

R -equivalence in adjoint classical groups over fields of virtual cohomological dimension 2

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Abstract

Let F be a field of characteristic not 2 whose virtual cohomological dimension is at most 2. Let G be a semisimple group of adjoint type defined over F . Let $RG(F)$ denote the normal subgroup of $G(F)$ consisting of elements R -equivalent to identity. We show that if G is of classical type not containing a factor of type D_n , $G(F)/RG(F) = 0$. If G is a simple classical adjoint group of type D_n , we show that if F and its multi-quadratic extensions satisfy strong approximation property, then $G(F)/RG(F) = 0$. This leads to a new proof of the R -triviality of F -rational points of adjoint classical groups defined over number fields.

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Introduction

In [Ma, Chapter II, §14] Manin introduced the notion of R -equivalence on a variety X over a field F as follows : two points $x, y \in X(F)$ are R -equivalent if there exist $x = x_0, x_1, x_2, \dots, x_n = y \in X(F)$ and F -rational maps $f_i : \mathbb{P}^1 \dashrightarrow X; 1 \leq i \leq n$, regular at 0 and 1 such that $f_i(0) = x_{i-1}$ and $f_i(1) = x_i$. If X is the underlying variety of a connected algebraic group G , then the set of elements of $G(F)$ which are

R -equivalent to 1, is a normal subgroup $RG(F)$ of $G(F)$. We denote the quotient $G(F)/RG(F)$ by $G(F)/R$. A connected algebraic group is called R -trivial, if for all field extensions E of F , we have $G(E)/R = 0$. Colliot-Thélène and Sansuc [CTS] proved that if the variety of a connected algebraic group G is stably rational then G is R -trivial. For example, if G is an adjoint classical group of type 1A_n , ${}^2A_{2n}$ [VK, pp. 240] or B_n , then G is rational, and hence R -trivial.

Let G be a classical group of adjoint type defined over a number field. The group $G(F)/R$ is then trivial. If \tilde{G} is a simply connected cover of G , the triviality of $G(F)/R$ can be deduced from the following results:

- (i) [Y] If \tilde{G} is of type 2A_n then $\tilde{G}(F)/R = 0$.
- (ii) [PR, Theorem 9.5] The group $\tilde{G}(F)$ is projectively simple provided \tilde{G} does not contain a factor of type A_n . In particular the non-central normal subgroup $R\tilde{G}(F)$ coincides with $\tilde{G}(F)$.
- (iii) [G, pp. 222] and [CTGP, proof of Cor. 4.11] The natural map $\tilde{G}(F)/R \rightarrow G(F)/R$ is surjective.

Number fields are examples of fields of virtual cohomological dimension two. The aim of this paper is to study the group $G(F)/R$ where G is a classical group of adjoint type defined over a field of virtual cohomological dimension two.

Let Γ_F be the Galois group $\text{Gal}(F_s/F)$, where F_s is the separable closure of F . The *cohomological dimension* of F is the least positive integer n such that for all discrete torsion Γ_F -modules M , the Galois cohomology groups $H^i(\Gamma_F, M)$ are zero for all $i \geq n + 1$. A field F is said to have *virtual cohomological dimension* n if the cohomological dimension of $F(\sqrt{-1})$ is n . We write $\text{cd}(F)$ to denote the cohomological dimension and $\text{vcd}(F)$ to denote the virtual cohomological dimension of F . We prove that $G(F)/R = 0$ for adjoint groups G of type 2A_n and C_n over a field F of virtual cohomological dimension at most 2. For classical groups of type D_n , we prove that if the cohomological dimension of F is at most 2 then $G(F)/R = 0$.

Further, if the virtual cohomological dimension of F is at most 2, then we show that $G(F)/R = 0$, provided F satisfies *certain* approximation properties. These results, in particular, lead to a new proof of the triviality of $G(F)/R$ for adjoint classical groups over number fields.

The main ingredients in the proof of our results are Merkurjev's computation of $G(F)/R$ for all adjoint groups of classical type [Me2, Th. 1], as well as results on the classification of hermitian forms over division algebras with involution over fields of virtual cohomological dimension two [BP2].

1 Some known results

In this section, we record some known results which are used in the paper. Let F be a field with $\text{char}(F) \neq 2$. Let $Z = F$, or a quadratic extension of F . Let A be a central simple algebra over Z and σ be an involution on A of either kind. If σ is of second kind, then let $Z^\sigma = F$. An element $a \in A^*$ is said to be a *similitude* of (A, σ) if $\sigma(a)a \in F^*$. The similitudes of (A, σ) form a group which we denote by $\text{Sim}(A, \sigma)$. The map $\mu(a) = \sigma(a)a$ is a homomorphism $\mu : \text{Sim}(A, \sigma) \rightarrow F^*$ whose image is denoted by $G(A, \sigma)$. Elements of $G(A, \sigma)$ are called *multipliers*. Let σ be adjoint to a hermitian form h . Then $\lambda \in G(A, \sigma)$ if and only if $\lambda h \simeq h$ [KMRT, Prop. 12.20]. Let $\mathbf{Sim}(A, \sigma)$ denote the algebraic group whose F rational points are given by $\text{Sim}(A, \sigma)$. Let $\mathbf{Sim}_+(A, \sigma)$ be the connected component of identity of $\mathbf{Sim}(A, \sigma)$. Let $\text{Sim}_+(A, \sigma)$ denote the F -rational points of $\mathbf{Sim}_+(A, \sigma)$. Elements of $\text{Sim}_+(A, \sigma)$ are called *proper similitudes*. We denote the group $\mu(\text{Sim}_+(A, \sigma))$ by $G_+(A, \sigma)$. Let $R_{Z/F}$ denote the Weil restriction to F . The group of *projective similitudes* is the quotient group

$$\mathbf{Sim}(A, \sigma)/R_{Z/F}(\mathbf{G}_m)$$

which we denote by $\mathbf{PSim}(A, \sigma)$. The group of F -rational points of $\mathbf{PSim}(A, \sigma)$ is $\text{Sim}(A, \sigma)/Z^*$. The connected component of the identity of the group $\mathbf{PSim}(A, \sigma)$ is denoted by $\mathbf{PSim}_+(A, \sigma)$

Let $N(Z) = F^{*2}$ or $N_{Z/F}(Z^*)$ according as σ is of first kind or second kind, respectively. Let $\text{Hyp}(A, \sigma)$ be the subgroup of F^* generated by the norms from all those finite extensions of F , where the involution σ becomes hyperbolic. If A is split, the involution σ is adjoint to a quadratic form q over F . The group $G_+(A, \sigma)$ is then denoted by $G_+(q)$ and the group $G(A, \sigma)$ is denoted by $G(q)$. In fact $G_+(q) = G(q)$, because of the existence of hyperplane reflections in the orthogonal group.

Theorem 1.1 ([Me2, Th. 1]) *With the notation as above, $N(Z) \cdot \text{Hyp}(A, \sigma)$ is a subgroup of $G_+(A, \sigma)$ and further,*

$$\mathbf{PSim}_+(A, \sigma)(F)/R \simeq G_+(A, \sigma)/N(Z) \cdot \text{Hyp}(A, \sigma).$$

□

We now record a lemma due to Dieudonné.

Lemma 1.2 (Dieudonné, [KMRT, Lemma 13.22]) *Let q be a quadratic form of even rank and $d = \text{disc}(q)$. Let $L = F(\sqrt{d})$. Then $G(q) \subseteq N_{L/F}(L^*)$.* □

The following result of Merkurjev-Tignol extends Dieudonné's lemma.

Lemma 1.3 ([MT, Th. A]) *Let A be a central simple algebra of even degree with an orthogonal involution σ . Let $d = \text{disc}(\sigma)$ and let $L = F(\sqrt{d})$. Then $G_+(A, \sigma) \subseteq N_{L/F}(L^*)$.* □

Let q be a non-degenerate quadratic form of rank r over F . Let τ_q be the adjoint involution on $M_r(F)$. Then $\text{Hyp}(M_r(F), \tau_q) = \text{Hyp}(q)$, the subgroup of F^* generated by $N_{L/F}(L^*)$; L varying over finite extensions of F where q becomes hyperbolic. If r is odd, then $\text{Hyp}(q) = 1$.

Theorem 1.4 ([Me2, pp. 200]) *Let A be a central simple algebra of odd degree with an orthogonal involution σ . Let q be a quadratic form over F such that σ is adjoint to q . Then $G_+(A, \sigma) = G(q) = \text{Hyp}(q) \cdot F^{*2} = \text{Hyp}(A, \sigma) \cdot F^{*2} = F^{*2}$.* □

We now record a result due to Knebusch which describes the group of spinor norms of a quadratic form. Let q be a quadratic form over F and $\text{sn}(q)$ denote the subgroup of F^* generated by F^{*2} and representatives of the square classes in the image of the spinor norm map $\text{sn} : \text{SO}(q) \rightarrow F^*/F^{*2}$. For a central simple algebra A over F , let $\text{Nrd} : A \rightarrow F$ denote the reduced norm map. For $S \subseteq F^*$, we denote by $\langle S \rangle$, the subgroup generated by S in F^* .

Theorem 1.5 (Knebusch's norm principle, [L, Theorem VII.5.1]) *For a quadratic form q over F we have:*

$$\text{sn}(q) = \langle \{N_{L/F}(L^*) : L/F \text{ is a quadratic extension over } F \text{ and } q_L \text{ is isotropic}\} \rangle.$$

□

The two results recorded below describe the group $G(A, \sigma)$ in the case when σ is unitary or symplectic under further assumptions on the degree of A .

Theorem 1.6 ([Me2, §2]) *Let F be a field with $\text{char}(F) \neq 2$. Let A be a central simple algebra over Z of odd degree with an involution σ of second kind with $Z^\sigma = F$. Then $G_+(A, \sigma) = G(A, \sigma) = \text{Hyp}(A, \sigma) = N(Z)$.* □

Theorem 1.7 ([Me2, §2, Lemma 3]) *Let F be a field with $\text{char}(F) \neq 2$. Let A be a central simple algebra over F of degree $2n$ with n odd. Let σ be a symplectic involution on A . Then $G_+(A, \sigma) = G(A, \sigma) = \text{Hyp}(A, \sigma) = \text{Nrd}(A)$.* □

Next results we record are local criteria for elements to be reduced norms or spinor norms for formally real fields F with $\text{vcd}(F) \leq 2$.

Theorem 1.8 ([BP2, Theorem 2.1]) *Let F be a formally real field with $\text{vcd}(F) \leq 2$. Let Ω denote the set of orderings on F . Let A be a central simple algebra over F and $A_v = A \otimes_F F_v$, F_v denoting the real closure of F at v . Let $\lambda \in F^*$ be such that $\lambda >_v 0$ at those orderings $v \in \Omega$ where A_v is non-split. Then $\lambda \in \text{Nrd}(A^*)$.* □

Theorem 1.9 ([BP2, Cor. 7.10]) *Let F be a formally real field with $\text{vcd}(F) \leq 2$. Let q be a quadratic form over F . Then $\text{sn}(q)$ consists of elements of F^* which are positive at each $v \in \Omega$ such that q is definite at F_v .* □

We say that a quadratic form q over F is *locally isotropic* if over each real closure F_v , $v \in \Omega$, the form q is isotropic.

Corollary 1.10 *With the notation as in 1.9, if q locally isotropic, then $sn(q) = F^*$. \square*

Let Γ_F denote the Galois group $\text{Gal}(F_s/F)$. For a discrete Γ_F -module M , let $H^n(F, M)$ denote the Galois cohomology group $H^n(\text{Gal}(F_s/F), M)$. We now record some results of Arason which we shall use in the paper.

Theorem 1.11 (Corollar 4.6, [A1]) *Let $Z = F(\sqrt{\delta})$ be a quadratic extension of F . Then we have a long exact sequence of abelian groups*

$$\cdots \rightarrow H^n(F, \mu_2) \xrightarrow{\text{res}} H^n(Z, \mu_2) \xrightarrow{\text{cores}} H^n(F, \mu_2) \xrightarrow{\cup_{n,1}(\delta)} H^{n+1}(F, \mu_2) \rightarrow \cdots$$

where res and cores denote the restriction and corestriction maps respectively. \square

In view of 1.11 and the isomorphism $H^2(F, \mu_2) \simeq {}_2\text{Br}(F)$, we have the following

Proposition 1.12 *Let $Z = F(\sqrt{\delta})$ be a quadratic extension of F and let A be a central simple algebra over Z with $\exp(A) = 2$ and $\text{cores}_{Z/F}([A]) = 0 \in H^2(F, \mu_2)$. Then there exists a central simple algebra A_0 over F such that $A_0 \otimes_F Z$ is Brauer equivalent to A . \square*

We say that a field extension L/F is a *quadratic tower over F* if there exist fields F_i such that $F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r = L$ and each F_i/F_{i-1} is a quadratic extension for $1 \leq i \leq r$. We denote by $\mathcal{F}_2(F)$ the set of quadratic towers of F in an algebraic closure of F . Let $I(F)$ denote the fundamental ideal of the Witt ring $W(F)$ of F . For each $n \geq 1$, we denote by $I^n(F)$ to denote the ideal $I(F)^n$.

Lemma 1.13 ([A1, Satz 3.6]) *Let $I^3(F) = 0$ and L/F be a quadratic tower. Then $I^3(L) = 0$. \square*

Theorem 1.14 ([A2, Prop. 2]) *Let F be a field with $\text{cd}(F) \leq 2$. Then $I^3(F) = 0$. \square*

A non-trivial element $\chi \in H^r(F, \mu_2)$ is called (-1) -torsion-free if for every $s \geq 1$, the element $\chi \cup (-1) \cup (-1) \cup \cdots \cup (-1) \in H^{r+s}(F, \mu_2)$ is non-trivial. The following is a consequence of 1.11

Proposition 1.15 *Let F be a field with $\text{vcd}(F) \leq n$. Then $H^{n+1}(F, \mu_2)$ is (-1) -torsion-free. \square*

The following lemma relates the conditions $\text{vcd}(F) \leq 2$ and $I^3(F)$ being torsion-free.

Lemma 1.16 ([BP2, Lemma 2.4]) *Let F be a field with virtual cohomological dimension at most two. Then $I^3(F)$ is torsion-free.*

Proof Since $\text{vcd}(F) \leq 2$, by [AEJ] the invariants $e_r : I^r(F) \rightarrow H^r(F, \mu_2)$ have kernel $I^{r+1}(F)$ for each $r \geq 0$ and $H^r(F(\sqrt{-1}), \mu_2) = 0$ for $r \geq 3$. Then it is evident from Arason exact sequence 1.11 for the quadratic extension $F(\sqrt{-1})/F$ that $H^r(F, \mu_2) \xrightarrow{\cup(-1)} H^{r+1}(F, \mu_2)$ is an isomorphism for $r \geq 3$. Let $q \in I^3(F)$ be a torsion-element. Then $2^s \cdot q = 0 \in W(F)$ for some integer $s \geq 0$. As a consequence $e_3(q) \cup (-1) \cup (-1) \cup \cdots \cup (-1) = 0 \in H^{3+s}(F, \mu_2)$. Since $H^r(F, \mu_2) \xrightarrow{\cup(-1)} H^{r+1}(F, \mu_2)$, $r \geq 3$, are isomorphisms, we conclude that $e_3(q) = 0$; i.e. $q \in \ker(e_3) = I^4(F)$. By a similar argument $q \in I^r(F)$ for each $r \geq 3$ and hence $q \in \bigcap_r I^r(F)$. By a theorem of Arason-Pfister [L, Cor. X.3.2], $q = 0 \in W(F)$ and hence $I^3(F)$ is torsion-free. \square

The following result is a weaker form of [Se, Prop. 10, §II.4.1].

