REDUCED K-THEORY OF AZUMAYA ALGEBRAS

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Abstract. In the theory of central simple algebras, often we are dealing with abelian groups which arise from the kernel or co-kernel of functors which respect transfer maps (for example K-functors). Since a central simple algebra splits and the functors above are "trivial" in the split case, one can prove certain calculus on these functors. The common examples are kernel or co-kernel of the maps $K_i(F) \to K_i(D)$, where K_i are Quillen K-groups, D is a division algebra and F its centre, or the homotopy fiber arising from the long exact sequence of above map, or the reduced Whitehead group SK_1. In this note we introduce an abstract functor over the category of Azumaya algebras which covers all the functors mentioned above and prove the usual calculus for it. This, for example, immediately shows that K-theory of an Azumaya algebra over a local ring is "almost" the same as K-theory of the base ring.

The main result is to prove that reduced K-theory of an Azumaya algebra over a Henselian ring coincides with reduced K-theory of its residue central simple algebra.

The note ends with some calculation trying to determine the homotopy fibers mentioned above.

1. Introduction

In the theory of central simple algebras, often we are dealing with abelian groups which arise from the kernel or co-kernel of functors which respect transfer maps (for example K-functors). Since a central simple algebra splits and the functors above are "trivial" in the split case, the abelian groups are n torsion (annihilated by a power of n) where n is the degree of the algebra. This immediately implies that extensions of degree relatively prime to n and decomposition of the algebra to its primary components are "understood" [1] by these groups.

For example consider a division algebra D with centre F. Let $K_i(D)$ for $i \geq 0$ denote the Quillen K-groups. The inclusion map $\mathrm{id} : F \to D$ gives rise to the long exact sequence

$$F_i \to K_i(F) \to K_i(D), \tag{1}$$

in K-theory where F_i are the homotopy fibers. If $\mathrm{ZK}_i(D)$ and $\mathrm{CK}_i(D)$ are the kernel and co-kernel of $K_i(F) \to K_i(D)$ respectively, then

$$1 \to \mathrm{CK}_{i+1}(D) \to F_i \to \mathrm{ZK}_i(D) \to 1. \tag{2}$$

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[1] This terminology is used by van der Kallen

The functors CK_i, F_i and ZK_i are all examples of the abelian groups mentioned above. From (2) it is clear that computation of CK_i and ZK_i leads us to the determination of the long exact sequence of K-theory. Let $i = 0$. Since $K_0(F) \cong K_0(D) = \mathbb{Z}$, one can immediately see from (2) that the homotopy fibre $F_0 \cong \mathrm{CK}_1(D) \cong D^*/F^*D'$. On the other hand $\mathrm{ZK}_1(D) = D' \cap F^* = Z(D')$ which is the centre of the commutator subgroup D'. In [8] Valuation theory of division algebras is used to compute the group CK_1 for some valued division algebras. For example this group is determined for totally ramified division algebras and in particular for any finite cyclic group G, a division algebra D such that $\mathrm{CK}_1(D) = G$ or $G \times G$ is constructed. It seems computation of the group CK_2 is a difficult task. It is very desirable to relate the group $\mathrm{CK}_i(D)$ to $\mathrm{CK}_i(\bar{D})$, where D is a valued division algebra and $\bar{D}$ is its residue division algebra.

On the other hand, since splitness characterises Azumaya algebras, one can work in the category of Azumaya algebras. Thus in Section 2 we define an abstract functor $\mathcal{D}$ which captures the properties of the functors mentioned above. We show that $\mathcal{D}$-functors are n^2 torsion. As the first application, this immediately implies that K-theory of an Azumaya algebra over a local ring is very close to the K-theory of the base ring itself (Corollary 2.3).

Another example of $\mathcal{D}$-functors are the reduced Whitehead groups which brings us to the main section of this note. In Section 3 we attempt to investigate the behaviour of the reduced Whitehead group $\mathrm{SK}_1(A)$, where A is an Azumaya algebra over a commutative ring R. Beginning with the work of V. Platonov, who answered the Bass-Tannaka-Artin problem ($\mathrm{SK}_1(D)$ is not trivial in general) there have been extensive investigation both on functorial behaviour and computational aspect of this group over the category of central simple algebras. On the other hand study of Azumaya algebras over commutative rings and extending some theorems from the theory of central simple algebras to Azumaya algebras, has shown that in certain cases these two objects behave similar (although there are also essential points of difference).

Here we initiate a study of the functor SK_1 over the category of Azumaya Algebras. We recall a construction of reduced norm in this setting. Let A be an Azumaya algebra over a commutative ring R. A commutative R-algebra S is called a *splitting ring* of A if there is a finitely generated faithfully projective S-module P, such that $A \otimes_R S \simeq \mathrm{Hom}_S(P,P)$. S is called a *proper* splitting ring if $R \subseteq S$. The existence of a proper splitting ring in general seems to be an open question. Let A be an Azumaya R-algebra with a proper splitting ring S, and consider $A \otimes_R S \simeq \mathrm{Hom}_S(P,P)$. Then for any $a \in A$,

$$\mathrm{Nrd}_A(a) = \det{}_P(a \otimes 1)$$
$$\mathrm{Trd}_A(a) = \mathrm{tr}_p(a \otimes 1)$$

where Nrd_A and Trd_A are called the *reduced norm* and the *reduced trace* of A respectively (see [3]). Now consider the kernel of the reduced norm $\mathrm{SL}(1, A)$ and form the *reduced Whitehead group* $\mathrm{SK}_1(A) = SL(1, A)/A'$ where A' is the derived subgroup of A^*, invertible elements of A.

