Motivic decomposition of a generalized Severi-Brauer variety

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Abstract

Let A and B be two central simple algebras of a prime degree n over a field F generating the same subgroup in the Brauer group $\mathrm{Br}(F)$. We show that the Chow motive of a Severi-Brauer variety $\mathrm{SB}(A)$ is a direct summand of the motive of a generalized Severi-Brauer variety $\mathrm{SB}_d(B)$ if and only if $[A] = \pm d[B]$ in $\mathrm{Br}(F)$. The proof uses methods of Schubert calculus and combinatorial properties of Young tableaux, e.g., the Robinson-Schensted correspondence.

Keywords: Severi-Brauer variety, Chow motive, Robinson-Schensted correspondence, Grassmannian

1 Introduction

Let X be a twisted flag G-variety for a linear algebraic group G. The main result of paper [CPSZ] says that under certain restrictions the Chow motive of X can be expressed in terms of motives of "minimal" flag varieties, i.e., those which correspond to maximal parabolic subgroups of G. A natural question arises: is it possible to decompose the motive of such a "minimal" flag?

A particular case of such a decomposition was already provided in [CPSZ]. More precisely, a "minimal flag" X was a generalized Severi-Brauer variety of ideals of reduced dimension 2 in a division algebra B of degree 5 and the decomposition was (see [CPSZ, Theorem 2.5])

$$\mathcal{M}(\mathrm{SB}_2(B)) \simeq \mathcal{M}(\mathrm{SB}(A)) \oplus \mathcal{M}(\mathrm{SB}(A))(2),$$

where $[B] = 2[A]$ in the Brauer group.

In the present paper we provide an affirmative answer on this question for any adjoint group G of inner type A_n of a prime rank n. Namely, we show that the motive of a generalized Severi-Brauer variety always contains (as a direct summand) the motive of a Severi-Brauer variety.

1.1 Theorem. *Let A and B be two central simple algebras of a prime degree n over a field F generating the same subgroup in the Brauer group* $\mathrm{Br}(F)$. *Then the motive of a Severi-Brauer variety* $\mathrm{SB}(A)$ *is a direct summand of the motive of a generalized Severi-Brauer variety* $\mathrm{SB}_d(B)$ *if and only if*

$$[A] = \pm d\,[B] \ \textit{in}\ \mathrm{Br}(F). \tag{1}$$

1.2 Remark. Theorem 1.1 may also be considered as a generalization of the result by N. Karpenko (see [Ka00, Criterion 7.1]) which says that the motives of Severi-Brauer varieties $\mathrm{SB}(A)$ and $\mathrm{SB}(B)$ of central simple algebras A and B are isomorphic if and only if $[A] = \pm[B]$ in $\mathrm{Br}(F)$.

1.3 Remark. We expect that Theorem 1.1 holds in the case when d and n are coprime (see Section 7). Observe that if d and n are not coprime, then the theorem fails. It can be already seen on the level of generating functions. Namely, consider the Grassmann variety $\mathbb{G}_2(4)$ of 2-planes in a 4-dimensional affine space ($n = 4$ and $d = 2$). Then for the generating functions we have

$$P(\mathbb{G}_2(4), t) = (t^2+1)(t^2+t+1) \text{ and } P(\mathbb{P}^3, t) = t^3+t^2+t+1.$$

Obviously $P(\mathbb{P}^3, t)$ doesn't divide $P(\mathbb{G}_2(4), t)$. Indeed, it can be shown that $P(\mathbb{P}^{n-1}, t)$ divides $P(\mathbb{G}_d(n), t)$ if and only if d and n are coprime.

1.4 Remark. For motives with $\mathbb{Z}/n\mathbb{Z}$-coefficients there is the following decomposition (see [CPSZ, Proposition 2.4])

$$\mathcal{M}(\mathrm{SB}_d(B)) \simeq \bigoplus_i \mathcal{M}(\mathrm{SB}(B))(i)^{\oplus a_i}, \tag{2}$$

where the integers a_i are coefficients of the quotient of Poincaré polynomials $\frac{P(\mathbb{G}_d(n),t)}{P(\mathbb{P}^{n-1},t)} = \sum_i a_i t^i$ (see 2.8). Note that in this case the motives of Severi-Brauer varieties corresponding to different classes of algebras generating the same subgroup in the Brauer group are isomorphic (see [Ka00, Section 7]), i.e., $\mathcal{M}(\mathrm{SB}(A)) \simeq \mathcal{M}(\mathrm{SB}(B))$.

The proofs are based on Rost Nilpotence Theorem for projective homogeneous varieties proved by V. Chernousov, S. Gille and A. Merkurjev in [CGM05]. Briefly speaking, this result reduces the problem of decomposing the motive of a variety X over F into the question about algebraic cycles in the Chow ring $\mathrm{CH}(X_s \times X_s)$ over the separable closure F_s. Namely, the motive of $\mathrm{SB}(A)$ is a direct summand of the motive of $\mathrm{SB}_d(B)$ if there exist two cycles f and g in $\mathrm{CH}(\mathbb{P}^{n-1} \times \mathbb{G}_d(n))$ such that both cycles belong to the image of the restriction map $\mathrm{CH}(\mathrm{SB}(A) \times \mathrm{SB}_d(B)) \to \mathrm{CH}(\mathbb{P}^{n-1} \times \mathbb{G}_d(n))$, and the correspondence product $g^t \circ f$ is the identity.

We define f and g to be the Schur functions of total Chern classes of certain bundles on $\mathbb{P}^{n-1} \times \mathbb{G}_d(n)$. Using the language and properties of Schur functions we show that the identity $g^t \circ f = \mathrm{id}$ is a direct consequence of the Robinson-Schensted correspondence, one of classical combinatorial facts about Young tableaux.

The paper is organized as follows. In section 2 we remind several definitions and notation used in the proofs. These include Chow motives, rational cycles, and generalized Severi-Brauer varieties. In section 3 we describe the subgroup of rational cycles of the Chow group of the product of two generalized Severi-Brauer varieties. Indeed, we provide an explicit set of generators for this subgroup modulo n in terms of Schur functions. In section 4 we use this description for proving some known results on motives of Severi-Brauer varieties. Section 5 is devoted to the proof of the main theorem. In section 6 we prove the crucial congruence used in the proof of the main theorem. In the last section we discuss the case of $\mathbb{G}_2(n)$, where n is an odd integer (not necessarily prime).