Theorem 1.17 *Let F be a field and $\text{cd}(F) \neq \text{vcd}(F)$. Then F has orderings. \square*

2 Some norm principles

Let F be a field with $\text{char}(F) \neq 2$ and $I^3(F) = 0$. Let A be a central simple algebra with $\exp(A) = 2$. Then by [Me1], there are quaternion algebras H_i ; $1 \leq i \leq r$, such that $A \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r$. We define an integer $r(A)$ associated to A as follows:

$$r(A) := \min\{r : A \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r\}.$$

If A is split, then we define $r(A) = 0$. Given a central simple algebra B over a field Z with $[Z : F] \leq 2$ and a field extension L of F , we set: $B_L = B \otimes_F L$.

Proposition 2.1 *Let $I^3(F) = 0$ and A be a central simple algebra over F . If $\exp(A)$ is a power of 2 then*

$$F^* = \langle \{N_{L/F}(L^*) : L \text{ is a quadratic tower of } F \text{ with } A_L \text{ split}\} \rangle = \text{Nrd}(A^*)$$

In fact, for each $\lambda \in F^$ there is a quadratic tower L/F and $\alpha \in L^*$ such that $\lambda = N_{L/F}(\alpha)$.*

Proof By the classical norm principle for reduced norms, over any field we have the inclusion

$$\langle \{N_{L/F}(L^*) : L \text{ is a quadratic tower of } F \text{ with } A_L \text{ split}\} \rangle \subseteq \text{Nrd}(A^*)$$

Thus to complete the proof, it suffices to show that under the assumption $I^3(F) = 0$,

$$F^* \subseteq \langle \{N_{L/F}(L^*) : L \text{ is a quadratic tower of } F \text{ with } A_L \text{ split}\} \rangle. \quad (1)$$

Let $\exp(A) = 2^m$. We prove the lemma by induction on m . Suppose $m = 1$. Then $\exp(A) = 2$ and hence by Merkurjev's Theorem [Me1], we write $A \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r$, where $r = r(A)$ and each H_i is a quaternion algebra over F . We proceed further by induction on r . If $r = 1$ the result holds by [BP2, Prop. 2.7]. Let $r \geq 2$ and $\lambda \in F^*$. By [BP2, Prop. 2.7] there exists a quadratic extension L of F which splits H_1 and $\lambda \in N_{L/F}(L^*)$. Then $r(A_L) < r$ and by 1.13 we have $I^3(L) = 0$. Induction on r leads to (1).

Suppose that $m \geq 2$. Then $\exp(A \otimes_F A) = 2^{m-1}$. Let $\lambda \in F^*$. By induction, there exists a quadratic tower L over F and $\alpha \in L^*$ such that $\lambda = N_{L/F}(\alpha)$ and $(A \otimes_F A)_L$ is split. Then $\exp(A_L) = 2$ and by 1.13, $I^3(L) = 0$. By the previous case, there exists a quadratic tower M of L with $\alpha \in N_{M/L}(M^*)$ and A_M is split. Thus M is a quadratic tower of F such that $\lambda \in N_{M/F}(M^*)$ and A_M is split. This completes the proof. \square

Proposition 2.2 *Let $I^3(F) = 0$ and Z be a quadratic extension of F . Let A be a central simple algebra over Z such that $\text{cores}_{Z/F}(A) = 0$ and $\exp(A) = 2^m$. Then for each $\lambda \in F^*$, there exists a quadratic tower L/F such that $\lambda \in N_{L/F}(L^*)$ and A_L is split.*

Proof We prove this by induction on m . Suppose $m = 1$. Since $\exp(A) = 2$ and $\text{cores}_{Z/F}(A) = 0$, by 1.12 there exists a central simple algebra A_0 of exponent 2 over F such that $A \sim A_0 \otimes_F Z$. Let $\lambda \in F^*$. Since $I^3(F) = 0$, by 2.1, there exists a quadratic tower L/F such that $(A_0)_L$ is split and $\lambda \in N_{L/F}(L^*)$. Clearly the extension L splits A and the proposition follows.

Suppose $m \geq 2$. Let $\lambda \in F^*$. Since $\exp(A \otimes_Z A) = 2^{m-1}$, by induction there exists a quadratic tower L/F such that $\lambda = N_{L/F}(\alpha)$ for some $\alpha \in L^*$, and $(A \otimes_Z A)_L$ splits. Clearly $\exp(A_L) = 2$, and by the previous case we have a quadratic tower M/L such that A_M splits and $\alpha \in N_{M/L}(M^*)$. Then M/F is a quadratic tower such that $\lambda \in N_{M/F}(M^*)$ and A_M is split. This completes the proof. \square

We shall now describe norm principles for fields F with $\text{vcd}(F) \leq 2$. If F has no orderings by 1.17, $\text{cd}(F) \leq 2$, and the results follow from the previous discussion. We shall assume in the rest of the section that F has orderings. We denote by Ω , the set of orderings on F . If A is a central simple algebra over F then A is said to be *locally split* if $A \otimes_F F_v = A_v$ is split for each $v \in \Omega$.

Proposition 2.3 *Let $\text{vcd}(F) \leq 2$ and A be a central simple algebra over F with $\exp(A) = 2^m$. Then*

$$F^* = \langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } \text{index}(A_M) \leq 2\} \rangle.$$

Proof We prove the proposition by induction on m . Let $m = 1$. Then $\exp(A) = 2$ and we proceed by further induction on $r(A)$. The statement is obvious if $r(A) \leq 1$. Let $r(A) \geq 2$ and $A \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r$ with $r = r(A)$ and $H_i, 1 \leq i \leq r$, quaternion algebras over F . Let $H_1 = (a, b)$ and $H_2 = (c, d)$. Then to the algebra $H_1 \otimes H_2$ is associated the Albert form (cf. [KMRT, §16.A])

$$q = \langle -a, -b, ab, c, d, -cd \rangle.$$

Since $\text{disc}(q) = 1$ and $\dim(q) = 6$, the form q is isotropic at F_v for each $v \in \Omega$. Thus by 1.10, $\text{sn}(q) = F^*$ and by 1.5 we have

$$F^* = \text{sn}(q) = \langle \{N_{L/F}(L^*) : L \text{ is a quadratic extension of } F \text{ and } q_L \text{ is isotropic}\} \rangle$$

Let L be a quadratic extension of F with q_L isotropic. By Albert's Theorem [KMRT, Th. 16.5], we have $r((H_1 \otimes H_2)_L) \leq 1$. Thus $r(A_L) < r(A)$ and by induction we have,

$$L^* = \langle \{N_{M/L}(M^*) : M \in \mathcal{F}_2(L) \text{ and } \text{index}(A_M) \leq 2\} \rangle.$$

and therefore taking norms from L to F we have,

$$\begin{aligned} F^* = \text{sn}(q) &= \langle \{N_{L/F}(L^*) : L \text{ is a quadratic extension of } F \text{ and } q_L \text{ is isotropic}\} \rangle \\ &\subseteq \langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } \text{index}(A_M) \leq 2\} \rangle. \end{aligned}$$

This completes the case $m = 1$. Now let $m \geq 2$. Then $\exp(A \otimes_F A) = 2^{m-1}$ and by induction

$$F^* = \langle \{N_{L/F}(L^*) : L \in \mathcal{F}_2(F) \text{ and } \text{index}((A \otimes_F A)_L) \leq 2\} \rangle. \quad (2)$$

Let $L \in \mathcal{F}_2(F)$ be such that $\text{index}((A \otimes_F A)_L) \leq 2$. Since the Brauer group of a real-closed field is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, it follows that $(A \otimes_F A)_L$ is locally split. Thus by 1.8, $\text{Nrd}((A \otimes_F A)_L) = L^*$. Since $\text{index}((A \otimes_F A)_L) \leq 2$, we have

$$\begin{aligned} &\text{Nrd}((A \otimes_F A)_L) \\ &\subseteq \langle \{N_{N/L}(N^*) : N \text{ is a quadratic extension of } L \text{ and } (A \otimes_F A)_N \text{ is split}\} \rangle. \end{aligned} \quad (3)$$

Let N be a quadratic extension of L such that $(A \otimes_F A)_N$ is split. Then $\exp(A_N) = 2$ and by the case $m = 1$

$$N^* = \langle \{N_{M/N}(M^*) : M \in \mathcal{F}_2(N) \text{ and } \text{index}(A_M) \leq 2\} \rangle. \quad (4)$$

Now it is clear from (2), (3) and (4) that

$$F^* = \langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } \text{index}(A_M) \leq 2\} \rangle.$$

□

We refine 2.3 to the following:

Proposition 2.4 *Let F be a field with $\text{vcd}(F) \leq 2$. Let A be a central simple algebra over F with $\exp(A) = 2^m$ for some $m \geq 1$. Then,*

$$F^* = \langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } A_M \sim (-1, -x) \text{ for some } x \in M^*\} \rangle.$$

Proof Let L be a quadratic tower over F such that $\text{index}(A_L) \leq 2$. Then $A_L \sim (a, b)$, $a, b \in L^*$. Let Ω_L denote the set of orderings on L . For each $w \in \Omega_L$, the quadratic form $q' = \langle -1, -a, -b, ab \rangle$ is isotropic over L_w , where L_w denotes the real closure of L at w . Therefore by [BP2, Prop. 7.7] we have $\text{sn}(q') = L^*$. Thus, in view of 1.5 we have:

$$L^* = \langle \{N_{M/L}(M^*) : M \text{ is a quadratic extension of } L \text{ and } q'_M \text{ is isotropic} \} \rangle.$$

Let M be a quadratic extension of L such that q'_M is isotropic. Then the form $\langle -a, -b, ab \rangle_M$ represents 1, and we can write: $\langle -a, -b, ab \rangle_M \simeq \langle 1, x, y \rangle_M$; with $x, y \in M^*$. Comparing the discriminants, we have $\langle -a, -b, ab \rangle_M \simeq \langle 1, x, x \rangle_M$. Thus $\langle 1, -a, -b, ab \rangle_M \simeq \langle 1, 1, x, x \rangle_M$ and $(a, b)_M \simeq (-1, -x)$. Thus,

$$\begin{aligned} L^* &= \langle \{N_{M/L}(M^*) : M \text{ is a quadratic extension of } F \text{ and } q'_M \text{ is isotropic} \} \rangle \\ &\subseteq \langle \{N_{M/L}(M^*) : M \in \mathcal{F}_2(L) \text{ and } A_M \sim (-1, -x) \text{ for some } x \in M^*\} \rangle \end{aligned}$$

and

$$N_{L/F}(L^*) \subseteq \langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } A_M \sim (-1, -x) \text{ for some } x \in M^*\} \rangle.$$

This together with 2.3 gives

$$F^* \subseteq \langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } A_M \sim (-1, -x) \text{ for some } x \in M^*\} \rangle.$$

This completes the proof. □

Corollary 2.5 *Let $\text{vcd}(F) \leq 2$. Let A_1 and A_2 be central simple algebras over F with $\exp(A_i)$ a power of 2 for $i = 1, 2$. Then we have:*

$$\begin{aligned} F^* &= \langle \{N_{M/F}(M^*) : M \in \mathcal{F}_2(F) \text{ and } A_{1M} \sim (-1, -x), \\ &\quad A_{2M} \sim (-1, -y) \text{ for some } x, y \in M^*\} \rangle \end{aligned}$$

□

The following is a refinement of the surjectivity of the reduced norm (Theorem 1.8) for locally split algebras with centre a quadratic extension of F .

Proposition 2.6 *Let $\text{vcd}(F) \leq 2$ and F has orderings. Let Ω denote the set of orderings on F . Let $Z = F(\sqrt{\delta})$ be a quadratic extension of fields. Let A be a central simple Z -algebra which is split at each $v \in \Omega$. Further assume that $\exp(A) = 2^m$ for some integer m and $\text{cores}_{Z/F}(A) = 0$. Then for each $\lambda \in F^*$, there exist extensions E_i over F and $\lambda_i \in E_i^*$ such that each $A \otimes_F E_i$ is split and $\lambda = \prod_i N_{E_i/F}(\lambda_i)$.*

Proof We proceed by induction on m . Let $m = 1$. Since $\text{cores}_{Z/F}(A) = 0$, by 1.12 there is a central simple algebra A_0 over F with $\exp(A_0) = 2$ and $A \sim A_0 \otimes_F Z$. By [Me1], there are quaternion algebras $H_i; 1 \leq i \leq r = r(A_0)$ over F such that $A_0 \sim H_1 \otimes H_2 \otimes \cdots \otimes H_r$. Suppose $r = 1$. Then $A_0 \sim H_1 = (a, b)$ for some $a, b \in F^*$. Let q denote the quadratic form $\langle 1, -a, -b, ab\delta \rangle$ over F . Then by [CTSk, Prop. 2.3] we have $\text{sn}(q) = \text{Nrd}((H_1 \otimes_F F(\sqrt{\delta}))^*) \cap F^*$. Since A is locally split, by 1.8 $\text{Nrd}((H_1 \otimes_F F(\sqrt{\delta}))^*) = \text{Nrd}(A^*) = Z^*$. Therefore $\text{sn}(q) = F^*$. Thus by 1.5, for each $\lambda \in F^*$, there exist quadratic extensions E_i/F and $\lambda_i \in E_i^*$ such that each q_{E_i} is isotropic and $\lambda = \prod_i N_{E_i/F}(\lambda_i)$. Further $A \otimes_F E_i \sim (a, b) \otimes_{E_i} E_i(\sqrt{\delta})$ and the norm form of $(a, b) \otimes_{E_i} E_i(\sqrt{\delta})$ is isometric to $q_{E_i(\sqrt{\delta})}$, which is isotropic. It follows therefore that each $A \otimes_F E_i$ is split. Thus F^* is generated by the norms from those extensions of F where the algebra A is split.

Now suppose $r \geq 2$. Then by 2.3 we have

$$F^* = \langle \{N_{L/F}(L^*) : \text{index}((A_0)_L) \leq 2\} \rangle.$$

The proposition follows immediately from the case $r = 1$.

Let $m \geq 2$. Then $\exp(A \otimes_Z A) = 2^{m-1}$ and at each $v \in \Omega$ the algebra $(A \otimes_Z A) \otimes_F F_v$ is split since $\text{Br}(F_v) = \mathbb{Z}/2\mathbb{Z}$. Thus by induction, F^* is generated by norms from extensions M_i over F such that the algebra $(A \otimes_Z A) \otimes_F M_i$ splits. It is clear that $\exp(A \otimes_F M_i) = 2$. Thus by the exponent 2 case, it follows that each M_i^* is generated by norms from extensions E_i of M_i such that $A \otimes_F E_i$ is split. We conclude therefore, that F^* is generated by norms from those extensions of F where A splits. \square

3 Fields with $\text{cd}(F) \leq 2$

In this section, we prove that if $\text{cd}(F) \leq 2$, then for adjoint classical groups G of type 2A_n , C_n and D_n , $G(F)/R = 0$. We begin with the result leading to the triviality of $G(F)/R$ in the C_n case.