A question naturally arises here is the following: What is the relation of the reduced Whitehead group of an Azumaya algebra to the reduced Whitehead groups of its extensions. In particular what is the relation of SK_1 of an Azumaya algebra to SK_1 of its residue central simple algebras. To be precise, let A be an Azumaya algebra over R and m a maximal ideal of R. Since $\overline{\mathrm{Nrd}_{A/R}(a)} = \mathrm{Nrd}_{\bar{A}/\bar{R}}(\bar{a})$ where $\bar{R} = R/m$ and $\bar{A} = A/mA$, it follows that there is a homomorphism $\mathrm{SK}_1(A) \to \mathrm{SK}_1(\bar{A})$. On the other direction, if S is a multiplicative closed subset of R, since the reduced norm respects the extension, there is a homomorphism $\mathrm{SK}_1(A) \to \mathrm{SK}_1(A \otimes_R S^{-1}R)$. In particular if R is an integral domain then $\mathrm{SK}_1(A) \to \mathrm{SK}_1(A_K)$ where K is a quotient field of R and A_K is the central simple algebra over K. The question arises here is, under what circumstances these homomorphisms would be mono or isomorphisms. The following observation shows that even in the case of a split Azumaya algebra, these two groups could differ: consider the split Azumaya algebra $A = M_n(R)$ where R is an arbitrary commutative ring. In this case the reduced norm coincide with the ordinary determinant and $\mathrm{SK}_1(A) = \mathrm{SL}_n(R)/[\mathrm{GL}_n(R), \mathrm{GL}_n(R)]$. There are examples of an integral domain R (even Dedekind domain) such that $\mathrm{SK}_1(A) \neq 1$. But in this setting, obviously $\mathrm{SK}_1(\bar{A}) = 1$ and $\mathrm{SK}_1(A_K) = 1$ (for some examples see [21], Chapter 2).

In Section 3 we study this question. We shall show that over a Henselian ring R, SK_1 of an Azumaya algebra coincides with SK_1 of its residue central simple algebra. Section 4 focuses on the functors CK_i. We will proide some examples where CK_i's are trivial and compute some K-sequences of these examples.

2. K-theory of Azumaya algebras

Definition 2.1. Let $\mathcal{R}$ a the category of rings, with morphisms as follow. Let A and B be two rings with centres R and S respectively such that S is an R-algebra (or $R \subseteq S$). Then an R-algebra homomorphism $f : A \to B$ is considered as a morphism of the category. Let $\mathcal{G} : \mathcal{R} \to \mathcal{A}b$ be an abelian group valued functor such that

(1) (Determinant Property) For any natural number m, there is a homomorphism $d_m : \mathcal{G}(M_m(A)) \to \mathcal{G}(A)$ such that $d_m i_m = \eta_m$ where $i_m : \mathcal{G}(A) \to \mathcal{G}(M_m(A))$ induced by the natural embedding $A \to M_m(A)$.
(2) (Torsion Property) For any $a \in \mathrm{Ker}(d_m)$, $a^m = 1$.
(3) (Transfer Map) For any ring A with centre R and any commutative R-algebra S free over R of rank $[S : R]$, there is a homomorphism $t :$

$\mathcal{G}(A\otimes_R S) \to \mathcal{G}(A)$ such that $ti = \eta_{[S:R]}$ where $i : \mathcal{G}(A) \to \mathcal{G}(A\otimes_R S)$ induced by the natural embedding $A \to A \otimes_R S$.

(4) For any commutative ring R, $\mathcal{G}(R) = 1$.

Then $\mathcal{G}$ is called a *$\mathcal{D}$-functor.*

Let A be a ring with centre R. Consider the Quillen K-groups $K_i(A)$ for $i \geq 0$. Let $\mathcal{P}(A)$ and $\mathcal{P}(M_m(A))$ be categories of finitely generated projective modules over A and $M_m(A)$ respectively. The natural embedding $A \to M_m(A)$ induces

$$\begin{gathered}\mathcal{P}(A) \longrightarrow \mathcal{P}(M_m(A))\\ p \mapsto M_m(A) \otimes_A p.\end{gathered}$$

Furthermore, if

$$\begin{gathered}\mathcal{P}(M_m(A)) \longrightarrow \mathcal{P}(A)\\ q \mapsto A^m \otimes_{M_m(A)} q\end{gathered}$$

then one can see that the above maps induces the sequence

$$K_i(A) \longrightarrow K_i(M_m(A)) \xrightarrow{\cong} K_i(A)$$

of K-groups where the composition is η_m. On the other hand the inclusion map id : $R \to A$ gives the following exact sequence

$$1 \to \mathrm{ZK}_i(A) \to K_i(R) \to K_i(A) \to \mathrm{CK}_i(A) \to 1 \tag{3}$$

where $\mathrm{ZK}_i(A)$ and $\mathrm{CK}_i(A)$ are the kernel and co-kernel of $K_i(R) \to K_i(A)$ respectively. The following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{ZK}_i(M_m(A)) & \longrightarrow & K_i(R) & \longrightarrow & K_i(M_m(A)) & \longrightarrow & \mathrm{CK}_i(M_m(A)) & \longrightarrow & 1\\ & & \downarrow & & \downarrow{\scriptstyle \eta_m} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{ZK}_i(A) & \longrightarrow & K_i(R) & \longrightarrow & K_i(A) & \longrightarrow & \mathrm{CK}_i(A) & \longrightarrow & 1 \end{array}$$

which shows that K_i functors and therefore CK_i functors satisfy the first condition of $\mathcal{D}$-functors.

Condition (2) follows from chasing the element $a \in \mathrm{Ker}(\mathrm{CK}_i(M_m(A)) \to \mathrm{CK}_i(A))$ in the diagram, and using the fact that $K_i(M_m(A)) \to K_i(A)$ is an isomorphism.

Since S is a free R-module, the regular representation $r : S \to \mathrm{End}_R(S) \cong M_n(R)$ where $[S : R] = n$ and thus the commutative diagram

$$\begin{array}{ccc} K_i(A) & \longrightarrow & K_i(A \otimes_R S)\\ & \searrow & \downarrow{\scriptstyle 1\otimes r}\\ & & K_i(M_n(A)) \end{array}$$

implies that K_i and therefore CK_i satisfy the condition (3) of $\mathcal{D}$-functors.

Therefore the functors CK_i and ZK_i are $\mathcal{D}$-functors (we shall later show that the reduced Whitehead group SK_1 is also a $\mathcal{D}$-functor on the category of Azumaya algebras over local rings).

In the rest of this section we assume that the functor $\mathcal{G}$ is a $\mathcal{D}$-functor. Our aim is to show that if a ring A splits, then $\mathcal{G}(A)$ is torsion of bounded exponent. Thus from now on, we shall restrict the category to the category of Azumaya algebras over local rings $\mathcal{A}z$.

