

Chow motives of twisted flag varieties

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May 11, 2005

Abstract

Let G be an adjoint simple algebraic group of inner type. We express the Chow motive (with integral coefficients) of some anisotropic projective G -homogeneous varieties in terms of motives of simpler G -homogeneous varieties, namely, those that correspond to maximal parabolic subgroups of G . We decompose the motive of a generalized Severi-Brauer variety $SB_2(A)$, where A is a division algebra of degree 5, into a direct sum of two indecomposable motives. As an application we provide another counter-example to the uniqueness of a direct sum decomposition in the category of motives with integral coefficients.

Keywords: anisotropic homogeneous variety, twisted flag variety, Chow motive, generalized Severi-Brauer variety

1 Introduction

Let G be an adjoint simple algebraic group of inner type over a field k . Let X be a twisted flag variety, i.e., a projective G -homogeneous variety over k . The main purpose of the paper is to express the Chow motive of X in terms of motives of “minimal” flags, i.e., those G -homogeneous varieties that correspond to maximal parabolic subgroups of G .

Observe that the motive of an *isotropic* G -homogeneous variety can be decomposed in terms of motives of simpler G -homogeneous varieties using the techniques developed by Chernousov, Gille, Merkurjev [CGM] and Karpenko [Ka01]. For G -varieties, when G is *isotropic*, one obtains a similar decomposition following the arguments of Brosnan [Br03]. In the case of G -varieties, where G is *anisotropic*, no general decomposition methods are known except

several particular cases of quadrics (see for example Rost [Ro98]) Severi-Brauer varieties (see Karpenko [Ka95]) and exceptional varieties of type F_4 (see [NSZ]).

In the present paper we provide methods that allow to decompose the motives of some *anisotropic* twisted flag G -varieties, where the root system of G is of types A_n , B_n , C_n , G_2 and F_4 , i.e., has a Dynkin diagram which does not branch.

As an application, we provide another counter-example to the uniqueness of a direct sum decomposition in the category of Chow motives with integral coefficients (see 2.6). Observe that such a counter-example was already constructed by Chernousov and Merkurjev (see [CM04, Example 9.4]) and is given by a G -homogeneous variety, where G is a product of two simple groups. Our example is given by a G -variety, where G is a *simple* group. Apart from this, our example involves an (indecomposable) object in the category of integral Chow motives which is isomorphic over $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}[\frac{1}{3}]$ to the motive of the Severi-Brauer variety of a central simple division algebra of degree 5, but not integrally.

The paper is organized as follows. In section 2 we state the main results. We then provide some technical facts that are extensively used in the proofs (section 3). In the other sections we give proofs of the results for varieties of type A_n (section 4), of types B_n and C_n (section 6), and exceptional varieties of types G_2 and F_4 (section 7). Section 5 is devoted to the motivic decomposition of generalized Severi-Brauer varieties.

Notation and Conventions By G we denote an adjoint simple algebraic group of inner type over k and by n its rank. All varieties that appear in the paper are projective G -homogeneous varieties over k . They can be considered as twisted forms of the varieties G/P , where P is a parabolic subgroup of G . The Chow motive of a variety X is denoted by $\mathcal{M}(X)$. By A we denote a central simple algebra over k of index $\text{ind}(A)$ and by $\text{SB}(A)$ the corresponding Severi-Brauer variety. I is always a right ideal of A and $\text{rdim } I$ stands for its reduced dimension. V is a vector space over k .

By $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ we denote a partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0$ with $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_l$. Integers d_1, d_2, \dots, d_k always satisfy the condition $1 \leq d_1 < d_2 < \dots < d_k \leq n$ and are the dimensions of some flag. For each $i = 0, \dots, k$ we define δ_i to be the difference $d_{i+1} - d_i$ (assuming here $d_0 = 0$ and $d_{k+1} = n + 1$).

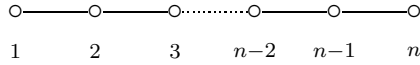
2 Statements of Results

We follow [MPW96, Appendix] and [CG05] for the description of projective G -homogeneous varieties that appear below. According to the type of the group G , we obtain the following results.

A_n : In this case $G = \mathrm{PGL}_1(A)$, where A is a central simple algebra of degree $n + 1$, $n > 0$, and the set of points of a projective G -homogeneous variety X can be identified with the set of flags of (right) ideals

$$X(d_1, \dots, d_k) = \{I_1 \subset I_2 \subset \dots \subset I_k \subset A\}$$

of fixed reduced dimensions $1 \leq d_1 < d_2 < \dots < d_k \leq n$. Observe that this variety is a twisted form of G/P , where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i .



The following result reduces the computation of the motive of X to the motives of “smaller” flags.

2.1 Theorem. *Suppose that $\mathrm{gcd}(\mathrm{ind}(A), d_1, \dots, \hat{d}_m, \dots, d_k) = 1$, then*

$$\mathcal{M}(X(d_1, \dots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \dots, \hat{d}_m, \dots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \dots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geq \lambda_1 \geq \dots \geq \lambda_{\delta_{m-1}} \geq 0$.

Proof. See 4.8. □

As a consequence, for the variety of complete flags we obtain

2.2 Corollary. *The motive of the variety $X = X(1, \dots, n)$ of complete flags is isomorphic to*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{n(n-1)/2} \mathcal{M}(\mathrm{SB}(A))(i)^{\oplus a_i},$$

where the a_i are the coefficients of the polynomial $\varphi_n(z) = \sum_i a_i z^i = \prod_{k=2}^n \frac{z^k - 1}{z - 1}$.

Proof. Apply Theorem 2.1 recursively to the sequence of varieties $X(1, \dots, n)$, $X(1, \dots, n-1)$, \dots , $X(1, 2)$ and $X(1) = \text{SB}(A)$. \square

Another interesting example is the “incidence” variety $X(1, n)$:

2.3 Corollary. *The motive of $X(1, n)$ is isomorphic to*

$$\mathcal{M}(X(1, n)) \simeq \bigoplus_{i=0}^{n-1} \mathcal{M}(\text{SB}(A))(i).$$

In order to complete the picture we need to know how to decompose the motive of a “minimal” flag, i.e., a generalized Severi-Brauer variety.

Note that for some rings of coefficients (fields, discrete valuation rings) one easily obtains the desired decomposition by using Krull-Schmidt Theorem (the uniqueness of a direct sum decomposition). More precisely, consider the subcategory $\mathcal{M}(G, R)$ of the category of motives with coefficients in a ring R that is the pseudo-abelian completion of the category of motives of projective G -homogeneous varieties (see [CM04, section 8]). Then we have the following

2.4 Proposition. *Let $X(d) = \text{SB}_d(A)$, $1 < d < n$, be a generalized Severi-Brauer variety for a central simple algebra A of degree $n+1$ such that $\gcd(\text{ind}(A), d) = 1$. Let R be a ring such that Krull-Schmidt Theorem holds in the category $\mathcal{M}(G, R)$. Then the motive of $\text{SB}_d(A)$ with coefficients in R is isomorphic to*

$$\mathcal{M}(\text{SB}_d(A)) \simeq \bigoplus_{i \in \mathcal{I}} \mathcal{M}(\text{SB}(A))(i)^{\oplus a_i},$$

where the integers a_i are the coefficients of the polynomial $\frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)}$ at terms z^i and the set of indices $\mathcal{I} = \{i \mid a_i \neq 0\}$.

Proof. See 4.10. \square

It turns out that the motives of some generalized Severi-Brauer varieties with integral coefficients can still be decomposed but in a slightly unexpected way.

2.5 Theorem. *Let $\text{SB}_2(A)$ be a generalized Severi-Brauer variety for a division algebra of degree 5. Then there is an isomorphism*

$$\mathcal{M}(\text{SB}_2(A)) \simeq \mathcal{H} \oplus \mathcal{H}(2),$$

where the motive $\mathcal{H}(i)$ taken with R -coefficients, where 2 or 3 is invertible in a ring R , becomes isomorphic to the motive of $\text{SB}(A)$; ii) is indecomposable over \mathbb{Z} ; iii) is not isomorphic to the motive of $\text{SB}(A)$ over \mathbb{Z} .