Theorem 3.1 *Let F be a field with $\text{char}(F) \neq 2$ and $I^3(F) = 0$. Let A be a central simple algebra of degree $2n$ over F and σ be a symplectic involution on A . Then $\text{Hyp}(A, \sigma) = F^*$.*

Proof Let $\lambda \in F^*$. Since exponent of A is 2 and $I^3(F) = 0$, by 2.1, there exists a quadratic tower L/F such that L splits A and $\lambda \in N_{L/F}(L^*)$. The involution σ_L is adjoint to a skew-symmetric form h_L over L which is hyperbolic. Therefore $\lambda \in \text{Hyp}(A, \sigma)$. \square

Let q be a quadratic form over F of rank $2n$. Let σ be the involution on $M_{2n}(F)$ which is adjoint to q . We denote by $C(q)$ the Clifford invariant of q .

Proposition 3.2 *If $I^3(F) = 0$, then $G(q) \subseteq \text{Hyp}(q)$.*

Proof We first assume that the discriminant of q is trivial. Let $\lambda \in F^*$. The algebra $C(q)$ has exponent 2 and by 2.1, there exists a quadratic tower M of F such that $C(q) \otimes_F M$ is split and $\lambda \in N_{M/F}(M^*)$. By 1.13, $I^3(M) = 0$. Since q_M is an even dimensional quadratic form with trivial discriminant and trivial Clifford invariant, in view of [EL, Th. 3] q_M is hyperbolic and hence $\lambda \in \text{Hyp}(q)$. Thus $\text{Hyp}(q) = F^*$.

Now suppose that $\text{disc}(q)$ is non-trivial, $d \in F^*$ a representative of the square class of $\text{disc}(q)$ in F^*/F^{*2} and $L = F(\sqrt{d})$. Let $\lambda \in G(q)$. By 1.2, $\lambda \in N_{L/F}(L^*)$. Since $\text{disc}(q_L) = 1$, by the previous case $L^* = \text{Hyp}(q_L)$. Taking norms we get $N_{L/F}(L^*) \subseteq \text{Hyp}(q)$. Thus $G(q) \subseteq \text{Hyp}(q)$. \square

We prove a similar result when A is not split.

Theorem 3.3 *Let $I^3(F) = 0$. Let A be a central simple algebra with an involution σ of orthogonal type. Let d be the discriminant of σ and $L = F[X]/(X^2 - d)$. Then we have $G_+(A, \sigma) = \text{Hyp}(A, \sigma) = N_{L/F}(L^*)$.*

Proof Since A supports an involution of first kind, $\exp(A) \leq 2$. Suppose first that $\text{disc}(\sigma)$ is trivial. Let M be a quadratic tower of F which splits A . By the proof of 3.2 we have $M^* = \text{Hyp}(A_M, \sigma_M)$. Thus $N_{M/F}(M^*) \subseteq \text{Hyp}(A, \sigma)$. This, together with 2.1 implies that $F^* = \text{Nrd}(A^*) \subseteq \text{Hyp}(A, \sigma)$. Hence $F^* = \text{Hyp}(A, \sigma) = G_+(A, \sigma)$. Since $L = F \times F$, we have $N_{L/F}(L^*) = F^*$. Thus $G_+(A, \sigma) = \text{Hyp}(A, \sigma) = N_{L/F}(L^*)$.

Suppose that $\text{disc}(\sigma)$ is not trivial. Let $d \in F^*$ represent the class of $\text{disc}(\sigma)$ in F^*/F^{*2} . Let $\lambda \in G_+(A, \sigma)$. Then by 1.3, we have $\lambda \in N_{L/F}(L^*)$ where $L = F(\sqrt{d})$. Clearly $\text{disc}(\sigma_L) = 1$ and by previous case $L^* = \text{Hyp}(A_L, \sigma_L)$. Thus $\lambda \in N_{L/F}(L^*) \subseteq \text{Hyp}(A, \sigma)$. Thus $G_+(A, \sigma) \subseteq N_{L/F}(L^*) \subseteq \text{Hyp}(A, \sigma)$. By 1.1, $\text{Hyp}(A, \sigma).F^{*2} \subseteq G_+(A, \sigma)$. Hence $G_+(A, \sigma) = \text{Hyp}(A, \sigma) = N_{L/F}(L^*)$. \square

Let Z be a quadratic extension of F and A be a central simple algebra over Z with an involution σ of second kind such that $Z^\sigma = F$. In the next lemma, we consider the case where A splits and the involution σ is adjoint to a Z/F -hermitian form h . In view of 1.6, we further assume that h has even rank; i.e. $\deg(A)$ is even.

Lemma 3.4 *Let $I^3(F) = 0$, let A be split and σ be an involution of second kind on A such that $Z^\sigma = F$. Then $\text{Hyp}(A, \sigma) = F^*$.*

Proof Let $Z = F(\sqrt{\delta})$. Let q_h be the quadratic form over F defined by $q_h(x) = h(x, x)$. Then $q_h \simeq \langle 1, -\delta \rangle \otimes q$ [S, pp. 349, Remark 1.3] for some quadratic form q over F having the same rank as h , which is even. Therefore $q_h \in I^2(F)$ and by a theorem of Jacobson [MH, pp. 114], the form h is hyperbolic over an extension M of F if and only if the quadratic form q_h is hyperbolic over M . Let C denote the Clifford invariant of q_h . Let $\lambda \in F^*$. By 2.1, there exists a quadratic tower M over F such that C_M is split and $\lambda \in N_{M/L}(M^*)$. Since $I^3(M) = 0$, by [EL, Th. 3], $(q_h)_M$ is hyperbolic and hence the hermitian form h_M is hyperbolic. Therefore $N_{M/F}(M^*) \subseteq \text{Hyp}(A, \sigma)$. Thus $\text{Hyp}(A, \sigma) = F^*$. \square

Theorem 3.5 *If $I^3(F) = 0$ and $\exp(A) = 2^m$ then $\text{Hyp}(A, \sigma) = F^*$.*

Proof Since A supports an involution σ of second kind, by [S, Th. 9.5] we have $\text{cores}_{Z/F}(A) = 0$. Therefore by 2.2, given $\lambda \in F^*$ there exists a quadratic tower L/F such that A_L splits and $\lambda \in N_{L/F}(L^*)$. Since A_L is split, by 3.4, $L^* = \text{Hyp}(A_L, \sigma_L)$. Taking norms we conclude that $\lambda \in \text{Hyp}(A, \sigma)$. Therefore $\text{Hyp}(A, \sigma) = F^*$. \square

Theorem 3.6 *Let $\text{cd}(F) \leq 2$ and Z be a quadratic extension of F . Let A be a central simple algebra of even degree over Z with an involution σ of second kind such that $Z^\sigma = F$. Then $\text{Hyp}(A, \sigma).F^{*2} = F^*$.*

Proof By [BP1, Lemma 3.3.1], there exists an odd degree extension L over F such that $\exp(A \otimes_F L)$ is a power of 2. Since the condition $\text{cd}(F) \leq 2$ is preserved under finite extensions of fields [Ar, Th 2.1], we have $\text{cd}(L) \leq 2$. By 1.14 $I^3(L) = 0$ and by 3.5, $\text{Hyp}(A_L, \sigma_L) = L^*$. Hence $N_{L/F}(L^*) \subseteq \text{Hyp}(A, \sigma)$. Let $\lambda \in F^*$ and $[L : F] = 2s + 1$. Then $\lambda^{2s+1} = N_{L/F}(\lambda) \in \text{Hyp}(A, \sigma)$ and we have $\lambda \in \text{Hyp}(A, \sigma).F^{*2}$. This implies that $\text{Hyp}(A, \sigma).F^{*2} = F^*$. \square

Theorem 3.7 *If $\text{cd}(F) \leq 2$ and G an adjoint group of classical type defined over F , then $G(F)/R = 0$.*

Proof A classical adjoint group G is a direct product of groups $R_{L_i/F}(G_i)$, where L_i/F are finite extensions and G_i are absolutely simple adjoint groups of classical type defined over L_i [T, 3.1.2]. Moreover, $G_i(L_i)/R = R_{L_i/F}(G_i)(F)/R$ and R -equivalence commutes with direct products [CTS, pp. 195]. In view of this, it suffices to prove the theorem for an absolutely simple classical adjoint group G defined over F . By [We] such an algebraic group is isomorphic to $\mathbf{PSim}_+(A, \sigma)$ for a central simple algebra A over a field Z , $[Z : F] \leq 2$, with an involution σ . In view of 1.1 and 1.14, the result follows in 2A_n case from 3.6 and 1.6, in B_n case from 1.4 in C_n case from 3.1 and 1.7, and in D_n case from 3.3. \square

Remark Theorem 3.7 for groups of type A_n and C_n also follows from [CTGP, Cor. 4.11], using the fact that $G(F)/R = 0$ if G is simply connected of type A_n or C_n , and [G, pp. 222].

4 Fields with $\text{vcd}(F) \leq 2$: Symplectic groups

In this section F denotes a formally real field with $\text{vcd}(F) \leq 2$, and Ω , the set of orderings on F . Let A be a central simple algebra over F of degree $2n$ and σ be an involution of symplectic type on A . In view of 1.7, we assume that n is even. We say that σ is *locally hyperbolic* if for each $v \in \Omega$, the involution σ_v on $A_v = A \otimes_F F_v$ is hyperbolic, F_v denoting the real closure of F at v .

Proposition 4.1 *Let A be a central simple algebra over F of degree $2n$, where n is an even integer. Let σ be a symplectic involution on A . If σ is locally hyperbolic then $\text{Hyp}(A, \sigma) = F^*$.*

Proof First assume that $A = M_n(H)$, where H is a quaternion algebra over F . Let bar denote the canonical involution on H and h a hermitian form of rank n over $(H, -)$ such that σ is adjoint to h . Since σ is locally hyperbolic, so is h and hence $\text{sgn}(h) = 0$. Thus h has even rank and trivial signature and by [BP2, Th. 6.2], the form h itself is hyperbolic. Thus $\text{Hyp}(A, \sigma) = F^*$.

Suppose A is arbitrary. Since A supports an involution, $\exp(A) = 2$ [S, Th. 8.4] and by 2.3, we have

$$F^* = \langle \{N_{M/F}(M^*) : \text{index}(A_M) \leq 2\} \rangle \quad (*)$$

Let M be a finite extension of F such that $\text{index}(A_M) \leq 2$. Then $A_M \simeq M_n(H)$ where H is a quaternion algebra over M . Since σ is locally hyperbolic, so is σ_M and by the previous case, $M^* = \text{Hyp}(A_M, \sigma_M)$. Therefore $N_{M/F}(M^*) \subseteq \text{Hyp}(A, \sigma)$ and in view of (*) we get $\text{Hyp}(A, \sigma) = F^*$. \square

Theorem 4.2 *Let F be a formally real field with $\text{vcd}(F) \leq 2$. Let A be a central simple algebra over F of degree $2n$ and σ be a symplectic involution on A . Then $G(A, \sigma) \subseteq \text{Hyp}(A, \sigma)$.*

Proof In view of 1.7, we assume that n is even. Let $\lambda \in G(A, \sigma)$ and $K = F(\sqrt{-\lambda})$. Let Ω_K denote the set of orderings on K . For each $w \in \Omega_K$, $\lambda \equiv -1$ modulo K_w^{*2} is a similarity factor for σ_K and hence $\text{sgn}(\sigma_K) = 0$. Further $\deg(A)$ is divisible by 4 and hence the involution σ_K is locally hyperbolic. Thus by 4.1, we have $\text{Hyp}(A_K, \sigma_K) = K^*$. Therefore

$$\lambda = N_{K/F}(\sqrt{-\lambda}) \in N_{K/F}(K^*) = N_{K/F}(\text{Hyp}(A_K, \sigma_K)) \subseteq \text{Hyp}(A, \sigma).$$

□

5 Fields with $\text{vcd}(F) \leq 2$: Unitary groups

Let F be an arbitrary field with $\text{char}(F) \neq 2$. Let $Z = F(\sqrt{\delta})$ be a quadratic extension of F . Let A be a central simple algebra over Z and σ be an involution on A such that $Z^\sigma = F$. In view of 1.6, we assume throughout this section that A has even degree.

Let $\deg(A) = 2m$ and $D = D(A, \sigma)$ denote the discriminant algebra of (A, σ) (*cf.* [KMRT, §10.E]). The algebra D is a central simple algebra over F and carries an involution $\bar{\sigma}$ of first kind, which is of symplectic type if m is odd and of orthogonal type if m is even [KMRT, Prop. 10.30]. For $1 \leq i \leq 2m$, let $\wedge^i A$ be the i^{th} \wedge -power of A (*cf.* [KMRT, §10 (10.4)]). By [KMRT, Prop. 14.3], there is a homogeneous polynomial map $\wedge^i : A \rightarrow \wedge^i A$ of degree i , $1 \leq i \leq 2m$. If $A = \text{End}_F(V)$ then $\wedge^i A = \text{End}_F(\wedge^i V)$ and $\wedge^i(f) = \wedge^i(f)$, the i^{th} exterior power of the linear map $f \in \text{End}_F(V)$.

Theorem 5.1 *Let F be a field with $\text{char}(F) \neq 2$. Let A be a central simple algebra of degree $2m$ over a field Z with m odd. Let σ be an involution of second kind on A such that $Z^\sigma = F$. Let $D = D(A, \sigma)$ be the discriminant algebra of (A, σ) . Then $G(A, \sigma) \subseteq \text{Nrd}(D^*) \cdot N_{Z/F}(Z^*)$.*

Proof Let $x \in G(A, \sigma)$ and $g \in \text{Sim}(A, \sigma)$ be such that $\mu(g) = \sigma(g)g = x$. Then $N_{Z/F}(\text{Nrd}(g)) = \mu(g)^{2m}$ and by Hilbert Theorem-90, there exists $\alpha \in Z^*$ such that

$\mu(g)^{-m} \text{Nrd}(g) = \alpha^{-1} \bar{\alpha}$, where bar denotes the non-trivial automorphism of Z over F . By [KMRT, Lemma 14.6], we have

$$\bar{\sigma}(\alpha^{-1} \wedge^m g) \alpha^{-1} \wedge^m g = N_{Z/F}(\alpha)^{-1} \mu(g)^m.$$

Since m is odd, $x = \mu(g) \in G(D, \bar{\sigma}).N_{Z/F}(Z^*)$. Thus

$$G(A, \sigma) \subseteq G(D, \bar{\sigma}).N_{Z/F}(Z^*) \quad (*)$$

Let $y \in G(D, \bar{\sigma})$ be arbitrary and $h \in \text{Sim}(D, \bar{\sigma})$ be such that $\mu(h) = \bar{\sigma}(h)h = y$. Since m is odd, the involution $\bar{\sigma}$ is of symplectic type and by [KMRT, Prop. 12.23] we have $\mu(h)^m = \text{Nrd}(h)$. Again, since m is odd, we have $y = \mu(h) \in \text{Nrd}(D^*).F^{*2}$. Thus

$$G(D, \bar{\sigma}) \subseteq \text{Nrd}(D^*).F^{*2} \quad (**)$$

and combining the inclusions (*) and (**) above, we get

$$G(A, \sigma) \subseteq \text{Nrd}(D^*).N_{Z/F}(Z^*).$$

This completes the proof. □

In this section, from now onwards we assume that $\text{vcd}(F) \leq 2$, and F has orderings and denote by Ω , the set of orderings on F . A quadratic form q over F is called *locally hyperbolic* if q is hyperbolic at every real closure F_v , $v \in \Omega$.