2 Preliminaries

In the present section we remind definition of the category of Chow motives over a field F following [Ma68] and [Ka01]. We recall the notion of a rational cycle and state the Rost Nilpotence Theorem for idempotents following [CGM05]. We recall several auxiliary facts concerning generalized Severi-Brauer varieties following [KMRT98].

2.1 (Chow motives). Let F be a field and $\mathcal{V}ar_F$ be the category of smooth projective varieties over F. We define the category $\mathcal{C}or_F$ of *correspondences* over F. Its objects are non-singular projective varieties over F. For morphisms, called correspondences, we set $\mathrm{Mor}(X, Y) := \mathrm{CH}^{\dim X}(X \times Y)$. For two correspondences $\alpha \in \mathrm{CH}(X \times Y)$ and $\beta \in \mathrm{CH}(Y \times Z)$ we define the composition $\beta \circ \alpha \in \mathrm{CH}(X \times Z)$

$$\beta \circ \alpha = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\alpha) \cdot \mathrm{pr}_{23}^*(\beta)),$$

where pr_{ij} denotes the projection on product of the i-th and j-th factors of $X \times Y \times Z$ respectively and ${\mathrm{pr}_{ij}}_*$, pr_{ij}^* denote the induced push-forwards and pull-backs for Chow groups. The composition $\circ$ induces a ring structure on the abelian group $\mathrm{CH}^{\dim X}(X \times X)$. The unit element of this ring is the class of diagonal cycle Δ_X.

The pseudo-abelian completion of $\mathcal{C}or_F$ is called the category of *Chow motives* and is denoted by $\mathcal{M}_F$. The objects of $\mathcal{M}_F$ are pairs (X, p), where X is a non-singular projective variety and p is a projector, that is, $p \circ p = p$. The motive (X, Δ_X) will be denoted by $\mathcal{M}(X)$.

By construction $\mathcal{M}_F$ is a self-dual tensor additive category, where the duality is given by the transposition of cycles $\alpha \mapsto \alpha^t$ and the tensor product is given by the usual fiber product $(X,p) \otimes (Y,q) = (X \times Y, p \times q)$. Moreover, the contravariant Chow functor $\mathrm{CH} : \mathcal{V}ar_F \to \mathbb{Z}\text{-}\mathcal{A}b$ (to the category of $\mathbb{Z}$-graded abelian groups) factors through $\mathcal{M}_F$, i.e., one has the commutative diagram of functors

$$\begin{tikzcd} \mathcal{V}ar_F \arrow[rr, "\mathrm{CH}"] \arrow[dr, "\Gamma"'] & & \mathbb{Z}\text{-}\mathcal{A}b \\ & \mathcal{M}_F \arrow[ur, "R"'] & \end{tikzcd}$$

where $\Gamma : f \mapsto \Gamma_f$ is the (contravariant) graph functor and $R : \mathcal{M}_F \to \mathbb{Z}\text{-}\mathcal{A}b$ is the (covariant) realization functor given by $R : (X, p) \mapsto \mathrm{im}(p^*)$, where p^* is the composition

$$p^* : \mathrm{CH}(X) \xrightarrow{\mathrm{pr}_1^*} \mathrm{CH}(X \times X) \xrightarrow{\cdot p} \mathrm{CH}(X \times X) \xrightarrow{\mathrm{pr}_{2*}} \mathrm{CH}(X).$$

Consider the morphism $(\mathrm{id}, e) : \mathbb{P}^1 \times \{pt\} \to \mathbb{P}^1 \times \mathbb{P}^1$. The image of the induced push-forward $(\mathrm{id}, e)_*$ doesn't depend on the choice of a point $e : \{pt\} \to \mathbb{P}^1$ and defines the projector in $\mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ denoted by p_1. The motive $L = (\mathbb{P}^1, p_1)$ is called *Lefschetz motive*. For a motive M and a nonnegative integer i we denote by $M(i) = M \otimes L^{\otimes i}$ its *twist*. Observe that

$$\mathrm{Mor}((X,p)(i), (Y,q)(j)) = q \circ \mathrm{CH}^{\dim X + i - j}(X \times Y) \circ p.$$

2.2 (Product of cellular varieties). Let G be a split linear algebraic group over a field F. Let X be a projective G-homogeneous variety, i.e., $X = G/P$, where P is a parabolic subgroup of G. The abelian group structure of $\mathrm{CH}(X)$, as well as its ring structure, is well-known. Namely, X has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the free additive generators of $\mathrm{CH}(X)$ (see [Ka01]). Note that the product of two projective homogeneous varieties $X \times Y$ has a cellular filtration as well, and $\mathrm{CH}^*(X \times Y) \cong \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y)$ as graded rings. The correspondence product of two cycles $\alpha = f_\alpha \times g_\alpha \in \mathrm{CH}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \mathrm{CH}(Y \times X)$ is given by

$$(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \deg(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta), \tag{3}$$

where $\deg : \mathrm{CH}(Y) \to \mathrm{CH}(\{pt\}) = \mathbb{Z}$ is the degree map.

2.3 (Rational cycles). Let X be a projective variety of dimension n over F. Let F_s denote the separable closure of F. Consider the scalar extension $X_s = X \times_F F_s$. We say a cycle $J \in \mathrm{CH}(X_s)$ is *rational* if it lies in the image of the pull-back homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(X_s)$. For instance, there is an obvious rational cycle Δ_{X_s} on $\mathrm{CH}^n(X_s \times X_s)$ that is given by the diagonal class.

Let E be a vector bundle over X. Then the total Chern class $c(E_s)$ of the pull-back induced by the scalar extension F_s/F is rational. Let L/F be a finite separable field extension which splits X, i.e., there is an induced isomorphism $\mathrm{CH}(X_L) \xrightarrow{\simeq} \mathrm{CH}(X_s)$. Then the cycle $\deg(L/F) \cdot J$, $J \in \mathrm{CH}(X_s)$, is rational. Observe that all linear combinations, intersections and correspondence products of rational cycles are rational.