Proof. See 5.11. □

As an immediate consequence of Theorems 2.1 and 2.5 we obtain

2.6 Corollary. *The Krull-Schmidt Theorem fails in the category of motives $\mathcal{M}(\text{PGL}_1(A), \mathbb{Z})$ where A is a division algebra of degree 5.*

Proof. Apply Theorem 2.1 recursively to the sequences of varieties $X(1, 2)$, $X(1)$ and $X(1, 2)$, $X(2)$, where $X(1, 2)$ is the twisted flag G -variety for $G = \text{PGL}_1(A)$. We obtain two decompositions of the motive of $X(1, 2)$

$$\bigoplus_{i=0}^3 \mathcal{M}(\text{SB}(A))(i) \simeq \mathcal{M}(X(1, 2)) \simeq \mathcal{M}(\text{SB}_2(A)) \oplus \mathcal{M}(\text{SB}_2(A))(1).$$

Apply now Theorem 2.5 to the components of the second decomposition. We obtain two different decompositions of the motive $\mathcal{M}(X(1, 2))$ into indecomposable objects

$$\bigoplus_{i=0}^3 \mathcal{M}(\text{SB}(A))(i) \simeq \mathcal{M}(X(1, 2)) \simeq \bigoplus_{i=0}^3 \mathcal{H}(i).$$

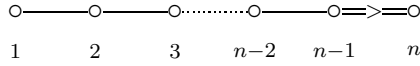
This finishes the proof of the corollary. □

2.7 Remark. Observe that the counter-example provided by Chernousov and Merkurjev (see [CM04, Ex.9.4]) is the product of two Severi-Brauer varieties $X = \text{SB}(A) \times \text{SB}(B)$ which is a G -homogeneous variety for the *semi-simple* group $G = \text{PGL}_1(A) \times \text{PGL}_1(B)$, where A and B are two division algebras satisfying some conditions. The example that we provide, i.e., the flag $X(1, 2)$, is a G -homogeneous variety for the *simple* group $G = \text{PGL}_1(A)$.

B_n: We assume that the characteristic of the base field k is different from 2. It is known that $G = \text{O}^+(V, q)$, where (V, q) is a regular quadratic space of dimension $2n + 1$, $n > 0$, and projective G -homogeneous varieties can be described as flags of totally q -isotropic subspaces

$$X(d_1, \dots, d_k) = \{V_1 \subset \dots \subset V_k \subset V\}.$$

of fixed dimensions $1 \leq d_1 < \dots < d_k \leq n$. Observe that this variety is a twisted form of G/P , where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i .



The following result shows that some motives of flag varieties can be decomposed into a direct sum of twisted motives of “smaller” flags.

2.8 Theorem. *Suppose that $m < k$, then*

$$\mathcal{M}(X(d_1, \dots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \dots, \hat{d}_m, \dots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \dots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geq \lambda_1 \geq \dots \geq \lambda_{\delta_{m-1}} \geq 0$.

Proof. See 6.5. □

In particular, for the variety of complete flags we obtain a formula similar to the one of Corollary 2.2.

2.9 Corollary. *The motive of the variety of complete flags $X = X(1, 2, \dots, n)$ is isomorphic to*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{n(n-1)/2} \mathcal{M}(X(n))(i)^{\oplus a_i},$$

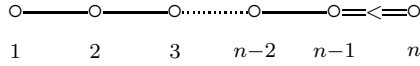
where the a_i are the coefficients of the polynomial $\varphi_n(z) = \sum_i a_i z^i = \prod_{k=2}^n \frac{z^k - 1}{z - 1}$ and $X(n)$ is the twisted form of the maximal orthogonal Grassmannian.

C_n: We assume that the characteristic of the base field k is different from 2. In this case $G = \text{Aut}(A, \sigma)$, where A is a central simple algebra of degree $2n$, $n \geq 2$, with an involution σ of symplectic type on A , and a projective G -homogeneous variety can be described as the set of flags of (right) ideals

$$X(d_1, \dots, d_k) = \{I_1 \subset \dots \subset I_k \subset A \mid I_i \subseteq I_i^\perp\}$$

of fixed reduced dimensions $1 \leq d_1 < \dots < d_k \leq n$, where $I^\perp = \{x \in A \mid \sigma(x)I = 0\}$ is the right ideal of reduced dimension $2n - \text{rdim } I$. Observe that

this variety is a twisted form of G/P , where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i .



Again, the motives of some flag varieties can be decomposed into a direct sum of twisted motives of “smaller” flags.

2.10 Theorem. *Suppose that d_i is odd for some $i < k$ and $d_k - d_{k-1} = 1$. Then*

$$\mathcal{M}(X(d_1, \dots, d_k)) \simeq \bigoplus_{i=0}^{2n-2d_{k-1}-1} \mathcal{M}(X(d_1, \dots, d_{k-1}))(i).$$

In particular, for the variety of complete flags we obtain

2.11 Corollary. *The motive of the variety of complete flags $X = X(1, 2, \dots, n)$ is isomorphic to*

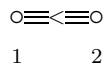
$$\mathcal{M}(X(1, \dots, n)) \simeq \bigoplus_{i=0}^{n(n-1)} \mathcal{M}(\text{SB}(A))(i)^{\oplus a_i},$$

where a_i are the coefficients of the polynomial $\psi_n(z) = \prod_{k=1}^{n-1} \frac{z^{2k}-1}{z-1}$.

G₂: We suppose that the characteristic of k is not 2. It is known that $G = \text{Aut}(C)$, where C is a Cayley algebra over k . By an i -space, where $i = 1, 2$, we mean an i -dimensional subspace V_i of C such that $uv = 0$ for every $u, v \in V_i$. The only flag variety corresponding to a non-maximal parabolic is the variety of complete flags $X(1, 2)$ which is described as follows

$$X(1, 2) = \{V_1 \subset V_2 \mid V_i \text{ is a } i\text{-subspace of } C\}.$$

We enumerate the simple roots on the Dynkin diagram as follows:



In this case we obtain

2.12 Theorem. *The motive of the variety of complete flags $X = X(1, 2)$ is isomorphic to*

$$\mathcal{M}(X) \simeq \mathcal{M}(X(2)) \oplus \mathcal{M}(X(2))(1).$$

Proof. See 7.5 □

Observe that by the result of Bonnet [Bo03] the motives of $X(1)$ and $X(2)$ are isomorphic (here $X(1)$ is a 5-dimensional quadric).

F₄: We suppose that the characteristic of k is not 2 and 3. It is known that $G = \text{Aut}(J)$, where J is an exceptional Jordan algebra of dimension 27 over k . Set $\mathcal{I} = \{1, 2, 3, 6\}$. By an i -space, $i \in \mathcal{I}$, we mean an i -dimensional subspace V of J such that every $u, v \in V$ satisfy the following condition:

$$\text{tr}(u) = 0, \quad u \times v = 0, \quad \text{and if } i < 6 \text{ then } u(va) = v(ua) \text{ for all } a \in J.$$

A projective G -homogeneous variety can be described as the set of flags of subspaces

$$X(d_1, \dots, d_k) = \{V_1 \subset \dots \subset V_k \mid V_i \text{ is a } d_i\text{-subspace of } J\}.$$

where the integers $d_1 < \dots < d_k$ are taken from the set \mathcal{I} . Observe that this variety is a twisted form of G/P , where P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i .

$$\begin{array}{cccc} \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & & & & & & \\ 1 & & 2 & & 3 & & 6 \end{array}$$

In this case we obtain

2.13 Theorem. *Suppose that $m < k$ and either $d_{m+1} < 6$ or $d_m = 1$, then*

$$\mathcal{M}(X(d_1, \dots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \dots, \hat{d}_m, \dots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \dots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geq \lambda_1 \geq \dots \geq \lambda_{\delta_{m-1}} \geq 0$.

Proof. See 7.10. □

3 Preliminaries

In the present section, we introduce the category of Chow motives following [Ma68]. We formulate and prove the Grassmann Bundle Theorem. At the end we recall the notion of functor of points following [Ka01, section 8] and provide some examples.

3.1 (Chow motives). Let k be a field and $\mathcal{V}ar_k$ be the category of smooth projective varieties over k . We define the category $\mathcal{C}or_k$ of *correspondences* over k . Its objects are smooth projective varieties over k . As morphisms, called correspondences, we set $\text{Mor}(X, Y) := \text{CH}^{\dim X}(X \times Y)$. For any two correspondences $\alpha \in \text{CH}(X \times Y)$ and $\beta \in \text{CH}(Y \times Z)$ we define the composition $\beta \circ \alpha \in \text{CH}(X \times Z)$

$$\beta \circ \alpha = \text{pr}_{13*}(\text{pr}_{12}^*(\alpha) \cdot \text{pr}_{23}^*(\beta)),$$

where pr_{ij} denotes the projection on the i -th and j -th factors of $X \times Y \times Z$ respectively and $\text{pr}_{ij*}, \text{pr}_{ij}^*$ denote the induced push-forwards and pull-backs for Chow groups. Observe that the composition \circ induces a ring structure on the abelian group $\text{CH}^{\dim X}(X \times X)$. The unit element of this ring is the class of the diagonal Δ_X .