Lemma 5.2 *If q is a locally hyperbolic quadratic form of even rank and trivial discriminant over F , $\text{Hyp}(q) = F^*$.*

Proof Since q is locally hyperbolic the Clifford algebra $C(q)$ of q is locally split. Thus by 1.8 we have $\text{Nrd}(C(q)^*) = F^*$. Let $\lambda \in F^*$ and let L/F be a finite extension such that $\lambda \in N_{L/F}(L^*)$ and $C(q)_L$ is split. Then q_L has even dimension, trivial discriminant, trivial Clifford invariant and $\text{sgn}(q_L) = 0$. Therefore by [EL, Th. 3], the form q_L is hyperbolic and $\lambda \in N_{L/F}(L^*) \in \text{Hyp}(q)$. □

Proposition 5.3 *Let $Z = F(\sqrt{\delta})$ be a quadratic extension. Let $A = M_r(Z)$, where r is an even positive integer, support a locally hyperbolic Z/F -involution σ . Then $\text{Hyp}(A, \sigma) = F^*$.*

Proof Let the involution σ be adjoint to a Z/F -hermitian form h . Then the rank of h is r . Let q_h be the quadratic form over F given by $q_h(x) = h(x, x)$. Then $q_h \simeq \langle 1, -\delta \rangle \otimes q$ [S, pp. 349, Remark 1.3], where q is a quadratic form over F of the same rank as that of h , which is even. Therefore $q_h \in I^2(F)$. By Jacobson's theorem [MH, p. 114], the form h_M is hyperbolic if and only if the quadratic form $(q_h)_M$ is hyperbolic. It follows that $\text{Hyp}(A, \sigma) = \text{Hyp}(q_h)$. Since h is locally hyperbolic, the form q_h is locally hyperbolic as well. By 5.2, we have $\text{Hyp}(q_h) = F^*$. Thus $\text{Hyp}(A, \sigma) = \text{Hyp}(q_h) = F^*$. \square

The following is a consequence of 5.3 and 2.6.

Proposition 5.4 *Let A be a locally split central simple Z -algebra and σ be a locally hyperbolic Z/F -involution on A . Let $\exp(A) = 2^m$. Then $\text{Hyp}(A, \sigma) = F^*$.*

Proof Since A supports an involution σ of second kind with $Z^\sigma = F$, by [S, Th. 9.5], $\text{cores}_{Z/F}(A) = 0$. Thus by 2.6 we have

$$F^* = \langle \{N_{L/F}(L^*) : A_L \text{ is split} \} \rangle.$$

Let L/F be an extension which splits A . By 5.3, $\text{Hyp}(A_L, \sigma_L) = L^*$ and taking norm from L/F , we conclude that $\text{Hyp}(A, \sigma) = F^*$. \square

Proposition 5.5 *Let A be a central simple algebra over Z of degree $2m$, where m is odd. Let σ be a Z/F -involution on A with $\text{sgn}(\sigma) = 0$. Let $D = D(A, \sigma)$ be the discriminant algebra of (A, σ) . Then $\text{Nrd}(D^*) \subseteq \text{Hyp}(A, \sigma).F^{*2}$. Further*

$$G(A, \sigma) = \text{Nrd}(D^*).N_{Z/F}(Z^*) = \text{Hyp}(A, \sigma).N_{Z/F}(Z^*).$$

Proof We first show that $\text{Nrd}(D^*) \subseteq \text{Hyp}(A, \sigma).F^{*2}$. Assume first that $\exp(A)$ is a power of 2. Since $\deg(A) = 2m$ with m odd, $\text{index}(A) = \exp(A) = 2$ and $A = M_m(H)$ for some quaternion algebra H over Z . By [KMRT, §10.4], [KMRT, Prop. 10.30] and the hypothesis that m is odd, it follows that

$$D \otimes_F Z \simeq \wedge^m(M_m(H)) \sim H^{\otimes m} \sim H.$$

Thus if M is a finite extension of F such that D_M is split, then H_M is split and $\text{sgn}(\sigma_M) = 0$. Thus by 5.3, $\text{Hyp}(A_M, \sigma_M) = M^*$ and taking norms, $N_{M/F}(M^*) \subseteq \text{Hyp}(A, \sigma)$. In view of the classical norm principle for reduced norms, $\text{Nrd}(D^*) \subseteq \text{Hyp}(A, \sigma)$.

Now suppose that $\exp(A)$ is arbitrary. By [BP1, Lemma 3.3.1], there exists an odd degree extension L/F such that $\exp(A_L)$ is a power of 2. Let $\lambda \in \text{Nrd}(D^*)$. Then $\lambda \in \text{Nrd}(D_L^*)$. By the previous case, $\lambda \in \text{Hyp}(A_L, \sigma_L)$. Taking norm from L/F and using the hypothesis that m is odd, we conclude that $\lambda \in \text{Hyp}(A, \sigma).F^{*2}$. This proves the first assertion of 5.5. It follows immediately that

$$\text{Nrd}(D^*).N_{Z/F}(Z^*) \subseteq \text{Hyp}(A, \sigma).N_{Z/F}(Z^*) \quad (*)$$

From 5.1, it is clear that

$$G(A, \sigma) \subseteq \text{Nrd}(D^*).N_{Z/F}(Z^*) \quad (**)$$

and further by 1.1, $\text{Hyp}(A, \sigma).N_{Z/F}(Z^*) \subseteq G(A, \sigma)$. In view of this and the inclusions (*) and (**) we conclude that

$$G(A, \sigma) = \text{Nrd}(D^*).N_{Z/F}(Z^*) = \text{Hyp}(A, \sigma).N_{Z/F}(Z^*).$$

□

Theorem 5.6 *Let F be a field with $\text{vcd}(F) \leq 2$ and Z be a quadratic extension of F . Let A be a central simple algebra over Z of degree $2m$, where m is odd. Let σ be a Z/F -involution on A . Then $G(A, \sigma) = \text{Hyp}(A, \sigma).N_{Z/F}(Z^*)$.*

Proof Let $\lambda \in G(A, \sigma)$. Let $D = D(A, \sigma)$ be the discriminant algebra of (A, σ) . By 5.1, $\lambda \in \text{Nrd}(D^*).N_{Z/F}(Z^*)$. Let $\lambda_1 \in \text{Nrd}(D^*)$ and $\alpha \in N_{Z/F}(Z^*)$ be such that

$\lambda = \lambda_1 \alpha$. Since $N_{Z/F}(Z^*) \subseteq \text{Hyp}(A, \sigma) \subseteq G(A, \sigma)$, it follows that $\lambda_1 \in G(A, \sigma)$. Let $K = F(\sqrt{-\lambda_1})$. Then $\text{sgn}(\sigma_K) = 0$ and by 5.5, $\text{Nrd}(D_K^*) \subseteq \text{Hyp}(A_K, \sigma_K).K^{*2}$. Further, since $\lambda_1 \in \text{Nrd}(D_K^*)$ and $\lambda_1 \equiv -1 \pmod{K^{*2}}$, D_K is locally split and by 1.8, $\text{Nrd}(D_K^*) = K^*$. Thus $\text{Hyp}(A_K, \sigma_K).K^{*2} = K^*$. Taking norms, we get

$$\lambda_1 \in N_{K/F}(K^*) = N_{K/F}(\text{Hyp}(A_K, \sigma_K)) \subseteq \text{Hyp}(A, \sigma).$$

Thus $\lambda = \lambda_1 \alpha \in \text{Hyp}(A, \sigma).N_{Z/F}(Z^*)$. This completes the proof. \square

Let $\Sigma(F)$ denote the set of elements of F which are positive at all orderings of F .

Lemma 5.7 *Let $\alpha, \delta \in F^*$. Then we have:*

$$F^* = \langle \{N_{L/F}(L^*) : L/F \text{ is a quadratic extension such that there exists } u_L \in L(\sqrt{\delta}) \text{ with } N_{L(\sqrt{\delta})/L}(u_L) = 1 \text{ and } \alpha u_L \in \Sigma(L(\sqrt{\delta}))\} \rangle.$$

Proof Since the quadratic form $\phi = \langle 1, \delta, -\alpha, \delta\alpha \rangle$ is locally isotropic, by 1.5 and 1.10,

$$F^* = \text{sn}(\phi) = \langle \{N_{L/F}(L^*) : L \text{ is a quadratic extension of } F \text{ such that } \phi_L \text{ is isotropic}\} \rangle \quad (*)$$

At an extension L/F where ϕ is isotropic, we choose $a, b, c, d \in L^*$ such that $a^2 + \delta b^2 - \alpha c^2 + \delta\alpha d^2 = 0$. If $c^2 + \delta d^2 = 0$ or $a^2 + \delta b^2 = 0$, clearly $L(\sqrt{\delta})$ has no ordering and thus $\Sigma(L(\sqrt{\delta})) = L(\sqrt{\delta})^*$. In this case, we may take $u_L = 1$. Otherwise, we let $\theta = c + d\sqrt{\delta}$ and $u_L = \theta^{-1}\bar{\theta}$, where $\bar{\theta} = c - d\sqrt{\delta}$. It is immediate that $\text{Tr}_{L(\sqrt{\delta})/L}(u_L) = 2(c^2 + \delta d^2)(c^2 - \delta d^2)^{-1}$ and $N_{L(\sqrt{\delta})/L}(u_L) = 1$. Since $a^2 + \delta b^2 - \alpha c^2 + \delta\alpha d^2 = 0$ and both $c^2 + \delta d^2$ and $c^2 - \delta d^2$ are units, it follows that

$$\alpha = ((a^2 + \delta b^2)(c^2 + \delta d^2)^{-1}) ((c^2 + \delta d^2)(c^2 - \delta d^2)^{-1})$$

Thus

$$2\alpha \text{Tr}_{L(\sqrt{\delta})/L}(u_L) = ((a^2 + \delta b^2)(c^2 + \delta d^2)^{-1}) \left(\text{Tr}_{L(\sqrt{\delta})/L}(u_L) \right)^2 \in N_{L(\sqrt{-\delta})/L}(L(\sqrt{-\delta}))$$

and hence the quaternion algebra $\left(2\alpha \text{Tr}_{L(\sqrt{\delta})/L}(u_L), -\delta \right)$ over L is split.

Let v be an ordering on L which extends to an ordering w on $L(\sqrt{\delta})$. Then $\delta >_v 0$ and hence $2\alpha\text{Tr}_{L(\sqrt{\delta})/L}(u_L) >_v 0$. Let bar denote the non-trivial automorphism of $L(\sqrt{\delta})$ over L . Since $\alpha u_L \overline{\alpha u_L} = \alpha^2 >_v 0$, both αu_L and $\overline{\alpha u_L}$ have same sign at w . But $\alpha\text{Tr}_{L(\sqrt{\delta})/L}(u_L) = \alpha(u_L + \overline{u_L}) >_v 0$. Thus $\alpha u_L >_w 0$. This is true for every ordering of $L(\sqrt{\delta})$. Thus $\alpha u_L \in \Sigma(L(\sqrt{\delta}))$ and $N_{L(\sqrt{\delta})/L}(u_L) = 1$. This completes the proof of the lemma. \square

Let D be a division algebra with centre Z and τ be an involution on D of the second kind. Let $Z^\tau = F$. Let (V, h) be a non-degenerate hermitian space over (D, τ) . Then the integer $\dim_D(V)$ is said to be the *rank* of h and is denoted by $\text{rank}(h)$. Let $\text{rank}(h) = n$. For a choice $\{e_1, e_2, \dots, e_n\}$ of a D -basis of V , the form h determines a matrix $M_h = (h(e_i, e_j)) \in M_n(D)$. The matrix M_h is τ -hermitian symmetric. Let $r = n \deg(D)$. We define the *discriminant* of h to be $(-1)^{r(r-1)/2} \text{Nrd}(M_h) \in F^*/N_{Z/F}(F^*)$ and denote it by $\text{disc}(h)$.

We refine the notion of discriminant to the notion of *Discriminant* as follows:

Let $M_h \in M_n(D)$ be a matrix as above, representing the hermitian form h . Let $M'_h \in M_n(D)$ also represent h . Then there exists an invertible matrix $T \in M_n(D)$ such that

$$\text{Nrd}(M'_h) = \text{Nrd}(M_h) \text{Nrd}(T) \tau(\text{Nrd}(T))$$

Thus we have the following well defined notion of Discriminant:

$$\text{Disc}(h) = (-1)^{r(r-1)/2} \text{Nrd}(M_h) \in F^*/N_{Z/F}(\text{Nrd}(D^*))$$

where $r = n \deg(D)$.

We now quote a classification result for hermitian forms over division algebras with an involution of second kind over fields with $\text{vcd}(F) \leq 2$.

Theorem 5.8 ([BP2, Theorem 4.8]) *Let F be a field with $\text{vcd}(F) \leq 2$ and D be a division algebra with an involution τ of second kind such that $(\text{centre}(D))^\tau = F$.*

Let h be a hermitian form over (D, τ) , Then h is hyperbolic if and only if $\text{rank}(h)$ is even, $\text{Disc}(h)$ is trivial and h has trivial signature. \square

Lemma 5.9 Let D be a central division algebra over Z , τ be a Z/F -involution over D and h be a hermitian of rank $2s$ over (D, τ) . Let $\text{disc}(h) = 1$. Then

$$F^* = \langle \{N_{M/F}(M^*) : \text{Disc}(h_M) = 1\} \rangle.$$

Proof Let $M_h \in M_{2s}(D)$ be a matrix representing h . Since $\text{disc}(h) = 1 \in F^*/N_{Z/F}(Z^*)$, we have $\text{Nrd}(M_h) = d \in N_{Z/F}(Z^*)$. Let $z \in Z$ be such that $d = N_{Z/F}(z)$. Let $\beta = \text{Tr}_{Z/F}(z)$ and $\gamma = z\beta^{-1}$. Let w be an ordering on Z which extends an ordering v of F such that D_w is not split. Then $\text{Nrd}(M_h) = d = N_{Z/F}(z) >_w 0$. Thus $\text{Tr}_{Z/F}(z) = \beta >_w 0$ if and only if $z >_w 0$. This implies that $\gamma = z\beta^{-1} >_w 0$ and thus by 1.8, $\gamma \in \text{Nrd}(D^*)$. Let $x \in D^*$ be such that $\text{Nrd}(x) = \gamma$. Let

$$M'_h = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cdot & \\ & & & x \end{pmatrix} M_h \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cdot & \\ & & & \tau(x) \end{pmatrix}^t$$

Then $\text{Nrd}(M'_h) = (d\beta^{-1})^2$ and we conclude that for a suitable choice of a matrix M_h representing the hermitian form h , $\text{Nrd}(M_h) = \alpha^2$, $\alpha \in F^*$. Let $\lambda \in F^*$. By 5.7, there exist quadratic extensions L_i/F , $\lambda_i \in L_i^*$ and $u_i \in L_i(\sqrt{\delta})$, $1 \leq i \leq r$, such that $\lambda = \prod_i N_{L_i/F}(\lambda_i)$, $\alpha u_i \in \Sigma(L(\sqrt{\delta}))$ and $N_{L_i(\sqrt{\delta})/L}(u_i) = 1$. Then

$$\alpha^2 = N_{L_i(\sqrt{\delta})/L_i}(\alpha u_i) \in N_{L_i(\sqrt{\delta})/L_i}(\text{Nrd}(D_{L_i(\sqrt{\delta})}))$$

and hence $\text{Disc}(h_{L_i}) = 1$ for $1 \leq i \leq r$. Thus $\lambda \in \langle \{N_{M/F}(M^*) : \text{Disc}(h_M) = 1\} \rangle$ and we conclude that $F^* = \langle \{N_{M/F}(M^*) : \text{Disc}(h_M) = 1\} \rangle$. \square

The following propositions are used in the proof of 5.13, which is the main result of this section.