Proposition 2.2. *Let A be an Azumaya algebra over a local ring R of rank n^2. Then $\mathcal{G}(A)$ is a torsion group of bounded exponent n^2.*

Proof. Since R is a local ring, there is a faithfully projective R-algebra S of rank n which splits A (e.g. S could be a maximal commutative subalgebra of A separable over R, see [12] Lemma 5.1.17). Since $\mathcal{G}$ is a $\mathcal{D}$-functor, there is a sequence $\mathcal{G}(A) \xrightarrow{i} \mathcal{G}(A \otimes_R S) \xrightarrow{t} \mathcal{G}(A)$ such that $ti(x) = x^n$. But $A \otimes_R S \simeq \mathrm{End}_S(S^n)$. Conditions 2 and 4 of being a $\mathcal{D}$-functor force that $\mathcal{G}(\mathrm{End}_S(S^n))$ to be a torsion group of exponent n. Now from the sequence above the assertion of the theorem follows. □

Corollary 2.3. *Let A be an Azumaya algebra over a local ring R of rank n^2. Then $K_i(R) \otimes \mathbb{Z}[n^{-1}] \cong K_i(A) \otimes \mathbb{Z}[n^{-1}], i \geq 0$.*

Proof. Since ZK_i and CK_i are $\mathcal{D}$-functors, from Proposition 2.2 follows that these groups are n^2-torsion. Tensoring the exact sequence (3) with $\mathbb{Z}[n^{-1}]$ the result follows. □

Remark 2.4. Corollary 2.3 implies that for a finite dimensional division algebra D over its centre F with index n, $K_i(F)\otimes\mathbb{Z}[n^{-1}] \cong K_i(D)\otimes\mathbb{Z}[n^{-1}], i \geq 0$ (Compare this with [10]).

Corollary 2.5. *Let A be an Azumaya algebra over a local ring R of rank n^2 and S a projective R-algebra such that $(n^2, [S : R]) = 1$. Then $\mathcal{G}(A) \xrightarrow{i} \mathcal{G}(A \otimes_R S)$ is a monomorphism.*

Proof. From 2.2 and Condition 2 of 2.1 the kernel of the map i is simultaneously n^2 and $[S : R]$ torsion. Since these numbers are relatively prime it follows that the kernel is trivial. □

Let A and B be R-Azumaya algebras where B has constant rank $[B : R]$. Consider the regular representation $B \to \mathrm{End}_R(B)$ and the sequence,

$$\mathcal{G}(A) \to \mathcal{G}(A \otimes_R B) \to \mathcal{G}(A \otimes_R \mathrm{End}_R(B)) \xrightarrow{d} \mathcal{G}(A).$$

The composition of the above homomorphisms is $\eta_{[B:R]}$. If we further assume that R is a local ring then $\mathcal{G}(A)$ and $\mathcal{G}(B)$ are torsion of exponent $[A : R]$ and $[B : R]$ respectively. Therefore if $[A : R]$ and $[B : R]$ are relatively prime, we can prove that $\mathcal{G}$ has a decomposition property similar to the reduced Whitehead group SK_1 for central simple algebras. The proof follows the same pattern as the one for SK_1 for central simple algebras.

Theorem 2.6. *Let A and B be Azumaya algebras over a local ring R such that $(i(A), i(B)) = 1$. Then $\mathcal{G}(A \otimes_R B) \simeq \mathcal{G}(A) \times \mathcal{G}(B)$.*

Proof. By Corollary 2.2, $\mathcal{G}(A\otimes_F B)$ is a torsion group of bounded exponent mn where $m=[A:R]$ and $n=[B:R]$. From the theory of torsion abelian groups, one can write $\mathcal{G}(A\otimes_F B)\simeq K\times H$, where K and H are torsion abelian groups such that $\exp(K)|m$ and $\exp(H)|n$. We shall prove that $\mathcal{G}(A)\simeq K$ and $\mathcal{G}(B)\simeq H$.

Consider the sequence of R-homomorphisms

$$A\longrightarrow A\otimes_R B\longrightarrow A\otimes B\otimes B^{op}\xrightarrow{\simeq} A\otimes M_n(R)\xrightarrow{\simeq} M_n(A) \tag{4}$$

and apply the functor $\mathcal{G}$ to the sequence to get

$$\mathcal{G}(A)\xrightarrow{\phi}\mathcal{G}(A\otimes_R B)\xrightarrow{\psi}\mathcal{G}(A\otimes B\otimes B^{op})\xrightarrow{\theta}\mathcal{G}(A) \tag{5}$$

so that $\theta\psi\phi=\eta_n$ by the property (1) of $\mathcal{D}$-functors. Then $\mathcal{G}(A)=\eta_n\eta_n(\mathcal{G}(A))=\eta_n\theta\psi\phi(\mathcal{G}(A))\subseteq\theta\psi\eta_n(K\times H)=\theta\psi(K)\subseteq\mathcal{G}(A)$. This clearly shows that $\theta\psi|_K:K\longrightarrow\mathcal{G}(A)$ is surjective. We need to show that $\theta\psi|_K$ is injective. Consider the sequence (4) above and replace B^{op} with its regular representation $B^{op}\longrightarrow M_n(R)$ as follows

$$\mathcal{G}(A\otimes_F B)\xrightarrow{\psi}\mathcal{G}(A\otimes B\otimes B^{op})\xrightarrow{\psi'}\mathcal{G}(A\otimes B\otimes M_n(R))\xrightarrow{\theta'}\mathcal{G}(A\otimes B)$$

where θ' is provided by the first property of $\mathcal{D}$-functor and thus $\theta'\psi'\psi=\eta_n$. Now let $w\in\mathcal{G}$ such that $w\neq 1$. Then $\theta'\psi'\psi(w)=\eta_n(w)=w^n\neq 1$ implying that $\psi|_K$ is injective. Rewrite the sequence (5) as follows:

$$\mathcal{G}(A\otimes B)\xrightarrow{\psi}\mathcal{G}(A\otimes B\otimes B^{op})\xrightarrow{\simeq}\mathcal{G}(M_n(A))\xrightarrow{d_n}\mathcal{G}(A).$$

Let $x\in K$ such that $\theta\psi(x)=1$. The above sequence and the property (3) of $\mathcal{D}$-functors show that $\psi(x)^n=1$. Since $\psi|_K$ is injective, it follows that $x^n=1$. Since m and n are relatively prime, $x=1$. This shows that $\theta\psi$ is an isomorphism and so $\mathcal{G}(A)\simeq K$. Similarly, it can be shown that $G(B)\simeq H$. This completes the proof. □

Example 2.7. SK_1 *is Morita equivalent*

Consider a local ring A with more than two elements. For any integer m, one can see that $\mathrm{GL}_m(A)=E_m(A)\delta(A^*)$ where $\delta(A^*)$ are matrices with elements of A^* in a_{11}-position, units in the rest of diagonal, and zero elsewhere. Plus $E_m(A)\cap\delta(A^*)=\delta(A')$ (These all follow from the fact that the Dieudonné determinant extends to local rings ([21], §2.2)).