2.4 (Rost nilpotence). We will use the following fact (see [CGM05, Cor. 8.3]) that follows from the Rost Nilpotence Theorem. Let X be a twisted flag G-variety for a semisimple group G of inner type over F. Let p_s be a non-trivial rational projector in $\mathrm{CH}^n(X_s \times X_s)$, i.e., $p_s \circ p_s = p_s$. Then there exists a non-trivial projector p on $\mathrm{CH}^n(X \times X)$ such that $p \times_F F_s = p_s$. Hence, existence of a non-trivial rational projector p_s on $\mathrm{CH}^n(X_s \times X_s)$ gives rise to the decomposition of the Chow motive of X

$$\mathcal{M}(X) \cong (X, p) \oplus (X, \Delta_X - p) \tag{4}$$

2.5 (Tautological and quotient bundles). Let A be a central simple algebra of degree n over F. Consider a generalized Severi-Brauer variety

$\mathrm{SB}_d(A)$ of ideals of reduced dimension d of A. Over the separable closure it becomes isomorphic to the Grassmannian $\mathbb{G}_d(n)$ of d-dimensional planes in a n-dimensional affine space.

There is a tautological vector bundle over $\mathrm{SB}_d(A)$ of rank dn denoted by τ_d^A and given by the ideals of A of reduced dimension d considered as vector spaces over F. Over the separable closure F_s this bundle becomes isomorphic to $Hom(\mathcal{E}_n, \tau_d) = \tau_d^{\oplus n}$, where $\mathcal{E}_n$ denotes the trivial bundle of rank n and τ_d the tautological bundle over the Grassmannian $\mathbb{G}_d(n)$.

The universal quotient bundle over $\mathrm{SB}_d(A)$, denoted by κ_d^A, is the quotient $\mathcal{E}_A/\tau_d^A$ of a trivial bundle $\mathcal{E}_A$ of rank n^2 modulo the tautological bundle τ_d^A. Clearly, κ_d^A is of rank $(n-d)n$. Over F_s this bundle becomes isomorphic to $Hom(\mathcal{E}_n, \kappa_d) = \kappa_d^{\oplus n}$, where κ_d is the universal quotient bundle over $\mathbb{G}_d(n)$.

We will extensively use the following fact

2.6 Lemma. *Let A be a central simple algebra over F, r a positive integer and B a division algebra which represents the class of the r-th tensor power of A in the Brauer group. Let $X = \mathrm{SB}(A) \times \mathrm{SB}_d(B^{\mathrm{op}})$. Then the bundle $T_s = \mathrm{pr}_1^*(\tau_1^{\otimes r}) \otimes \mathrm{pr}_2^*(\tau_d)$ over X_s is a pull-back of some bundle T over X. As a consequence, any Chern class of T_s is a rational cycle.*

Proof. According to [Pa94, 10.2] the image of the restriction map on K_0

$$K_0(\mathrm{SB}(A) \times \mathrm{SB}_d(B^{\mathrm{op}})) \to K_0(\mathbb{P}^{n-1} \times \mathbb{G}_d(n))$$

is generated by classes of bundles $\mathrm{ind}(A^i \otimes B^{-j}) \cdot [\mathrm{pr}_1^*(\tau_1^{\otimes i}) \otimes \mathrm{pr}_2^*(\tau_d^{\otimes j})]$. □

2.7 Remark. Observe that the similar fact holds if one replaces the tensor power by an exterior (lambda) power. This is due to the fact that $[\Lambda^r A] = [A^{\otimes r}]$ in $\mathrm{Br}(F)$ (see [KMRT98, 10.A.]).

2.8 (Poincaré polynomial). By [Fu97] the Poincaré polynomial of a Chow group of a Grassmannian $\mathbb{G}_d(n)$ is given by the Gaussian polynomial

$$P(\mathbb{G}_d(n), t) = \binom{n}{d}(t) = \frac{(1-t^n)(1-t^{n-1})\dots(1-t^{n-d+1})}{(1-t)(1-t^2)\dots(1-t^d)}.$$

Observe that for a projective space, i.e., for $d = 1$, the respective polynomial takes the most simple form

$$P(\mathbb{P}^{n-1}, t) = \frac{1-t^n}{1-t}.$$

Observe also that if n is a prime integer, then the polynomial $P(\mathbb{P}^{n-1}, t)$ always divides $P(\mathbb{G}_d(n), t)$.

3 The subgroup of rational cycles

The goal of the present section is to provide an explicit set of generators for the image of the restriction map $\mathrm{CH}(X) \to \mathrm{CH}(X_s)$ modulo n, where X is a product of a Severi-Brauer variety by a generalized Severi-Brauer variety corresponding to division algebras of a prime degree n.

3.1 (Grassmann bundle structure). Let A be a division algebra of degree n over F, r a positive integer and B a division algebra which represents the class of the r-th tensor power of A in $\mathrm{Br}(F)$. According to [IK00, Proposition 4.3] the product $\mathrm{SB}(A) \times \mathrm{SB}_d((A^{\otimes r})^{\mathrm{op}})$ can be identified with the Grassmann bundle $\mathbb{G}_d(\mathcal{V})$ over $\mathrm{SB}(A)$, where $\mathcal{V} = (\tau_1^A)^{\otimes r}$ is a locally free sheaf of (right) $A^{\otimes r}$-modules. By Morita equivalence we may replace $A^{\otimes r}$ by B and, hence, obtain that the product $X = \mathrm{SB}(A) \times \mathrm{SB}_d(B^{\mathrm{op}})$ is the Grassmann bundle $\mathbb{G}_d(\mathcal{W})$ over $\mathrm{SB}(A)$ for a locally free sheaf of (right) B-modules $\mathcal{W}$. The tautological bundle T over $\mathbb{G}_d(\mathcal{W})$ is the bundle

$$T = \mathrm{pr}_1^*(\mathcal{W}) \otimes_B \mathrm{pr}_2^*(\tau_B^{\mathrm{op}}).$$

Let $Q = \mathrm{pr}_1^*(\mathcal{W})/T$ denote the universal quotient bundle over $\mathbb{G}_d(\mathcal{W})$. Over the separable closure it can be identified with

$$Q_s = \mathrm{pr}_1^*(\tau_1^{\otimes r}) \otimes \mathrm{pr}_2^*(\kappa_d)$$

where τ_1 and κ_d are the tautological and quotient bundles over $\mathbb{P}^{n-1}$ and $\mathbb{G}_d(n)$ respectively. We shall write this bundle simply as $\tau_1^{\otimes r} \otimes \kappa_d$ meaning the respective pull-backs.