Proposition 5.10 Let $A \simeq M_r(D)$ where D is a division algebra over Z and r is even. Let σ be a locally hyperbolic Z/F -involution on A . Then $\text{Hyp}(A, \sigma) = F^*$.

Proof Let σ be adjoint to a hermitian form h of rank r . Let $d \in F^*/N_{Z/F}(Z^*)$ denote the discriminant of h . Since σ is locally hyperbolic, for each $v \in \Omega$, the quaternion algebra (δ, d) splits at F_v . Thus by 1.8, $\text{Nrd}((\delta, d)) = F^*$. Let $\lambda \in F^*$. There exists a finite extension E/F such that $\lambda \in N_{E/F}(E^*)$ and (δ, d) splits over E . Then $\text{disc}(h_E)$ is trivial. By 5.9 we have

$$E^* = \langle \{N_{M/E}(M^*) : \text{Disc}(h_M) = 1\} \rangle \quad (*)$$

Let M/E be an extension such that $\text{Disc}(h_M) = 1$. Since σ is locally hyperbolic, $\text{sgn}(h_M) = 0$. Thus by 5.8, the form h_M is hyperbolic and $\text{Hyp}(h_M) = M^*$. Hence by (*), $\text{Hyp}(h_E) = E^*$ and

$$\lambda \in N_{E/F}(E^*) = N_{E/F}(\text{Hyp}(h_E)) \subseteq \text{Hyp}(h) = \text{Hyp}(A, \sigma)$$

which implies that $\text{Hyp}(A, \sigma) = F^*$. This completes the proof. \square

Proposition 5.11 *Let A be a central simple algebra over Z with $\deg(A) \equiv 0(4)$. Let $\exp(A) = 2^m$ for some positive integer m . Let σ be a locally hyperbolic Z/F -involution on A . Then $\text{Hyp}(A, \sigma) = F^*$.*

Proof Suppose $m = 1$. Since $\text{cores}_{Z/F}(A) = 0$, by 1.12 $A \sim A_0 \otimes_F Z$ for some central simple F -algebra A_0 with $\exp(A_0) = 2$. Let M be a finite extension of F such that $A_{0M} \sim H$ for some quaternion algebra H over M . Then $A_M = M_r(H \otimes_F Z)$. Since $\deg(A) \equiv 0(4)$, the integer r is even. Thus by 5.10, $\text{Hyp}(A_M, \sigma_M) = M^*$. In view of 2.4 we have

$$F^* = \langle \{N_{M/F}(M^*) : A_{0M} \sim H \text{ for some quaternion algebra } H \text{ over } M\} \rangle.$$

It follows that $\text{Hyp}(A, \sigma) = F^*$.

Suppose $m \geq 2$. Since $\text{Br}(Z_w) = \mathbb{Z}/2\mathbb{Z}$ for each ordering $w \in \Omega_Z$, the algebra $A \otimes_Z A$ splits locally. Clearly $\exp(A \otimes_Z A) = 2^{m-1}$ and $\text{cores}_{Z/F}(A \otimes_Z A) = 0$. Let $\lambda \in F^*$. By 2.6, there exist extensions L_i/F , $1 \leq i \leq s$ and $\lambda_i \in L_i^*$ such that each $(A \otimes_Z A) \otimes_F L_i$ is split and $\lambda = \prod_i N_{L_i/F}(\lambda_i)$. Then $\exp(A \otimes_F L_i) = 2$ for each i and by the case $m = 1$, $\lambda_i \in \text{Hyp}(A_{L_i}, \sigma_{L_i})$. Hence

$$\lambda = \prod_i N_{L_i/F}(\lambda_i) \in \prod_i N_{L_i/F}(\text{Hyp}(A_{L_i}, \sigma_{L_i})) \subseteq \text{Hyp}(A, \sigma)$$

and it follows that $\text{Hyp}(A, \sigma) = F^*$. \square

Proposition 5.12 *Let A be a central simple algebra over Z with $\deg(A) \equiv 0(4)$. Let σ be a locally hyperbolic Z/F -involution on A . Then we have $\text{Hyp}(A, \sigma).F^{*2} = F^*$.*

Proof By [BP1, Lemma 3.3.1], there exists an odd degree extension M of F such that $\exp(A_M)$ is a power of 2 and by 5.11, $\text{Hyp}(A_M, \sigma_M) = M^*$. Taking norm from M/F and using that $[M : F]$ is odd, we conclude that $\text{Hyp}(A, \sigma).F^{*2} = F^*$. \square

Theorem 5.13 *Let F be a field with $\text{vcd}(F) \leq 2$ and let Z be a quadratic extension over F . Let A be a central simple algebra over Z and σ be a Z/F -involution on A . Then, $G(A, \sigma) \subseteq \text{Hyp}(A, \sigma).N_{Z/F}(Z^*)$.*

Proof The cases where $\deg(A)$ is odd or $\deg(A) \equiv 2(4)$ are covered by 1.6 and 5.6 respectively. We assume that $\deg(A) \equiv 0(4)$. Let $\lambda \in G(A, \sigma)$. At each $v \in \Omega$, the involution σ_v is adjoint to an even rank hermitian form which is hyperbolic if and only if $\text{sgn}(\sigma_v) = 0$. Therefore $\lambda >_v 0$ at those $v \in \Omega$, where σ_v is not hyperbolic. Let $K = F(\sqrt{-\lambda})$. Then $\deg(A_K) \equiv 0(4)$ and σ_K is locally hyperbolic. Thus by 5.12, we have $\text{Hyp}(A_K, \sigma_K).K^{*2} = K^*$. Let $\sqrt{-\lambda} = \alpha\beta^2$, where $\alpha \in \text{Hyp}(A_K, \sigma_K)$ and $\beta \in K^*$. Then $\lambda = N_{K/F}(\sqrt{-\lambda}) = N_{K/F}(\alpha)(N_{K/F}(\beta))^2$. But $N_{K/F}(\alpha) \in N_{K/F}(\text{Hyp}(A_K, \sigma_K)) \subseteq \text{Hyp}(A, \sigma)$. Thus $\lambda \in \text{Hyp}(A, \sigma).F^{*2}$. This completes the proof. \square

6 Fields with $\text{vcd}(F) \leq 2$: Orthogonal groups

Let F be an arbitrary field with $\text{char}(F) \neq 2$. Let D be a central division algebra over F with an orthogonal involution τ . We first recall from [BP2], certain invariants associated to hermitian forms over (D, τ) .

Discriminant: Let D and τ be as above and h be a hermitian form of even rank over (D, τ) . Let $\text{rank}(h) = 2m$ and let $M_h \in M_{2m}(D)$ represent the hermitian form h . Let

$$\text{Disc}(h) = (-1)^{r(r-1)/2} \text{Nrd}(M_h) \in F^*/(\text{Nrd}(D^*))^2,$$

where $r = 2m \deg(D)$. If $M'_h \in M_{2m}(D)$ is another matrix representing h then there exists an invertible matrix $T \in M_{2m}(D)$ such that $M_h = TM_h(\tau(T)^t)$. Thus $\text{Nrd}(M'_h) = \text{Nrd}(M_h)\text{Nrd}(T)^2$ and $\text{Disc}(h)$ is well defined. We call $\text{Disc}(h)$ the *Discriminant* of h .

Clifford invariant: We recall from [KMRT, §8.B], the notion of the *Clifford algebra* $C(A, \sigma)$ associated to a central simple algebra A over a field F with an involution σ of orthogonal type. If A is split and σ is adjoint to a quadratic form q then $C(A, \sigma)$ is the even Clifford algebra $C_0(q)$ of the quadratic form q . If $\text{disc}(\sigma)$ is trivial, $C(A, \sigma)$ decomposes into a product $C_+(A, \sigma) \times C_-(A, \sigma)$, each of the factors being a central simple algebra over F such that

$$[C_+(A, \sigma)] + [C_-(A, \sigma)] = [A] \in \text{Br}(F).$$

Let D, τ and h be as above. Let $\text{disc}(h)$ be trivial and $A = M_{2m}(D)$. Let τ_h be the orthogonal involution on A which is adjoint to h . We define the *Clifford invariant* of h as follows:

$$\mathcal{C}\ell(h) = [C_+(M_{2m}(D), \tau_h)] \in \text{Br}(F)/[D]$$

Let H_{2m} denote the matrix $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} \in M_{2m}(D)$ where I_m is the identity matrix of size m . The matrix H_{2m} represents the hyperbolic form of rank $2m$ over (D, τ) . Let $\text{U}_{2m}(D, \tau)$, $\text{SU}_{2m}(D, \tau)$ and $\text{Spin}_{2m}(D, \tau)$ denote respectively, the unitary, special unitary group and spin group with respect to the hyperbolic form H_{2m} over (D, τ) . We have an exact sequence

$$1 \rightarrow \mu_2 \rightarrow \text{Spin}_{2m}(D, \tau) \rightarrow \text{SU}_{2m}(D, \tau) \rightarrow 1$$

from which one gets the exact sequence of pointed sets

$$\rightarrow H^1(F, \text{Spin}_{2m}(D, \tau)) \rightarrow H^1(F, \text{SU}_{2m}(D, \tau)) \xrightarrow{\delta} H^2(F, \mu_2)$$

Let \mathfrak{S} denote the set of ordered pairs (X, a) , where $X \in \text{GL}_{2m}(D)$ and $a \in F^*$ satisfy $\tau(X) = X^t$ and $\text{Nrd}(X) = \text{Nrd}(H_{2m})a^2$. The elements of $H^1(F, \text{SU}_{2m}(D, \tau))$

are equivalence classes of \mathfrak{S} under the following equivalence relation: $(X, a) \sim (X', a')$ if and only if there exists $Y \in \mathrm{GL}_{2m}(D)$ with $X' = YX\bar{Y}^t$ and $a' = \mathrm{Nrd}(Y)a$.

Let h be a hermitian form over (D, τ) with $\mathrm{rank}(h) = 2m$ and $\mathrm{disc}(h) = 1$. Let M_h be a matrix which represents h and $\mathrm{Nrd}(M_h) = a^2$, $a \in F^*$. The two elements $\xi_a = (M_h, a)$ and $\xi_{-a} = (M_h, -a)$ in $H^1(F, \mathrm{SU}_{2m}(D, \tau))$ map to $[h]$ under $H^1(F, \mathrm{SU}_{2m}(D, \tau)) \rightarrow H^1(F, \mathrm{U}_{2m}(D, \tau))$. Let $C_+(h) = \delta(\xi_a)$ and $C_-(h) = \delta(\xi_{-a})$. We recall the following lemma from [BMPS, Lemma 3.1].

Lemma 6.1 *If F is a formally real field and v is an ordering on F such that D_v is not split, then the algebra $C_+(h)$ is split at v if and only if $a >_v 0$. \square*

Rost invariant: Let h be a hermitian form over (D, τ) with $\mathrm{rank}(h) = 2m$, trivial discriminant and trivial Clifford invariant. Consider the exact sequence

$$1 \rightarrow \mathrm{SU}_{2m}(D, \tau) \rightarrow \mathrm{U}_{2m}(D, \tau) \rightarrow \mu_2 \rightarrow 1.$$

This gives rise to the following exact sequence of pointed sets

$$\rightarrow \mathrm{U}_{2m}(D, \tau)(F) \rightarrow \{\pm 1\} \rightarrow H^1(F, \mathrm{SU}_{2m}(D, \tau)) \rightarrow H^1(F, \mathrm{U}_{2m}(D, \tau)) \rightarrow$$

Since $\mathcal{C}\ell(h) = 0$, there exists $\xi \in H^1(F, \mathrm{SU}_{2m}(D, \tau))$ mapping to $[h] \in H^1(F, \mathrm{U}_{2m}(D, \tau))$ such that $\delta(\xi) = 0$. Let $\tilde{\xi} \in H^1(F, \mathrm{Spin}_{2m}(D, \tau))$ be a preimage of $\xi \in H^1(F, \mathrm{SU}_{2m}(D, \tau))$. Let $G = \mathrm{Spin}_{2m}(D, \tau)$ and $\mathcal{R}_G : H^1(F, G) \rightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ denote the Rost invariant of G [Me3]. The *Rost invariant of h* is defined as follows: [BP2, pp. 664]

$$R(h) = \mathcal{R}_G(\tilde{\xi}) \in \frac{H^3(F, \mathbb{Q}/\mathbb{Z}(2))}{F^* \cup [D]}$$

The element $\mathcal{R}_G(\tilde{\xi})$ takes values in $H^3(F, \mathbb{Z}/4)$ [BP2, Remark 1], where $\mathbb{Z}/4$ has the trivial Galois module structure. We now recall a proposition which we shall use often.

Proposition 6.2 ([BP2, Cor. 2.6]) *Let F be a formally real field and let $I^3(F)$ be torsion-free. Let Ω be the set of orderings on F . Then the natural map*

$$H^3(F, \mathbb{Z}/4) \rightarrow \prod_{v \in \Omega} H^3(F_v, \mathbb{Z}/4)$$

is injective. \square

We now record a classification result for hermitian forms over central division algebras with orthogonal involutions over fields with $\text{vcd}(F) \leq 2$.

Theorem 6.3 ([BP2, Th. 7.3]) *Let F be a field with $\text{vcd}(F) \leq 2$ and D be a central division algebra over F with an orthogonal involution τ . Let h be a hermitian form over (D, τ) . Then h is hyperbolic if and only if h has even rank, trivial Discriminant, trivial Clifford and Rost invariant and trivial signature.* \square

Let F be a field with $\text{vcd}(F) \leq 2$ and (A, σ) be a central simple algebra over F with orthogonal involution. If A is split, $\deg(A)$ is even, σ is locally hyperbolic and $\text{disc}(\sigma) = 1$, so that by 5.2 we have $\text{Hyp}(A, \sigma) = F^*$. We now consider the case where A is locally split.