Now assume that A is a local Azumaya algebra. Since the splitting ring of A, is a splitting ring for the matrix algebra $M_m(A)$, one can easily see that $\mathrm{Nrd}_{M_m(A)}(\delta(a))=\mathrm{Nrd}_A(a)$. This immediately implies that SK_1 is Morita equivalent for local Azumaya algebras. On the other hand over a Henselian ring, any Azumaya algebra is a full matrix algebra over a local Azumaya algebra ([2], Theorem 26). Thus it follows from the above argument that over Henselian rings, SK_1 is Morita equivalent.

Example 2.8. SK_1 *as* $\mathcal{D}$*-functors*

Recently Vaserstein showed that the analogue of the Dieudonné determinant exists for a semilocal ring A which is free of $M_2(\mathbb{Z}_2)$ components and have at most one copy of $\mathbb{Z}_2$ in the decomposition into simple rings in $A/J(A)$ [25]. Using this result, one can extend Example 2.7 to all Azumaya algebras A over local rings (R,m) such that either $R/m \neq \mathbb{Z}_2$ or $rank A > 2$.

Now consider the full subcategory of Azumaya algebras over local rings of $\mathcal{A}z$ and exclude the exceptional cases above. As the determinant is present in this category, take d_m just the usual determinant for semi local rings. It is then easy to see that the group SK_1 satisfy all the conditions of being a $\mathcal{D}$-functor.

3. SK_1 of Azumaya algebras over Henselian rings

The aim of this section is to show that the reduced Whitehead group SK_1 of a *tame* Azumaya algebra over a Henselian ring coincides with the SK_1 of its residue division algebra. As indicated in Example 2.7, the reduced Whitehead group is Morita equivalent for Azumaya algebras over Henselian rings. Thus it is enough to work with local Azumaya algebras. Therefore throughout this section we assume the ring R is local with maximal ideal m and A is a local Azumaya algebra, i.e., $\bar{A} = A/mA$ is a division ring unless stated otherwise. The *index* of A is defined to be the square root of the rank of A over R.

Theorem 3.1. *Let A be a local Azumaya algebra over a local ring R of rank n^2. Consider the sequence*

$$K_1(A) \xrightarrow{\mathrm{Nrd}_A} K_1(R) \xrightarrow{i} K_1(A).$$

Then $i \circ \mathrm{Nrd}_A = \eta_n$ where $\eta_n(a) = a^n$.

Proof. Since A is an Azumaya algebra over a local ring R, there exist a maximal commutative subalgebra S of A which splits A as follows $A \otimes_R S \simeq M_n(S)$ (see Lemma 5.1.13 and 5.1.17 and their proofs in [12]). Consider the sequence of R-algebra homomorphism

$$\phi : A \longrightarrow A \otimes_R S \xrightarrow{\simeq} M_n(S) \hookrightarrow M_n(A)$$

and also the embedding $i : A \to M_n(A)$. For any Azumaya algebra over a semi-local ring the Skolem-Noether theorem is valid, i.e. if B is an Azumaya subalgebra of A, then any R-algebra homomorphism $\phi : B \to A$ can be extended to an inner automorphism of A. Thus there is a $g \in \mathrm{GL}_n(A)$ such that $\phi(x) = gi(x)g^{-1}$ for any $x \in A$. Now taking the Dieudonné determinant from the both sides of the above equality, we have $\mathrm{Nrd}_{A/R}(x) = x^n d_x$ where $d_x \in A'$. This completes the proof. □

Corollary 3.2. *Let A be a local Azumaya algebra over a local ring R. Then for any $x \in A^*$, $x^n = \mathrm{Nrd}_{A/R}(x) d_x$ where n^2 is the rank of A over R and $d_x \in A'$. In particular the reduced Whitehead group $\mathrm{SK}_1(A)$ is torsion of bounded exponent n.*

In order to establish a relation between SK_1 of an Azumaya algebra to that of a central simple algebra, we need a version of Platonov's congruence theorem in the setting of Azumaya algebras. The original proof of congruence theorem is quite complicated and seems to be not possible to adopt it in the setting of Azumaya algebras. It is now two short proofs of the theorem in the case of a tame division algebra available, one due to Suslin (buried in [23]) and one due to the author [7]. Here it seems the Suslin proof is the suitable one to adopt for this category. Before we state the congruence theorem we establish some useful facts needed later in this note.

From now on we assume that the base ring is Henselian. Recall that a commutative local ring R is *Henselian* if $f(x) \in R[x]$ and $\bar{f} = g_0 h_0$ with g_0 monic and g_0 and h_0 coprime in $\bar{R}[x]$, then f factors as $f = gh$ with g monic and $\bar{g} = g_0$, $\bar{h} = h_0$. Here $\bar{R} = R/m$ and $\bar{f}$ is the reduction of $f(x)$ with respect to m. (see §30, [17]). Now let A be an Azumaya algebra over R. Thus $A = M_m(B)$ where B is a local Azumaya algebra. We say A is *tame* if $\mathrm{Char}\,\bar{R}$ does not divide the index of B.

Proposition 3.3. *Let A be a tame local Azumaya algebra of rank n^2 over a Henselian ring R with maximal ideal m. Then*

1. *The map $\eta_n : 1+m \longrightarrow 1+m$ where $\eta_n(x) = x^n$ is an automorphism.*
2. $\mathrm{Nrd}_A(1 + mA) = 1 + m$.
3. *$1 + mA$ is a n-divisible group.*

Proof. 1. Take $a \in 1 + m$. Since R is Henselian and $\mathrm{Char}\,\bar{R}$ does not divide n, the polynomial $x^n - a$ has a simple root in $\bar{R}[x]$, hence a simple root b for $x^n - a$. This shows η_n is epimorphism. But if $1 + a' \in (1+m) \cap \mathrm{SL}(1, A)$ then $(1 + a')^n = 1$. Since R is local and $\mathrm{Char}\,\bar{R} \nmid n$, it follows that $a = 0$ and this shows that η_n is also monomorphism.

