l is a quadratic extension of k. Let $U(A,\sigma)$ be the unitary k-group scheme. Let $U_l(1)$ be an algebraic tori given by $N_{l/k} = 1$. One has a functor

$$R \mapsto \mathrm{U}_l(1)(R)/\mathrm{Nrd}(\mathrm{U}_{A,\sigma}(R)) = \{\alpha \in R_l^{\times} \mid N_{l/k}(\alpha) = 1\}/\mathrm{Nrd}(\mathrm{U}_{A,\sigma}(R)) \tag{23}$$

where Nrd is the reduced norm map. Purity for this functor was originally proved in [Z, Thm.3.3].

(7) With the notation of example (5) choose an integer d and consider the k-group scheme morphism $\mu : \mathrm{GL}_{1,A} \times \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ given by $(\alpha, a) \mapsto \mathrm{Nrd}(\alpha) \cdot a^d$. One has a functor

$$R \mapsto R^\times/\mu[\mathrm{GL}_{1,A} \times \mathbb{G}_{m,k})(R)] = R^\times/\mathrm{Nrd}(A_R^\times) \cdot R^{\times d} \tag{24}$$

Purity for this functor was originally proved in [Z, Thm.3.2].

(8) With the notation of example (5) choose an integer d and consider the functor

$$R \mapsto U_l(1)(R)/[\mathrm{Nrd}(U_{A,\sigma}(R)) \cdot \{\alpha \in R_l^\times | N_{l/k}(\alpha) = 1\}^d] \tag{25}$$

Purity for this functor was originally proved in [Z, Thm.3.2].

(9) Let G_1,G_2,C be affine k-group schemes. Assume that C is commutative and let $\mu_1 : G_1 \to C$, $\mu_2 : G_2 \to C$ be k-group scheme morphisms and $\mu : G_1 \times G_2 \to C$ be given by $\mu(g_1, g_2) = \mu_1(g_1) \cdot \mu_2(g_2)$. One has a functor

$$R \mapsto C(R)/\mu[(G_1 \times G_2)(R)] = C(R)/[\mu_1(G_1(R)) \cdot \mu_2(G_2(R))] \tag{26}$$

In this style one could get a lot of curious examples of functors, one of which is given here.

(10) Let (A_1, σ_1) be a finite separable k-algebra with an orthogonal involution. Let A_2 be a csa over k. Let μ_1 be the multiplier map for (A_1, σ_1) and Nrd_2 be the reduced norm for A_2. One has a functor

$$R \mapsto R^\times/[\mu_1(\mathrm{GO}^+(A_1, \sigma_1)(R)) \cdot \mathrm{Nrd}_2(A_{2,R}^\times) \cdot R^{\times d}]. \tag{27}$$

(10) Let A be a csa of degree 3 over k, Nrd the reduced norm and Trd be the reduced trace. Consider the cubic form on the 27-dimensional k-vector space $A \times A \times A$ given by $N := \mathrm{Nrd}(x) + \mathrm{Nrd}(y) + \mathrm{Nrd}(z) - \mathrm{Trd}(xyz)$. Let $\mathrm{Iso}(A, N)$ be the k-group scheme of isometries of N and $\mathrm{Sim}(N)$ be the k-group scheme of similitudes of N. It is known that $\mathrm{Iso}(N)$ is a normal algebraic subgroup in $\mathrm{Sim}(A, N)$ and the factor group coincides with $\mathbb{G}_{m,k}$. So we have a canonical k-group morphism (the multiplier) $\mu : \mathrm{Sim}(N) \to \mathbb{G}_{m,k}$. Now one has a functor

$$R \mapsto R^\times/\mu(\mathrm{Sim}(N)(R)). \tag{28}$$

Note that the connected component of $\mathrm{Iso}(N)$ is a simply connected algebraic k-group of the type E_6.

(11) Let (A, σ) be a csa of degree 8 over k with a symplectic involution. Let $V \subset A$ be the subspace of all skew-symmetric elements. It is of dimension 28. Let Pfd be the reduced Pfaffian on V and Trd be the reduced trace on A. Consider the degree 4 form on the space $V \times V$ given by $F := \mathrm{Pfr}(x) + \mathrm{Pfr}(y) - 1/4\mathrm{Trd}((xy)2) - 1/16\mathrm{Trd}(xy)^2$. Consider the symplectic form on $V \times V$ given by $\phi((x_1, y_1), (x_2, y_2)) = \mathrm{Trd}(x_1y_2 - x_2y_1)$. Let $\mathrm{Iso}(F)$ (resp. $\mathrm{Iso}(\phi)$) be the k-group scheme of isometries of the pair F

(resp. of ϕ)). Let $\mathrm{Sim}(F)$ (resp. $\mathrm{Sim}(\phi)$) be the k-group scheme of similitudes of F (resp. of ϕ). Set $G = \mathrm{Iso}(F) \cap \mathrm{Iso}(\phi)$ and $G^+ = \mathrm{Sim}(F) \cap \mathrm{Sim}(\phi)$. It is known that G is a normal algebraic subgroup in G^+ and the factor group coincides with $\mathbb{G}_{m,k}$. So we have a canonical k-group morphism $\mu : G^+ \to \mathbb{G}_{m,k}$. Now one has a functor

$$R \mapsto R^\times/\mu(G^+(R)). \tag{29}$$

Note that G is a simply-connected group of the type E_7.

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