The pseudo-abelian completion of $\mathcal{C}or_k$ is called the category of *Chow motives* and is denoted by \mathcal{M}_k . The objects of \mathcal{M}_k are pairs (X, p) , where X is a smooth projective variety and p is a projector, that is, $p \circ p = p$. The motive (X, Δ_X) will be denoted by $\mathcal{M}(X)$.

3.2. By the construction \mathcal{M}_k is a self-dual tensor additive category, where the duality is given by the transposition of cycles $\alpha \mapsto \alpha^t$ and the tensor product is given by the usual fiber product $(X, p) \otimes (Y, q) = (X \times Y, p \times q)$. Moreover, the Chow functor $\text{CH} : \mathcal{V}ar_k \rightarrow \mathbb{Z}\text{-Ab}$ (to the category of \mathbb{Z} -graded abelian groups) factors through \mathcal{M}_k , i.e., one has the commutative diagram of functors

$$\begin{array}{ccc} \mathcal{V}ar_k & \xrightarrow{\text{CH}} & \mathbb{Z}\text{-Ab} \\ & \searrow \Gamma & \nearrow R \\ & & \mathcal{M}_k \end{array}$$

where $\Gamma : f \mapsto \Gamma_f$ is the graph and $R : (X, p) \mapsto \text{im}(p^*)$ is the realization.

3.3. Consider the morphism $(\text{id}, e) : \mathbb{P}^1 \times \{pt\} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Its image by means of the induced push-forward $(\text{id}, e)_*$ doesn't depend on the choice of a

point $e : \{pt\} \rightarrow \mathbb{P}^1$ and defines the projector in $\mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ denoted by p_1 . The motive $L = (\mathbb{P}^1, p_1)$ is called the Lefschetz motive. For a motive M and an nonnegative integer i we denote by $M(i) = M \otimes L^{\otimes i}$ its twist. Observe that

$$\mathrm{Mor}((X, p)(i), (Y, q)(j)) = q \circ \mathrm{CH}^{\dim X + i - j}(X, Y) \circ p.$$

We will extensively use the following fact that easily follows from Manin's Identity Principle and Grassmann Bundle Theorem for Chow groups [Ful].

3.4 Proposition (Grassmann Bundle Theorem). *Let X be a variety over k and \mathcal{E} be a vector bundle over X of rank n . Then the motive of the Grassmann bundle $\mathrm{Gr}(d, \mathcal{E})$ over X is isomorphic to*

$$\mathcal{M}(\mathrm{Gr}(d, \mathcal{E})) \simeq \bigoplus_{\lambda} \mathcal{M}(X)(d(n-d) - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \dots, \lambda_d)$ such that $n - d \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0$.

Proof. We follow notations of [Ma68]. Denote $\mathrm{Gr}(d, \mathcal{E})$ by Y and the canonical projection of Y to X by ψ . It is known (see [Ful, Prop. 14.6.5]) that $\mathrm{CH}^*(Y)$ as $\mathrm{CH}^*(X)$ -module (via ψ^*) has a basis consisting of elements $\Delta_{\lambda} \in \mathrm{CH}^{|\lambda|}(Y)$ parametrized by partitions λ . For any partition λ denote by λ^{op} the partition defined by $\lambda_i^{\mathrm{op}} = n - d - \lambda_{d+1-i}$. Then for every two partitions λ and μ with $|\lambda| + |\mu| \leq d(n-d)$ and any element $a \in \mathrm{CH}^*(X)$ the following duality formula holds (see [Ful, Prop. 14.6.3]):

$$\psi_*(\Delta_{\lambda} \Delta_{\mu} \psi^*(a)) = \begin{cases} a, & \mu = \lambda^{\mathrm{op}}, \\ 0, & \text{otherwise.} \end{cases}$$

For every partition λ , we define an element f_{λ} of $\mathrm{End}(\mathcal{M}(Y))$ inductively. For the unique partition λ_{\max} with $|\lambda_{\max}| = d(n-d)$ set $f_{\lambda_{\max}} = c(\psi) \circ c(\psi)^t$. Now, by decreasing induction on $|\lambda|$, set

$$f_{\lambda} = f_{\lambda_{\max}} \circ c_{\Delta_{\lambda^{\mathrm{op}}}} \circ (\Delta_Y - \sum_{|\mu| > |\lambda|} c_{\Delta_{\mu}} \circ f_{\mu}).$$

Finally, set $p_{\lambda} = c_{\Delta_{\lambda}} \circ f_{\lambda}$.

Now, the duality formula implies that

$$(p_{\lambda})_{\epsilon} \left(\sum_{\mu} \Delta_{\mu} \psi^*(a_{\mu}) \right) = \Delta_{\lambda} \psi^*(a_{\lambda}).$$

Therefore, by Manin's identity principle, the p_λ form a complete orthogonal system of projectors. Identification of their images with twisted motives of X can be done as in [Ma68, §7] and we omit it. \square

3.5 (Functors of Points). In sections 4, 6 and 7 we use the functorial language, that is consider k -schemes as functors from the category of k -algebras to the category of sets. Fix a scheme X . By an X -algebra we mean a pair (R, x) , where R is a k -algebra and x is an element of $X(R)$. X -algebras form a category with obvious morphisms. Morphisms $\varphi: Y \rightarrow X$ can be considered as functors from the category of X -algebras to the category of sets, by sending a pair (R, x) to its preimage in $Y(R)$.

3.6. Let X be a variety over k . To any vector bundle \mathcal{F} over X we can associate the Grassmann bundle $Y = \text{Gr}(d, \mathcal{F})$. Fix an X -algebra (R, x) . The value of the functor corresponding to $\text{Gr}(d, \mathcal{F})$ at (R, x) is the set of direct summands of rank d of the projective R -module $\mathcal{F}_x \otimes_k R$ (cf. [Ka01, section 9]), where $\mathcal{F}_x = \mathcal{F}(R, x)$.

4 Groups of type A_n

The goal of the present section is to prove Theorem 2.1 and Proposition 2.4. We use the notation of section 2.

4.1. Let G be an adjoint group of inner type A_n defined over a field k . It is well known that $G = \text{PGL}_1(A)$, where A is a central simple algebra of degree $n + 1$ and points of projective G -homogeneous varieties are flags of (right) ideals of A

$$X(d_1, \dots, d_k) = \{I_1 \subset \dots \subset I_k \subset A \mid \text{rdim } I_i = d_i\}.$$

For convenience we set $d_0 = 0$, $d_{k+1} = n + 1$, $I_0 = 0$, $I_{k+1} = A$.

4.2. The value of the functor of points corresponding to the variety $X(d_1, \dots, d_k)$ at a k -algebra R (see 3.5) equals the set of all flags $I_1 \subset \dots \subset I_k$ of right ideals of $A_R = A \otimes_k R$ having the following properties (see [IK00, section 4])

- the injection of A_R -modules $I_i \hookrightarrow A_R$ splits;
- $\text{rdim } I_i = d_i$.

4.3. On the scheme $X = X(d_1, \dots, d_k)$ there are “tautological” vector bundles \mathcal{J}_i , $i = 0, \dots, k+1$, of ranks $(n+1)d_i$. The value of \mathcal{J}_i on an X -algebra (R, x) , where $x = (I_1, \dots, I_k)$, is the ideal I_i considered as a projective R -module. The bundle \mathcal{J}_i also has a structure of right A_X -module, where A_X is the constant sheaf of algebras on X determined by A .

For every $m \in \{1, \dots, k\}$ there exists an obvious morphism

$$\begin{aligned} X(d_1, \dots, d_k) &\rightarrow X(d_1, \dots, \hat{d}_m, \dots, d_k) \\ (I_1, \dots, I_k) &\mapsto (I_1, \dots, \hat{I}_m, \dots, I_k) \end{aligned}$$

that turns $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ -scheme.