2. This follows from the fact that $\overline{\mathrm{Nrd}_{A/R}(a)} = \mathrm{Nrd}_{\bar{A}/\bar{R}}(\bar{a})$ and the first part of this Proposition.

3. We shall show that for any $a \in 1+mA$, there is a $b \in 1+mA$ such that $a = b^n$. Since A is finite over its centre R, any element of A is integral over R. Take $a \in 1 + mA$. Consider the ring $R[a]$ generated by a and R. $R[a]$ is a local ring. In general any commutative subalgebra S of A is a local ring. For consider the R/m subalgebra $S/(mA \cap S)$ of A/mA. Since A/mA is a division ring, $mA \cap S$ is a maximal ideal of S. But any element of $mA \cap S$ is quasi-regular in A and therefore in S (see [2], Corollary to Theorem 9). It follows that $mA \cap S$ is a unique maximal ideal of S. So the ring $R[a]$ is local and finitely generated as a R-module. Since any local ring which is integral over a Henselian ring R is Henselian ([17] Corollary 43.13), it follows that $R[a]$ is Henselian. Now because $a \in 1 + mA$, thus $a \in 1 + m_{R[a]}$ where $m_{R[a]}$ is a unique maximal ideal of $R[a]$. The first part of this Proposition guarantee an element $b \in 1 + m_{R[a]}$, such that $b^n = a$. From this it follows that $1 + mA$ is a n-divisible group. □

Consider the commutative diagram

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & \mathrm{SK}_1(A) & \longrightarrow & K_1(A) & \xrightarrow{\mathrm{Nrd}_A} & K_1(R) & \longrightarrow & \mathrm{SH}^0(A) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{SK}_1(\bar{A}) & \longrightarrow & K_1(\bar{A}) & \xrightarrow{\mathrm{Nrd}_{\bar{A}}} & K_1(\bar{R}) & \longrightarrow & \mathrm{SH}^0(\bar{A}) & \longrightarrow & 1 \end{array}$$

where the vertical maps are canonical epimorphism and SH^0 are the cokernel of the reduced norm maps.

Theorem 3.4. *Let R be a Henselian ring and A a tame Azumaya algebra over R. Then* $\mathrm{SH}^0(A) \simeq \mathrm{SH}^0(\bar{A})$.

Proof. Since the image of the reduced norm is Morita equivalent (see Example 2.7), it is enought to prove the theorem for A a local Azumaya algebra. The canonical homomorphism

$$R^*/\operatorname{Nrd}_A(A^*) \to \bar{R}^*/\operatorname{Nrd}_{\bar{A}}(\bar{A}^*)$$

is clearly an epimorphism. Let $a \in R^*$ such that $\bar{a} \in \operatorname{Nrd}_{\bar{A}}(\bar{A}^*)$. Thus there is a $\bar{b}$ in $\bar{A}^*$ such that $\operatorname{Nrd}_A(b)a^{-1} \in 1+m$ where m is the maximal ideal of R. Now by Part one of the Proposition 3.3, there is an $c \in 1+m$ such that $\operatorname{Nrd}_A(bc^{-1}) = a$. This completes the proof. □

Theorem 3.5. (Congruence Theorem). *Let A be a tame local Azumaya algebra over a Henselian ring R. Then $(1+mA) \cap SL(1,A) \subseteq A'$.*

Proof. Let $a \in (1+mA) \cap \mathrm{SL}(1,A)$. By the third part of the Proposition 3.3 there is a $b \in 1+mA$ such that $b^n = a$. Now $\operatorname{Nrd}_{A/R}(b)^n = 1$, so by the first and second part of the Proposition, $\operatorname{Nrd}_{A/R}(b) = 1$. Thus $b \in \mathrm{SL}(1,A)$. But by Corollary 3.2, $\mathrm{SK}_1(A)$ is n-torsion, thus $a = b^n \in A'$. □

Now we are ready to state our main theorem.

Theorem 3.6. *Let R be a Henselian ring and A a tame Azumaya algebra over R. Then for any ideal I of R,* $\mathrm{SK}_1(A) \simeq \mathrm{SK}_1(A/IA)$. *In particular* $\mathrm{SK}_1(A) \simeq \mathrm{SK}_1(\bar{A})$.

Proof. We can assume that A is a local Azumaya algebra (see Example 2.7). It is enough to prove the theorem for $I = m$. Since $\overline{\operatorname{Nrd}_{A/R}(a)} = \operatorname{Nrd}_{\bar{A}/\bar{R}}(\bar{a})$, the restriction of the reduction map to $\mathrm{SL}(1,A)$ gives the well define homomorphism $\mathrm{SL}(1,A) \longrightarrow \mathrm{SL}(1,\bar{A}), a \mapsto \bar{a}$. This map is epimorphism. For if $\bar{a} \in \mathrm{SL}(1,\bar{A})$ then $\overline{\operatorname{Nrd}_{A/R}(a)} = 1$. Thus $\operatorname{Nrd}_{A/R}(a) \in 1+m$. From the second part of the Proposition 3.3 follows that there is a $b \in 1+mA$ such that $\operatorname{Nrd}_{A/R}(a) = \operatorname{Nrd}_{A/R}(b)$. Therefore $ab^{-1} \mapsto \bar{a}$. Thus we have the following isomorphism,

$$\mathrm{SL}(1,A)/(1+mA) \cap \mathrm{SL}(1,A) \longrightarrow \mathrm{SL}(1,\bar{A}).$$

Now since A is local, $\overline{A'} = \bar{A}'$, and the Congruence Theorem implies that $\mathrm{SK}_1(A) \simeq \mathrm{SK}_1(\bar{A})$. □

Example 3.7. Let D be a division algebra over its centre F such that $\operatorname{Char} F$ does not divide the index of D. Consider the Azumaya algebra $D \otimes_F F[[x]]$ over the Henselian ring $F[[x]]$. Then Theorem 3.6 guarantees that $\mathrm{SK}_1(D[[x]]) \simeq \mathrm{SK}_1(D)$.