3.2 (Grassmann bundle theorem). According to Grassmann bundle theorem the Chow ring $\mathrm{CH}(X)$ is a free $\mathrm{CH}(\mathrm{SB}(A))$-module with the basis $\Delta_\lambda(c(Q))$, where λ runs through the set of all partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $n - d \geq \lambda_1 \geq \ldots \geq \lambda_d \geq 0$, $c(Q)$ is the total Chern class of the quotient bundle Q and Δ_λ is the Schur function. In other words, for any k we have the decomposition

$$\mathrm{CH}^k(X) \cong \bigoplus_\lambda \Delta_\lambda(c(Q)) \cdot \mathrm{pr}_1^*(\mathrm{CH}^{k-|\lambda|}(\mathrm{SB}(A))) \tag{5}$$

3.3. Observe that the decomposition (5) is compatible with the scalar extension F_s/F. Hence, any rational cycle $\alpha \in \mathrm{CH}^k(X_s)$ can be represented uniquely as the sum of cycles

$$\alpha = \sum_{\lambda} \Delta_\lambda(c(Q_s)) \cdot \alpha_\lambda, \tag{6}$$

where $\alpha_\lambda \in \mathrm{CH}^{k-|\lambda|}(\mathbb{P}^{n-1})$ is rational. If n is a prime integer, then all rational cycles of positive codimensions in $\mathrm{CH}(\mathbb{P}^{n-1})$ are divisible by n (see [Ka95, Corollary 4]). Hence, considering (6) modulo n we obtain

3.4 Lemma. *If n is a prime integer, the cycles $\Delta_\lambda(c(Q_s))$, where λ runs through the set of all partitions, generate the subgroup of rational cycles of $\mathrm{CH}(X_s)$ modulo n. In particular case $d = 1$ the basis of the subgroup of rational cycles of $\mathrm{CH}^k(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})$ modulo n consist of Chern classes $c_k(\tau_1^{\otimes r}\otimes\kappa_1)$, where $k = 0 \dots n-1$.*

4 Applications to Severi-Brauer varieties

The following two lemmas were proven by N. Karpenko (see [Ka96, Theorem 2.2.1] and [Ka00, Criterion 7.1]). In the present section we provide short proofs of these results restricting to algebras of prime degrees.

4.1 Lemma. *The motive of a Severi-Brauer variety of a division algebra A of a prime degree n is indecomposable.*

Proof. Consider the product $X = \mathrm{SB}(A) \times \mathrm{SB}(A)$. It can be identified with the product $\mathrm{SB}(A)\times\mathrm{SB}(B^{\mathrm{op}})$, where $[B] = [A^{n-1}]$. Apply Lemma 3.4 to the case $d = 1$ and $r = n-1$. We obtain that in codimension $k = n-1$ there is only one basis element of the subgroup of rational cycles modulo n

$$\Delta_{n-1} = c_{n-1}(\tau_1^{\otimes n-1} \otimes \kappa_1) = \sum_{i=0}^{n-1} (1-n)^i \cdot H^i \times H^{n-1-i} \in \mathrm{CH}^{n-1}(X_s)$$

which is congruent modulo n to the diagonal cycle $\Delta = \sum_{i=0}^{n-1} H^i \times H^{n-1-i}$.

If the motive of $\mathrm{SB}(A)$ splits, then there must exist a rational projector $p \in \mathrm{CH}^{n-1}(X_s)$ such that $\Delta \pm p$ and p are non-trivial. By composition rule (3) if p is a projector, then $p = \sum_i \pm H^i \times H^{n-1-i}$, where the index i runs through a subset of $\{0 \dots n-1\}$. From the other hand, since p is rational, it must be a multiple of Δ modulo n. So the only possibility for p is to coincide either with $\pm\Delta$ or with 0, contradiction. ∎

4.2 Lemma. *Let A and B be two division algebras of a prime degree n generating the same subgroup in the Brauer group. Then the motives of* $\mathrm{SB}(A)$ *and* $\mathrm{SB}(B^{\mathrm{op}})$ *are isomorphic iff $A = B$ or B^{op}.*

Proof. Take r such that $[B] = [A^{\otimes r}]$ in $\mathrm{Br}(F)$. and apply Lemma 3.4 for $X = \mathrm{SB}(A) \times \mathrm{SB}(B^{\mathrm{op}})$. We obtain that the subgroup of rational cycles of codimension $k = n-1$ is generated by the cycle

$$\Delta_{n-1} = c_{n-1}(\tau_1^{\otimes r} \otimes \kappa_1) = \sum_{i=0}^{n-1} (-r)^i \cdot H^i \times H^{n-1-i} \in \mathrm{CH}^{n-1}(X_s).$$

According to composition rule (3) and Rost nilpotence theorem (see 2.4) any motivic isomorphisms between $\mathrm{SB}(A)$ and $\mathrm{SB}(B^{\mathrm{op}})$ is given by a lifting of a rational cycle of the kind $\sum_{i=0}^{n-1} \pm H^i \times H^{n-1-i} \in \mathrm{CH}^{n-1}(X_s)$ and vice versa. To finish the proof observe that a rational cycle of this kind is a multiple of Δ_{n-1} modulo n iff $r \equiv \pm 1 \mod n$. $\square$

5 Generalized Severi-Brauer varieties

In the present section we prove the main theorem of the paper which is formulated as follows

5.1 Theorem. *Let A and B be two division algebras of a prime degree n generating the same subgroup in the Brauer group. Take an integer r such that $[B] = [A^{\otimes r}]$. Then the motive of a Severi-Brauer variety* $\mathrm{SB}(A)$ *is a direct summand of the motive of a generalized Severi-Brauer variety* $\mathrm{SB}_d(B)$ *if and only if*

$$d \cdot r \equiv \pm 1 \mod n.$$

The cases $d = 1$ and $d = n-1$ were considered in Lemma 4.2. From now on we assume $1 < d < n-1$. Moreover, by duality we may assume $d \leq \left[\frac{n}{2}\right]$.

