CONNECTEDNESS OF CLASSES OF FIELDS AND ZERO CYCLES ON PROJECTIVE HOMOGENEOUS VARIETIES

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1. Introduction

Let X be a proper scheme of finite type over a field F. A *zero-cycle* on X is the formal sum $\sum n_i[x_i]$ where $n_i \in \mathbb{Z}$ and x_i are closed (zero-dimensional) points of the variety X. The factor group of the group of zero-cycles modulo rational equivalence is called *Chow group of dimension zero* and is denoted by $\mathrm{CH}_0(X)$. The assignment $x \mapsto \deg(x)$ extends to the *degree homomorphism*

$$\deg : \mathrm{CH}_0(X) \to \mathbb{Z}.$$

The image of deg coincides with $n(X)\mathbb{Z}$ where $n(X)$ is the greatest common divisor of the degrees $\deg(x) = [F(x) : F]$ over all closed points $x \in X$. We denote the kernel of deg by $\overline{\mathrm{CH}}_0(X)$.

The main purpose of the paper is to present a characteristic free uniform method of computing the group $\overline{\mathrm{CH}}_0(X)$ for projective homogeneous varieties of semisimple algebraic groups. The method is based on the idea of parametrization of fields over which X has a point. We illustrate the method by proving that in many cases the group $\overline{\mathrm{CH}}_0(X)$ is trivial and give examples of varieties when this group is not trivial. The main results of the paper can be summarized as follows.

Let X be a scheme over F. We denote by $\mathcal{A}(X)$ the class of all field extensions L/F such that $X(L) \neq \emptyset$. We say that two fields $L_0, L_1 \in \mathcal{A}(X)$ of the same degree n over F are simply X-equivalent if they are members of a continuous family of fields $L_t \in \mathcal{A}(X)$, $t \in \mathbb{A}^1$, of degree n over F (for precise definition see Section 6). We say that L and L' are X-equivalent if they can be connected by a chain of fields $L_0 = L, L_1, \dots, L_r = L'$ such that L_i and L_{i+1} are simply X-equivalent for $i = 0, \dots, r-1$. Furthermore we say that the class $\mathcal{A}(X)$ is *connected* if every two fields in $\mathcal{A}(X_E)$ of degree $n(X_E)$ over E are X_E-equivalent over any special extension E/F (see Section 6).

Our first result (Theorem 6.5) asserts that if X is an arbitrary proper scheme over F such that the class $\mathcal{A}(X)$ is connected and $\overline{\mathrm{CH}}_0(X_L) = 0$ for any field $L \in \mathcal{A}(X)$, then $\overline{\mathrm{CH}}_0(X) = 0$. Note that the condition $\overline{\mathrm{CH}}_0(X_L) = 0$ always holds for projective homogeneous varieties X. Thus the connectedness of the class $\mathcal{A}(X)$ for such X implies $\overline{\mathrm{CH}}_0(X) = 0$.

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We prove the connectedness of $\mathcal{A}(X)$ for various classes of projective homogeneous varieties. These include: Severi-Brauer varieties, certain generalized Severi-Brauer varieties, quadrics, involution varieties, projective homogeneous varieties related to groups of exceptional types ${}^{3,6}\mathsf{D}_4, \mathsf{G}_2, \mathsf{F}_4, {}^{1,2}\mathsf{E}_6, \mathsf{E}_7$ with trivial Tits algebras. As an application we get that $\overline{\mathrm{CH}}_0(X) = 0$, i.e. the Chow group $\mathrm{CH}_0(X)$ is infinite cyclic for all above mentioned varieties. We borrow from [11] the idea of using symplectic involutions in the case of generalized Severi-Brauer varieties and involution varieties.

Some of our results were known before under certain restrictions on characteristic of the ground field F. Triviality of $\overline{\mathrm{CH}}_0(X)$ in the case of the Severi-Brauer variety $X = SB(A)$ was proven by Panin in [15] if $\operatorname{char} F$ does not divide $\operatorname{ind}(A)$. Quadrics over fields of characteristic $\neq 2$ were considered by Swan [27] and Karpenko [8]. The cases of certain generalized Severi-Brauer varieties and involution varieties were treated by Krashen in [11] under the assumption $\operatorname{char} F = 0$. Involution varieties of algebras of index at most 2 were considered in [13] by the second author under assumption $\operatorname{char} F \neq 2$. Petrov, Semenov and Zainoulline [19], independently, have recently shown that $\overline{\mathrm{CH}}_0(X) = 0$ for projective homogeneous varieties X related to groups of types $\mathsf{G}_2, \mathsf{F}_2, {}^1\mathsf{E}_6$ over fields of characteristic 0 with trivial Tits algebras.

The notion of X-equivalence used in the paper as the main technical tool is formulated in terms of field extensions and discrete valuations, so that we avoid symmetric power constructions used in [11] to describe closed points. Flexibility of the notion of X-equivalence allows us not to impose any characteristic restrictions on F (except for the trialitarian D_4 and E_6, E_7 where we assume that characteristic is $\neq 2, 3$). Another advantage of our method is its transparency and shortness. Even for those varieties where the result was already known our proofs are simpler.

Finally we remark that most likely our results on algebraic groups of exceptional types are close to optimal. It looks hopeless to weaken restrictions on Tits algebras and prove that $\mathrm{CH}_0(X)$ is infinite cyclic for larger classes of projective homogeneous varieties of exceptional groups.

At the end of the paper we give two examples of projective homogeneous varieties X with $\overline{\mathrm{CH}}_0(X) \neq 0$ related to algebraic groups of types $\mathsf{A}_1+\mathsf{A}_1+\mathsf{A}_1$ and B_3 with nontrivial Tits algebras. Note that our first result is minimal possible since $\overline{\mathrm{CH}}_0(X) = 0$ for all projective homogeneous varieties X of dimension at most 2 (see Proposition 4.5).

2. Preliminary facts on algebraic groups

2.1. **Parabolic subgroups.** Let G be a simple simply connected algebraic group over a field F. Fix a maximal torus $T \subset G$ over F and a basis Δ of the root system $\Sigma = \Sigma(G, T)$ of G with respect to T. Recall that for each subset $S \subset \Delta$ one can associate the parabolic subgroup P_S in G, whose semisimple part is generated by the corresponding root subgroups $U_{\pm\alpha}$ of G for all roots $\alpha \in S$. It is defined over a separable closure F_{sep} of F and is called the *standard*

parabolic subgroup of type S. Every parabolic subgroup P in G over F_{sep} is conjugate to a unique standard parabolic subgroup P_S. We say that P *is of type* S.

Let X_S be the variety of all parabolic subgroups of G of type S. If S is stable with respect to the so-called $*$-action of the Galois group $\mathrm{Gal}\,(F_{sep}/F)$ (see [28]), the variety X_S is defined over F. It has a rational point if and only if G contains an F-defined parabolic subgroup of type S. If $X_S(F) \neq \emptyset$, then we say that X_S is *isotropic*.

2.2. **Tits indices.** Recall [28] that given G one can associate a geometric picture called the *Tits index* of G. It consists of the corresponding Dynkin diagram of G with some vertices circled. The set S_0 of uncircled vertices corresponds to an F-subgroup of G called a *semisimple anisotropic kernel*. This subgroup is the semisimple part of the centralizer $C_G(T_1)$ where $T_1 \subset G$ is a maximal F-split subtorus in G. All maximal F-split subtori are conjugate over F. Hence a semisimple anisotropic kernel of G is defined uniquely up to conjugation.

All vertices in Tits index of G are circled if and only if G is quasi-split over F and none of them is circled if and only if G is F-anisotropic. The last occurs if and only if G contains no nontrivial F-split subtorus.

Let S be stable with respect to the $*$-action of $\mathrm{Gal}\,(F_{sep}/F)$. Then the variety X_S of parabolic subgroups in G of type S is defined over F and it has a rational point if and only if S contains the subset S_0 of all uncircled vertices. Hence an F-defined parabolic subgroup $P \subset G$ of type S is minimal if and only if $S = S_0$; in particular all minimal parabolic subgroups in G have the same type and they are conjugate over F.

2.3. **Basic types.** We say that S is a *basic* type for G if there is a form H of G over a field extension L/F such that H is L-isotropic and its minimal parabolic subgroups have type S. In other words, a type S is basic if in the classification table [28] of Tits indices there is one whose set of all uncircled vertices coincides with S.

3. Strongly inner forms

One knows that G is a form of a quasi-split simply connected group G_0, i.e. there exists a cocycle $\xi \in Z^1(F, \mathrm{Aut}\, G_0)$ such that G is isomorphic over F to the twisted group ${}^{\xi}G_0$. One says that G is of the same *inner type* as G_0 or an *inner form* of G_0 if the class $[\xi]$ is contained in $\mathrm{Im}\,[H^1(F, \overline{G}_0) \to H^1(F, \mathrm{Aut}\, G_0)]$ where $\overline{G}_0 = G_0/Z$ is the corresponding adjoint group and $Z \subset G_0$ is the center of G_0. Otherwise G is an *outer form* of G_0. One says that G is a *strongly inner form* of G_0 if $[\xi]$ is contained in $\mathrm{Im}\,[H^1(F, G_0) \to H^1(F, \mathrm{Aut}\, G_0)]$.

Remark 3.1. The notion of strongly inner forms was first introduced by Tits in [30].

For future reference we need some structure facts on strongly inner forms of a quasi-split simple simply connected group G_0. In what follows $\xi \in Z^1(F, G_0)$ and $G = {}^{\xi}G_0$ is the corresponding twisted group.