Lemma 6.4 *Let $\text{vcd}(F) \leq 2$ and A be a central simple algebra of even degree over F . Let σ be an orthogonal involution on A . If A is locally split then*

$$G_+(A, \sigma) = \text{Hyp}(A, \sigma).F^{*2}.$$

Proof Let $\lambda \in G_+(A, \sigma)$ and $K = F(\sqrt{-\lambda})$. Clearly $\lambda \in N_{K/F}(K^*)$. Let $\text{disc}(\sigma) = d$ and $L = F(\sqrt{d})$. By 1.3, $\lambda \in N_{L/F}(L^*)$. Let $M = F(\sqrt{-\lambda}, \sqrt{d})$. Using [W, Lemma 2.14] for the biquadratic extension M/F , there exist $x \in M^*$ and $y \in F^*$ such that $\lambda = N_{M/F}(x)y^2$. Further A_M is locally split and by 1.8, $\text{Nrd}(A_M) = M^*$. Let E/M be an extension such that $x = N_{E/M}(\alpha)$ for some $\alpha \in E^*$ and A_E be split. Clearly $\text{disc}(\sigma_E) = 1$, σ_E is locally hyperbolic and A_E is split. Thus by 5.2, $\text{Hyp}(A_E, \sigma_E) = E^*$ and hence $x = N_{E/M}(\alpha) \in \text{Hyp}(A_M, \sigma_M)$. Thus $\lambda = N_{M/F}(x)y^2 \subseteq \text{Hyp}(A, \sigma).F^{*2}$. We conclude that $G_+(A, \sigma) \subseteq \text{Hyp}(A, \sigma).F^{*2}$. In view of 1.1 we have $G_+(A, \sigma) = \text{Hyp}(A, \sigma).F^{*2}$. \square

We continue with some lemmas which will be used in the proofs of the main results of this section.

Lemma 6.5 *Let $\text{vcd}(F) \leq 2$ and $\chi \in H^3(F, \mu_2)$. Then*

$$F^* = \langle \{N_{L/F}(L^*) : L \in \mathcal{F}_2(F) \text{ and } \chi_L = (-1) \cup (-1) \cup (-x) \text{ for some } x \in L^*\} \rangle.$$

Proof Since $\text{vcd}(F) \leq 2$, $H^3(F(\sqrt{-1}), \mu_2) = 0$ and in view of the Arason exact sequence 1.11, the map $H^2(F, \mu_2) \xrightarrow{\cup(-1)} H^3(F, \mu_2)$ is surjective. Let $\xi \in H^2(F, \mu_2)$ be such that $(-1) \cup \xi = \chi$. Let D_ξ be a central division algebra over F , whose Brauer class is represented by ξ . Then $\exp(D_\xi) = 2$. Let $L \in \mathcal{F}_2(F)$ be such that $(D_\xi)_L \sim (-1) \cup (-x)$ for some $x \in L$. Then

$$\chi_L = (-1) \cup \xi_L = (-1) \cup (D_\xi)_L = (-1) \cup (-1) \cup (-x).$$

In view of this and 2.4, we have that

$$\begin{aligned} F^* &= \langle \{N_{L/F}(L^*) : L \in \mathcal{F}_2(F) \text{ and } (D_\xi)_L = (-1) \cup (-x) \text{ for some } x \in L^*\} \rangle \\ &\subseteq \langle \{N_{L/F}(L^*) : L \in \mathcal{F}_2(F) \text{ and } \chi_L = (-1) \cup (-1) \cup (-x) \text{ for some } x \in L^*\} \rangle \end{aligned}$$

□

For $\chi \in H^r(F, \mu_2)$, we set $N(\chi) = \langle \{N_{L/F}(L^*) : \chi_L = 0\} \rangle$.

Lemma 6.6 *Let $\text{vcd}(F) \leq 2$ and $\chi \in H^r(F, \mu_2)$, $r \geq 2$. Then following three groups coincide:*

- (i) $N(\chi)$
- (ii) $\{\lambda \in F^* : \lambda >_v 0 \text{ at those } v \in \Omega \text{ where } \chi_v \neq 0\}$.
- (iii) $\{\lambda \in F^* : (\lambda) \cup \chi = 0\}$.

Proof Since $\text{vcd}(F) \leq 2$, in view of 1.15 the cohomology groups $H^{r+1}(F, \mu_2)$ are (-1) -torsion-free for $r \geq 2$ and thus the groups (ii) and (iii) coincide. We show that $N(\chi) \subseteq \{\lambda \in F^* : (\lambda) \cup \chi = 0\}$. Let $\lambda \in N(\chi)$ be such that $\lambda = N_{L/F}(\mu)$ for an extension L/F with $\chi_L = 0$. Then $((\mu) \cup \chi)_L = 0$ and thus we have

$$\text{cores}_{L/F}((\mu) \cup \chi)_L = (\lambda) \cup \chi = 0.$$

Hence $N(\chi) \subseteq \{\lambda \in F^* : (\lambda) \cup \chi = 0\}$. To complete the proof, we show that $\{\lambda \in F^* : \lambda >_v 0 \text{ at those } v \in \Omega \text{ where } \chi_v \neq 0\} \subseteq N(\chi)$. Let $\lambda \in F^*$ be such that $\lambda >_v 0$ at those $v \in \Omega$ where $\chi_v \neq 0$. Let $L = F(\sqrt{-\lambda})$. Then it

follows that $\chi_w = 0$ for each ordering w of L . It follows from [Ar, Th. 2.1] that $\text{vcd}(L) \leq 2$ and thus by 1.15, $H^3(L, \mu_2)$ is (-1) -torsion-free. Therefore $\chi_L = 0$. Thus $\lambda = N_{L/F}(\sqrt{-\lambda}) \in N(\chi)$. This completes the proof. \square

In 6.7, 6.8 and 6.9 below, the only restriction on F is that $\text{char}(F) \neq 2$. Let D be a central division algebra over F and τ be an orthogonal involution on D . Let h be a hermitian form of rank $2m$ and trivial discriminant over (D, τ) . Let $a \in F^*$ be such that $\text{Nrd}(M_h) = a^2$, where M_h is a matrix representing the form h . Since $\text{disc}(h) = 1$, we recall from [MT, Prop. 1.12] that $G_+(h) = G(h)$.

Lemma 6.7 *Let D be a central division algebra over a field F of characteristic different from 2 with an orthogonal involution τ . Let h and h' be two even rank hermitian forms of trivial discriminant over (D, τ) . Then we have the following additive property for Clifford invariants:*

$$\mathcal{C}l(h \perp h') = \mathcal{C}l(h) + \mathcal{C}l(h') \in H^2(F, \mu_2)/[D].$$

Proof We extend the scalars to the function field of the Brauer-Severi variety of D . Using the fact that the invariant e_2 of quadratic forms is additive on forms of trivial discriminant and that the kernel of the scalar extension map $H^2(F, \mu_2) \rightarrow H^2(F(X_D), \mu_2)$ is generated by the class of D in $H^2(F, \mu_2)$ [MT, Cor. 2.7], the lemma follows. \square

From this lemma and the fact that two similar hermitian forms with even rank and trivial discriminant have the same Clifford invariants [BP1, pp. 204], we immediately have

Corollary 6.8 *Let D , τ and h be as in 6.7. Then for each $\lambda \in F^*$, the Clifford invariant $\mathcal{C}l(h \perp -\lambda h)$ is trivial. \square*

In the following lemma, we compute the Rost invariant of the hermitian form $h \perp -\lambda h$, where h is as in 6.7 and $\lambda \in F^*$ is an arbitrary scalar.

Lemma 6.9 *Let D be a central division algebra of even degree over a field F of characteristic different from 2. Let τ be an orthogonal involution on D and h be a hermitian form over (D, τ) of even rank and trivial discriminant. Let $\lambda \in F^*$. Then,*

$$R(h \perp -\lambda h) = (\lambda) \cup [C_+(h)] \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))/F^* \cup [D].$$

Proof Let $\text{rank}(h) = 2m$ and $A = M_{2m}(D)$. Let τ_h be the involution on A which is adjoint to h . We denote by $\mathbf{PGO}_+(h)$ the group $\mathbf{PSim}_+(A, \tau_h)$ of similitudes. We have an exact sequence

$$1 \rightarrow \mu_2 \rightarrow \text{SU}(h) \rightarrow \mathbf{PGO}_+(h) \rightarrow 1$$

which induces a map on the cohomology sets $H^1(F, \mu_2) \rightarrow H^1(F, \text{SU}(h))$. We claim that under this map $(\lambda) \in H^1(F, \mu_2)$ is mapped to an element $\xi_\lambda \in H^1(F, \text{SU}(h))$ which corresponds to the class of the hermitian form λh in $H^1(F, \text{U}(h))$. In fact, the cocycle $(\lambda) \in Z^1(F, \mu_2)$ given by $s \mapsto s(\sqrt{\lambda})(\sqrt{\lambda})^{-1}$ for $s \in \text{Gal}(F_s/F)$, when treated as a cocycle with values in $\text{U}(h)$, represents $[\lambda h]$ in $H^1(F, \text{U}(h))$.

Since $\deg(A) \equiv 0(4)$, the centre of $\text{Spin}(h)$ is $\mu_2 \times \mu_2$ and the kernel of the map $\text{Spin}(h) \rightarrow \text{SU}(h)$ is (ϵ, ϵ) , where $\epsilon = \pm 1$. The quotient of $\mu_2 \times \mu_2$ by μ_2 under the diagonal embedding maps isomorphically onto the centre of $\text{SU}(h)$. By [MPT, Th. 1.14], the Rost invariant of the image $\tilde{\xi}_\lambda$ of $(1, \lambda) \in H^1(F, \mu_2 \times \mu_2)$ in $H^1(F, \text{Spin}(h))$ is $(\lambda) \cup [C_+(h)]$. Thus $\tilde{\xi}_\lambda \in H^1(F, \text{Spin}(h))$ maps to $\xi_\lambda \in H^1(F, \text{SU}(h))$, which in turn maps to the class of λh in $H^1(F, \text{U}(h))$ as is seen above. Thus we conclude that the hermitian form λh admits a lift $\tilde{\xi}_\lambda$ such that $\mathcal{R}(\tilde{\xi}_\lambda) = (\lambda) \cup [C_+(h)]$.

We now compute $R(h \perp -\lambda h)$. Let $i : \text{Spin}(-h) \rightarrow \text{Spin}(-h \perp h)$ be the natural map and $\tilde{i} : H^1(F, \text{Spin}(-h)) \rightarrow H^1(F, \text{Spin}(-h \perp h))$ the induced map on the cohomology sets. In view of [BP2, Lemma 3.6], $\mathcal{R}(\tilde{i}(\xi)) = \mathcal{R}(\xi)$ for every $\xi \in H^1(F, \text{Spin}(-h))$. The group $\text{Spin}(-h \perp h)$ maps isomorphically onto $\text{Spin}_{4m}(D, \tau)$ preserving the Rost invariant. Further, the image of $(1, \lambda)$ in the cohomology set $H^1(F, \text{Spin}(-h))$ maps to the isometry class of $-\lambda h$ in $H^1(F, \text{U}(h))$ and to the isometry class of $-\lambda h \perp h$ in $H^1(F, \text{U}_{4n}(D, \tau))$. This implies that the Rost invariant $R(h \perp -\lambda h)$ is equal to $(\lambda) \cup [C_+(h)] \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))/F^* \cup [D]$. This completes

the proof. □

From now on, we assume that $\text{vcd}(F) \leq 2$. Let L/F be a formally real extension and Ω_L be the set of orderings on L . Let h be a hermitian form over (D, τ) - a central division algebra D over F with an orthogonal involution τ . We define $\mathcal{S}_{\ell, L}(h)$ as follows:

$$\mathcal{S}_{\ell, L}(h) = \{\lambda \in L^* : h_{L_w} \simeq \lambda h_{L_w} \text{ for all } w \in \Omega_L\}.$$

If $L = F$ then we simply write $\mathcal{S}_{\ell}(h)$ to denote $\mathcal{S}_{\ell, F}(h)$. In 6.10 - 6.14 below, h denotes an even rank hermitian form over (D, τ) with trivial discriminant and $a \in F^*$ denotes a scalar which satisfies $\text{Nrd}(M_h) = a^2$ for a choice M_h of a matrix representing h . Further, for $z \in F^*$ and a central simple algebra B over F with $\exp(B) = 2$, we denote by $z \cup B$ the element $(z) \cup [B] \in H^3(F, \mu_2)$.

Proposition 6.10 *We have $(N(a \cup D)N(-a \cup D)) \cap \mathcal{S}_{\ell}(h)$.*

Proof We first prove that $(N(a \cup D)N(-a \cup D)) \cap \mathcal{S}_{\ell}(h) \subseteq G(h)$. Let $\lambda \in (N(a \cup D)N(-a \cup D)) \cap \mathcal{S}_{\ell}(h)$. We show that $h \perp -\lambda h$ is hyperbolic. It is clear that $h \perp -\lambda h$ has even rank. Further $\text{Nrd}(M(h \perp -\lambda h)) = a^4$, and since a^2 is totally positive, by 1.8 it belongs to $\text{Nrd}(D^*)$. Thus it follows that $h \perp -\lambda h$ has trivial Discriminant. Moreover since $\text{disc}(h)$ is trivial, by 6.8, it follows that the Clifford invariant of $h \perp -\lambda h$ is trivial. Since $\lambda \in \mathcal{S}_{\ell}(h)$, the form $h \perp -\lambda h$ has trivial signature as well.

By 6.9 we see that the Rost invariant $R(h \perp -\lambda h) = [(\lambda) \cup C_+(h)]$. We show that $[(\lambda) \cup C_+(h)]$ is trivial in $H^3(F, \mathbb{Z}/4)/F^* \cup [D]$. Let $x \in F^*$ be such that $x \in N(-a \cup D)$ and $\lambda x^{-1} \in N(a \cup D)$. We claim that $(\lambda) \cup [C_+(h)] = (x) \cup [D]$. In view of 6.2, it suffices to check that at each $v \in \Omega$, we have $(\lambda) \cup [C_+(h)_v] = (x) \cup [D_v]$.

Suppose $v \in \Omega$ is such that $\lambda >_v 0$ and $x >_v 0$. In this case $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are both trivial.

Suppose $\lambda >_v 0$ and $x <_v 0$. Then $\lambda x^{-1} <_v 0$. Since $\lambda x^{-1} \in N(a \cup D)$ and $x \in N(-a \cup D)$, in view of 6.6 both $(a) \cup [D]$ and $(-a) \cup [D]$ are split at v . Thus $-1 \in \text{Nrd}(D_v)$ and hence D is split at v . Thus both $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are trivial in this case as well.

Now suppose that $\lambda <_v 0$ and D_v is split. Since $\lambda \in \mathcal{S}_l(h)$, we conclude that h_v is hyperbolic. Thus the Clifford invariant $\mathcal{C}l(h)_v = 0$. Further, since D_v is split and h_v is hyperbolic, we have $C_+(h)_v = C_-(h)_v = 0$. Thus we conclude that $C_+(h)_v$ is split and thus $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are both zero.

Next, suppose that $\lambda <_v 0$, D_v is not split and $x <_v 0$. Since $x \in N(-a \cup D)$, by 6.6 $(-a) \cup [D_v] = 0$; i.e. $-a \in \text{Nrd}(D_v)$. Hence $a <_v 0$. Since D_v is not split and $a <_v 0$, by 6.1 $C_+(h)_v$ is not split. Thus we conclude in this case that both $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are non-zero and hence equal.

Now the only remaining case is when $\lambda <_v 0$, D_v is not split and $x >_v 0$. In that case, $\lambda x^{-1} <_v 0$ and since $\lambda x^{-1} \in N(a \cup D)$, by 6.6 we have that $(a) \cup [D_v] = 0$. Thus $a \in \text{Nrd}(D_v)$ and hence $a >_v 0$. Since D_v is non-split, by 6.1 $C_+(h)_v$ is split. Thus both $(\lambda) \cup [C_+(h)_v]$ and $(x) \cup [D_v]$ are zero in this case.