4.4 Lemma. Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ by X . Assume there exists a vector bundle \mathcal{E} over X such that $A_X \simeq \text{End}_{\mathcal{O}_X}(\mathcal{E})$. Consider the vector bundle

$$\mathcal{F} = \mathcal{J}_{m+1}\mathcal{E}/\mathcal{J}_{m-1}\mathcal{E} = \mathcal{J}_{m+1}/\mathcal{J}_{m-1} \otimes_{A_X} \mathcal{E}$$

of rank $d_{m+1} - d_{m-1}$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z = \text{Gr}(d_m - d_{m-1}, \mathcal{F})$ over X .

Proof. We use essentially the same method as in [IK00, Proposition 4.3].

Fix an X -algebra (R, x) where $x = (I_1, \dots, \hat{I}_m, \dots, I_k)$. The fiber of Y over x , i.e., the value at (R, x) , can be identified with the set of all ideals I_m satisfying the conditions 4.2 such that $I_{m-1} \subset I_m \subset I_{m+1}$. The fiber of Z over x is the set of all R -submodules N of $\mathcal{F}_y = \mathcal{F}(R, y)$ such that the injection $N \hookrightarrow \mathcal{F}_y$ splits and $\text{rk}_R N = d_m - d_{m-1}$.

We define a natural bijection between the fibers of Y and Z over x as follows.

Consider the following mutually inverse bijections between the set of all right ideals of reduced dimension r in A_R (satisfying 4.2) and the set of all direct summands of rank r of the R -module \mathcal{E}_x

$$\begin{aligned} \Phi : I &\mapsto I\mathcal{E}_x \\ \Psi : N &\mapsto \text{Hom}_R(\mathcal{E}_x, N) \subset \text{End}_R(\mathcal{E}_x) \simeq A_R \end{aligned}$$

Observe that these bijections preserve the respective inclusions of ideals and modules. So ideals of reduced dimension d_m between I_{m-1} and I_m correspond to submodules of rank d_m between $I_{m-1}\mathcal{E}_x$ and $I_{m+1}\mathcal{E}_x$, and, therefore, to submodules of rank $d_{m+1} - d_{m-1}$ in $I_{m+1}\mathcal{E}_x/I_{m-1}\mathcal{E}_x = \mathcal{F}_x$. This gives the desired natural bijection on the fibers. \square

4.5 Lemma. *Suppose that $\gcd(\text{ind}(A), d_1, \dots, d_k) = 1$. Then there exists a vector bundle \mathcal{E} over $X = X(d_1, \dots, d_k)$ of rank $n + 1$ such that $A_X \simeq \text{End}_{\mathcal{O}_X}(\mathcal{E})$.*

Proof. We have to prove that the class $[A_X]$ in $\text{Br}(X)$ is trivial. Since X is a regular Noetherian scheme the canonical map

$$\text{Br}(X) \rightarrow \text{Br}(K)$$

where $K = k(X)$ is the function field of X , is injective by [Gr, 1.10] and [AG60, Theorem 7.2]. So it is enough to prove that $A \otimes_k K$ splits. But the generic point of X defines a flag of ideals of $A \otimes_k K$ of reduced dimensions d_1, \dots, d_k . Since the index $\text{ind}(A \otimes_k K)$ divides d_1, \dots, d_k and $\text{ind} A$, by the assumption of the lemma it must be equal to 1. So $A \otimes_k K$ is split and this finishes the proof of the lemma. \square

4.6 Remark. In the case $d_1 = 1$ one can take $\mathcal{E} = \mathcal{J}_1^\vee$.

4.7 Remark. It can be shown using the Index Reduction Formula (see [MPW96]) that the condition on the gcd is necessary and sufficient for the central simple algebra $A_{k(X)}$ to be split.

We are now ready to finish the proof of 2.1.

4.8 (Proof of Theorem 2.1). By Lemma 4.5 there exists a vector bundle \mathcal{E} over $X = X(d_1, \dots, \hat{d}_m, \dots, d_k)$ of rank $n + 1$ such that $A_X \simeq \text{End}_{\mathcal{O}_X}(\mathcal{E})$. By Lemma 4.4 we conclude that $Y = X(d_1, \dots, d_k)$ is a Grassmann bundle over X . Now by Proposition 3.4 we obtain the isomorphism of 2.1.

4.9 Remark. Note that the assumption of Theorem 2.1 on the reduced dimensions d_1, \dots, d_k is essential. Indeed, suppose the Theorem holds for any twisted flag variety. Consider the flag $X = X(1, d)$ with $\gcd(\text{ind}(A), d) > 1$. Then we have an isomorphism of motives

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{d-1} \mathcal{M}(\text{SB}_d(A))(i)$$

which appears after applying Theorem to the flags $X(1, d)$ and $X(d)$. Consider the group $\text{CH}_0(X) = \text{Mor}_{\mathcal{M}_k}(\mathcal{M}(pt), \mathcal{M}(X))$. The isomorphism above induces the isomorphism of groups

$$\begin{aligned} \text{Coker}(\text{CH}_0(X) \xrightarrow{\text{res}} \text{CH}_0(X_{k_s})) &\cong \text{Coker}(\text{CH}_0(\text{SB}_d(A)) \xrightarrow{\text{res}} \text{CH}_0(\text{Gr}(d, n+1))) \\ &\cong \mathbb{Z} / \left(\frac{\text{ind}(A)}{\gcd(\text{ind}(A), d)} \right) \mathbb{Z}, \end{aligned}$$

where res is the pull-back induced by the scalar extension k_s/k (here k_s denotes the separable closure of k) and the last isomorphism follows by [B191, Theorem 3]. On the other hand, applying Theorem 2.1 to the flags $X(1, d)$ and $X(1)$ we obtain the isomorphism

$$\mathcal{M}(X) \simeq \bigoplus_{\lambda} \mathcal{M}(\text{SB}(A))((n+1-d)(d-1) - |\lambda|)$$

which induces the isomorphism of groups

$$\begin{aligned} \text{Coker}(\text{CH}_0(X) \xrightarrow{\text{res}} \text{CH}_0(X_{k_s})) &\cong \text{Coker}(\text{CH}_0(\text{SB}(A)) \xrightarrow{\text{res}} \text{CH}_0(\mathbb{P}^n)) \\ &\cong \mathbb{Z}/\text{ind}(A)\mathbb{Z}, \end{aligned}$$

that leads to a contradiction.

We now prove Proposition 2.4.

4.10 (Proof of Proposition 2.4). Let $G = \text{PGL}_1(A)$ and let $\mathcal{M}(G, R)$ be the symmetric tensor category of motives of G -homogeneous varieties with coefficients in a ring R for which the Krull-Schmidt theorem holds. It is the case, e.g., when R is a field or, more general, a discrete valuation ring (see [CM04, Theorem 9.6]).

Consider the G -homogeneous variety $X(1, d)$, $1 < d < n$. Apply Theorem 2.1 to the sequences of flags $X(1, d)$, $X(d)$ and $X(1, d)$, $X(1)$. We obtain two isomorphisms in $\mathcal{M}(G, R)$

$$\bigoplus_{i=0}^{d-1} \mathcal{M}(\text{SB}_d(A))(i) \simeq \mathcal{M}(X) \simeq \bigoplus_{\lambda} \mathcal{M}(\text{SB}(A))((n+1-d)(d-1) - |\lambda|), \quad (*)$$

where the sum on the right hand side is taken over all partitions $\lambda = (\lambda_1, \dots, \lambda_{d-1})$ such that $n+1-d \geq \lambda_1 \geq \dots \geq \lambda_{d-1} \geq 0$. Since Krull-Schmidt Theorem holds in $\mathcal{M}(G, R)$, the motive $\text{SB}(A)$ has a unique decomposition into the direct sum of indecomposable objects H_i , $i \in \mathcal{I}$, and their twists

$$\mathcal{M}(\text{SB}(A)) \simeq \bigoplus_{i \in \mathcal{I}} (\oplus_{j \in \mathcal{J}_i} H_i(j)).$$

Consider the subcategory $\mathcal{M}(G, R)_{\mathcal{I}}$ additively generated by the motives H_i , $i \in \mathcal{I}$, and their twists. The abelian group of isomorphism classes of objects of this category can be equipped with a structure of a free module

over the polynomial ring $R[z]$. Namely, multiplication by z is given by the twist. Clearly, the classes $[H_i]$, $i \in \mathcal{I}$, form the basis of this $R[z]$ -module.