3.1. **Quasi-splitness criterion.**

Lemma 3.2. *G is quasi-split if and only if $[\xi] = 1$.*

Proof. Let G be quasi-split. Since $H^1(F, \operatorname{Aut} G_0)$ classifies all forms of G_0,

$$[\xi] \in \operatorname{Ker}[H^1(F, G_0) \to H^1(F, \operatorname{Aut} G_0)].$$

The group $\operatorname{Aut} G_0$ is the semi-direct product of the adjoint group $\overline{G}_0$ and the automorphism group $\operatorname{Aut} D$ of the Dynkin diagram D of G_0. It follows that $\operatorname{Aut} G_0(F) \to \operatorname{Aut} D(F)$ is surjective, hence the mapping $H^1(F, \overline{G}_0) \to H^1(F, \operatorname{Aut} G_0)$ has trivial kernel. This implies that $[\xi]$ is contained in the kernel of $H^1(F, G_0) \to H^1(F, \overline{G}_0)$. Consider now an exact sequence $1 \to Z \to G_0 \to \overline{G}_0 \to 1$. It gives rise to an exact sequence

$$H^1(F, Z) \to H^1(F, G_0) \to H^1(F, \overline{G}_0).$$

It follows that ξ up to equivalence takes values in Z. As G_0 is simply connected and quasi-split, it contains a maximal quasi-split torus $T \subset G_0$. Since $Z \subset T$ and $H^1(F, T) = 1$, by Hilbert Theorem 90, the result follows. □

3.2. **Application of Steinberg's Theorem.** Let $C_1 \subset G$ be a maximal F-split torus. The centralizer $C_G(C_1)$ is a reductive group over F whose central torus C contains C_1. Note that if G_0 is split, we have $C = C_1$. However in general case $C \neq C_1$. The semisimple part $H = [C_G(C_1), C_G(C_1)]$ of $C_G(C_1)$ is an F-anisotropic simply connected group, not necessarily simple. Its Tits index coincides with the subindex of G generated by uncircled vertices. The group $C_G(C_1)$ is an almost direct product of H and C, i.e $H \cap C$ is a finite central subgroup of H and $C_G(S_1) = H \cdot C$.

Theorem 3.3. *Assume that $H^1(F, C/H \cap C) = 1$. Then H is a strongly inner form of a quasi-split simply connected group H_0 over F.*

Proof. Let $C_2 \subset H$ be a maximal torus over F and let $T = C \cdot C_2$. By dimension count, T is a maximal torus in G over F. It easily follows from arguments contained in [25, § 10] (see also [20, Propositions 6.18, 6.19, p. 338–339]) that there is an F-embedding $\phi : T \to G_0$ such that the class $[\xi]$ is contained in $\operatorname{Im}[H^1(F, T) \to H^1(F, G_0)]$. Since $C \cap H = C \cap C_2$ and $H^1(F, C/C \cap H) = 1$, the mapping $H^1(F, C_2) \to H^1(F, T)$ is surjective. Thus, up to equivalence, we may assume that ξ takes values in C_2.

The centralizer $C_{G_0}(C_1)$ is a reductive F-subgroup in G_0 and from the construction of ϕ (see the proof of in [20, Proposition 6.18, p. 339]) it follows that $C_{G_0}(C_1)$ and $C_G(C_1)$ are isomorphic over F_{sep}. Hence their semisimple parts H_0 and H are isomorphic over F_{sep}. As all maximal F-split tori in G_0 are conjugate, $C_{G_0}(C_1)$ (and hence $H_0 = [C_{G_0}(C_1), C_{G_0}(C_1)]$) is a quasi-split group. By construction H_0 contains C_2, hence ξ can be viewed as an element in $Z^1(F, H_0)$ and then clearly we have $H \simeq {}^{\xi}H_0$. □

3.3. **Groups isotropic over a quadratic extension.** Assume that F is infinite and perfect. Let L/F be a quadratic extension making G isotropic. We denote the generator of $\mathrm{Gal}(L/F)$ by σ. If $P \subset G$ is an L-defined standard parabolic subgroup of type $S \subset \Delta$ we denote by P^- the opposite parabolic subgroup (it is generated by root subgroups $U_{-\alpha}$, $\alpha \in \Delta \setminus S$, and $U_{\pm\alpha}$, $\alpha \in S$). The class $\mathcal{P}$ of parabolic subgroups conjugate to P is called self-opposite if it contains P^-. Clearly, $P \cap P^-$ is a reductive part of P.

Lemma 3.4. *If $\mathcal{P}$ is self-opposite, then it contains an L-parabolic subgroup $Q \in \mathcal{P}$ such that $Q \cap \sigma(Q)$ is a reductive part of Q.*

Proof. See [20, Lemma 17′, p. 383]. □

4. Comparison of two homogeneous varieties

Proposition 4.1. *Let X be a scheme over F and Y be a projective homogeneous variety over F such that $Y_{F(x)}$ is isotropic for every $x \in X$. Then the projection $f : X \times Y \to X$ induces an isomorphism $f_* : \mathrm{CH}_0(X \times Y) \to \mathrm{CH}_0(X)$.*

Proof. Consider the spectral sequence [23, §8]

$$E^1_{p,q} = \coprod_{x \in X_{(p)}} A_q(Y_{F(x)}, K_p) \Rightarrow A_{p+q}(X \times Y, K_0)$$

for the projection f, where $A_*(X \times Y, K_*)$ are the K-homology groups defined in [23]. It gives an exact sequence

$$E^1_{1,0} \xrightarrow{\partial} E^1_{0,0} \to \mathrm{CH}_0(X \times Y) \to 0.$$

We shall identify the terms $E^1_{n,0}$. It is proven in [3, Th. 7.5] that the Chow motive of the isotropic variety $Y_{F(x)}$ is isomorphic over $F(x)$ to a direct sum of the motive $\mathbb{Z}$ and twisted motives of the form $Z(r)$ for some projective homogeneous variety Z and $r > 0$. By the dimension consideration, $A_0(Z(r), K_n) = A_{-r}(Z, K_{n+r}) = 0$, hence

$$A_0(Y_{F(x)}, K_n) \simeq A_0(\mathrm{Spec}\, F(x), K_n) = K_n F(x)$$

where the isomorphism is the push-forward map with respect to the structure morphism $Y_{F(x)} \to \mathrm{Spec}\, F(x)$. Thus

$$E^1_{n,0} = \coprod_{x \in X_{(0)}} K_n F(x),$$

and therefore $\mathrm{CH}_0(X \times Y) = \mathrm{Coker}(\partial) \simeq \mathrm{CH}_0(X)$. □

Corollary 4.2. *Let X and Y be two projective homogeneous varieties over F such that each of them is isotropic over residue field of every point of another. Then $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y)$ and $\overline{\mathrm{CH}}_0(X) \simeq \overline{\mathrm{CH}}_0(Y)$.*

Taking $Y = \mathrm{Spec}\, F$ in Corollary 4.2 we get

Corollary 4.3. *Let X be an isotropic projective homogeneous variety. Then $\overline{\mathrm{CH}}_0(X) = 0$.*

Let X be a variety over a field F. Consider the Brown-Gersten-Quillen spectral sequence

$$E_2^{p,q} = A^p(X, K_{-q}) \Rightarrow K_{-p-q}(X), \tag{4.4}$$

where $A^p(X, K_{-q})$ is the K-cohomology group and $K_n(X)$ is the K-group of X (see [21, §7]). If X is projective, the group $E_2^{0,-1} = A^0(X, K_1)$ of regular invertible functions on X is equal to $K_1(F) = F^\times$. Since the composite $K_1(F) \to K_1(X) \xrightarrow{e} A^0(X, K_1)$ is the identity, the edge homomorphism e is surjective. In particular, all the differentials going from $E_*^{0,-1}$ are trivial. If $d = \dim X \le 2$, all differentials coming to $E_*^{d,-d}$ are trivial, in particular, the edge homomorphism $e' : \mathrm{CH}_0(X) \to K_0(X)$ is injective.

If X is projective, the degree homomorphism for $\mathrm{CH}_0(X)$ is the composition

$$\mathrm{CH}_0(X) \xrightarrow{e'} K_0(X) \xrightarrow{f_*} K_0(\operatorname{Spec} F) = \mathbb{Z},$$

where the f_* is the push-forward homomorphism with respect to the structure morphism $f : X \to \operatorname{Spec}(F)$. Therefore, $\operatorname{Ker}(e') \subset \overline{\mathrm{CH}}_0(X)$. If X is a projective homogeneous variety, the group $K_0(X)$ is torsion free (see [16]), hence $\overline{\mathrm{CH}}_0(X) \subset \operatorname{Ker}(e')$ since $\overline{\mathrm{CH}}_0(X)$ is a torsion group. Thus $\overline{\mathrm{CH}}_0(X) = \operatorname{Ker}(e')$. In particular we have proven

Proposition 4.5. *Let X be a projective homogeneous variety of dimension at most 2. Then $\overline{\mathrm{CH}}_0(X) = 0$.*

4.1. **Reduction to a basic type.** Let G be a simple simply connected group over F. We fix a maximal torus $T \subset G$ over F, the corresponding root system $\Sigma = \Sigma(G, T)$, a basis $\Delta \subset \Sigma$ and a subset $S \subset \Delta$. Consider the variety X_S of parabolic subgroups in G of type S. Let $L = F(X_S)$ be the field of rational functions of X_S. Since $X_S(L) \neq \emptyset$, there is an L-defined parabolic subgroup P in G of type S. It contains a minimal parabolic subgroup $P' \subset P$ of G over L whose type S' is a subset in S.

Proposition 4.6. *There are canonical isomorphisms $\mathrm{CH}_0(X_S) \simeq \mathrm{CH}_0(X_{S'})$ and $\overline{\mathrm{CH}}_0(X_S) \simeq \overline{\mathrm{CH}}_0(X_{S'})$.*

Proof. We show that X_S and $X_{S'}$ satisfy conditions of Corollary 4.2. As $S' \subset S$, there is a natural map $f : X_{S'} \to X_S$ over F. Therefore X_S is isotropic over $F(x')$ for every $x' \in X_{S'}$.

Conversely, let $x \in X_S$ and set $E = F(x)$. The set S' consists of all uncircled vertices of the Tits index of G over $F(X)$. Therefore S' contains all uncircled vertices of the Tits index of G over $E(X)$. Since $E(X)/E$ purely transcendental, and the Tits index does not change under such extensions, the set S' contains all uncircled vertices of the Tits index of G over E, i.e., $X_{S'}(E) \neq \emptyset$. □

Remark 4.7. The proposition allows us to consider basic subsets $S \subset \Delta$ only while computing $\mathrm{CH}_0(X_S)$. In fact we proved even more: it is sufficient to compute $\mathrm{CH}_0(X_S)$ for those basic subsets S that appear as sets of uncircled

vertices of the Tits indices of G over all field extensions L/F. Thus we can narrow the set of basic types by considering Tits indices of strongly inner forms of G_0 only.

5. Special fields

Let p be a prime integer. A field F is called *p-special* if the degree of every finite field extension of F is a power of p. We say that F is *special* if F is p-special for some p. The following properties of special fields follow directly from the definition.

(1) An algebraic extension of a p-special field is p-special.
(2) For every field F and every prime p there is an algebraic field extension F_p/F such that the field F_p is p-special and the degree of any finite sub-extension L/F of F_p/F is prime to p.