We conclude therefore that $(\lambda) \cup [C_+(h)_v] = (x) \cup [D_v]$ for all $v \in \Omega$. Thus by 6.2, we have $(\lambda) \cup [C_+(h)] = (x) \cup [D]$ and

$$R(h \perp -\lambda h) = (\lambda) \cup [C_+(h)] = 0 \in H^3(F, \mathbb{Z}/4)/F^* \cup [D].$$

Since $\text{vcd}(F) \leq 2$, by 6.3 we have that $h \perp -\lambda h$ is hyperbolic. Thus $\lambda \in G(h)$.

We now show the inclusion $G(h) \subseteq (N(a \cup D)N(-a \cup D)) \cap \mathcal{S}_\ell(h)$. It is clear that $G(h) \subseteq \mathcal{S}_\ell(h)$. We thus show that $G(h) \subseteq N(a \cup D)N(-a \cup D)$. Let $\lambda \in G(h)$. Then the form $h \perp -\lambda h$ is hyperbolic and hence its Rost invariant $(\lambda) \cup [C_+(h)]$ is trivial. Thus there exists $x \in F^*$ such that $(\lambda) \cup [C_+(h)] = (x) \cup [D]$. By reading this equality locally at each $v \in \Omega$ and observing the sign pattern, we conclude that $x \in N(-a \cup D)$ and $\lambda x \in N(a \cup D)$. Therefore, $\lambda \in N(a \cup D)N(-a \cup D)$. This completes the proof. \square

The following lemma will be used in the proof of 6.12.

Lemma 6.11 *Let D be a central division algebra over F and let τ be an orthogonal involution on D . Let h be an even rank locally hyperbolic hermitian form over (D, τ) with $\text{Disc}(h) = 1$ and $\mathcal{C}l(h) = 0$. Then $\text{Hyp}(h) = F^*$.*

Proof Since the hermitian form h has even rank, trivial discriminant and trivial Clifford invariant, there exists $\tilde{\xi} \in H^1(F, \text{Spin}_{2m}(D, \tau))$ which maps to $[h] \in H^1(F, \text{U}_{2m}(D, \tau))$. Let $\mathcal{R}(\tilde{\xi}) \in H^3(F, \mu_2)$ be the Rost invariant of $\tilde{\xi}$. Let $L \in \mathcal{F}_2(F)$ be such that $\mathcal{R}(\tilde{\xi}_L) = (-1) \cup (-1) \cup (-x)$ for some $x \in L^*$. We claim that $\mathcal{R}(\tilde{\xi}_L) = (-x) \cup D_L$.

Let Ω_L be the set of orderings on L and $w \in \Omega_L$ be such that D_{Lw} is split. Then $\mathcal{R}(\tilde{\xi}_{Lw}) = e_3(h_{Lw})$, where e_3 is the Arason invariant of quadratic forms. Since h_w is hyperbolic by hypothesis, we have $e_3(h_{Lw}) = 0$. Thus $\mathcal{R}(\tilde{\xi}_L)$ and $(-x) \cup D_L$ are both zero at w .

Now suppose D_{Lw} is not split. Then $D_{Lw} = (-1) \cup (-1)$ and thus $\mathcal{R}(\tilde{\xi}_{Lw}) = (-1) \cup (-1) \cup (-x) = (-x) \cup D_{Lw}$. Thus $\mathcal{R}(\tilde{\xi}_L) = (-x) \cup D_L$ at each $w \in \Omega_L$ and by 6.2, $R(h_L) = 0$. Therefore h_L is a locally hyperbolic form with even rank, trivial Discriminant, trivial Clifford invariant and trivial Rost invariant. By 6.3 the form h_L is hyperbolic. In view of this and 6.5 we conclude that $\text{Hyp}(h) = F^*$. \square

The following proposition gives an explicit description of the group $\text{Hyp}(h)$.

Proposition 6.12 *We have $\text{Hyp}(h) = (N(a \cup D) \cap \mathcal{S}_\ell(h)).(N(-a \cup D) \cap \mathcal{S}_\ell(h))$.*

Proof We first prove that $\text{Hyp}(h) \subseteq (N(a \cup D) \cap \mathcal{S}_\ell(h)).(N(-a \cup D) \cap \mathcal{S}_\ell(h))$. Let L/F be a finite extension such that h_L is hyperbolic. Then $\text{Disc}(h_L)$ is trivial in $L^*/\text{Nrd}(D_L^*)^2$ and hence either $a \in \text{Nrd}(D_L^*)$ or $-a \in \text{Nrd}(D_L^*)$; i.e. either $N(a \cup D_L) = L^*$ or $N(-a \cup D_L) = L^*$. We clearly have $\mathcal{S}_{\ell,L}(h) = L^*$. Thus

$$(N(a \cup D_L) \cap \mathcal{S}_{\ell,L}(h)).(N(-a \cup D_L) \cap \mathcal{S}_{\ell,L}(h)) = L^*.$$

Clearly $N_{L/F}(N(a \cup D_L)) \subseteq N(a \cup D)$ and $N_{L/F}(N(-a \cup D_L)) \subseteq N(-a \cup D)$. Further as in [KMRT, Prop. 12.21], $N_{L/F}(\mathcal{S}_{\ell,L}(h)) \subseteq \mathcal{S}_\ell(h)$. Thus

$$\begin{aligned} N_{L/F}(L^*) &\subseteq N_{L/F}((N(a \cup D_L) \cap \mathcal{S}_{\ell,L}(h)).(N(-a \cup D_L) \cap \mathcal{S}_{\ell,L}(h))) \\ &\subseteq (N(a \cup D) \cap \mathcal{S}_\ell(h)).(N(-a \cup D) \cap \mathcal{S}_\ell(h)). \end{aligned}$$

Since $N_{L/F}(L^*)$ generate $\text{Hyp}(h)$ as L runs over extensions where h is hyperbolic, it follows that $\text{Hyp}(h) \subseteq (N(a \cup D) \cap \mathcal{S}_\ell(h)).(N(-a \cup D) \cap \mathcal{S}_\ell(h))$.

To complete the proof, we show that $N(a \cup D) \cap \mathcal{S}_\ell(h) \subseteq \text{Hyp}(h)$. The inclusion $N(-a \cup D) \cap \mathcal{S}_\ell(h) \subseteq \text{Hyp}(h)$ follows in the similar manner. Let $\lambda \in N(a \cup D) \cap \mathcal{S}_\ell(h)$. By 6.10, $\lambda \in G(h)$. Let $K = F(\sqrt{-\lambda})$. Since $\lambda \in N(a \cup D)$, by 6.6 $(\lambda) \cup (a) \cup [D] = 0 \in H^4(F, \mu_2)$. Thus $(-1) \cup (a) \cup [D_K] = 0 \in H^4(K, \mu_2)$. By 1.15, $(a) \cup [D_K] = 0 \in H^3(K, \mu_2)$. Hence $a \in \text{Nrd}(D_K)$ and $\text{Disc}(h)_K = 1$.

Let w be an ordering on K . Since $\lambda <_w 0$ and $\lambda \in G(h_K)$, the form h_K is locally hyperbolic. Thus the Clifford invariant $\mathcal{C}\ell(h)_K$ is trivial at w . Therefore, if D_{Kw} is split, then $C_+(h)_{Kw} = C_-(h)_{Kw} = 0$. If D_{Kw} is not split, then in view of 1.8, $a >_w 0$ as $a \in \text{Nrd}(D_K)$. By 6.1, $C_+(h)_{Kw}$ is split. We have thus shown that $C_+(h_K)$ is locally split. By 1.8, it follows that $\text{Nrd}(C_+(h_K)) = K^*$. Let L/K be a finite extension and $\alpha \in L^*$ be such that $\sqrt{-\lambda} = N_{L/K}(\alpha)$ and $C_+(h_L) = 0$. Then h_L is an even rank locally hyperbolic form with $\text{Disc}(h_L) = 1$ and $C_+(h_L) = 0$. By 6.11, $\text{Hyp}(h_L) = L^*$. Thus

$$\sqrt{-\lambda} = N_{L/K}(\alpha) \in N_{L/K}(\text{Hyp}(h_L)) \subseteq \text{Hyp}(h_K).$$

Taking norm from K/F we have $\lambda \in \text{Hyp}(h)$. Thus $N(a \cup D) \cap \mathcal{S}_\ell(h) \subseteq \text{Hyp}(h)$. \square

With the notation as above, we have following corollaries.

Corollary 6.13 *If h is locally hyperbolic then*

$$\text{Hyp}(h) = N(a \cup D).N(-a \cup D) = G(h).$$

Proof Since h is locally hyperbolic, $\mathcal{S}_\ell(h) = F^*$. From 6.10 and 6.12, it is clear that $\text{Hyp}(h) = N(a \cup D).N(-a \cup D) = G(h)$. \square

Corollary 6.14 *If h has trivial Discriminant, then $\text{Hyp}(h) = \mathcal{S}_\ell(h) = G(h)$.*

Proof Since $\text{Disc}(h) = 1$, it follows that either $N(a \cup D) = F^*$ or $N(-a \cup D) = F^*$. In either case, it is immediate from 6.10 and 6.12, that $\text{Hyp}(h) = \mathcal{S}_\ell(h) = G(h)$. \square

Let A be a central simple algebra over F with an orthogonal involution σ . Suppose $\text{disc}(\sigma) = 1$ and $C(A, \sigma) = C_+(A, \sigma) \times C_-(A, \sigma)$. We have the following extension of 6.13

Proposition 6.15 *Let F be a field with $\text{char}(F) \neq 2$. Let (A, σ) be a central simple algebra of even degree over F with an orthogonal involution. Let $\deg(A) \equiv 0(4)$ and $\text{disc}(\sigma) = 1$. Then $\text{Hyp}(A, \sigma) \subseteq \text{Nrd}(C_+(A, \sigma)) \text{Nrd}(C_-(A, \sigma))$. Further if $\text{vcd}(F) \leq 2$ and σ is locally hyperbolic then*

$$\text{Hyp}(A, \sigma) = \text{Nrd}(C_+(A, \sigma)) \text{Nrd}(C_-(A, \sigma)) = G(A, \sigma).$$

Proof The first assertion follows from the fact that over any extension L/F where σ is hyperbolic, either $C_+(A_L, \sigma_L)$ or $C_-(A_L, \sigma_L)$ is split [KMRT, Prop. 12.21].

Suppose $\text{vcd}(F) \leq 2$ and σ locally hyperbolic. Let $\lambda \in \text{Nrd}(C_+(A, \sigma))$. Let L/F be a finite extension such that $\lambda \in N_{L/F}(L^*)$ and $C_+(A_L, \sigma_L)$ is split. We show that $\text{Hyp}(A_L, \sigma_L) = L^*$. In view of 2.3, replacing L by a quadratic tower, we may assume that $A_L \simeq M_{2r}(H)$ for some quaternion algebra H over L . Let τ be an orthogonal involution on H and h be a hermitian form over (H, τ) such that σ_L is adjoint to h . Let $M_h \in M_{2r}(H)$ represent h and $\text{Nrd}(M_h) = a^2$ for some $a \in L^*$. Let w be an ordering on L such that H_w is not split. Since $C_+(h) = C_+(A_L, \sigma_L)$ is split, by 6.1 $a >_w 0$. Thus $(a) \cup [H] = 0 \in H^3(L, \mu_2)$ and $N(a \cup H) = L^*$. In view of 6.13, $\text{Hyp}(A_L, \sigma_L) = \text{Hyp}(h) = N(a \cup H).N(-a \cup H) = L^*$. Thus $\text{Hyp}(A_L, \sigma_L) = L^*$. Taking norms from L/F we have $\lambda \in N_{L/F}(L^*) = N_{L/F}(\text{Hyp}(A_L, \sigma_L)) \subseteq \text{Hyp}(A, \sigma)$. Thus $\text{Nrd}(C_+(A, \sigma)) \subseteq \text{Hyp}(A, \sigma)$.

The inclusion $\text{Nrd}(C_-(A, \sigma)) \subseteq \text{Hyp}(A, \sigma)$ follows from a similar argument. We therefore conclude that $\text{Hyp}(A, \sigma) = \text{Nrd}(C_+(A, \sigma)) \text{Nrd}(C_-(A, \sigma))$

To complete the proof we show that $G(A, \sigma) \subseteq \text{Nrd}(C_+(A, \sigma)) \text{Nrd}(C_-(A, \sigma))$. Let $\lambda \in G(A, \sigma)$. Then the hermitian form $\langle 1, -\lambda \rangle$ is hyperbolic. Hence the Rost invariant $R(\langle 1, -\lambda \rangle)$ is trivial. As in the proof of 6.9, $R(\langle 1, -\lambda \rangle) = (\lambda) \cup [C_+(A, \sigma)]$. Since the Rost invariant is trivial, there exists $x \in F^*$ such that $(\lambda) \cup [C_+(A, \sigma)] = (x) \cup [A]$. If for an ordering v on F , the algebra A_v is split, then h_v being hyperbolic, $C_+(A, \sigma)_v$ and $C_-(A, \sigma)_v$ are both split. If A_v is not split and $x <_v 0$ then $C_+(A, \sigma)_v$ is not split. Hence $C_-(A, \sigma)_v$ is split. Thus $x \in \text{Nrd}(C_-(A, \sigma))$ and a similar argument gives $\lambda x \in \text{Nrd}(C_+(A, \sigma))$. Hence $\lambda = \lambda x.x^{-1} \in \text{Nrd}(C_+(A, \sigma)) \text{Nrd}(C_-(A, \sigma))$. We have thus shown that

$$G(A, \sigma) \subseteq \text{Hyp}(A, \sigma) = \text{Nrd}(C_+(A, \sigma)) \text{Nrd}(C_-(A, \sigma)).$$

The inclusion $\text{Hyp}(A, \sigma) \subseteq G(A, \sigma)$ follows from 1.1 and this completes the proof. \square

Theorem 6.16 *Let $\text{vcd}(F) \leq 2$ and let A be a central simple algebra over F with $\deg(A)$ even and an involution σ of orthogonal type. If $\text{disc}(\sigma) = 1$ and σ is locally hyperbolic then $G(A, \sigma) = \text{Hyp}(A, \sigma).F^{*2}$.*

Proof Let $\deg(A) = 2n$. Suppose n is odd. Since σ is locally hyperbolic, the algebra A is locally split and by 6.4 the results holds. We can thus assume that n is even. In this case we are through by 6.15. \square

7 Fields with $\text{vcd}(F) \leq 2$ satisfying SAP

Let F be a field with orderings and let Ω denote the set of orderings on F . Given $a \in F^*$, we define the corresponding *Harrison set* Ω_a as follows:

$$\Omega_a := \{v \in \Omega : a >_v 0\}.$$

The set Ω has *Harrison topology* for which $\{\Omega_a : a \in F^*\}$ is a sub-basis. With this topology, Ω is a Hausdorff, compact and totally disconnected space. We say that F has *strong approximation property* (SAP), if every closed and open set of Ω is of the form Ω_a for some $a \in F^*$. A quadratic form q is said to be *weakly isotropic*, if for some positive integer s , the s -fold orthogonal sum $s.q = \perp_s q$ is isotropic. Combining [ELP, Th. C] and [P, Satz. 3.1] we have the following

Theorem 7.1 *A field F with orderings has SAP if and only if for every $a, b \in F^*$, the quadratic form $\langle 1, a, b, -ab \rangle$ is weakly isotropic.* \square

In what follows, for $a_1, a_2, \dots, a_r \in F^*$ the notation $\langle\langle a_1, a_2, \dots, a_r \rangle\rangle$ will denote the r -fold Pfister form $\langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \dots \otimes \langle 1, -a_r \rangle$. For a quadratic form q , we denote by $D(q)$ the set of elements of F^* represented by q . We remark that if $q = \langle 1, -a, -b, ab \rangle$ then $D(q) = \text{Nrd}(H^*)$, where H is the quaternion algebra (a, b) over F . Set $\Omega(H) = \{v \in \Omega : H \otimes_F F_v \text{ is split } \}$. The following lemma is recorded in [Ga].