By (*) we have $\mathcal{M}(\text{SB}_d(A)) \in \mathcal{M}(G, R)_{\mathcal{I}}$ and the isomorphisms (*) can be rewritten as

$$\begin{aligned} \frac{z^d - 1}{z - 1} [\text{SB}_d(A)] &= \frac{\varphi_n(z)}{\varphi_{d-1}(z)\varphi_{n+1-d}(z)} [\text{SB}(A)] \\ &= \frac{z^d - 1}{z - 1} \frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)} [\text{SB}(A)] \end{aligned}$$

where $\varphi_n(z) = \prod_{k=2}^n \frac{z^k - 1}{z - 1}$. This immediately implies the equality

$$[\text{SB}_d(A)] = \frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)} [\text{SB}(A)],$$

i.e., the isomorphism in $\mathcal{M}(G, R)_{\mathcal{I}}$ between $\mathcal{M}(\text{SB}_d(A))$ and the respective sum of twists of $\mathcal{M}(\text{SB}(A))$. This finishes the proof of the proposition.

5 Motivic decomposition of $\text{SB}_2(A)$

This section is devoted to the proof of Theorem 2.5.

First, we need to recall some properties of rational cycles on projective homogeneous varieties.

5.1. Let G be a split linear algebraic group over k . Let X be a projective G -homogeneous variety, i.e., $X = G/P$, where P is a parabolic subgroup of G . The abelian group structure of $\text{CH}(X)$, as well as its ring structure, is well-known. Namely, X has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the free additive generators of $\text{CH}(X)$ (see [Ka01]). Note that the product of two projective homogeneous varieties $X \times Y$ has a cellular filtration as well, and $\text{CH}^*(X \times Y) \cong \text{CH}^*(X) \otimes \text{CH}^*(Y)$ as graded rings. The correspondence product of two cycles $\alpha = f_\alpha \times g_\alpha \in \text{CH}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \text{CH}(Y \times X)$ is given by (cf. [Bo03, Lem. 5])

$$(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \deg(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta),$$

where $\deg : \text{CH}(Y) \rightarrow \text{CH}(\{pt\}) = \mathbb{Z}$ is the degree map.

5.2. Let X be a projective variety of dimension n over a field k . Let k_s be the separable closure of the field k . Consider the scalar extension $X_s = X \times_k k_s$. We say a cycle $J \in \text{CH}(X_s)$ is *rational* if it lies in the image of the pull-back homomorphism $\text{CH}(X) \rightarrow \text{CH}(X_s)$. For instance, there is an obvious rational cycle Δ_{X_s} on $\text{CH}^n(X_s \times X_s)$ that is given by the diagonal class. Clearly, linear combinations, intersections and correspondence products of rational cycles are rational.

5.3. We will use the following fact (see [CGM, Cor. 8.3]) that follows from the Rost Nilpotence Theorem. Let p_s be a non-trivial rational projector in $\text{CH}^n(X_s \times X_s)$, i.e., $p_s \circ p_s = p_s$. Then there exists a non-trivial projector p on $\text{CH}^n(X \times X)$ such that $p \times_k k_s = p_s$. Hence, the existence of a non-trivial rational projector p_s on $\text{CH}^n(X_s \times X_s)$ gives rise to the decomposition of the Chow motive of X

$$\mathcal{M}(X) \cong (X, p) \oplus (X, \Delta_X - p)$$

Our goal is to find such a projector in the case $X = \text{SB}_d(A)$.

5.4. An isomorphism between $(X, p)(m)$ and $(Y, q)(l)$ can be given by correspondences $j_1 \in \text{CH}^{\dim X - l + m}(X \times Y)$ and $j_2 \in \text{CH}^{\dim Y - m + l}(Y \times X)$ such that $q \circ j_1 = j_1 \circ p$, $p \circ j_2 = j_2 \circ q$, $j_1 \circ j_2 = q$, $j_2 \circ j_1 = p$. If X and Y lie in the category $\mathcal{M}(G, \mathbb{Z})$ then by the Rost nilpotence theorem (see [CM04, Theorem 8.2] and [CGM, Corollary 8.4]) it suffices to give a rational j_1 and some j_2 satisfying these conditions over separable closure (note that j_2 will automatically be rational).

We now recall some properties of Grassmann varieties and describe their Chow rings.

5.5. Consider the Grassmann variety $\text{Gr}(d, n+1)$, $1 \leq d \leq n$, of d -planes in the $(n+1)$ -dimensional affine space. It has the dimension $d(n+1-d)$. The twisted form of it is a generalized Severi-Brauer variety $\text{SB}_d(A)$, where A is a central simple algebra of degree $n+1$. For any two integers d and d' , $1 \leq d, d' \leq n$, there is the fiber product diagram

$$\begin{array}{ccc} \text{Gr}(d, n+1) \times \text{Gr}(d', n+1) & \xrightarrow{\text{Seg}} & \text{Gr}(dd', (n+1)^2) \\ \downarrow & & \downarrow \simeq \\ \text{SB}_d(A) \times \text{SB}_{d'}(A^{\text{op}}) & \xrightarrow{\text{Seg}} & \text{SB}_{dd'}(A \otimes A^{\text{op}}) \end{array} \quad (1)$$

where the horizontal arrows are Segre embeddings given by the tensor product of ideals (resp. linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension k_s/k (here k_s is the separable closure of k). Observe that the right vertical arrow is an isomorphism since $A \otimes A^{\text{op}}$ splits.

5.6. The diagram (1) induces the commutative diagram of rings

$$\begin{array}{ccc} \text{CH}(\text{Gr}(d, n+1) \times \text{Gr}(d', n+1)) & \xleftarrow{\text{Seg}^*} & \text{CH}(\text{Gr}(dd', (n+1)^2)) \\ \uparrow \text{res} & & \simeq \uparrow \text{res} \\ \text{CH}(\text{SB}_d(A) \times \text{SB}_{d'}(A^{\text{op}})) & \xleftarrow{\text{Seg}^*} & \text{CH}(\text{SB}_{dd'}(A \otimes A^{\text{op}})) \end{array} \quad (2)$$

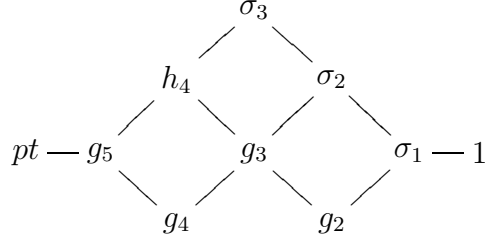
where all maps are the induced pull-backs. Consider a vector bundle E over $\text{Gr}(dd', (n+1)^2)$. It is easy to see that the pull-back of the total Chern class $\text{Seg}^*(c(E))$ is a rational cycle on $\text{CH}(\text{Gr}(d, n+1) \times \text{Gr}(d', n+1)) = \text{CH}(\text{Gr}(d, n+1)) \otimes \text{CH}(\text{Gr}(d', n+1))$. In particular, if $E = \tau_{dd'}$ is the tautological bundle of $\text{Gr}(dd', (n+1)^2)$ we obtain the following

5.7 Lemma. *The total Chern class $c(\text{pr}_1^* \tau_d \otimes \text{pr}_2^* \tau_{d'})$ of the tensor product of the pull-backs (induced by the projection maps) of the tautological bundles τ_d and $\tau_{d'}$ of $\text{Gr}(d, n+1)$ and $\text{Gr}(d', n+1)$ respectively is rational.*

From now on we restrict ourselves to the case $n = 4$, $d = 2$ and $d' = 1$, i.e., to the Grassmannian $\text{Gr}(2, 5)$ and the projective space $\mathbb{P}^4 = \text{Gr}(1, 5)$.

5.8. We describe the generators and relations of the Chow ring $\text{CH}(\text{Gr}(2, 5))$ following [Ful, section 14.7]. Set $\sigma_m = c_m(Q)$, $m = 1, 2, 3$, where $Q = \mathcal{O}^5/\tau_2$ is the universal quotient bundle of rank 3 over $\text{Gr}(2, 5)$. It is known that the elements σ_m generate the Chow ring $\text{CH}(\text{Gr}(2, 5))$. More precisely, as an abelian group this ring is generated by the Schubert cycles $\Delta_\lambda(\sigma)$ that are parametrized by all partitions $\lambda = (\lambda_1, \lambda_2)$ such that $3 \geq \lambda_1 \geq \lambda_2 \geq 0$. In particular, $\sigma_m = \Delta_{(m,0)}$, $m = 1, 2, 3$. For other generators we set the following notation $g_2 = \Delta_{(1,1)}$, $g_3 = \Delta_{(2,1)}$, $h_4 = \Delta_{(3,1)}$, $g_4 = \Delta_{(2,2)}$, $g_5 = \Delta_{(3,2)}$, $pt = \Delta_{(3,3)}$. These generators corresponds to the vertices of the Hasse

diagram of $\text{Gr}(2, 5)$



The multiplication rules can be determined using Pieri formulae

$$\Delta_\lambda \cdot \sigma_m = \sum_{\mu} \Delta_\mu,$$

where the sum is taken over all partitions $\mu = (\mu_1, \mu_2)$ such that $3 \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq 0$.