Proof ($\Rightarrow$) Consider the product $X = \mathrm{SB}(A) \times \mathrm{SB}_d(B^{\mathrm{op}})$. According to Theorem 3.4 the subgroup of rational cycles of $\mathrm{CH}^k(X_s)$ is generated (modulo prime n) by the cycles

$$\Delta_\lambda = \Delta_\lambda(c(\tau_1^{\otimes r} \otimes \kappa_d)),$$

where λ runs through all partitions with $|\lambda| = k$. By [Fu98, Example A.9.1] we have

$$\Delta_\lambda = \sum_{\mu\subset\lambda} d_{\tilde\lambda,\tilde\mu} \cdot c_1(\tau_1^{\otimes r})^{k-|\mu|} \cdot \Delta_\mu(c(\kappa_d)) =$$

$$= \sum_{i=0}^{k} (-r)^{k-i} \cdot H^{k-i} \times \left(\sum_{\substack{\mu\subset\lambda \\ |\mu|=i}} d_{\tilde\lambda,\tilde\mu}\omega_\mu \right), \tag{7}$$

where $\tilde\lambda$ denotes the conjugate partition for λ , i.e., obtained by interchanging rows and columns in the respective Young diagram, H is the class of a hyperplane section of $\mathbb{P}^{n-1}$, ω_μ denotes the additive generator of $\mathrm{CH}^{|\mu|}(\mathbb{G}_d(n))$ corresponding to a partition μ and the coefficients $d_{\tilde\lambda,\tilde\mu}$ are the binomial determinants

$$d_{\tilde\lambda,\tilde\mu} = \left| \binom{\tilde\lambda_i + n - d - i}{\tilde\mu_j + n - d - j} \right|_{1\le i,j\le n-d}$$

Let $k = N$, where $N = d(n-d)$ is the dimension of $\mathbb{G}_d(n)$. In this codimension there is only one partition λ with $|\lambda| = N$, namely, the maximal one $\lambda = (n-d, \ldots, n-d)$. Let g denote the cycle (7) corresponding to this maximal partition, i.e.,

$$g = \sum_{m=0}^{n-1} (-r)^m \cdot H^m \times \left(\sum_{|\mu|=N-m} d_{\tilde\lambda,\tilde\mu}\omega_\mu \right), \tag{8}$$

where $\tilde\lambda = (d, d, \ldots, d)$ and the coefficients $d_{\tilde\lambda,\tilde\mu}$ are given by

$$d_{\tilde\lambda,\tilde\mu} = \left| \binom{n-i}{\tilde\mu_j + n - d - j} \right|_{1\le i,j\le n-d}$$

From now on we denote the coefficient $d_{\tilde\lambda,\tilde\mu}$ by $d_{\mu'}$, where μ' is the dual partition $(n-d-\mu_d, \ldots, n-d-\mu_1)$. Observe that $|\mu'| = N - |\mu|$.

For an integer m denote by $g^{(m)}$ the summand of (8) for the chosen index m and by $d^{(m)}_{\mu'}$ the respective coefficients, i.e.,

$$g^{(m)} = (-r)^m \cdot H^m \times \left(\sum_{|\mu'|=m} d^{(m)}_{\mu'}\omega_\mu \right)$$

Consider the summands $g^{(0)}$ and $g^{(1)}$. Since the Chow group $\mathrm{CH}(\mathbb{G}_d(n))$ has only one additive generator in the last two codimensions N and $N-1$ denoted by ω_N and ω_{N-1} respectively, we obtain that

$$g^{(0)} = 1 \times \omega_N \quad \text{and} \quad g^{(1)} = -rd \cdot H \times \omega_{N-1} \tag{9}$$

Now we are ready to finish the **(⇒)** part of the theorem.

Assume that the motive of $\mathrm{SB}_d(B^{\mathrm{op}})$ contains the motive of $\mathrm{SB}(A)$ as a direct summand. Then there must exist two rational cycles

$$\alpha \in \mathrm{CH}^{n-1}(\mathbb{P}^{n-1} \times \mathbb{G}_d(n)) \quad \text{and} \quad \beta \in \mathrm{CH}^N(\mathbb{G}_d(n) \times \mathbb{P}^{n-1})$$

such that $\beta \circ \alpha = \mathrm{id}$. According to the composition rule (3) this can happen only if the coefficients before the monomials $\omega_N \times 1$ and $\omega_{N-1} \times H$ of the cycle β are equal to ± 1. But all rational cycles in codimension N are generated (modulo n) by the transposed cycle g^t which has coefficients 1 and $-rd$ before the respective monomials (see (9)). This can only be possible if $rd \equiv \pm 1 \mod n$.

Proof (⇐) Assume that the congruence $rd \equiv \pm 1 \mod n$ holds. We want to produce two rational cycles

$$\alpha \in \mathrm{CH}^{n-1}(\mathbb{P}^{n-1} \times \mathbb{G}_d(n)) \quad \text{and} \quad \beta \in \mathrm{CH}^N(\mathbb{G}_d(n) \times \mathbb{P}^{n-1})$$

such that $\beta \circ \alpha \in \mathrm{CH}^{n-1}(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ is the identity morphism, i.e., the diagonal cycle $\sum_{m=0}^{n-1} H^{n-1-m} \times H^m$. This will show that $\mathcal{M}(\mathrm{SB}_d(B^{\mathrm{op}}))$ contains $\mathcal{M}(\mathrm{SB}(A))$ as a direct summand.

Assume $rd \equiv 1 \mod n$. Consider the bundle $\kappa_1 \otimes \Lambda^d(\tau_d)$ of rank $n-1$ on the product $\mathbb{P}^{n-1} \times \mathbb{G}_d(n)$ and define the cycle f as

$$f = c_{n-1}(\kappa_1 \otimes \Lambda^d(\tau_d)) = \sum_{m=0}^{n-1} c_{n-1-m}(\kappa_1) c_1(\tau_d)^m =$$

$$= \sum_{m=0}^{n-1} (-1)^m \cdot H^{n-1-m} \times \omega_1^m = \sum_{m=0}^{n-1} (-1)^m \cdot H^{n-1-m} \times \left(\sum_{|\rho|=m} c_\rho^{(m)} \omega_\rho \right).$$

Observe that f is a rational cycle, since $[A] = [B^{\otimes d}]$. If $rd \equiv -1 \mod n$, then we take the bundle $\Lambda^{n-d}(\kappa_d)$ instead of $\Lambda^d(\tau_d)$ and obtain the same formulae but without the coefficient $(-1)^m$.