If X is a scheme over a p-special field F, then $n(X)$ is a power of p and there is a closed point $x \in X$ such that $\deg x = n(X)$.

The following statement reduces the problem of computing $\mathrm{CH}_0(X)$ to the case of schemes over p-special fields.

Lemma 5.1. *Let X be a proper scheme over F. Assume that for any p-special field extension L/F, the group $\overline{\mathrm{CH}}_0(X_L)$ is trivial. Then $\overline{\mathrm{CH}}_0(X) = 0$.*

Proof. Let $a \in \overline{\mathrm{CH}}_0(X)$ and fix a prime integer p. As the image of a in $\overline{\mathrm{CH}}_0(X_{F_p})$ is trivial, a is trivial over some finite sub-extension L/F in F_p/F. Applying the push-forward homomorphism with respect to the morphism $X_L \to X$, we get $[L : F] \cdot a = 0$. In particular, the order of a is not divisible by p. Since this holds for every prime p, we have $a = 0$. □

The following statement enables us to consider only closed points of minimal degree.

Lemma 5.2. *Let X be a proper scheme over a p-special field F. Assume that for any finite field extension L/F, every two closed points of X_L of the smallest degree $n(X_L)$ are rationally equivalent. Then $\overline{\mathrm{CH}}_0(X) = 0$.*

Proof. Let x be a closed point of X of the smallest degree $p^a = n(X)$ and let y be a closed point of X of degree p^b. We claim that $[y] = p^{b-a}[x]$ in $\mathrm{CH}_0(X)$.

We shall prove the claim by induction on $p^b = \deg(y)$. We may assume that $p^b > p^a$. Choose an intermediate field $F \subset L \subset F(y)$ such that $[L : F] = p$ and a point $y' \in X_L$ over y of degree p^{b-1}. Note that $n(X_L)$ divides $n(X)$ and $n(X)$ divides $p \cdot n(X_L)$. Therefore we have the following two cases.

Case 1: $n(X_L) = n(X) = p^a$. Then there is a point $x' \in X_L$ over x of degree p^a. By the induction hypothesis applied to X_L, $[y'] = p^{b-1-a}[x']$ in $\mathrm{CH}_0(X_L)$. Taking the push-forward homomorphism with respect to $f : X_L \to X$ we get

$$[y] = f_*([y']) = p^{b-1-a} f_*([x']) = p^{b-1-a} \cdot p\ [x] = p^{b-a}[x].$$

Case 2: $n(X_L) = p^{a-1}$. Choose a closed point $z' \in X_L$ of degree p^{a-1} and denote by z its image in X. The degree of z is p^a, in particular, $[z] = [x]$ in

$\mathrm{CH}_0(X)$. By the induction hypothesis applied to X_L, $[y'] = p^{(b-1)-(a-1)}[z']$ in $\mathrm{CH}_0(X_L)$. Therefore

$$[y] = f_*([y']) = p^{(b-1)-(a-1)} f_*([z']) = p^{b-a}[z] = p^{b-a}[x].$$

The claim is proven. It follows that every 0-cycle α on X is equivalent to $m[x]$ for some $m \in \mathbb{Z}$. If $\deg \alpha = 0$, then $m = 0$, i.e., $\alpha = 0$ in $\mathrm{CH}_0(X)$. □

6. Equivalences of field extensions

6.1. **X-equivalence.** Let F be a field and let $\mathcal{A}$ be the class of all field extensions of F. For any subclass $\mathcal{B} \subset \mathcal{A}$ and every field $L \in \mathcal{A}$ we write $\mathcal{B}_L$ for the subclass of $\mathcal{B}$ of all fields containing L. Let $n(\mathcal{B})$ denote the greatest common divisor of the degrees $[E : F]$ over all fields $E \in \mathcal{B}$ of finite degree over F. (We set $n(\mathcal{B}) = 0$ if $\mathcal{B}$ has no finite degree extensions.) In particular, $n(\mathcal{A}) = 1$. If F is p-special then $n(\mathcal{B})$ is a power of p or 0 and if $n(\mathcal{B}) > 0$ then there is a field $E \in \mathcal{B}$ such that $[E : F] = n(\mathcal{B})$.

For every $a \in F$, let v_a be the discrete valuation of the rational function field $F(t)$ over F corresponding to the irreducible polynomial $f(t) = t - a$.

Let $\mathcal{B} \subset \mathcal{A}$ be a subclass of field extensions of F. Two fields L and L' in $\mathcal{B}$ of the same degree n over F are called *simply $\mathcal{B}$-equivalent* if there is a degree n field extension $E/F(t)$ such that $E \in \mathcal{B}$ and two discrete valuations of E over v_0 and v_1 with residue fields isomorphic to L and L' respectively over F. Two fields L and L' in $\mathcal{B}$ are called *$\mathcal{B}$-equivalent* if there is a chain of fields $L_0 = L, L_1, \dots, L_r = L'$ in $\mathcal{B}$ such that L_i is simply $\mathcal{B}$-equivalent to L_{i+1} for all $i = 0, 1, \dots, r-1$.

Note that if two fields L and L' in $\mathcal{B}$ contain a common subfield $K \in \mathcal{A}$, i.e., $L, L' \in \mathcal{B}_K$, then $L \sim_{\mathcal{B}_K} L'$ implies $L \sim_{\mathcal{B}} L'$.

Example 6.1. Let X be a scheme of finite type over F. Denote by $\mathcal{A}(X)$ the class of all fields $E \in \mathcal{A}$ such that $X(E) \neq \emptyset$. The number $n\big(\mathcal{A}(X)\big)$ coincides with $n(X)$, and $\mathcal{A}(X)$-equivalent fields will be called *X-equivalent.*

Example 6.2. In the notation of Proposition 4.6 the classes $\mathcal{A}(X_S)$ and $\mathcal{A}(X_{S'})$ coincide.

We shall need the following

Lemma 6.3. *Let X be a proper scheme of finite type over F such that $\overline{\mathrm{CH}}_0(X_L) = 0$ for any field $L \in \mathcal{A}(X)$ (e.g., X is a projective homogeneous variety by Corollary 4.3). Let x and x' be two closed points of X such that residue fields $F(x)$ and $F(x')$ are isomorphic over F. Then $[x] = [x']$ in $\mathrm{CH}_0(X)$.*

Proof. Let $L = F(x)$. By assumption there are rational points y and y' of X_L over x and x' respectively. As $[y] = [y']$ in $\mathrm{CH}_0(X_L)$, taking the push-forward homomorphism with respect to $X_L \to X$ we get $[x] = f_*([y]) = f_*([y']) = [x']$. □

Lemma 6.4. *Let X be a proper scheme of finite type over F such that $\overline{\mathrm{CH}}_0(X_L) = 0$ for any field $L \in \mathcal{A}(X)$. Let x and x' be two rational points of X of degree*

$n(X)$ such that residue fields $F(x)$ and $F(x')$ are X-equivalent. Then $[x] = [x']$ in $\mathrm{CH}_0(X)$.

Proof. We may assume that $F(x)$ and $F(x')$ are simply X-equivalent. Let E be the field extension of $F(t)$ of degree $n = n(X)$ in the definition of the simple equivalence between $F(x)$ and $F(x')$. We have the dominant morphism $\operatorname{Spec} E \to \operatorname{Spec} F(t) \to \mathbb{P}^1_F$. Since $E \in \mathcal{A}(X)$, there is a morphism $\operatorname{Spec} E \to X$ over F. Denote by C the closure of the image of the diagonal morphism $\operatorname{Spec} E \to X \times \mathbb{P}^1_F$. Thus C is an integral proper curve with a surjective morphism $f : C \to \mathbb{P}^1_F$. By assumption there are two discrete valuations v and v' of E over the valuations v_0 and v_1 of $F(t)$ respectively. Let v and v' dominate points c and c' on C respectively, so that $f(c) = 0$ and $f(c') = 1$. Note that the residue field of c is isomorphic to a subfield of the residue field of v, which in its turn is isomorphic to the field $F(x)$ of degree n over F. On the other hand, $C \subset X \times \mathbb{P}^1_F$, hence $n(C) \geq n(X) = n$. Therefore, c is a closed point of degree n, $f^{-1}(0) = \operatorname{Spec} F(c)$ and $F(c) \simeq F(x)$. Similarly, $f^{-1}(1) = \operatorname{Spec} F(c')$ and $F(c') \simeq F(x')$.

The morphism f is flat, hence $[c] = f^*([0]) = f^*([1]) = [c']$ in $\mathrm{CH}_0(C)$. Consider the projection $g : C \to X$. Let $z = g(c)$ and $z' = g(c')$. As $F(z)$ is isomorphic to a subfield of $F(c)$ and $n(X) = n$, we have $F(z) \simeq F(c) \simeq F(x)$. Similarly $F(z') \simeq F(c') \simeq F(x')$. Therefore, $[z] = g_*([c]) = g_*([c']) = [z']$. By Lemma 6.3, $[x] = [z] = [z'] = [x']$. □

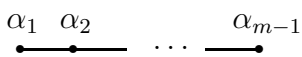
A class $\mathcal{B} \subset \mathcal{A}$ is called *connected* if for any special field $E \in \mathcal{A}$ every two fields in $\mathcal{B}_E$ of degree $n(\mathcal{B}_E)$ over E are $\mathcal{B}_E$-equivalent. For example, the whole class $\mathcal{A}$ is connected, and therefore for every X with $X(F) \neq 0$ the class $\mathcal{A}(X) = \mathcal{A}$ is connected.

Theorem 6.5. *Let X be a proper scheme of finite type over F such that $\overline{\mathrm{CH}}_0(X_L) = 0$ for any field $L \in \mathcal{A}(X)$. Suppose that the class $\mathcal{A}(X)$ is connected. Then $\overline{\mathrm{CH}}_0(X) = 0$.*

Proof. By Lemma 5.1 we may assume that F is p-special for some prime p. It follows from Lemma 6.4 that for any finite field extension L/F, every two closed point of X_L of degree $n(X_L)$ are rationally equivalent. By Lemma 5.2, $\overline{\mathrm{CH}}_0(X) = 0$. □

7. Severi-Brauer varieties

Let A be a central simple algebra of degree m over a field F and let X be the corresponding Severi-Brauer variety of right ideals of A of dimension $\deg(A)$ (see [10, §1]). The variety X is projective homogeneous of the group $\mathbf{SL}_1(A)$ of type $S = \{\alpha_2, \ldots, \alpha_{m-1}\}$ where simple roots $\alpha_1, \ldots, \alpha_{m-1}$ of the Dynkin diagram A_{m-1} are numbered as follows:



Here $m-1$ is the rank of $\mathbf{SL}_1(A)$. A field extension L/F belongs to $\mathcal{A}(X)$ if and only if the algebra $A_L = A \otimes_F L$ is split. In particular $n(X) = \mathrm{ind}(A)$. If A is split, i.e., $A = \mathrm{End}(V)$ for some vector space V, the Severi-Brauer variety X is the projective space $\mathbb{P}(V)$.