5.9. Consider the tautological bundle τ_2 of the Grassmannian $\text{Gr}(2, 5)$. Its total Chern class is

$$c(\tau_2) = c(Q)^{-1} = \frac{1}{1 + \sigma_1 + \sigma_2 + \sigma_3} = 1 - \sigma_1 - \sigma_2 + \sigma_1^2 + \dots$$

where the rest consists of the summands of degree greater than 2. Hence, we obtain $c_1(\tau_2) = -\sigma_1$ and $c_2(\tau_2) = -\sigma_2 + \sigma_1^2 = g_2$.

5.10. The Chow ring of the projective space \mathbb{P}^4 can be identified with the factor ring $\mathbb{Z}[H]/(H^5)$, where $H = c_1(\mathcal{O}(1))$ is the class of a hyperplane section. Thus, the first Chern class of the tautological bundle of \mathbb{P}^4 equals to $c_1(\tau_1) = c_1(\mathcal{O}(-1)) = -H$.

We are now ready to prove Theorem 2.5.

5.11 (Proof of Theorem 2.5). By Lemma 5.7 we obtain the following rational cycles in $\text{CH}^*(\text{Gr}(2, 5) \times \mathbb{P}^4)$

$$\begin{aligned} r &= c_1(\text{pr}_1^*(\tau_2) \otimes \text{pr}_2^*(\tau_1)) = c_1(\text{pr}_1^*(\tau_2) + 2c_1(\text{pr}_2^*(\tau_1))) = -\sigma_1 \times 1 - 2(1 \times H), \\ \rho &= c_2(\text{pr}_1^*(\tau_2) \otimes \text{pr}_2^*(\tau_1)) = c_2(\text{pr}_1^*(\tau_2)) + c_1(\text{pr}_1^*(\tau_2))c_1(\text{pr}_2^*(\tau_1)) + c_1(\text{pr}_2^*(\tau_1))^2 \\ &= g_2 \times 1 + \sigma_1 \times H + 1 \times H^2 \end{aligned}$$

For two cycles x and y we shall write $x =_5 y$ if there exists a cycle z such that $x - y = 5z$. Note that $=_5$ is an equivalence relation that preserves rationality of cycles. Then the following cycles are rational

$$\begin{aligned}\rho^2 &= 1 \times H^4 + 2\sigma_1 \times H^3 + (\sigma_2 + 3g_2) \times H^2 + 2g_3 \times H + g_4 \times 1, \\ \rho^3 &= (3\sigma_2 + g_2) \times H^4 + (\sigma_3 + 3g_3) \times H^3 + (g_4 + 3h_4) \times H^2 + 3g_5 \times H + pt \times 1.\end{aligned}$$

and a direct computation shows that $\rho^3 \circ (\rho^2)^t =_5 \Delta_{\mathbb{P}^4}$ is the class of the diagonal in $\text{CH}^4(\mathbb{P}^4 \times \mathbb{P}^4)$.

Consider the composition

$$\begin{aligned}(\rho^2)^t \circ \rho^3 &= (3\sigma_2 + g_2) \times g_4 + (2\sigma_3 + g_3) \times g_3 \\ &\quad + (g_4 + 3h_4) \times (\sigma_2 - 2g_2) + g_5 \times \sigma_1 + pt \times 1.\end{aligned}$$

Note that the right-hand side is a rational projector (over \mathbb{Z}) and, therefore, by Rost Nilpotence Theorem (see [CGM, Corollary 8.3]) has a form $p \times_k k_s$ where p is a projector in $\text{End}(\mathcal{M}(\text{SB}_2(A)))$. The latter determines an object $(\text{SB}_2(A), p)$ in the category of motives (actually in $\mathcal{M}(G, \mathbb{Z})$) which we denoted by \mathcal{H} .

Set $q = \Delta_{\text{SB}_2(A)} - p$. We then show that

$$(\mathcal{M}(\text{SB}_2(A)), q) \simeq (\mathcal{M}(\text{SB}_2(A)), p^t) \simeq \mathcal{H}(2),$$

which gives the claimed decomposition $\mathcal{M}(\text{SB}_2(A)) \simeq \mathcal{H} \oplus \mathcal{H}(2)$.

Observe that an isomorphism $(\text{SB}_2(A), q) \simeq (\text{SB}_2(A), p^t)$ is given by the two mutually inverse motivic isomorphisms $p_s^t \circ q_s$ and $q_s \circ p_s^t$ over k_s which are rational. An isomorphism $\mathcal{H}(2) \simeq (\mathcal{M}(\text{SB}_2(A)), p^t)$ is given by the following two cycles

$$\begin{aligned}j_1 &= (3\sigma_2 + g_2) \times pt - (2\sigma_3 + g_3) \times g_5 \\ &\quad + (g_4 + 3h_4) \times (g_4 + 3h_4) - g_5 \times (2\sigma_3 + g_3) + pt \times (3\sigma_2 + g_2), \\ j_2 &= 1 \times g_4 - \sigma_1 \times g_3 + (\sigma_2 - 2g_2) \times (\sigma_2 - 2g_2) - g_3 \times \sigma_1 + g_4 \times 1.\end{aligned}$$

Note that j_1 is rational, since $j_1 =_5 (1 \times (3\sigma_2 + g_2))p$, and $1 \times (3\sigma_2 + g_2) =_5 3(\rho + r^2)^t \circ \rho^2$ is rational.

We now check the properties of the motive \mathcal{H} .

i) To prove that \mathcal{H} is isomorphic to $\mathcal{M}(\text{SB}(A))$ over R it is enough to provide rational cycles $\alpha \in \text{CH}^4(\mathbb{P}^4 \times \text{Gr}(2, 5))$ and $\beta \in \text{CH}^6(\text{Gr}(2, 5) \times \mathbb{P}^4)$ such that $\alpha \circ \beta = p_s$ and $\beta \circ \alpha = \Delta_{\mathbb{P}^4}$. If 2 is invertible in R , set

$$\alpha = (\rho^2)^t, \quad \beta = \rho^3 - \frac{5}{2}(g_5 \times H + g_3 \times H^3).$$

If 3 is invertible, set

$$\alpha = (\rho^2)^t - \frac{5}{3}(H \times g_3 + H^3 \times \sigma_1) - 5H^2 \times g_2, \quad \beta = \rho^3.$$

ii) We show that \mathcal{H} is indecomposable over \mathbb{Z} . It is enough to prove that it is indecomposable modulo 5. But over $\mathbb{Z}/5\mathbb{Z}$ \mathcal{H} is isomorphic to $\mathcal{M}(\text{SB}(A))$ by item i). Let μ be a non-trivial projector in $\text{CH}(\text{SB}(A) \times \text{SB}(A))/5$. Then $\mu_s = \sum_i \pm H^i \times H^{5-i}$. Chose any inverse image ν of μ in $\text{CH}(\text{SB}(A) \times \text{SB}(A))$. Then $\nu_s =_5 \sum_i \pm H^i \times H^{5-i}$ is rational, and so we have a rational non-trivial projector over \mathbb{Z} , that contradicts [Ka95, theorem 2.2.1].

iii) We claim that $\mathcal{H} \not\cong \mathcal{M}(\text{SB}(A))$ over \mathbb{Z} . Indeed, assume there is a motivic isomorphism $\varphi: \mathcal{M}(\text{SB}(A)) \xrightarrow{\cong} \mathcal{H}$. Since \mathcal{H} is a direct summand of $\mathcal{M}(\text{SB}_2(A))$, there exist maps α', β' such that $\alpha' \circ \beta' = p$, $\beta' \circ \alpha' = \Delta_{\mathcal{H}}$ (see the diagram).