The coefficients $c_\rho^{(m)}$ appearing in the presentation of ω_1^m in terms of additive generators ω_ρ of $\mathrm{CH}^m(\mathbb{G}_d(n))$ have the following nice property

5.2 Lemma. *For any partition ρ with $m = |\rho| < n$ the coefficient $c_\rho^{(m)}$ is coprime with n.*

Proof. According to [Fu98, Example 14.7.11.(ii)] we have

$$c_\rho^{(m)} = \deg(a_0, \dots, a_{d-1}) = \frac{m!}{a_0!\, a_1! \dots a_{d-1}!} \prod_{i>j} (a_i - a_j), \tag{10}$$

where the set $a = (a_0, a_1, \dots, a_{d-1})$ is defined by $a_i = \rho_{d-i} + i$, $i = 0, \dots, d-1$. Observe that the set a has the property $0 \le a_0 < \dots < a_{d-1} \le n-1$ and corresponds to a Schubert variety of dimension m which is dual to the Schubert variety of codimension m corresponding to the partition ρ (see [Fu98, 14.7]). ☐

5.3 Remark. The constructed cycle $f \in \mathrm{CH}^{n-1}(\mathbb{P}^{n-1} \times \mathbb{G}_d(n))$ has the following geometric interpretation. Consider the Plücker embedding $\mathrm{SB}_d(B) \to \mathrm{SB}(\Lambda^d(B))$. Its graph defines a correspondence $\Gamma \in \mathrm{CH}(\mathrm{SB}(\Lambda^d(B)) \times \mathrm{SB}_d(B))$. It is known that the motive of $\mathrm{SB}(\Lambda^d(B))$ splits as a direct sum of twisted motives of $\mathrm{SB}(A)$ (see [Ka96, Corollary 1.3.2]). Let $i : \mathcal{M}(SB(A)) \to \mathcal{M}(\mathrm{SB}(\Lambda^d(B)))$ be the respective splitting. Then over the separable closure the composition $\Gamma \circ i$ coincides with the cycle f corresponding to the case $rd \equiv -1 \mod n$. After replacing B by B^{op} the respective composition $\Gamma \circ i$ will give the cycle f corresponding to the case $rd \equiv 1 \mod n$.

Consider the transposed cycle g^t introduced in the first part of the proof

$$g^t = \sum_{m=0}^{n-1} (-1)^m \cdot \left(\sum_{|\mu'|=m} (r^m d_{\mu'}) \cdot \omega_\mu \right) \times H^m$$

Consider the m-th summand of the composition $g^t \circ f$

$$(g^t \circ f)^{(m)} = \left(r^m \sum_{|\mu'|=m} c_{\mu'}^{(m)} \cdot d_{\mu'}^{(m)} \right) \cdot H^{n-1-m} \times H^m$$

Note that if $rd \equiv -1 \mod n$, then the coefficient $(-1)^m$ will appear on the right hand side.

Assume that the following formulae holds

$$\sum_{|\mu'|=m} c_{\mu'}^{(m)} \cdot d_{\mu'}^{(m)} \equiv d^m \mod n. \tag{11}$$

Then the coefficient of $(g^t \circ f)^{(m)}$ is congruent to 1 modulo n. We claim that it is possible to modify the cycles g^t and f by adding cycles divisible by n in such a way that the coefficient of $(g^t \circ f)^{(m)}$ becomes equal to 1 for each m.

First, we modify the cycle f. For each m, $0 \leq m \leq n-1$, we do the following procedure. Consider the m-th summand

$$f^{(m)} = (-1)^m \sum_{|\mu'|=m} c_{\mu'} \cdot H^{n-1-m} \times \omega_{\mu'}$$

and its coefficients $c_{\mu'}^{(m)}$. For $m = 0$ and 1 there is only one additive generator of $\mathrm{CH}^m(\mathbb{G}_d(n))$ (ω_0 and ω_1) and the respective coefficients are $c^{(0)} = c^{(1)} = 1$. So we set $\alpha^{(0)} = f^{(0)}$ and $\alpha^{(1)} = f^{(1)}$. For $1 < m \leq n-1$ the number of generators of $\mathrm{CH}^m(\mathbb{G}_d(n))$ is greater than 1 and all the coefficients $c_{\mu'}^{(m)}$ are coprime with n, in particular, they are all non-zero. In this case we can modify each $c_{\mu'}^{(m)}$ modulo n by adding cycles of the kind $a \cdot n\omega_{\mu'}$, $a \in \mathbb{Z}$, to the cycle $f^{(m)}$ in such a way that the greatest common divisor of resulting coefficients, denoted by $\alpha_{\mu'}^{(m)}$, becomes equal to 1. As a result, we obtain a new cycle $\alpha^{(m)}$ having the coefficients $\alpha_{\mu'}$ instead of $c_{\mu'}$.

5.4 Definition. Define a cycle α as $\alpha = \sum_{m=0}^{n-1} \alpha^{(m)}$. By construction of $\alpha^{(m)}$ we have

- $\alpha^{(0)} = \alpha^{(1)} = 1$;
- α is rational (congruent modulo n to f);
- all the coefficients $\alpha_{\mu'}$ are coprime with n;
- for each m the g.c.d of coefficients $\alpha_{\mu'}^{(m)}$ is 1;
- for each m the coefficient of $(g^t \circ \alpha)^{(m)}$ is congruent to 1 modulo n.

Next we modify the second cycle g^t. For each m we apply the following obvious observation

5.5 Lemma. *Let $a_1, \ldots, a_l$ be a finite set of integers with g.c.d. $= 1$. Assume $\sum_i a_i b_i \equiv 1 \mod n$ for some integers b_i. Then there exist integers $b'_i \equiv b_i \mod n$ such that $\sum_i a_i b'_i = 1$.*

to the congruence

$$\sum_{|\mu'|=m} c_{\mu'}^{(m)}(r^m d_{\mu'}^{(m)}) \equiv 1 \mod n,$$

where $a_i = \alpha_{\mu'}^{(m)}$ are coefficients of the cycle $\alpha^{(m)}$ and $b_i = r^m d_{\mu'}^{(m)}$ are coefficients of the cycle $(g^t)^{(m)}$. As a result, we obtain a new cycle $\beta^{(m)}$ having coefficients b_i' instead of b_i.

5.6 Definition. Define a cycle β as $\beta = \sum_{m=0}^{n-1} \beta^{(m)}$. By construction of $\beta^{(m)}$ we have

- β is rational (congruent modulo n to g^t)
- for each m the coefficient of $(\beta \circ \alpha)^{(m)}$ is equal to 1.

To finish the proof observe that the constructed cycles α and β are rational and the composition $\beta \circ \alpha$ is the diagonal cycle. This implies that the composition $\alpha \circ \beta$ is a rational projector which gives rise to the decomposition of motives with integral coefficients (see (4))

$$\mathcal{M}(\mathrm{SB}_d(B^{\mathrm{op}})) \simeq \mathcal{M}(\mathrm{SB}(A)) \oplus H$$

for some motive H.