Let D be a (unique up to isomorphism) central division algebra F-algebra Brauer equivalent to A. Every maximal subfield of D has degree $n = n(X)$ over F and it is a splitting field for D and A and it belongs to $\mathcal{A}(X)$. Conversely, every field in $\mathcal{A}(X)$ of the smallest degree n is isomorphic over F to a maximal subfield of D [6, §4.4].

For every element $d \in D$ denote by $\mathrm{Prd}_d = \mathrm{Prd}_d(s) \in F[s]$ the reduced characteristic polynomial of d. The polynomial Prd_d is monic of degree n and Prd_d is irreducible if and only if d generates a maximal subfield of D. In this case $F(d) \simeq F[s]/(\mathrm{Prd}_d)$.

Recall that a finite field extension L/F is called *simple* if L is generated over F by one element. In particular every separable finite extension is simple.

Lemma 7.1. *Let $L \subset D$ be a simple maximal subfield and let $d \in D$. Then there is a maximal subfield $K \subset D$ containing d such that $L \sim_X K$.*

Proof. We have $L = F(a)$ for some $a \in D$. In particular the polynomial Prd_a is irreducible. Consider the element $c = td + (1-t)a \in D[t]$ and its reduced characteristic polynomial $\mathrm{Prd}_c \in F[s,t]$. Note that $\mathrm{Prd}_c(s,0) = \mathrm{Prd}_a$ and $\mathrm{Prd}_c(s,1) = \mathrm{Prd}_d$. In particular Prd_c is irreducible.

Set $R = F[s,t]/(\mathrm{Prd}_c)$. Since Prd_c is irreducible, R is a domain. The quotient field E of R is the maximal subfield of $D_{F(t)}$ generated by c and therefore belongs to $\mathcal{A}(X)$. Let Q be the principal ideal of R generated by t. The factor ring R/Q is isomorphic to $F[s]/(\mathrm{Prd}_a) \simeq L$. In particular, Q is a maximal ideal of R. As the localization R_Q is a Noetherian local domain of dimension one with principal maximal ideal, R_Q is a discrete valuation ring. The corresponding discrete valuation of E with residue field L restricts to v_0 on $F(t)$.

Let v be a discrete valuation of E restricting to v_1. Since R is integral over $F[t]$, R is contained in the valuation ring O of v. As $R/(1-t) = F[s]/(\mathrm{Prd}_d)$ and Prd_d is a power of the minimal polynomial $h = h(s)$ of d, the intersection of R with the maximal ideal of O is generated by $1-t$ and h. In particular residue field K of O contains $F[s]/(h) = F(d)$.

On the other hand, since X is proper, a morphism $\mathrm{Spec}\, E \to X$ extends to $\mathrm{Spec}\, O \to X$. Therefore K contains $F(x)$ for some $x \in X$. But $[K:F] \le [E : F(t)] = n$, therefore $K = F(x)$ and $[K:F] = n$. Thus K is isomorphic to a maximal subfield of D containing d and $L \sim_X K$. □

Theorem 7.2. *Let A be a central simple F-algebra and let X be the Severi-Brauer variety $SB(A)$. Then the class $\mathcal{A}(X)$ is connected.*

Proof. It is sufficient to show that every two maximal subfields of D are X-equivalent. We proceed by induction on $\deg D$ and may assume that $\deg(D) > 1$. Choose a separable maximal subfield L of D. It is sufficient to show that L is

X-equivalent to any maximal subfield $E \subset D$. Choose an element $d \in E \setminus F$. By Lemma 7.1, $L \sim_X K$ for some maximal subfield K containing d. Let $M = F(d)$ and consider the centralizer D' of M in D. The fields E and K are maximal subfields of D'. By the induction hypothesis, $K \sim_{X_M} E$ and therefore, $K \sim_X E$. Finally, $L \sim_X K \sim_X E$. □

Theorem 6.5 yields

Corollary 7.3. (Cf., [15]) *Let X be a Severi-Brauer variety. Then* $\overline{\mathrm{CH}}_0(X) = 0$.

8. Generalized Severi-Brauer varieties

Let A be a central simple F-algebra. Denote by $X = SB(2, A)$ the generalized Severi-Brauer variety of right ideals of A of dimension $2\deg(A)$ (see [10, §1]). It is a projective homogeneous variety of the algebraic group $\mathbf{SL}_1(A)$ of type $S = \{\alpha_1, \alpha_3, \dots, \alpha_m\}$ where simple roots are numbered as in Section 7. A field extension L/F belongs to $\mathcal{A}(X)$ if and only if $\operatorname{ind} A_L \leq 2$. If A is split, i.e., $A = \mathrm{End}(V)$ for some vector space V, the variety $SB(2, A)$ is the Grassmannian variety of planes in V.

If index of A is odd, then a field extension L/F belongs to $\mathcal{A}(X)$ if and only if L splits A. Therefore the class $\mathcal{A}(X)$ coincides with $\mathcal{A}(SB(A))$. Hence the class $\mathcal{A}(X)$ is connected, by Theorem 7.2.

Suppose now that $\operatorname{ind} A = 2n$ is even. We have $n(X) = n$. Let D be a central division F-algebra of degree $2n$ Brauer equivalent to A. In particular $\mathcal{A}(X) = \mathcal{A}\big(SB(2, D)\big)$. Every finite extension of F of degree n in $\mathcal{A}(X)$ is isomorphic to a subfield of D and conversely every subfield of D of degree n is contained in $\mathcal{A}(X)$.

Suppose that D has a symplectic involution σ. Denote by $\mathrm{Symd}(D, \sigma)$ the subspace of D of all elements $d + \sigma(d)$ for $d \in D$. The characteristic polynomial of an element $a \in \mathrm{Symd}(D, \sigma)$ is the square of the monic *pfaffian characteristic polynomial* $\mathrm{Prp}_a(s)$ of degree n [10, Prop. 2.9]. In particular, $[F(a) : F] \leq n$, moreover the polynomial $\mathrm{Prp}_a(s)$ is irreducible if and only if $[F(a) : F] = n$.

Lemma 8.1. *For any subfield K of D of degree n there is a symplectic involution τ of D such that $K \subset \mathrm{Symd}(D, \tau)$.*

Proof. The centralizer Q of K in D is a quaternion algebra over K. By [10, Th. 4.14], the canonical involution of Q can be extended to a symplectic involution τ on D. Choose an element $y \in Q$ such that $y + \tau(y) = 1$. Then $x = xy + \tau(xy)$ for every $x \in K$, i.e., $K \subset \mathrm{Symd}(D, \tau)$. □

Lemma 8.2. *Let σ and τ be two distinct symplectic involutions on D and let L be a simple subfield of $\mathrm{Symd}(D, \sigma)$ of degree n. Then*

1. *There exists a subfield K of D of degree n such that $L \sim_X K$ and $K \cap \mathrm{Symd}(D, \tau)$ contains a non-constant element.*
2. *For every $d \in \mathrm{Symd}(D, \sigma)$ there exists a subfield K of D of degree n such that $d \in K$ and $L \sim_X K$.*

Proof. Let $L = F(a)$ for some $a \in \mathrm{Symd}(D, \sigma)$.

(1) Choose $b \in D^\times$ such that $\sigma = \mathrm{Int}(b) \circ \tau$ [10, Prop. 2.7]. We have $ba \in \mathrm{Symd}(D, \tau)$. Replacing a by $a + 1$ if necessary we assume that $ba \neq F$.

Consider the element $c = t + (1 - t)b \in D[t]$ and the involution $\sigma_t = Int(c) \circ \sigma_{F(t)}$ on $D_{F(t)}$. Then $ca \in \mathrm{Symd}(D_{F(t)}, \sigma_t)$. The pfaffian characteristic polynomial Prp_{ca} belongs to $F[s, t]$. We have $\mathrm{Prp}_{ca}(s, 0) = \mathrm{Prp}_{ba}$ and $\mathrm{Prp}_{ca}(s, 1) = \mathrm{Prp}_a$, in particular Prp_{ca} is irreducible. Considering the quotient field E of the factor ring $F[s, t]/(\mathrm{Prp}_{ca})$ as in the proof of Lemma 7.1 we find two discrete valuations of E over v_1 and v_0 respectively with residue fields isomorphic to L and a subfield K of D containing the nonconstant element ba. Thus $L \sim_X K$.

(2) Let E be the degree n subfield of $D_{F(t)}$ generated by $ta + (1 - t)d$. As in the proof of Lemma 7.1 we see that there are two discrete valuations of E with residue fields $L = F(a)$ and K respectively, where K is a subfield of D of degree n and $d \in K$. □

Theorem 8.3. *Let X be the generalized Severi-Brauer variety $SB(2, A)$ of a central simple algebra A of exponent not divisible by 4. Then the class $\mathcal{A}(X)$ is connected.*

Proof. By definition of connectedness we may assume that F is a p-special field. If p is odd and we already know that the class $\mathcal{A}(X)$ is connected. Thus we may assume that $p = 2$, $\exp(A) = 2$ and $\mathrm{ind}(A) = 2n$ is even. By [10, Th. 3.8], the algebra A has a symplectic involution.

We use the notation above. It is sufficient to show that every two subfields of D of degree n are X-equivalent.

Let L and E be subfields of D of degree n. We may assume that L/F is separable (see [29, Prop. 5]) and therefore is simple. We prove that $L \sim_X E$ by induction on n. By Lemma 8.1, we can choose symplectic involutions σ and τ of D such that $L \subset \mathrm{Symd}(A, \sigma)$ and $E \subset \mathrm{Symd}(A, \tau)$.

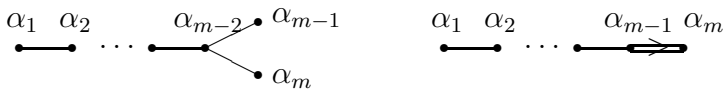
Case 1: $\sigma = \tau$. Choose an element $d \in E \setminus F$. By Lemma 8.2(2), there is a subfield K of D of degree n containing d such that $L \sim_X K$. Let D' be the centralizer of $M = F(d)$ in D. By the induction hypothesis applied to D' with the involution $\tau|_{D'}$ we have $K \sim_{X_M} E$, therefore $K \sim_X E$ and $L \sim_X K \sim_X E$.