It seems that in the above example the condition $\operatorname{Char} F \nmid i(D)$ is not necessary. This suggest that there might be a weaker condition for being tame as it is available for valued division algebras observed by Ershov in [4].

We can obtain one of the functorial properties of the reduced Whitehead group, namely the stability of SK_1 under the reduction for unramified division algebra from the Theorem above.

Example 3.8. Let D be a tame unramified division algebra over a Henselian field F. Jacob and Wadsworth observed that V_D is an Azumaya algebra over its centre V_F (Example 2.4 [11]). Since $D^* = F^* U_D$ and $V_D \otimes_{V_F} F \simeq D$, it can be seen that $\mathrm{SK}_1(D) = \mathrm{SK}_1(V_D)$. On the other hand our main Theorem 3.6 states that $\mathrm{SK}_1(V_D) \simeq \mathrm{SK}_1(\bar{D})$. Comparing these, we conclude the stability of SK_1 under reduction, namely $\mathrm{SK}_1(D) \simeq \mathrm{SK}_1(\bar{D})$ (compare this with the original proof, Corollary 3.13 [18]).

Example 3.9. Let F be a Henselian and discrete valued field and V_F its valuation ring. It is known that any Azumaya algebra A over V_F is of the form $M_r(V_D)$ where V_D is a valuation ring of D, a division algebra over F. It follows that D is an unramified division algebra over F. Thus from Example 2.7 and Example 3.8 it follows that if $\operatorname{Char} \bar{F}$ is prime to index of D then $\mathrm{SK}_1(A) \simeq \mathrm{SK}_1(V_D) \simeq \mathrm{SK}_1(\bar{D}) \simeq \mathrm{SK}_1(D)$.

Let A be a local Azumaya algebra over a local ring R. Consider R^h the *Henselisation ring* of R and the extension $\phi : \mathrm{SK}_1(A) \to \mathrm{SK}_1(A \otimes_R R^h)$. Since $A \otimes_R R^h$ is an Azumaya algebra over the Henselian ring R^h, by Theorem 3.6, $\mathrm{SK}_1(A \otimes_R R^h) \cong \mathrm{SK}_1(\overline{A \otimes_R R^h})$ and we have the following

$$\mathrm{SK}_1(A) \xrightarrow{\phi} \mathrm{SK}_1(A \otimes_R R^h) \cong \mathrm{SK}_1(\overline{A \otimes_R R^h}) \cong \mathrm{SK}_1(\bar{A}).$$

The following example shows that ϕ is not injective in general and in particular Theorem 3.6 does not hold if the base ring R is not Henselian. I learned this example from David Saltman.

Example 3.10. Consider a field F with p^2-th primitive root of unity ρ and $a, b, c, d \in F$ such that $(a, b)_{(F,p)} \otimes_F (c, d)_{(F,p)}$ is a cyclic division algebra, where $(a, b)_{(F,p)}$ is a symbol algebra, i.e., an algebra generated by symbols α and β subjected to the relations, $\alpha^p = a, \beta^p = b$ and $\beta\alpha = \rho^p \alpha\beta$. Consider the polynomial ring $F[x_1, x_2, x_3, x_4]$ where x_i are indeterminant, the local ring $R = F[x_1, x_2, x_3, x_4]_{(x_1 - a, x_2 - b, x_3 - c, x_4 - d)}$ and function field $L = F(x_1, x_2, x_3, x_4)$. Now the Azumaya algebra

$$A = (x_1, x_2)_{(R,p)} \otimes_R (x_3, x_4)_{(R,p)}$$

is contained in the central simple algebra

$$D = (x_1, x_2)_{(L,p)} \otimes_L (x_3, x_4)_{(L,p)}.$$

The algebra D is well understood (see [22], Example 3.6). In particular one knows that $\mathrm{Nrd}_D(\rho) = \rho^{p^2} = 1$ but ρ is not in derived subgroup D'. That is ρ is a non trivial element of $\mathrm{SK}_1(D)$. Since $A \subseteq D$, $\rho \notin A'$. Consider the residue central simple algebra $\bar{A}$ which is $(a,b)_{(F,p)} \otimes_F (c,d)_{(F,p)}$. Since $\bar{A}$ is cyclic, $\bar{\rho}$ is in $\bar{A}'$, and this shows that the map $\mathrm{SK}_1(A) \to \mathrm{SK}_1(\bar{A})$ and in particular ϕ is not injective.

For $i = 1$ one has the following commutative diagram,

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & A' \cap R^* & \longrightarrow & K_1(R) & \xrightarrow{i} & K_1(A) & \longrightarrow & \mathrm{CK}_1(A) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \bar{A}' \cap \bar{R}^* & \longrightarrow & K_1(\bar{R}) & \xrightarrow{i} & K_1(\bar{A}) & \longrightarrow & \mathrm{CK}_1(\bar{A}) & \longrightarrow & 1 \end{array}$$

where the vertical maps are canonical epimorphisms. In Section 4 we shall compute the groups CK_i for certain Azumaya algebras. Here we show that following the same pattern as SK_1, the group $\mathrm{CK}_1(A)$ also coincides with its residue division algebra. Note that $\mathrm{CK}_1(A) = A^*/R^*A'$. The conjecture that CK_1 of a division algebra D is not trivial if D is not a quaternion algebra still remains open. This has connections with the study of normal and maximal subgroups of D^* (see [9] and references there).

Theorem 3.11. *Let R be a Henselian ring and A an Azumaya algebra over R of rank n^2. If* $\mathrm{Char}\,\bar{R}$ *does not divide n, then* $\mathrm{CK}_1(A) \simeq \mathrm{CK}_1(\bar{A})$.