Lemma 7.2 *Let F be a field with orderings. Let $a, b \in F^*$. Let $q_1 = \langle\langle -1, -a \rangle\rangle$, $q_2 = \langle\langle -1, a \rangle\rangle$ and $H = (a, b)$. Suppose there does not exist $c \in F^*$ such that $\Omega_c = \Omega(H)$, then $-b \notin D(q_1)D(q_2)$.*

Proof Suppose $-b \in D(q_1)D(q_2)$ and let $x_1 \in D(q_1)$ and $x_2 \in D(q_2)$ be such that $-b = x_1x_2$. Then $q_1 \perp bq_2$ is isotropic and hence $2\langle 1, a, b, -ab \rangle \simeq q_0 \perp \mathbb{H}$ for some Albert form q_0 and $\mathbb{H} \simeq \langle 1, -1 \rangle$. We have

$$C(q_0) = C(q_1 \perp bq_2) = (-1, -1) = C(4\langle 1 \rangle \perp \mathbb{H}) \in \text{Br}(F).$$

Therefore by [KMRT, Prop. 16.3], $q_0 \simeq 4\langle c \rangle \perp \mathbb{H}$ for some $c \in F^*$. It is easy to see that $\Omega_c = \Omega(H)$, which contradicts the hypothesis. Thus $-b \notin D(q_1)D(q_2)$. \square

Lemma 7.3 *Let F be a field for which $I^3(F)$ is torsion-free. Let $a, b \in F^*$. Let $q_1 = \langle\langle -1, -a \rangle\rangle$, $q_2 = \langle\langle -1, a \rangle\rangle$ and $H = (a, b)$. If $-b \notin D(q_1)D(q_2)$ then there is no element $c \in F^*$ with $\Omega_c = \Omega(H)$.*

Proof Suppose there is an element $c \in F^*$ such that $\Omega_c = \Omega(H)$. Let $q' = \langle 1, a, b, -ab, -c, -c \rangle$. For $v \in \Omega$ if $c <_v 0$, then by the choice of c we have $a <_v 0$ and $b <_v 0$. This implies that the form q' is hyperbolic at v . If $c >_v 0$, then again by the choice of c , either $a >_v 0$ or $b >_v 0$ and in either case q' is hyperbolic at v . We thus conclude that q' is locally hyperbolic. Clearly $q' \in I^2(F)$, therefore $2q' \in I^3(F)$. Since $2q'$ is an even rank quadratic form with trivial signature, it is hyperbolic at each F_v , $v \in \Omega$. Thus by Pfister's local-global principle [L, Th. VIII.4.1], $2q'$ is a torsion element in the Witt group $W(F)$. By the hypothesis, $I^3(F)$ is torsion-free. Thus $2q' = 2\langle 1, a, b, -ab, -c, -c \rangle$ is hyperbolic. Therefore the form $q_1 \perp bq_2$ is isotropic, which implies that $-b \in D(q_1)D(q_2)$. This is a contradiction to the hypothesis. \square

Combining 7.2 and 7.3 above, we get the following

Corollary 7.4 *Let $I^3(F)$ be torsion-free. Let $a, b \in F^*$ and $q_1 = \langle\langle -1, -a \rangle\rangle$, $q_2 = \langle\langle -1, a \rangle\rangle$ and $H = (a, b)$. Then $-b \in D(q_1)D(q_2)$ if and only if there exists $c \in F^*$ such that $\Omega_c = \Omega(H)$.* \square

Using the results above, we have thus derived

Corollary 7.5 *Let F be a field with $I^3(F)$ torsion-free. Then the following statements are equivalent:*

- (i) *For all $a \in F^*$ we have $D(\langle\langle -1, -a \rangle\rangle) D(\langle\langle -1, a \rangle\rangle) = F^*$.*
- (ii) *The field F has SAP.*
- (iii) *Given a quaternion algebra $H = (a, b)$ over F , there exists an element $c \in F^*$ such that $\Omega_c = \Omega(H)$. □*

From now on, in this section the field F satisfies $\text{vcd}(F) \leq 2$. We say that a field F satisfies SAP for quadratic towers if each quadratic tower $L \in \mathcal{F}_2(F)$ has SAP.

Lemma 7.6 *Let $\text{vcd}(F) \leq 2$, F has orderings and has SAP for quadratic towers. Then for every $a, b \in F^*$ we have $\text{Nrd}(a, b)^* \cdot \text{Nrd}(-a, b)^* = F^*$.*

Proof Let $H_1 = (a, b)$ and $H_2 = (-a, b)$. Let $L \in \mathcal{F}_2(F)$ be such that $(a, b)_L = (-1, -x)$ and $(-a, b)_L = (-1, -y)$ for some $x, y \in L^*$. It is clear that at a given $w \in \Omega_L$ at least one of H_{1L_w} and H_{2L_w} is split, which means either $x <_w 0$ or $y <_w 0$. Thus if $-x <_w 0$ then $y <_w 0$. In other words, if $(-1, -x)_{L_w}$ is non-split then so is $(-1, y)_{L_w}$. Thus by 1.8, we have $\text{Nrd}(-1, y)^* \subseteq \text{Nrd}(-1, -x)^*$. Thus $\text{Nrd}(-1, y)^* \text{Nrd}(-1, -y)^* \subseteq \text{Nrd}(-1, -x)^* \text{Nrd}(-1, -y)^*$. Since F has SAP for quadratic towers, L has SAP and thus by 7.5, we have $\text{Nrd}(-1, -y)^* \text{Nrd}(-1, y)^* = L^*$. Therefore,

$$\text{Nrd}(H_{1L})^* \text{Nrd}(H_{2L})^* = \text{Nrd}(-1, -x)^* \text{Nrd}(-1, -y)^* = L^*.$$

In view of this and 2.5 we have that $\text{Nrd}(H_1)^* \cdot \text{Nrd}(H_2)^* = F^*$. This completes the proof. □

Proposition 7.7 *Let $\text{vcd}(F) \leq 2$ and F has SAP for quadratic towers. Let h be a locally hyperbolic hermitian form of even rank over a central-division algebra D with an orthogonal involution τ . Let $\text{disc}(h) = 1$. Then we have $\text{Hyp}(h) = F^*$.*

Proof Let $L \in \mathcal{F}_2(F)$ be such that $D_L \sim (-1, -x)$ for some $x \in L^*$. Since h_L is locally hyperbolic, by 6.13 we have $\text{Hyp}(h_L) = N(a \cup D_L).N(-a \cup D_L)$. Clearly $N(a \cup D_L) = \text{Nrd}(a, -x)^*$ and $N(-a \cup D_L) = \text{Nrd}(-a, -x)^*$. Since F has SAP for quadratic towers, so does L . Thus by 7.6, we have

$$\text{Nrd}(a, -x)^* . \text{Nrd}(-a, -x)^* = L^*$$

and we conclude that $\text{Hyp}(h_L) = L^*$. In view of this and 2.4 we have $\text{Hyp}(h) = F^*$. \square

Proposition 7.8 *Let $\text{vcd}(F) \leq 2$ and F has SAP with respect to quadratic towers. Let A be a central simple algebra of even degree over F and σ be a locally hyperbolic involution on A with $\text{disc}(\sigma) = 1$. Then $\text{Hyp}(A, \sigma) = F^*$.*

Proof Let $L \in \mathcal{F}_2(F)$ be such that $A_L \sim (-1, -x)$ for some $x \in L^*$. Let $H = (-1, -x)$. Then $A_L = M_r(H)$ for some positive integer r . Let τ be an orthogonal involution on H and h be a hermitian form over (H, τ) such that σ_L is adjoint to h . Then $\text{Hyp}(A_L, \sigma_L) = \text{Hyp}(h)$.

First assume that r is even. Then it follows from [KMRT, Prop. 7.3(1)] that $\text{disc}(\sigma_L) = \text{disc}(h) = 1$. Since σ_L is locally hyperbolic, the hermitian form h is locally hyperbolic. Thus h is a locally hyperbolic form of even rank and trivial discriminant over (H, τ) . Therefore in view of 7.7 we have $\text{Hyp}(A_L, \sigma_L) = \text{Hyp}(h) = L^*$.

Now suppose r is odd. Since σ_L is locally hyperbolic, the hermitian form h is locally hyperbolic. Thus the quaternion algebra H is locally split and by 1.8 we have $\text{Nrd}(H)^* = L^*$. Let $\lambda \in L^*$. Let $L_i \in \mathcal{F}_2(L)$ be such that each H_{L_i} is split and $\lambda = \prod_i N_{L_i/L}(\lambda_i)$, where $\lambda_i \in L_i^*$. Then h_{L_i} is a locally hyperbolic quadratic form over L_i with even rank and trivial discriminant. Thus by 5.2, we have $\text{Hyp}(h_{L_i}) = L_i^*$. Therefore

$$\lambda \in \prod_i N_{L_i/L}(\text{Hyp}(H_{L_i})) \subseteq \text{Hyp}(h)$$

and hence $\text{Hyp}(h) = L^*$.

Thus it follows that if $L \in \mathcal{F}_2(F)$ is such that $A_L \sim (-1, -x)$ for some $x \in L^*$ then $\text{Hyp}(A_L, \sigma_L) = \text{Hyp}(h) = L^*$. Therefore in view of 2.4, we have $\text{Hyp}(A, \sigma) = F^*$. This completes the proof. \square

Theorem 7.9 *Let $\text{vcd}(F) \leq 2$ and F has SAP for quadratic towers. Let A be a central simple algebra of degree $2n$ over F and σ be an orthogonal involution on A with $\text{disc}(\sigma) = 1$. Then $\text{Hyp}(A, \sigma).F^{*2} = G_+(A, \sigma)$.*

Proof Let $\lambda \in G_+(A, \sigma)$. Then $\lambda = \sigma(a)a$ for some $a \in A^*$ with $\text{Nrd}(a) = \lambda^n$. Let $K = F(\sqrt{-\lambda})$. First suppose that n is odd. Then $\lambda \in \text{Nrd}(A^*)$ and thus $\lambda >_v 0$ at those $v \in \Omega$ where A_v is not split. If A_v is split then $\lambda >_v 0$ at those $v \in \Omega$ where $\text{sgn}(\sigma_v) \neq 0$. Then A_K is locally split and σ_K is locally hyperbolic.

Now suppose that n is even let and $v \in \Omega$ is such that σ_v is not hyperbolic. Then by (Theorem 3.7, Chapter 10, [S]), A_v is split and σ_v is adjoint to a non-hyperbolic quadratic form q over F_v such that $q \simeq \lambda q$. Thus we conclude that $\lambda >_v 0$ at those orderings $v \in \Omega$ where σ_v is not hyperbolic. Then σ_K is locally hyperbolic.

Thus in either case, by 7.8 we have that $\text{Hyp}(A_K, \sigma_K) = K^*$. Taking norm of $\sqrt{-\lambda}$ from K/F , we conclude that $\lambda \in \text{Hyp}(A, \sigma)$ and therefore $G_+(A, \sigma) \subseteq \text{Hyp}(A, \sigma)$. By 1.1 we have $\text{Hyp}(A, \sigma).F^{*2} \subseteq G_+(A, \sigma)$, and hence we conclude that $G_+(A, \sigma) = \text{Hyp}(A, \sigma).F^{*2}$. \square

Theorem 7.10 *Let $\text{vcd}(F) \leq 2$ and (A, σ) be an algebra of type 2D_n over F . Let F have SAP for quadratic towers. Then we have $G_+(A, \sigma) = \text{Hyp}(A, \sigma).F^{*2}$.*

Proof Let $\text{disc}(\sigma) = d$. Let $\lambda \in G_+(A, \sigma)$. By 1.3, we have $G_+(A, \sigma) \subseteq N_{L/K}(L^*)$, where $L = F(\sqrt{d})$. As in the proof of the 7.9, $\sigma_{F(\sqrt{-\lambda})}$ is locally hyperbolic. Let $M = L(\sqrt{-\lambda})$. By the biquadratic lemma [W, Lemma 2.14], it follows that there exist $x \in M^*$ and $y \in F^*$ such that $\lambda = N_{M/F}(x)y^2$. It is clear that $\text{disc}(\sigma_M) = 1$ and σ_M is locally hyperbolic. Thus by 7.8 we have $\text{Hyp}(A_M, \sigma_M) = M^*$ and we easily see that $\lambda y^{-2} \in \text{Hyp}(A, \sigma)$. Thus $G_+(A, \sigma) \subseteq \text{Hyp}(A, \sigma).F^{*2}$. Since by 1.1 $\text{Hyp}(A, \sigma).F^{*2} \subseteq G_+(A, \sigma)$, we conclude that $G_+(A, \sigma) = \text{Hyp}(A, \sigma).F^{*2}$. \square

Corollary 7.11 *Let $\text{vcd}(F) \leq 2$, F has SAP for quadratic towers and let $\mathbf{PSim}_+(A, \sigma)$ be of type 1D_n or 2D_n . Then $\mathbf{PSim}_+(A, \sigma)(F)/R = 0$. \square*

Since number fields satisfy the conditions of 7.11, we have

Corollary 7.12 *Let F be a number field and let $\mathbf{PSim}_+(A, \sigma)$ be of type 1D_n or 2D_n . Then $\mathbf{PSim}_+(A, \sigma)(F)/R = 0$. \square*

Remark It is a well known fact that SAP is not preserved under field extensions. As Detlev Hoffmann has pointed out to us, there are examples of fields F with $\text{vcd}(F) = 2$ and quadratic extensions E/F such that F satisfies SAP but not E . Thus the condition ‘SAP with respect to quadratic towers’ is not redundant in 7.10.

Hoffmann’s example is the following: One can construct a formally real field k with the following properties: (i) k has no extension of odd degree. (ii) There is only one ordering on k . (iii) The u -invariant of k is 2. Let $\alpha \in k^* \setminus k^{*2}$ be a sum of two squares. Let $F = k((X))$ and $E = F(\sqrt{\alpha})$. Then $\text{vcd}(F) = 2$ and F has SAP but not E .

Combining together the results of §4, §5 and §7, we have

Theorem 7.13 *Let F be a field with $\text{vcd}(F) \leq 2$. Let G is a classical group of adjoint type defined over F . Then,*

(i) *If G does not contain a factor of type D_n then $G(F)/R = 0$.*

(ii) *If F satisfies SAP for quadratic towers then $G(F)/R = 0$.*

Proof If F does not have orderings, by 1.17 we have $\text{cd}(F) \leq 2$ and we are through by 3.7. Thus we assume that F has orderings. As in the proof of 3.7, it suffices to prove the theorem for absolutely simple adjoint groups defined over F . In view of 1.1, the Assertion (i) follows from 1.4, 4.2 and 5.13. The Assertion (ii) of the theorem follows immediately from Assertion (i) and 7.11. \square

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