Case 2: $\sigma \neq \tau$. By Lemma 8.2(1), there exists a subfield K of D of degree n such that $L \sim_X K$ and $K \cap \mathrm{Symd}(D, \tau)$ contains a non-constant element d. Choose a subfield $H \subset \mathrm{Symd}(D, \tau)$ of degree n containing d. Let D' be the centralizer of $M = F(d)$ in D. By the induction hypothesis applied to D', $K \sim_{X_M} H$ and therefore $K \sim_X H$. By the first case, $H \sim_X E$, hence $L \sim_X K \sim_X H \sim_X E$. □

Corollary 8.4. (Cf., [11]) *Let X be the generalized Severi-Brauer variety $SB(2, A)$ of a central simple algebra A of exponent not divisible by 4. Then $\overline{\mathrm{CH}}_0(X) = 0$.*

9. Quadrics

Let q be a quadratic form on an F-vector space V of finite dimension. Denote by X the projective quadric hypersurface in $\mathbb{P}(V)$ defined by the equation $q = 0$. If X is smooth, it is a projective homogeneous variety of the special orthogonal group $\mathbf{O}^+(q)$ of type $S = \{\alpha_2, \dots, \alpha_m\}$ where simple roots $\alpha_1, \dots \alpha_m$ of the Dynkin diagrams D_m and B_m are numbered as follows:



Here the first and second diagrams correspond to the cases $\dim V = 2m$ and $\dim V = 2m + 1$ respectively.

The quadric X has a rational point if and only if q is isotropic. If q is anisotropic, then $n(X) = 2$ by Springer's theorem.

Example 9.1. Let q be a nonzero 2-dimensional quadratic form. Then we have $\dim X = 0$, moreover $X = \operatorname{Spec} C_0(q)$, where $C_0(q)$ is the even Clifford algebra of q. The form q is non-degenerate (resp. anisotropic) if and only if $C_0(q)$ is an étale quadratic F-algebra (resp. a quadratic field extension of F).

Theorem 9.2. *The class $\mathcal{A}(X)$ for a projective quadric X is connected.*

Proof. We may assume that q is anisotropic. Let L/F be a quadratic field extension making q isotropic and let $x \in X$ be a point with $F(x) \simeq L$. Let $\{1, \alpha\}$ be a basis of L over F and let $w = u + \alpha v$, $u, v \in V$, be an isotropic vector in V_L. Then $w \in U_L$, where U is the 2-dimensional subspace of V spanned by v and u. Moreover, by Example 9.1, $F(x) \simeq C_0(q|_U)$.

Conversely, if $U \subset V$ is a 2-dimensional subspace, then

$$X \cap \mathbb{P}(U) = \operatorname{Spec} F(x) \simeq C_0(q|_U),$$

where x is a closed point of degree 2. Thus it is sufficient to show that for every two 2-dimensional subspaces U and U', the quadratic field extensions $L = C_0(q|_U)$ and $L' = C_0(q|_{U'})$ are X-equivalent. We may assume that $U \cap U' \neq 0$.

Let $\{v, u\}$ and $\{v, u'\}$ be bases of U and U' respectively. Consider the $F[t]$-submodule W of $V_{F[t]}$ generated by v and $tu + (1-t)u'$. Let $R = C_0(q_{F[t]}|_W)$. The quotient field E of R belongs to $\mathcal{A}(X)$. Since $R/(t) = L'$ is a field, the principal ideal Q generated by t is maximal. Therefore the localization R_Q is a Noetherian local domain of dimension one with principal maximal ideal, i.e., R_Q is a discrete valuation ring. The corresponding discrete valuation of E with residue field L' restricts to v_0 on $F(t)$. Similarly, there is a discrete valuation of E with residue field L restricting to v_1, i.e., L and L' are simply X-equivalent. □

Corollary 9.3. (Cf., [8], [27]) *Let X be a smooth projective quadric. Then $\overline{\mathrm{CH}}_0(X) = 0$.*

10. Involution varieties

Let A be a central simple algebra of degree $2m$ with a quadratic pair (σ, f), where σ is an involution on A of the first kind and $f : \mathrm{Sym}(A, \sigma) \to F$ is a linear map satisfying certain properties [10, §5.B]. If A is split, i.e., $A \simeq \mathrm{End}(V)$ for a $2n$-dimensional vector space V over F, the space $\mathrm{Sym}(A, \sigma)$ is identified with the subspace of symmetric elements in $V \otimes_F V$ and $f(v \otimes v) = q(v)$ for a non-degenerate quadratic form q on V.

Consider the morphism

$$\varphi : SB(A) \to \mathbb{P}\big(\mathrm{Sym}(A, \sigma)\big)$$

taking an ideal I in A to the (1-dimensional) subspace $I \cdot \sigma(I)$ of $\mathrm{Sym}(A, \sigma)$ [10, p. 122]. Let $W = \mathrm{Ker}(f)$. The inverse image $X = \varphi^{-1}\mathbb{P}(W)$ is called the *involution variety of the quadratic pair*. Thus X is a hypersurface in the Severi-Brauer variety $SB(A)$. In the split case as above, X is a smooth projective quadric hypersurface in $\mathbb{P}(V)$ given by the quadratic form q. In other words, X is a twisted form of a projective quadric. It is a projective homogeneous variety of the special orthogonal group $\mathbf{O}^+(A, \sigma, f)$ [10, §26] of type $S = \{\alpha_2, \dots, \alpha_m\}$, where S is the subset of vertices of the Dynkin diagram D_m. In fact, any twisted form of a smooth projective quadric is isomorphic to an involution variety.

If A is not split, then $\mathrm{ind}\, A$ is even. Consider the case $\mathrm{ind}(A) = 2$. Let Q be a quaternion division algebra Brauer equivalent to A. Then $A \simeq \mathrm{End}_Q(V)$ for some right Q-module V. The Severi-Brauer variety $SB(A)$ can be identified with the variety $\mathbb{P}_Q(V)$ of reduced rank 1 Q-submodules in V. Note that for any Q-submodule $U \subset V$ of rank 1, the subvariety $\mathbb{P}_Q(U)$ of $\mathbb{P}_Q(V)$ is a conic curve. Then either the conic $\mathbb{P}_Q(U)$ is contained in X or the intersection of $\mathbb{P}_Q(U)$ with X is a point of degree 2. Therefore $n(X) = 2$.

Theorem 10.1. *If* $\mathrm{ind}(A) = 2$ *then the class* $\mathcal{A}(X)$ *is connected.*

Proof. As $n(X) = 2$ it is sufficient to show that every two closed points x and x' of degree 2 in X are X-equivalent. By [13, Lemma 1], there are Q-submodules $U \subset V$ and $U' \subset V$ of rank 1 such that $x \in \mathbb{P}_Q(U)$ and $x' \in \mathbb{P}_Q(U')$. In fact $F(x)$ and $F(x')$ are isomorphic to the even Clifford algebras of the restrictions of the quadratic pair to U and U' respectively. Let v and v' be nonzero vectors of U and U' respectively. Consider $w = tv + (1-t)v' \in V_{F[t]}$ and let $W = Q[t]w$. Let R be the even Clifford algebra of the restriction of the quadratic pair $(\sigma, f) \otimes F[t]$ on W. As in the proof of Theorem 9.2 we see that the quotient field E of R is a quadratic extension of $F(t)$ equipped with two discrete valuations over v_0 and v_1 respectively and residue fields isomorphic to $F(x)$ and $F(x')$, i.e., $F(x)$ and $F(x')$ are simply X-equivalent. □

Consider the general case when $\mathrm{ind}(A) = 2n$. Assume that F is a 2-special field. We claim that $n(X) = 2n$. Indeed, $n(X) \geq 2n$ since X is a hypersurface in $SB(A)$ and $n(SB(A)) = 2n$. On the other hand, choose a field extension

K/F of degree n such that $\operatorname{ind} A_K = 2$. As $n(X_K) = 2$, we conclude that $n(X) \leq 2n$.

Set $Y = SB(2, A)$. We know that $n(Y) = n$.

Lemma 10.2. *Let y_0 and y_1 be two closed points of Y of degree n such that $F(y_0)$ and $F(y_1)$ are simply Y-equivalent. Then there exist points x_0 and x_1 of X of degree $2n$ such that $F(x_0)$ and $F(x_1)$ are simply X-equivalent and $F(y_i)$ is isomorphic to a subfield of $F(x_i)$ for $i = 0, 1$.*

Proof. Let $E/F(t)$ be an extension of degree n such that $Y(E) \neq \emptyset$ and let w_i, $i = 0, 1$, be two discrete valuations of E over the valuations v_i of $F(t)$ with $F(w_i) \simeq F(y_i)$.

Since $\operatorname{ind} A_E = 2$, there is a quadratic field extension K/E such that $X(K) \neq \emptyset$. Let u_i be a discrete valuation of K extending w_i and let L_i be the residue field of u_i. As X is projective, a morphism $\operatorname{Spec} K \to X$ factors through a morphism $\operatorname{Spec} O_i \to X$, where O_i is the valuation ring of u_i. In particular $X(L_i) \neq \emptyset$, therefore

$$2n = [K : F(t)] \geq [L_i : F] \geq n(X) = 2n.$$

Hence there is a closed point $x_i \in X$ of degree $2n$ with $F(x_i) \simeq L_i$. Note that L_0 and L_1 are simply X-equivalent and $F(y_i) \simeq F(w_i) \subset L_i$. □

Theorem 10.3. *Let A be a central simple algebra of even degree with a quadratic pair and let X be the corresponding involution variety. Then the class $\mathcal{A}(X)$ is connected.*

Proof. We may assume that F is p-special. If $p \neq 2$, then $X(F) \neq \emptyset$ and the class $\mathcal{A}(X)$ is connected. Let F be a 2-special field. If A splits, then X is a smooth projective quadric, and the result follows from Theorem 9.2. Thus we may assume that $\operatorname{ind} A = 2n$ for some $n \geq 1$.