Proof. First assume that A is a local Azumaya algebra. Thus A is tame. We show that $1 + mA \subseteq (1+m)A'$. Let $a \in 1 + mA$. By part 2 of Proposition 3.3, $\mathrm{Nrd}_{A/R}(a) = b \in 1 + m$. Since $1 + m$ is n-divisible, there is a $c \in 1 + m$ such that $b = c^n$. It follows that $\mathrm{Nrd}_{A/R}(ac^{-1}) = 1$. Thus $ac^{-1} \in \mathrm{SL}(1, A) \cap (1 + mA)$. Now by the congruence Theorem $ac^{-1} \in A'$ and the claim follows. Now consider the following sequence

$$\bar{A}^* \xrightarrow{\simeq} A^*/(1 + mA) \longrightarrow A^*/(1 + m)A' \longrightarrow A^*/R^*A'.$$

But the kernel of the composition of this sequence is $\bar{R}^*\bar{A}'$. Thus $\mathrm{CK}_1(A) \simeq \mathrm{CK}_1(\bar{A})$. Now assume A is a matrix algebra $M_s(B)$ where B is a local Azumaya algebra of rank t^2. Since $\mathrm{Char}\,\bar{R}$ does not divide t and s, it is not hard to see (repeating the same argument with $(1+m)^s = 1+m$) that also in this setting the CK_1's coincide. ☐

Example 3.12. Let A be an Azumaya algebra $(\frac{-1,x-1}{F[[x]]})$ where F is a formally real Pythagorean field (i.e. F has an ordering and the sum of two square elements is a square). Then by Theorem 3.11, $\mathrm{CK}_1(\frac{-1,x-1}{F[[x]]}) \simeq \mathrm{CK}_1(\frac{-1,-1}{F})$. It is not hard to see that the latter group is trivial ([9]). Thus $A^* = F[[x]]^*A'$.

On the other direction, the question of when the reduced Whitehead group $\mathrm{SK}_1(A)$ coincides with its extension $\mathrm{SK}_1(A \otimes_R K)$ where K is the field of fraction of R would follow if the Gersten complex of K-groups is exact as we will observe below. It is a conjecture that the Gersten complex is exact for an Azumaya algebra over a regular semilocal ring ([13] and the references there). Thus it is plausible to make the following conjecture:

Conjecture 3.13. Let A be an Azumaya algebra over a regular semilocal ring R. Then $\mathrm{SK}_1(A) \cong \mathrm{SK}_1(A \otimes_R K)$ where K is the field of fraction of R.

Using results of [13], one can observe that the above conjecture holds in the case of semilocal ring of geometric type. Recall that such a ring is obtained by localising a finite type algebra R over a field with respect to finitely many primes q, such that the ring R_q is regular.

In [13], Panin and Suslin prove that the following Gersten complex is exact where R is a semilocal ring of geometric type and A an Azumaya algebra over R:

$$0 \to K_i(A) \to K_i(A \otimes_R K) \to \bigoplus_{ht(p)=1} K_{i-1}(A_p/pA_p) \to \bigoplus_{ht(p)=2} K_{i-2}(A_p/pA_p) \to \cdots$$

For $i = 1$ one arrives at

$$0 \to K_1(A) \to K_1(A \otimes_R K) \to \bigoplus_{ht(p)=1} K_0(A_p/pA_p) \to 0.$$

Considering the same exact sequence for $A = R$, since the reduced norm map is compatible with the rest of maps, one gets the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A \otimes_R K) & \longrightarrow & \bigoplus_{ht(p)=1} K_0(A_p/pA_p) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle Nrd} & & \downarrow{\scriptstyle Nrd} & & \downarrow{\scriptstyle Nrd} & & \\ 1 & \longrightarrow & K_1(R) & \longrightarrow & K_1(K) & \longrightarrow & \bigoplus_{ht(p)=1} K_0(R_p/pR_p) & \longrightarrow & 1 \end{array}$$

Now the snake lemma immediately gives $\mathrm{SK}_1(A) = \mathrm{SK}_1(A \otimes_R K)$ thanks to the fact that K_0's in the above sequence are $\mathbb{Z}$ and $K_0(A_p/pA_p) \to K_0(R_p/pR_p)$ is injective.

4. On lower K-groups of Azumaya algebras

In the light of Theorem 3.11, in this section we try to determine the groups CK_i (and ZK_i) for Azumaya algebras over fields for $i = 1$ and 2. It would be interesting, among other things, to find out when these groups are trivial.

Recall that F is *real Pythagorean* if $-1 \notin F^{*2}$ and sum of any two square elements is a square in F. It follows immediately that F is an ordered field. F is called *Euclidean* if F^{*2} is an ordering of F. Let $D = (\frac{-1,-1}{F})$ be an ordinary quaternion division algebra over a field F. It is not hard to see that $\mathrm{CK}_1(D)$ is trivial if and only if F is a real Pythagorean field (see the proof of Theorem 4.1). The main result of [1] is to show that $\mathrm{CK}_2(D)$ is trivial for a Euclidean field F. Using results of Dennis and Rehmann we can observe that in fact $\mathrm{CK}_2(D)$ is trivial for any ordinary quaternion division algebra over a real Pythagorean field (Theorem 4.1). This enables one to construct interesting examples of CK_2. Before we start, let us remind a generalisation of Matsumoto's theorem for division rings due to Rehmann [19].

Let D be a division ring and $\mathrm{St}_1(D)$ a group generated by $\{u, v\}$ where $u, v \in D^*$ subjected to the relations

U1 $\{u, 1-u\} = 1$ where $1-u \neq 0$

U2 $\{uv, w\} = \{{}^u v, {}^u w\}\{u, w\}$

U3 $\{u, vw\}\{v, wu\}\{w, uv\} = 1$

then there is an exact sequence

$$1 \rightarrow K_2(D) \rightarrow \mathrm{St}_1(D) \rightarrow D' \rightarrow 1,$$

where $\{u, v\} \in \mathrm{St}_1(D)$ maps to the commutator $[u, v]$. Furthermore it is observed that ([20], Proposition 4.1) for a quaternion division algebra D, $K_2(D)$ is generated by elements of the form $\{u, v\}$ where u and v commutes. We are ready to observe,

Theorem 4.1. *Let $D = (\frac{-1,-1}{F})$ be the ordinary quaternion division algebra over a real Pythagorean field F. Then $\mathrm{CK}_2(D) = 1$ and we have the following long exact sequence of K-theory,*

$$K_2(F) \rightarrow K_2(D) \rightarrow \mathbb{Z}_2 \rightarrow K_1(F) \rightarrow K_1(D) \rightarrow 1.$$