6 The proof of (11)

In the present section we prove the following

6.1 Lemma.

$$\sum_{|\mu'|=m} c_{\mu'}^{(m)} \cdot d_{\mu'}^{(m)} \equiv d^m \mod n,$$

Proof. First, we express the coefficients $c_{\mu'}$ and $d_{\mu'}$ in terms of binomial determinants. According to [Fu98, Example 14.7.11] we have

$$c_{\mu'} = \deg(\omega_1^m \cdot \omega_\mu) = \frac{m! \cdot b_0!\, b_1! \dots b_{d-1}!}{a_0!\, a_1! \dots a_{d-1}!} \cdot \left| \binom{a_i}{b_j} \right|_{0 \le i,j \le d-1} \tag{12}$$

where $a_i = n{-}d{+}i{-}\mu_{i+1}$ and $b_j = j$ for $i, j = 0 \dots d{-}1$. Observe that the sets $a = (a_0, \dots, a_{d-1})$ and $b = (b_0, \dots, b_{d-1})$ with $0 \le a_0 < \dots < a_{d-1} \le n{-}1$ and

$0 \le b_0 < \ldots < b_{d-1} \le n-1$ correspond to the classes of Schubert varieties ω_μ and $\omega_N = \{pt\}$ of dimensions $\sum_{i=0}^{d-1}(a_i - i) = m$ and $\sum_{j=0}^{d-1}(b_j - j) = 0$ respectively.

From another hand side, we have

$$d_{\mu'} = \left| \binom{n-i}{\tilde{\mu}_j + n - d - j} \right|_{1 \le i.j \le n-d} = \left| \binom{\tilde{a}_i}{\tilde{b}_j} \right|_{0 \le i,j, \le n-d-1}$$

where $\tilde{a}_i = d + i$ and $\tilde{b}_j = \tilde{\mu}_{n-d-j} + j$ for $i, j = 0 \ldots n-d-1$. Here the sets $\tilde{a} = (\tilde{a}_0, \ldots, \tilde{a}_{n-d-1})$ and $\tilde{b} = (\tilde{b}_0, \ldots, \tilde{b}_{n-d-1})$ with $0 \le \tilde{a}_0 < \ldots \tilde{a}_{n-d-1} \le n-1$ and $0 \le \tilde{b}_0 < \ldots < \tilde{b}_{n-d-1} \le n-1$ correspond to the classes of Schubert varieties on the dual Grassmannian $\mathbb{G}_{n-d}(n)$ of dimensions N and $N-m$ respectively. Observe that the Schubert variety corresponding to the set $\tilde{b}$ is dual to the Schubert variety corresponding to the partition $\tilde{\mu}$. Hence, we have

$$\omega_1^m = \frac{m! \cdot \tilde{b}_0! \, \tilde{b}_1! \ldots \tilde{b}_{n-d-1}!}{\tilde{a}_0! \, \tilde{a}_1! \ldots \tilde{a}_{n-d-1}!} d_{\mu'} \cdot \omega_{\tilde{b}}$$

And by duality we obtain

$$\frac{m! \cdot \tilde{b}_0! \, \tilde{b}_1! \ldots \tilde{b}_{n-d-1}!}{\tilde{a}_0! \, \tilde{a}_1! \ldots \tilde{a}_{n-d-1}!} d_{\mu'} = \frac{m! \cdot b_0! \, b_1! \ldots b_{d-1}!}{a_0! \, a_1! \ldots a_{d-1}!} \cdot \left| \binom{a_i}{b_j} \right|_{0 \le i,j \le d-1}$$

Since the integers $a_0, \ldots, a_{d-1}$ form the complement of $\tilde{b}_0, \ldots, \tilde{b}_{n-d-1}$ in the set of integers from 0 to $n-1$ (see [Fu98, Example 14.7.5]), we obtain

$$d_{\mu'} = \left| \binom{a_i}{b_j} \right|_{0 \le i,j \le d-1} \tag{13}$$

According to [Fu98, Example 14.7.11] we express the binomial determinant appearing in (12) and (13) in terms of Vandermonde determinant $D_a = \prod_{i<j}(a_j - a_i)$ and get

$$c_{\mu'} = \frac{m! \cdot D_a}{a_0! \, a_1! \ldots a_{d-1}!}, \qquad d_{\mu'} = \frac{D_a}{0! \, 1! \ldots (d-1)!}. \tag{14}$$

Then the formulae (11) we want to prove turns into

$$\frac{m!}{0! \, 1! \ldots (d-1)!} \cdot \sum_a \frac{D_a^2}{a_0! a_1! \ldots a_{d-1}!} \equiv d^m \mod n, \tag{15}$$

where the sum is taken over all sets of integers $a = (a_0, \ldots, a_{d-1})$ such that $0 \le a_0 < \ldots < a_{d-1} \le n-1$ and $\sum_{i=0}^{d-1} a_i = m + \frac{d(d-1)}{2}$.

The case $d = 2$. In this case (15) follows from the following elementary fact

$$\frac{m!}{2} \cdot \sum_{\substack{0 \le x_1, x_2 \\ x_1 + x_2 = m+1}} \frac{(x_1 - x_2)^2}{x_1! x_2!} = 2^m \tag{16}$$

To prove it consider the following chain of identities

$$\frac{m! \cdot (x_1 - x_2)^2}{2 \cdot x_1! x_2!} = \frac{(m+1)!}{x_1!(m+1-x_1)!} \cdot \left(\frac{m+1}{2} - \frac{2x_1(m+1-x_1)}{m+1} \right) =$$

$$= \frac{m+1}{2}\binom{m+1}{x_1} - 2m\binom{m-1}{x_1 - 1}$$

Then taking the sum we obtain the desired equality

$$\sum_{x_1=0}^{m+1} \frac{m+1}{2}\binom{m+1}{x_1} - 2m\binom{m-1}{x_1-1} = \frac{m+1}{2} \cdot 2^{m+1} - 2m \cdot 2^{m-1} = 2^m$$

Observe that for $m < n - 1$ the left hand side of (15) coincides with the left hand side of (16), hence, we obtain the equality in (15) (not just a congruence modulo n). For $m = n - 1$ the left hand side of (15) is equal to $2^{n-1} - n$.