$$\mathcal{M}(\text{SB}_2(A)) \begin{array}{c} \xleftarrow{\alpha'} \\ \xrightarrow{\beta'} \end{array} \mathcal{H} \begin{array}{c} \xleftarrow{\varphi} \\ \xrightarrow{\varphi^{-1}} \end{array} \mathcal{M}(\text{SB}(A)).$$

Consider the cycles $\alpha = \alpha' \circ \varphi$ and $\beta = \varphi^{-1} \circ \beta'$. Clearly $\alpha \circ \beta = p$. Consider this equality over the separable closure k_s . We have two rational cycles

$$\alpha_s = \sum_{i=0}^4 H^i \times a_{4-i} \in \text{CH}^4(\mathbb{P}^4 \times \text{Gr}(2, 5)) \text{ and}$$

$$\beta_s = \sum_{j=0}^6 b_{6-j} \times H^j \in \text{CH}^6(\text{Gr}(2, 5) \times \mathbb{P}^4)$$

for some elements $a_i \in \text{CH}^i(\text{Gr}(2, 5))$ and $b_j \in \text{CH}^j(\text{Gr}(2, 5))$. Their composition can be written as

$$\alpha_s \circ \beta_s = \sum_{i=0}^4 b_{6-i} \times a_i = p_s.$$

Therefore,

$\alpha_s = \varepsilon_1 1 \times g_4 + \varepsilon_2 H \times g_3 + \varepsilon_3 H^2 \times (\sigma_2 - 2g_2) + \varepsilon_4 H^3 \times \sigma_1 + H^4 \times 1$ and
 $\beta_s = pt \times 1 + \varepsilon_4 g_5 \times H + \varepsilon_3 (g_4 + 3h_4) \times H^2 + \varepsilon_2 (2\sigma_3 + g_3) \times H^3 + \varepsilon_1 (3\sigma_2 + g_2) \times H^4$,
 where $\varepsilon_i = \pm 1$. Now let c denotes the rational cycle $(\rho^2)^t - \alpha_s$ and b the rational cycle $\rho^3 - \beta_s$. Then the cycle

$$3(b \circ c) =_5 H \times H^3 + H^3 \times H + 3(1 - \varepsilon_3)^2 H^2 \times H^2 + 3(1 - \varepsilon_1)^2 1 \times H^4 = d$$

is rational in $\mathrm{CH}^4(\mathbb{P}^4 \times \mathbb{P}^4)$. Now $d^{\circ 4}$ gives a non-trivial rational projector on \mathbb{P}^4 , i.e., the motive $\mathcal{M}(\mathrm{SB}(A))$ is decomposable. This leads to a contradiction with indecomposability of $\mathcal{M}(\mathrm{SB}(A))$ over \mathbb{Z} , and finishes the proof of the theorem.

6 Groups of types B_n and C_n

The goal of the present section is to prove Theorems 2.8 and 2.10.

6.1. Let G be an adjoint group of type B_n . From now on we suppose that the characteristic of k is not 2. It is known that $G = \mathrm{O}^+(V, q)$, where (V, q) is a regular quadratic space of dimension $2n + 1$ and projective G -homogeneous varieties can be described as flags of q -totally isotropic subspaces

$$X(d_1, \dots, d_k) = \{V_1 \subset \dots \subset V_k \subset V \mid \dim V_i = d_i\}.$$

6.2. The value of the functor corresponding to the variety $X(d_1, \dots, d_k)$ at a k -algebra R equals the set of all flags $V_1 \subset \dots \subset V_k$, where V_i is a q_R -totally isotropic direct summand of V_R of rank d_i .

For convenience we set $d_0 = 0, V_0 = 0$.

6.3. On the scheme $X = X(d_1, \dots, d_k)$ there are “tautological” vector bundles \mathcal{V}_i of ranks d_i . The value of \mathcal{V}_i on an X -algebra (R, x) is V_i , where $x = (V_1, \dots, V_k)$. For every m there exists an obvious morphism

$$\begin{aligned} X(d_1, \dots, d_k) &\rightarrow X(d_1, \dots, \hat{d}_m, \dots, d_k) \\ (V_1, \dots, V_k) &\mapsto (V_1, \dots, \hat{V}_m, \dots, V_k) \end{aligned}$$

which makes $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ -scheme.

6.4 Lemma. Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ by X . Suppose that $m < k$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z = \mathrm{Gr}(d_m - d_{m-1}, \mathcal{V}_{m+1}/\mathcal{V}_{m-1})$ over X .

Proof. Fix an X -algebra (R, x) , where $x = (V_1, \dots, \hat{V}_m, \dots, V_k)$. We define a natural bijection between the fibers over the point x of Y and Z as follows. The fiber of Y over x can be identified with the set of all direct summands V_m of V_R of rank d_m such that $V_{m-1} \subset V_m \subset V_{m+1}$ (note that V_m is automatically q_R -isotropic since V_{m+1} is so). This fiber is clearly isomorphic to the fiber of Z over x which is the set of all direct summands of $(\mathcal{V}_{m+1}/\mathcal{V}_{m-1})_x = V_{m+1}/V_{m-1}$ of rank d_m . \square

6.5 (Proof of Theorem 2.8). Apply Lemma 6.4 to the varieties $Y = X(d_1, \dots, d_k)$ and $X = X(d_1, \dots, \hat{d}_m, \dots, d_k)$. We obtain that Y is a Grassmann bundle over X . To finish the proof apply Proposition 3.4.

6.6. Let G be an adjoint group of type C_n over k . It is known that $G = \text{Aut}(A, \sigma)$, where A is a central simple algebra of degree $2n$ with an involution σ of symplectic type on A , and projective G -homogeneous varieties can be described as flags of (right) ideals of A

$$X(d_1, \dots, d_k) = \{I_1 \subset \dots \subset I_k \subset A \mid I_i \subseteq I_i^\perp, \text{rdim } I_i = d_i\}.$$

Here $I^\perp = \{x \in A \mid \sigma(x)I = 0\}$ is a right ideal of reduced dimension $2n - \text{rdim } I$.

6.7. The value of the functor corresponding to the variety $X(d_1, \dots, d_k)$ at a k -algebra R equals to the set of all flags $I_1 \subset \dots \subset I_k$ of right ideals of $A_R = A \otimes_k R$ having the following properties

- the injection of A_R -modules $I_i \hookrightarrow A_R$ splits;
- $I_i \subseteq I_i^\perp$;
- $\text{rdim } I_i = d_i$.

For convenience we set $I_0 = 0$.

6.8. On the scheme $X = X(d_1, \dots, d_k)$ there are “tautological” vector bundles \mathcal{J}_i of ranks $2nd_i$ and their “orthogonal complements” \mathcal{J}_i^\perp of rank $2n(2n - d_i)$. The value of \mathcal{J}_i (resp. \mathcal{J}_i^\perp) on an X -algebra (R, x) , where $x = (I_1, \dots, I_k)$, is I_i (resp. I_i^\perp) considered as a projective R -module. The bundles \mathcal{J}_i and \mathcal{J}_i^\perp also have structures of right A_X -modules, where A_X is a constant sheaf of algebras on X determined by A . There exists an obvious morphism

$$\begin{aligned} X(d_1, \dots, d_k) &\rightarrow X(d_1, \dots, d_{k-1}) \\ (I_1, \dots, I_k) &\mapsto (I_1, \dots, I_{k-1}), \end{aligned}$$

which makes $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, d_{k-1})$ -scheme.

6.9 Lemma. Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, d_{k-1})$ by X . Suppose that $d_k = d_{k-1} + 1$ and there exists a vector bundle \mathcal{E} over X such that $A_X \simeq \text{End}_{\mathcal{O}_X}(\mathcal{E})$. Consider the vector bundle

$$\mathcal{F} = \mathcal{J}_{k-1}^\perp \mathcal{E} / \mathcal{J}_{k-1} \mathcal{E} = \mathcal{J}_{k-1}^\perp / \mathcal{J}_{k-1} \otimes_{A_X} \mathcal{E}$$

of rank $2(n - d_{k-1})$. Then Y as a scheme over X can be identified with the projective bundle $Z = \mathbb{P}(\mathcal{F}) = \text{Gr}(1, \mathcal{F})$ over X .

Proof. Fix an X -algebra (R, x) , where $x = (I_1, \dots, I_{k-1})$. We define a natural bijection between the fibers over the point x of Y and Z . The fiber of Y can be identified with the set of all ideals I_k containing I_{k-1} and satisfying the conditions 6.7. The fiber of Z is the set of all direct summands of $\mathcal{F}_x = \mathcal{F}(R, x)$ of rank 1.