Proof. Consider the long exact sequence (1). Since $K_0(D) \cong K_0(F) = \mathbb{Z}$ one immediately deduce $F_0 \simeq \mathrm{CK}_1(D)$. On the other hand, since $\mathrm{SK}_1(D)$ is trivial, $\mathrm{CK}_1(D) \cong \mathrm{Nrd}(D^*)/F^{*2}$ and the reduced norm of elements of $(\frac{-1,-1}{F})$ are the sum of four squares, one can easily see that F is a real Pythagorean if and only if $\mathrm{CK}_1(D)$ is trivial. Thus F_0 is trivial. We are left to calculate the homotopy fiber F_1. $K_2(D)$ is generated by {u,v}, such that [u,v]=1. On the other hand if E is a quadratic extension of F, one can easily see that $K_2(E)$ is generated by $\{\alpha, a\}$ where $\alpha \in F^*, a \in E^*$. It follows that $K_2(D)$ is generated by $\{\alpha, a\}$ where $\alpha \in F^*$ and $a \in D^*$. But $\mathrm{CK}_1(D) = 1$ thus $D^* = F^* D'$. It follows that $K_2(D)$ is generated by $\{\alpha, \beta\}$ where $\alpha, \beta \in F^*$. Thus $\mathrm{CK}_2(D) = 1$. One can easily see that $\mathrm{ZK}_1(D) = D' \cap F^* = \mathbb{Z}_2$. Thus from (2), it follows that $F_1 = \mathbb{Z}_2$. This completes the proof. □

Remark 4.2. It is well-known that F is a real Pythagorean field if and only if $D = (\frac{-1,-1}{F})$ is a division algebra and every maximal subfield of D is F-isomorphic to $F(\sqrt{-1})$ (see [5]). Combining this fact with the similar argument as in Theorem 4.1, one can see that for any maximal subfield E of D, the map $K_2(E) \rightarrow K_2(D)$ is an epimorphism. (compare this with §6 [15]).

Example 4.3. We are ready to present an example of a F division algebra D which contains a F subdivision algebra A, such that $\mathrm{CK}_i(D) \cong \mathrm{CK}_i(A)$

for $i = 1, 2$. For this we need the Fein-Schacher-Wadsworth example of a division algebra of index $2p$ over a Pythagorean field F [6]. We briefly recall the construction. Let p be an odd prime and K/F be a cyclic extension of dimension p of real Pythagorean fields, and let σ be a generator of $\mathrm{Gal}(K/F)$. Then $K((x))/F((x))$ is a cyclic extension where $K((x))$ and $F((x))$ are the Laurant power series fields of K and F, respectively. The algebra

$$D = \Big(\frac{-1,-1}{F((x))}\Big) \otimes_{F((x))} \Big(K((x))/F((x)), \sigma, x\Big)$$

was shown to be a division algebra of index $2p$. Since F is real Pythagorean, so is $F((x))$. Now by Theorem 2.6,

$$\mathrm{CK}_i(D) \cong \mathrm{CK}_i\Big(\frac{-1,-1}{F((x))}\Big) \times \mathrm{CK}_i(A)$$

where $A = \big(K((x))/F((x)), \sigma, x\big)$. By Theorem 4.1, $\mathrm{CK}_i(\frac{-1,-1}{F((x))}) = 1$ for $i = 1, 2$. Thus $\mathrm{CK}_i(D) \cong \mathrm{CK}_i(A)$. This in particular shows that the exponent of the group $\mathrm{CK}_i(D)$ does not follow the same pattern as exponent of D.

In the following theorem we first show that if D is a (unique) quaternion division algebra over a real closed field (a Euclidean field such that every polynomial of odd degree has a zero, e.g. $\mathbb{R}$), then $\mathrm{CK}_2(D(x)) = 1$; The first deviation from the functor CK_1.

Theorem 4.4. *Let D be a quaternion division algebra over F a real closed field. Then $\mathrm{CK}_2(D(x)) = 1$ and we have the following long exact sequence of K-theory,*

$$K_2(F(x)) \to K_2(D(x)) \to \mathbb{Z}_2 \to K_1(F(x)) \to K_1(D(x)) \to \oplus_\infty \mathbb{Z}_2 \to 1.$$

Proof. Consider the following commutative diagram which is obtained from the localisation exact sequence of algebraic K-theory (see [24], Lemma 16.6),

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(F) & \longrightarrow & K_2(F(x)) & \longrightarrow & \prod_{p\in F[x]} K_1(\frac{F[x]}{p}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_2(D) & \longrightarrow & K_2(D(x)) & \longrightarrow & \prod_{p\in F[x]} K_1(D\otimes_F \frac{F[x]}{p}) & \longrightarrow & 1 \end{array}$$

where p runs over the irreducible monic polynomial of $F[x]$. The snake lemma immediately gives

$$\mathrm{CK}_2(D) \longrightarrow \mathrm{CK}_2(D(x)) \longrightarrow \prod_{p\in F[x]} \mathrm{CK}_1(D\otimes_F \frac{F[x]}{p}) \longrightarrow 1.$$

Now, since F is real closed, in particular Euclidean, Theorem 4.1 implies that $\mathrm{CK}_2(D) = 1$. Considering the fact that the irreducible polynomials of $F[x]$ have either degree one or two, and the quadratic extension of F is algebraically closed, a simple calculation shows that

$$\prod_{p\in F[x]} \mathrm{CK}_1(D\otimes_F \frac{F[x]}{p}) \cong \mathrm{CK}_1(D)\oplus \mathrm{CK}_1(M_2(\bar{F})) = 1.$$

Thus $\mathrm{CK}_2(D(x)) = 1$ and $F_1 = D(x)' \cap F(x)$. One can easily see that $D(x)'\cap F(x) \cong \mathbb{Z}_2$. It remains to compute $F_0 = \mathrm{CK}_1(D(x))$. One way to compute this group is to consider again the commutative diagram which is obtained from the localisation exact sequence (see [8] Theorem 2.10),

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_1(F) & \longrightarrow & K_1(F(x)) & \longrightarrow & \prod_{p\in F[x]}\mathbb{Z} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_1(D) & \longrightarrow & K_1(D(x)) & \longrightarrow & \prod_{p\in F[x]}\frac{n_p}{n}\mathbb{Z} & \longrightarrow & 1
\end{array}$$

where p runs over the irreducible monic polynomials of $F[x]$ and n_p is the index of $D\otimes_F F[x]/p$. Considering the fact that the irreducible polynomials of $F[x]$ have either degree one or two, the snake lemma immediately implies that $\mathrm{CK}_1(D(x)) = \oplus_\infty \mathbb{Z}_2$. This completes the proof. □

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