Let x and x' be two closed points of X of degree $2n$. We would like to show that $F(x)$ and $F(x')$ are X-equivalent. As F is 2-special, we can choose closed points y and $y' \in Y$ of degree n such that $F(y) \subset F(x)$ and $F(y') \subset F(x')$. By Theorem 8.3, $F(y)$ and $F(y')$ are Y-equivalent, i.e., there are closed points $y_0 = y, y_1, \ldots, y_r = y'$ of degree n such that $F(y_i)$ and $F(y_{i+1})$ are simply Y-equivalent for $i = 0, \ldots, r-1$. By Lemma 10.2 applied to each pair (y_i, y_{i+1}) there are closed points x'_i, x_{i+1} of X of degree $2n$ such that $F(x'_i)$ and $F(x_{i+1})$ are simply X-equivalent and $F(y_i) \subset F(x'_i)$, $F(y_{i+1}) \subset F(x_{i+1})$.

Set for convenience $x_0 = x$ and $x'_r = x'$. For every $i = 0, \ldots, r$, the fields $F(x_i)$ and $F(x'_i)$ contain $F(y_i)$. The index of A over $F(y_i)$ is equal to 2. By Theorem 10.1, the fields $F(x_i)$ and $F(x'_i)$ are $X_{F(y_i)}$-equivalent, and therefore are X-equivalent. Finally we have

$$F(x) = F(x_0) \sim F(x'_0) \sim F(x_1) \sim F(x'_1) \sim \cdots \sim F(x_r) \sim F(x'_r) = F(x'). \quad \square$$

Corollary 10.4. (Cf., [11]) *Let A be a central simple algebra of even degree with a quadratic pair and let X be the corresponding involution variety. Then $\overline{\mathrm{CH}}_0(X) = 0$.*

11. Cayley-Dickson and Jordan algebras

Let C be a Cayley-Dickson algebra over F. Denote by N the norm form of C given by $N(x) = x\bar{x}$, where $x \mapsto \bar{x}$ is the canonical involution of C. It is known [10, Prop. 33.18, Th. 33.19] that N is a 3-fold Pfister form, and C is split if and only if N is split. Therefore, Theorem 9.2 applied to the quadric given by N yields

Theorem 11.1. *The class of splitting fields of C is connected.*

Remark 11.2. One can give a direct proof of Theorem 11.1 along the lines of the proof of Lemma 7.1 using the fact that a quadratic field extension of L/F splits C if and only if L can be embedded into C.

Let J be an exceptional simple 27-dimensional Jordan algebra over a field F of degree 3 arising from the first Tits construction [7, Ch. IX, §12]. Recall that there is an associated central simple F-algebra A of degree 3 and an element $a \in F^\times$. The algebra J is split if and only if a is a reduced norm for A.

Theorem 11.3. *Let J be an exceptional simple 27-dimensional Jordan algebra over a 3-special field F of degree 3 arising from the first Tits construction. Then the class of all splitting fields of J is connected.*

Proof. We may assume that J is not split. Therefore for the class $\mathcal{B}$ of all splitting fields of J we have $n(\mathcal{B}) = 3$. Let L and L' be two cubic extensions of F splitting J. We shall show that $L \sim_{\mathcal{B}} L'$. By [17, Cor. 3] we may assume that L and L' are subfields of J. Choose generators x and x' of L and L' over F respectively and let E be the subfield of $J_{F(t)}$ generated by $tx + (1-t)x'$. As in the proof of Lemma 7.1 one shows using the cubic minimal polynomials of x and x' that there are two discrete valuations of E over v_0 and v_1 with residue fields L and L' respectively, i.e., L and L' are simply $\mathcal{B}$-equivalent. □

12. Groups of exceptional types

In what follows G_0 denotes split or quasi-split simple simply connected algebraic group over F of one of the following types: $\mathsf{G}_2, \mathsf{F}_4, \mathsf{D}_4, \mathsf{E}_6, \mathsf{E}_7$. Let $\xi \in Z^1(F, G_0)$ be a cocycle and let $G = {}^\xi G_0$ be the corresponding strongly inner form of G_0.

Theorem 12.1. *Let $X = X_S$ be a projective homogeneous variety of the group G as above. If G is an outer form of type E_6 of of type E_7 we assume that $S \neq \emptyset$. If G is a trialitarian group or of types $\mathsf{E}_6, \mathsf{E}_7$ we assume that* $\mathrm{char}(F) \neq 2, 3$. *Then the class $\mathcal{A}(X)$ is connected. In particular, we have $\overline{\mathrm{CH}}_0(X) = 0$.*

If $S = \Delta$, then $X_S = \operatorname{Spec} F$ and there is nothing to prove. Thus, by Proposition 4.6 we may assume that S is a proper basic subset of Δ. By Lemma 5.1, we may also assume that F is p-special.

If $p \neq 2, 3$, then we have $H^1(F, G_0) = 1$ for all types under the consideration. Indeed, by Steinberg's theorem [25], ξ is equivalent to a cocycle with coefficients in a maximal torus $T \subset G_0$ and, by [20, Proposition 6.21, p. 375], we have

$H^1(F,T) = 1$ for p-special fields with $p \neq 2,3$. It follows that G is split or quasi-split and hence $X_S(F) \neq \emptyset$. Thus we may assume that $p = 2,3$.

To finish the proof we shall either show directly that $\mathcal{A}(X)$ is connected or construct a projective homogeneous variety Y (under some algebraic simple group) such that

(i) *The class* $\mathcal{A}(Y)$ *is connected;*
(ii) $\mathcal{A}(X_S) = \mathcal{A}(Y)$.

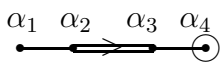
We shall construct Y using case-by-case consideration.

13. Types G_2 and F_4

By [28], each isotropic group of type G_2 is split. Hence the only basic type for G is $S = \emptyset$. Since $X_\emptyset$ is a variety of Borel subgroups of G, the class $\mathcal{A}(X_\emptyset)$ consists of all field extensions making G split.

Recall that the correspondence $C \mapsto G = \mathrm{Aut}(C)$ gives rise to a bijection between the set of isomorphism classes of Cayley-Dickson algebras and the set of isomorphism classes of simple groups of type G_2 [10, Theorem 26.19]. In particular the classes of splitting fields for C and G coincide. The connectedness of $\mathcal{A}(X_\emptyset)$ follows from Theorem 11.1.

We pass to type F_4. According to [28], if G is isotropic over a field extension L/F but not L-split, then its Tits index is of the form



It follows that there exist two basic types only:

$$S_1 = \emptyset \ \text{ and } \ S_2 = \{\alpha_1, \alpha_2, \alpha_3\}.$$

The class $\mathcal{A}(X_{S_1})$ consists of all field extensions L/F making G split and $\mathcal{A}(X_{S_2})$ consists of all field extensions L/F making G isotropic.

The correspondence $J \mapsto G = \mathrm{Aut}(J)$ gives rise to a bijection between the set of isomorphism classes of exceptional simple 27-dimensional Jordan algebra of degree 3 and the set of isomorphism classes of simple groups of type F_4 [10, Theorem 26.18]. In particular the classes of splitting fields for J and G coincide.

If $p = 3$, then J arises from the first Tits construction (cf. [7, Ch. IX, §12]). We claim that the diagram above cannot appear as the Tits index of G over any extension L/F of F. Indeed, assume the contrary. Since G is not split over L, so is J. The anisotropic kernel of G over L has type B_3, hence there is a finite extension E/L of degree $[E : L] = 2^a$ for some positive integer a making G (and hence J) split. On the other side, the extension E/L does not split J since it arises from the first Tits construction – a contradiction. Thus we showed that the classes $\mathcal{A}(X_{S_1})$, $\mathcal{A}(X_{S_2})$ and the class of all splitting fields of J coincide and we are done by Theorem 11.3.

If $p = 2$, then J is reduced. Recall (cf. [4], [18]) that given J one can associate a 3-Pfister form $f_3(J)$ and a 5-Pfister form $f_5(J)$ with the following properties (cf. [26], [22]):

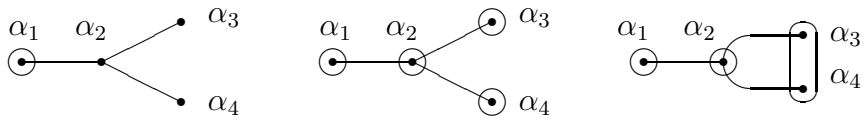
(*a*) *The group G is split over a field extension L/F if and only if $f_3(J)$ is hyperbolic over L;*
(*b*) *The group G is isotropic over a field extension L/F if and only if $f_5(J)$ is hyperbolic over L.*

Let Y_1 and Y_2 be the projective quadric defined by the equations $f_3(J) = 0$ and $f_5(J) = 0$ respectively. By (a) and (b), we have $\mathcal{A}(X_{S_1}) = \mathcal{A}(Y_1)$ and $\mathcal{A}(X_{S_2}) = \mathcal{A}(Y_2)$. The connectedness of $\mathcal{A}(X_{S_1})$ and $\mathcal{A}(X_{S_2})$ follows from Theorem 9.2.

14. Type D_4

Suppose first that G is anisotropic of type ${}^1D_4$ or ${}^2D_4$. Then $G \simeq \mathbf{Spin}(f)$, where f is an 8-dimensional quadratic form with trivial even Clifford algebra. Let $dF^{\times 2}$ be the discriminant of f. The form $f \perp \langle 1, -d\rangle$ has trivial discriminant and Clifford algebra, therefore it is represented by a 3-Pfister form g in the Witt ring $W(F)$.

As g can have Witt index 0 or 4 over a field extension L/F, it follows that if f becomes isotropic over L/F, then its Witt index over L is either 1 or f is quasi-split (i.e., its Witt index is either 3 or 4). Hence for the Tits index of G over L there are the following possibilities only:



By Remark 4.7, we need to consider the following basic types only:

$$S_1 = \{\alpha_2, \alpha_3, \alpha_4\} \quad \text{and} \quad S_2 = \emptyset.$$

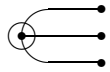
The variety X_{S_1} is isomorphic to the quadric Y defined by $f = 0$. The class $\mathcal{A}(Y)$ is connected by Theorem 9.2.

The second case S_2 corresponds to the variety of Borel subgroups of G. A field extension L/F makes $X_\emptyset$ isotropic if and only if it splits g. Let Z be the projective quadric defined by $g = 0$. Then the classes $\mathcal{A}(X_\emptyset)$ and $\mathcal{A}(Z)$ coincide, and we are done again by Theorem 9.2.

We now turn to trialitarian cases ${}^3D_4$ and ${}^6D_4$. We need to consider the case $p = 3$ only. Let E/F be a cubic extension over which G is a classical group.