The involution σ induces an isomorphism $h: \mathcal{E}_x \otimes \mathcal{L} \rightarrow \mathcal{E}_x^*$ for some invertible R -module \mathcal{L} (see [Knus, Lemma III.8.2.2]) such that

$$\begin{aligned}\sigma(f) \otimes 1 &= h^{-1} f^* h \text{ for all } f \in A \\ h^* \text{can} \otimes 1 &= -h\end{aligned}$$

where $\text{can}: \mathcal{E}_x \rightarrow \mathcal{E}_x^{**}$ is the canonical isomorphism.

Let U_1 and U_2 be direct summands of \mathcal{E}_x . We write $U_2 \subseteq U_1^\perp$ if $h(u \otimes l)(v) = 0$ for all $u \in U_1, v \in U_2, l \in \mathcal{L}$. We call a direct summand U of \mathcal{E}_x *totally isotropic* if $U \subseteq U^\perp$. Note that any direct summand of rank 1 is totally isotropic (it can be proved easily using localization).

Define Φ and Ψ as in the proof of Theorem 4.4. Direct computations show that $I_1 \subseteq I_2^\perp$ if and only if $\Phi(I_1) \subseteq \Phi(I_2)^\perp$.

So the fiber of Y over x is naturally isomorphic to the set of all totally isotropic direct summands U_k of \mathcal{E}_x of rank d_k containing $U_{k-1} = \Phi(I_k)$. One can represent U_k as the direct sum $U_{k-1} \oplus U$ where U is a direct summand of rank 1 (since $d_k = d_{k-1} + 1$). This U is totally isotropic and, therefore, U_k is totally isotropic if and only if $U_k \subseteq U_{k-1}^\perp$. Hence the set of all U_{k-1} is naturally isomorphic to the set of all direct summands of $\Phi(I_{k-1}^\perp)$ of rank d_k containing $\Phi(I_{k-1})$. The latter can be identified with $\mathbb{P}(\mathcal{F}_x)$. This finishes the proof of the lemma. \square

We are now ready to prove Theorem 2.10.

6.10 (Proof of Theorem 2.10). Consider the flag varieties $Y = X(d_1, \dots, d_k)$ and $X(d_1, \dots, d_{k-1})$. Since $\text{ind } A = 2^r$ for some r and there is an odd d_i , we have $\text{gcd}(\text{ind}(A), d_1, \dots, d_{k-1}) = 1$. By Lemma 4.5 there exists a bundle \mathcal{E} over X such that $A_X = \text{End}_{\mathcal{O}_X}(\mathcal{E})$. Applying Lemma 6.9 to the varieties X, Y and the bundle \mathcal{E} , we obtain that Y is a projective bundle over X . Now we use Proposition 3.4 to finish the proof.

7 Groups of types G_2 and F_4

This section is devoted to proofs of Theorems 2.12 and 2.13.

7.1. Let G be a group of type G_2 . We suppose that characteristic of k is not 2. It is known that $G = \text{Aut}(C)$ where C is a Cayley algebra over k . By i -space where $i = 1, 2$ we mean an i -dimensional subspace V_i of C such that $uv = 0$ for every $u, v \in V_i$.

The only flag variety corresponding to a non-maximal parabolic is the full flag variety $X(1, 2)$ which is described as follows (see [Bo03]):

$$X(1, 2) = \{V_1 \subset V_2 \mid V_i \text{ is a } i\text{-subspace of } C, \dim V_i = i\}.$$

Similarly one can describe the homogeneous flag variety corresponding to the maximal parabolic

$$X(2) = \{V \mid V \text{ is a 2-subspace of } C, \dim V = 2\}.$$

7.2. Let R be a k -algebra. By an i -submodule in $C_R = C \otimes_k R$ we mean a direct summand V_i of C_R of rank i such that $uv = 0$ for every two elements $u, v \in V_i$. The value of the functor corresponding to the variety $X(1, 2)$ (respectively $X(2)$) at a k -algebra R equals the set of all flags $V_1 \subset V_2$ (respectively submodules V_2) where V_i is an i -submodule of C_R .

7.3. On the scheme $X = X(2)$ there is a “tautological” vector bundle \mathcal{V} of rank 2. The value of \mathcal{V} on an X -algebra (R, x) is V , where $x = V$.

There exists an obvious morphism

$$\begin{aligned} X(1, 2) &\rightarrow X(2) \\ (V_1, V_2) &\mapsto V_2 \end{aligned}$$

which makes $X(1, 2)$ into a $X(2)$ -scheme.

7.4 Lemma. $X(1, 2)$ as a scheme over $X(2)$ can be identified with the projective bundle $\mathbb{P}(\mathcal{V}) = \text{Gr}(1, \mathcal{V})$ over $X(2)$.

Proof. The proof goes as in B_n -case (note that if V_2 is a 2-submodule then each of its direct summands of rank 1 is a 1-submodule). \square

7.5 (Proof of Theorem 2.12). Apply Lemma 7.4 and Proposition 3.4.

7.6. Let G be a group of type F_4 . We suppose that characteristic of k is not 2, 3. It is known that $G = \text{Aut}(J)$ where J is an exceptional Jordan algebra of dimension 27 over k . Set $I = \{1, 2, 3, 6\}$. By i -space where $i \in I$ we mean an i -dimensional subspace V_i of J such that every $u, v \in V_i$ satisfy the following condition:

$$\text{tr}(u) = 0, \quad u \times v = 0, \quad \text{and if } i < 6 \text{ then } u(va) = v(ua) \text{ for all } a \in J.$$

It is known that projective G -homogeneous varieties are parametrized by sequences of numbers $d_1 < \dots < d_k$ from I and can be described as follows:

$$X(d_1, \dots, d_k) = \{V_1 \subset \dots \subset V_k \mid V_i \text{ is a } d_i\text{-subspace of } A, \dim V_i = d_i\}.$$

7.7. Let R be a k -algebra. By an i -submodule in $J_R = J \otimes_k R$ we mean a direct summand V_i of J_R of rank i such that every two elements $u, v \in V_i$ satisfy the conditions above. The value of the functor corresponding to the variety $X(d_1, \dots, d_k)$ at a k -algebra R equals the set of all flags $V_1 \subset \dots \subset V_k$ where V_i is a d_i -submodule of J_R .

For convenience we set $d_0 = 0, V_0 = 0$.

7.8. On the scheme $X = X(d_1, \dots, d_k)$ there are ‘‘tautological’’ vector bundles \mathcal{V}_i of rank d_i . The value of \mathcal{V}_i on an X -algebra (R, x) is V_i , where $x = (V_1, \dots, V_k)$.

There exists an obvious morphism

$$\begin{aligned} X(d_1, \dots, d_k) &\rightarrow X(d_1, \dots, \hat{d}_m, \dots, d_k) \\ (V_1, \dots, V_k) &\mapsto (V_1, \dots, \hat{V}_m, \dots, V_k) \end{aligned}$$

which makes $X(d_1, \dots, d_k)$ into a $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ -scheme.

7.9 Lemma. Denote $X(d_1, \dots, d_k)$ by Y and $X(d_1, \dots, \hat{d}_m, \dots, d_k)$ by X . Suppose that $m < k$ and either $d_{m+1} < 6$ or $d_m = 1$. Then Y as a scheme over X can be identified with the Grassmann bundle $Z = \text{Gr}(d_m - d_{m-1}, \mathcal{V}_{m+1}/\mathcal{V}_{m-1})$ over X .

Proof. The proof goes as in B_n -case (note that under our restrictions if V_{m+1} is a d_{m+1} -submodule then each of its direct summands of rank d_m is a d_m -submodule). \square

7.10 (Proof of Theorem 2.13). Apply Lemma 7.9 to the varieties $Y = X(d_1, \dots, d_k)$ and $X = X(d_1, \dots, \hat{d}_m, \dots, d_k)$. We obtain that Y is a Grassmann bundle over X . To finish the proof apply Proposition 3.4.

Acknowledgements We are sincerely grateful to N. Karpenko for the comments concerning rational cycles and Severi-Brauer varieties.

The first and the last authors would like to thank U. Rehmann and RTN-project for hospitality and support. The last author is also grateful to Alexander von Humboldt foundation for support.

The second and the third authors highly appreciate A. Bak's hospitality and sympathy with their activity. The second author also would like to thank DAAD *Jahresstipendium* program and INTAS project 03-51-3251.

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