The general case. It turns out that the identity (16) is a particular case of the following combinatorial identity known as Robinson-Schensted correspondence (see [Fu97, 4.3.(5)])

$$\sum_{|\xi| = m} d_\xi(d) \cdot f^\xi = d^m, \tag{17}$$

where the sum is taken over all partitions $\xi = (\xi_1 \ge \xi_2 \ge \ldots \ge \xi_d \ge 0)$ with $|\xi| = m$, $d_\xi(d)$ denote the number of Young tableaux on the shape ξ whose entries are taken from the set $(1, \ldots, d)$ and f^ξ denote the number of standard tableaux on the shape ξ.

By using Hook length formulae (see [Fu97, 4.3, Exercise 9]) we obtain

$$f^\xi = \frac{m! \cdot D_l}{l_0! \ldots l_{d-1}!}, \tag{18}$$

where $l = (l_0, \ldots, l_{d-1})$ is a strictly increasing set of non-negative integers defined from the partition $\xi = (\xi_1, \ldots, \xi_d)$ by $l_{d-i} = \xi_i + d - i$ and D_l is the Vandermonde determinant for l. By definition we have $\sum_{i=0}^{d-1}(l_i - i) = m$.

By [GV85, Corollary 13] we have

$$d_\xi(d) = \left| \binom{l_i}{j} \right|_{0 \le i,j, \le d-1} = \frac{D_l}{0!\,1! \ldots (d-1)!} \tag{19}$$

Observe that if the set l is bounded by $n-1$, i.e., $l_{d-1} \le n-1$, the expressions (18) and (19) coincide with the expressions (14) defining the coefficients $c_{\mu'}$ and $d_{\mu'}$ respectively (take $\xi = \mu'$ and $a = l$).

Assume that $l_{d-1} \ge n$. Then $l_{d-2} \le n-1$, i.e., l_{d-1} is the only element of l which exceeds $n-1$. Indeed, if this is not the case then we have a sequence of inequalities

$$n - 1 + \tfrac{d(d-1)}{2} \ge m + \tfrac{d(d-1)}{2} = 0 + 1 + \ldots + l_{d-3} + l_{d-2} + l_{d-1} \ge$$

$$\ge 0 + 1 + \ldots + (d-3) + n + (n+1) = \tfrac{(d-3)(d-2)}{2} + 2n + 1$$

which can be rewritten as

$$d \ge \frac{n+5}{2}.$$

But we have assumed from the beginning that $d \le \left[\frac{n}{2}\right]$, contradiction.

Moreover, by the similar arguments one can check that $l_{d-1} < 2n$.

Now consider the product $x = d_\xi(d) \cdot f^\xi$, when ξ is a partition for which the respective $l_{d-1} \ge n$. Since n is prime, the denominator of x is divisible by n but not by n^2. Since x is an integer, the numerator of x must be divisible by n as well. From this we conclude that D_l is divisible by n. But the numerator of x is the product of the square of D_l by something, hence, it is divisible by n^2. So we obtain that x must be divisible by n. This means that modulo n the left hand side of (17) is congruent to the left hand side of (15). $\square$

7 Grassmannian $\mathbb{G}(2, n)$

The goal of the present is to extend Theorem 1.1 to the case of algebras of an arbitrary odd degree $n \ge 5$ and $d = 2$

7.1 Theorem. *Let A and B be two central simple algebras of an odd degree n over a field F generating the same subgroup in the Brauer group* $\mathrm{Br}(F)$. *Then the motive of a Severi-Brauer variety* $\mathrm{SB}(A)$ *is a direct summand of the motive of a generalized Severi-Brauer variety* $\mathrm{SB}_2(B)$ *if and only if*

$$[A] = \pm 2\,[B] \text{ in } \mathrm{Br}(F).$$

Proof Consider the cycles g and f defined in Section 5. Clearly, g and f are rational, since Lemma 2.6 holds for any algebras.

($\Rightarrow$) Repeating the arguments of 3.3 for an odd integer n and $d = 2$ one obtains (in view of [Ka95, Corollary 4]) that the subgroup of rational cycles of $\mathrm{CH}(X_s)$ modulo n is generated in codimension $N = \dim \mathbb{G}_d(n)$ by the cycles

$$\tfrac{n}{(n,N-|\lambda|)} \cdot H^{N-|\lambda|} \times 1 \cdot \Delta_\lambda(c(Q_s)),$$

for all partitions λ with $N - (n-1) \le |\lambda| \le N$. Observe that in the case $|\lambda| = N$ one obtains precisely the cycle g.

Since the coefficient $\frac{n}{(n,N-|\lambda|)}$ is divisible by n when $|\lambda| = N-1$, we may exclude the cycles with $|\lambda| = N-1$ from the set of generators. The latter implies that the last two non-trivial monomials (see (9)) of any rational cycle in $\mathrm{CH}^N(X_s)$ come only from the cycle g. And we finish the proof as in Section 5.

($\Leftarrow$) We claim that it is still possible to modify cycles f and g^t modulo n in such a way that the obtained rational cycles α and β will satisfy the property $\beta \circ \alpha = \mathrm{id}$, hence, providing the desired motivic decomposition for $\mathcal{M}(\mathrm{SB}_2(B))$.

The following lemma allows us to take $\alpha = f$ (see Definition 5.4).

7.2 Lemma. *Fix a codimension m, $0 \le m \le n-1$. For any partition ρ with $|\rho| = m$ let $c_\rho^{(m)}$ be the coefficient appearing in Lemma 5.2. Then the greatest common divisor of all the coefficients $c_\rho^{(m)}$ is* 1.

Proof. Since $d = 2$, for any codimension m which is less than $n-1$, there is a partition $\rho = (m, 0)$ with $c_\rho^{(m)} = 1$. In the last codimension $m = n-1$ consider two partitions $\rho = (1, n-2)$ and $\rho' = (2, n-3)$. By (10) the respective coefficients $c_\rho^{(n-1)}$ and $c_{\rho'}^{(n-1)}$ are equal to $\deg(1, n-1) = n-2$ and $\deg(2, n-2) = \frac{(n-1)(n-4)}{2}$. Since n is odd, $n-2$ and $\frac{(n-1)(n-4)}{2}$ are coprime. $\square$

Choose β as in the Section 5. To finish the proof we have to prove the congruence (11). But this was done already in Section 6, where we treated the case $d = 2$. Namely, we proved that for $m < n - 1$ there is an exact equality (not just a congruence modulo n) and for $m = n-1$ the left hand side of (11) is, indeed, equal to $2^{n-1} - n$. This finishes the proof of Theorem 7.1.

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