Lemma 14.1. *Let F be 3-special. If L/F is an extension such that G is L-isotropic, then G is quasi-split over L.*

Proof. If G is not quasi-split over L, then, by [28], its Tits index over L is of the form



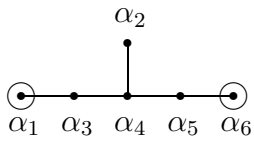
Hence the semisimple L-anisotropic kernel of G is isomorphic to $\mathrm{R}_{\mathrm{E}'/\mathrm{L}}(\mathbf{SL}_1(D))$, where D is a quaternion algebra over E' and $E' = E \cdot L$. Note that such a group is still anisotropic over E'. On the other side, since F is 3-special, G is quasi-split over E and hence over $E' = E \cdot L$ – a contradiction. □

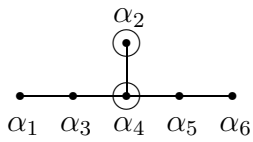
Lemma 14.1 implies that we need to consider the basic type $S = \emptyset$ only. Let $R_{G_0} : H^1(F, G_0) \to H^3(F, \mathbb{Z}/3)$ be the Rost invariant. We refer to [4] for the definition and properties of the Rost invariant. It takes values in $\mathbb{Z}/3$ since F is 3-special. By [2, Theorem 6.14], $\operatorname{Ker} R_{G_0} = 1$ and $R_{G_0}([\xi])$ is a 3-symbol, say $(a) \cup (b) \cup (c)$, where $a, b, c \in F^\times$.

Let J be the Jordan algebra arising from the first Tits construction corresponding to the central simple algebra $A = (a, b)$ over F of degree 3 and element c. Recall that it is split over a field extension L/F of F if and only if $c \in \operatorname{Nrd} A_L$ or equivalently if and only if $R_{G_0}([\xi])$ is trivial over L, by [14, Th. 12.2]. As $\operatorname{Ker} R_{G_0} = 1$, the last holds if and only if $[\xi_L] = 1$. Thus, by Lemma 3.2, the classes of splitting fields of J and $\mathcal{A}(X_\emptyset)$ coincide and the connectedness of $X_\emptyset$ follows from Theorem 11.3.

15. Inner forms of type ${}^1\mathsf{E}_6$

According to [28] all admissible Tits' indices of type ${}^1\mathsf{E}_6$ corresponding to isotropic (not split) groups are as follows:

(a) 

(b) 

In case (b) the corresponding anisotropic kernel has type $\mathsf{A}_2 + \mathsf{A}_2$. Since $\mathbf{SL}_3 \times \mathbf{SL}_3$ has no nontrivial strongly inner forms, Theorem 3.3 implies that diagram (b) can not appear as the Tits index of a strongly inner form of type ${}^1\mathsf{E}_6$ (note that in the notation of that theorem the condition $H^1(F, C/C \cap H) = 1$ holds, since $C = C_1$ is an F-split torus). Thus, by Remark 4.7, the only basic types to be consider are

$$S_1 = \emptyset \quad \text{and} \quad S_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}.$$

Lemma 15.1. *Assume that G has diagram (a) over an extension L/F. Then the anisotropic L-kernel of G is isomorphic to $\mathbf{Spin}(f)$ where f is a 3-Pfister form over L.*

Proof. The anisotropic L-kernel is a strongly inner form of type ${}^1\mathsf{D}_4$, by Theorem 3.3. So the result is clear. □

Lemma 15.2. *Let F be 3-special. Then $\mathcal{A}(X_{S_1}) = \mathcal{A}(X_{S_2})$.*

Proof. The inclusion $\mathcal{A}(X_{S_1}) \subset \mathcal{A}(X_{S_2})$ is clear. Conversely, let $L \in \mathcal{A}(X_{S_2})$. If $X_{S_1}(L) = \emptyset$, then G has Tits index (a) over L. The corresponding anisotropic kernel of G over L is isomorphic to $\mathbf{Spin}(f)$ where f is a 3-Pfister form, by Lemma 15.1. Hence it can be split by a quadratic extension E/L. On the other side, the Rost invariant R_{G_0} has trivial kernel (cf. [2], [5]), takes values in $H^3(F, \mathbb{Z}/3)$ and $R_{G_0}([\xi])$ is a symbol, by [2]. Let J be a Jordan algebra corresponding to the symbol $R_{G_0}([\xi])$. As $\operatorname{Ker} R_{G_0} = 1$, J is nontrivial over L and remains so over its quadratic extension E/L since J arises from the first Tits construction. However J, $[\xi]$ (and hence G) have the same splitting fields – a contradiction. □

If $p = 3$, then by Lemma 15.2, $\mathcal{A}(X_{S_1}) = \mathcal{A}(X_{S_2})$ and the classes of splitting fields of J and $\mathcal{A}(X_{S_1})$ coincide where J is the Jordan algebra constructed in Lemma 15.2. The connectedness of $\mathcal{A}(X_{S_1})$, $\mathcal{A}(X_{S_2})$ follows from Theorem 11.3.

Let now $p = 2$.

Lemma 15.3. *If F is 2-special, then G is F-isotropic.*

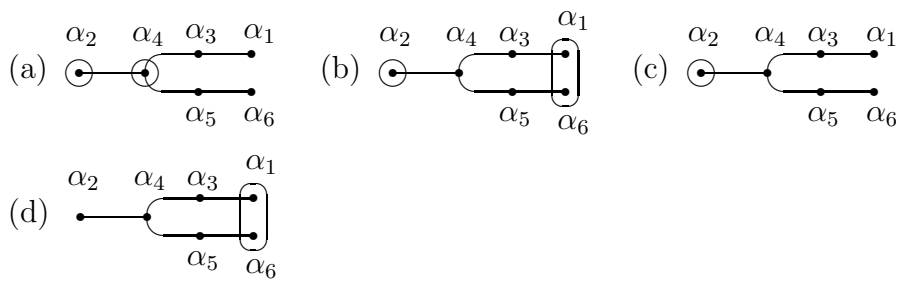
Proof. There exists the canonical embedding $H_0 \to G_0$ where H_0 is a split group of type F_4. It induces the mapping $H^1(F, H_0) \to H^1(F, G_0)$ which is surjective, by [4, page 51, Exercise 22.9], since F is 2-special. Each class in $H^1(F, H_0)$ is split over a quadratic extension E/F, by property (a) in Section 13, hence so is $[\xi]$. By [2, Lemma 6.4], G is isotropic over F. □

Lemma 15.3 implies that $X_{S_2}(F) \neq \emptyset$ and hence $\mathcal{A}(X_{S_2})$ is connected. Furthermore, if G is not split, then, by Lemma 15.1, its anisotropic kernel is isomorphic to $\mathrm{Spin}(f)$ where f is a 3-Pfister form f. Hence $\mathcal{A}(X_{S_1}) = \mathcal{A}(Y)$ where Y is the quadric defined by $f = 0$.

16. Outer forms of type E_6

If $p \neq 2$, then G_0 and G are inner forms of type E_6. So we may assume that $p = 2$. Then F is perfect since by our assumption $\mathrm{char}(F) \neq 2$.

16.1. **Basic types.** Let $K = F(\sqrt{d})$ be the quadratic extension which makes G_0 (and hence G) an inner form of type E_6. We denote by σ a unique nontrivial automorphism of K/F. According to Tits' classification [28] if G is F-isotropic but not quasi-split, then its F-index is one of the following:



Lemma 16.1. *The Tits index of G is not of the form* (a).

Proof. In case (a) the anisotropic kernel of G has type $\mathsf{A}_2 + \mathsf{A}_2$ and a maximal F-split torus C_1 has dimension 2. By dimension count C_1 coincides with the center C of its centralizer; in particular C and all its quotients have trivial cohomology in dimension 1. Hence, by Theorem 3.3, the anisotropic kernel of G is a strongly inner form of $R_{K/F}(\mathbf{SL}_3)$. As $H^1(F, R_{K/F}(\mathbf{SL}_3)) = 1$, the group $R_{K/F}(\mathbf{SL}_3)$ has no nontrivial strongly inner forms. □

Thus there are four basic types only:

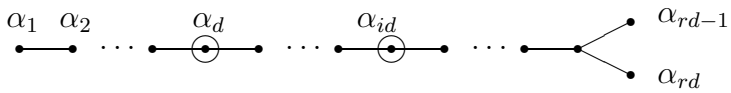
$$S_1 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\},\ S_2 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\},\ S_3 = \{\alpha_3, \alpha_4, \alpha_5\},\ S_4 = \emptyset.$$

According to Lemma 15.3, G is isotropic over K. If it is not K-split, then its Tits index is given by diagram (a) in Section 15 and hence its anisotropic kernel over K is isomorphic to $\mathbf{Spin}(f)$ where f is a 3-Pfister form over K. Since f can be split by a quadratic extension of K, we obtain that $n(X_{S_4})$ divides 4.

To compute the numbers $n(X_S)$ we need information on groups of type D_n.

16.2. **Classical groups of type D_n.** Recall that a simple simply connected F-group of type ${}^1\mathsf{D}_n$ can be realized as a spinor group $\mathbf{Spin}(A, \tau)$ associated to a central simple algebra A over F equipped with an involution τ of the first kind, orthogonal type and trivial discriminant. If A is represented by a skew field D in the Brauer group $\mathrm{Br}\,(F)$, then $A \simeq \mathrm{End}_D(V)$ where V is a right D-module, and τ corresponds to a non-degenerate skew-hermitian form h of trivial discriminant on V with respect to a suitable involution τ' on D of symplectic type.

$\mathbf{Spin}(A, \tau)$ is isotropic over F if and only if h is isotropic, i.e. there is a vector $v \in V$ such that $h(v, v) = 0$. Tits index of $\mathbf{Spin}(A, \tau)$ over F is of the form



Here $d = \deg(D)$, i is the Witt index of h and $rd = n$.

16.3. **Trialitarian effect.** We keep the above notation. Assume that we are given a pair (A, τ) such that A has index 2 and degree 8. Consider the even Clifford algebra of (A, τ). It is of the form $A_1 \times A_2$ where A_1, A_2 are central simple algebras of degree 8 equipped with involutions τ_1, τ_2 of orthogonal type and $A \cdot A_1 \cdot A_2 = 1$ in the Brauer group $\mathrm{Br}\,(F)$ (see [10]). The center Z of $\mathbf{Spin}(A, \tau)$ is isomorphic to $Z \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. Three nontrivial elements in Z give rise to the quotient morphisms $\phi_i : \mathbf{Spin}(A, \tau) \to \mathbf{O}^+(A_i, \tau_i)$, $i = 1, 2$ and $\phi : \mathbf{Spin}(A, \tau) \to \mathbf{O}^+(A, \tau)$. If $\mathbf{Spin}(A, \tau)$ is isotropic over F, there are two possibilities for its Tits index:

In the first case all algebras A, A_1, A_2 have index 2 and in the second case one of the algebras A_1, A_2 is split and a quadratic form f corresponding to its involution is isotropic. Note that according to the above picture isotropy of f implies automatically that its Witt index is 2.

Lemma 16.2. *Assume that A, A_1, A_2 have indices at most 2. Then there is a quadratic extension L/F such that $\mathbf{Spin}(A, \tau)$ has rank at least 2 over L.*

Proof. If all algebras A, A_1, A_2 are matrix algebras, then $G \simeq \mathbf{Spin}(f)$ where f is a 3-Pfister form. Such a form can be split by a quadratic extension.

Assume that A has index 2. Consider the pair (A_1, τ_1) and the corresponding involution variety Y. As $n(Y) = 2$, there is a quadratic extension L/F belonging to $\mathcal{A}(Y)$. The remark above the lemma shows that L-rank of $\mathbf{Spin}(A, \tau)$ is at least 2. □

16.4. **Computing $n(X_S)$.**

Proposition 16.3. *We have $n(X_{S_1}) = n(X_{S_2}) = n(X_{S_3}) = 2$.*

Proof. We consider the most difficult case when G is anisotropic over F. Isotropic cases are reduced to groups of classical types and can be treated much easier.

We may assume that G is not split over K, since otherwise $X_{S_i}(K) \neq \emptyset$, $i = 1, \dots, 4$ and we are done. Then its Tits K-index is given by diagram (a) in Section 15 and so the variety of parabolic subgroups in G of type $S_1 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ contains a K-point. By Lemma 3.4, there is a parabolic subgroup P in G over K of type S_1 such that $H_1 = P \cap \sigma(P)$ is a reductive part of P.

H_1 is defined over F and is an almost direct product of a simple simply connected F-group of classical type D_4 and a 2-dimensional central torus T_1. Since T_1 is F-anisotropic and split over K, σ acts on the character lattice of T_1 by multiplication -1. Hence each 1-dimensional subtorus in T_1 is F-defined.

Let $\tilde{\alpha}$ be the highest root in Σ with respect to the ordering given on diagram (a) in Section 15. The restriction mapping $\tilde{\alpha}|_{T_1} : T_1 \to G_m$ is a nontrivial homomorphism. Let T_2 be its kernel. It has dimension 1 and, by construction, T_2 commutes with $U_{\pm\tilde{\alpha}}$. As T_2 is defined over F, so is $H_2 = C_G(T_2)$ which is a reductive group whose semisimple part $[H_2, H_2]$ is a simple simply connected group of classical type D_5 generated by roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \tilde{\alpha}$.

Let us describe the structure of $[H_2, H_2]$. Note first that it has K-rank 1 (and hence K-isotropic), since T_1 is contained in $H_2 = C_G(T_2)$ and T_1 is a maximal K-split torus in G of dimension 2. Furthermore, up to isogeny, we have $[H_2, H_2] \simeq \mathbf{SU}(D, h)$ where D is a skew field over F equipped with a symplectic involution τ and h is a skew hermitian form over D with respect to τ. As $[H_2, H_2]$ is K-isotropic and its anisotropic semisimple kernel over K coincides with that of G (it is a group of type D_4 generated by $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$), Theorem 3.3 applied to G over K shows that this anisotropic kernel has trivial Tits algebras over K. It follows that the discriminant of h is trivial over K, the algebra D splits over K and hence D is either F or a quaternion skew field.

We claim that the discriminant of h is d. Indeed, we know that it is trivial over K, hence we need only to exclude the possibility for it to be trivial over F. Let Z be the variety of Borel subgroups of $[H_2, H_2]$. If the discriminant of h is trivial over F, then $[H_2, H_2]$ is a group of inner type and hence it has rank 5 over a field extension $F(Z)/F$. This implies automatically that G is split over $F(Z)$. But F is algebraically closed in $F(Z)$, hence K is not contained in $F(Z)$. It follows that G is still a group of outer type over $F(Z)$ – a contradiction.

Assume now that D is a quaternion skew field. By [13, Lemma 1], h can be written in the form $h = h_1 \oplus h_2$ where h_2 is a 1-dimensional skew-hermitian form over D isotropic over K and h_1 is a 4-dimensional form with trivial discriminant. Lemma 16.2 applied to (D, h_1) shows that there is a quadratic extension L/F such that L-rank of $\mathbf{SU}(D, h_1)$ is at least 2. Note that the conditions of this lemma are satisfied since the anisotropic kernel of $\mathbf{SU}(D, h_1)$ over K is a group of classical type D_4 with trivial Tits algebras. An inspection of possible Tits indices of G over L shows that G is either quasi-split over L or has Tits index (b) in the above list of Tits indices of outer type E_6. In all cases we have $X_{S_1}(L) \neq \emptyset$, $X_{S_2}(L) \neq \emptyset$, $X_{S_3}(L) \neq \emptyset$ and hence $n(X_{S_1}) = n(X_{S_2}) = n(X_{S_3}) = 2$.

If $D = F$ and h is a 10-dimensional quadratic form over F, we can apply the same argument as above. Namely, since h is isotropic over K, we can write it in the form $h = h_1 \oplus h_2$ where h_2 is a 2-dimensional quadratic form splitting over K and h_1 is an 8-dimensional quadratic form with trivial discriminant. Arguing as in Lemma 16.2 we easily see that h_1 has rank at least 2 over a proper quadratic extension L/F. So the result follows. □

16.5. **Basic type S_1.** Let $L \in \mathcal{A}(X_{S_1})$ and $[L : F] = 2$. As $K \in \mathcal{A}(X_{S_1})$, it suffices to show that L and K are X_{S_1}-equivalent. We shall construct an F-subgroup $H \subset G$ of classical type D_5 with the properties:

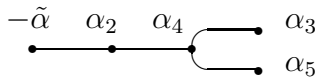
1) *there is an F-embedding $Y \to X_{S_1}$, where Y is the involution variety corresponding to H;*
2) $L, K \in \mathcal{A}(Y)$.

Property 2) implies that L and K are Y-equivalent, by Theorem 10.3, and property 1) implies that L and K are X_{S_1}-equivalent.

To construct H with the required properties we repeat the argument in 16.4 by verbatim. Namely, G contains a parabolic subgroup P of type S_1 over L. We may assume that P is standard in the corresponding ordering of the root system of G. Let τ be the nontrivial automorphism of L/F. Without lost of generality we may assume that $H_1 = P \cap \tau(P)$ is a reductive part of P, by Lemma 3.4. It is an F-reductive group whose semisimple part is a simple simply connected group of type D_4 generated by roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ and whose central torus T_1 has dimension 2. This torus is isotropic over L and splits over $K \cdot L$.

Let $T_2 \subset T_1$ be the kernel of the restriction mapping $\tilde{\alpha}|_{T_1}$ where $\tilde{\alpha}$ is the highest root in the chosen ordering of the root system of G. Below we shall show that T_2 is F-defined. Then the centralizer $H_2 = C_G(T_2)$ is a reductive F-group whose semisimple part $H = [H_2, H_2]$ is a simple simply connected group of type D_5 over F. We claim that it has the required properties.

Let us check property 1). The Dynkin diagram of H is of the form



The corresponding involution variety Y is the variety of parabolic subgroups in H of type $S_1 = \{\alpha_2, \alpha_4, \alpha_3, \alpha_5\}$. The subgroup $P_H = P \cap H$ of H is parabolic of type S_1. Then the required embedding is given by $H/P_H \to G/P$, $hP_H \to hP$. It is well defined and it is easy to check that it is F-defined using the twisted argument.

To check 2) we need information about the structure of T_1. It is proven in [1, Theorem 2.12], that T_1 viewed over L is isomorphic to $T_1 \simeq R_{E/L}(\mathbf{G}_{m,E})$ where $E = L \cdot K$. Thus T_1 viewed over L is an almost direct product of a 1-dimensional split L-subtorus, say V_1, and a 1-dimensional L-anisotropic torus $V_2 = R^{(1)}_{E/L}(\mathbf{G}_{m,E})$. It follows that $\mathrm{Gal}(L/F)$ preserves V_1, V_2 and hence both of them are F-defined. Note that computation in [1] shows that V_1 is contained in H and commutes with the subgroup of H generated by $\alpha_2, \alpha_4, \alpha_3, \alpha_5$. The last implies that H is L-isotropic and $L \in \mathcal{A}(Y)$.

Lemma 16.4. *We have $V_2 = T_2$ and T_2 splits over K.*

Proof. The 2-torsion part of T_1 is of the form $R_{E/L}(\mu_2)$. One easily checks the centralizer of $-1 \in R_{E/L}(\mu_2)$ in G is $H \cdot T_2$, hence T_2 is L-defined. But T_1 contains only two L-defined subtori, namely V_1 and V_2. As $V_1 \subset H$, we get $T_2 = V_2$.

Assume that T_2 is not split over K. Let Z be the variety Z of Borel subgroups of H over K. The group H being quasi-split over $K(H/B)$ contains a split torus over $K(H/B)$, say T_3, of dimension at least 4. One easily checks that $C_G(T_3)$ is a maximal torus in G containing T_2. As K is algebraically closed in $K(Z)$, the torus T_2 is still anisotropic over $K(Z)$ and hence $C_G(T_3)$ does not split over $K(Z)$.

On the other side, as rank of G over $K(H/B)$ is at least 4, Tits classification implies that G is split over $K(H/B)$ (see the diagrams in Section 15) and hence so is the torus $C_G(T_3)$ because any $K(Z)$-split torus in G is contained in a maximal $K(Z)$-split torus of G. This contradicts our assumption that T_2 is anisotropic over $K(H/B)$. □

As T_2 is K-split, the K-anisotropic kernel of G is contained in $H_2 = C_G(T_2)$, hence G and H have the same K-anisotropic kernel. Then Tits classification of groups of inner type E_6 shows that $K \in \mathcal{A}(Y)$.

16.6. **Basic type** S_2**.** Assume that $L \in \mathcal{A}(X_{S_2})$ and $[L : F] = 2$. Let τ be the nontrivial automorphism of L/F. Choose a parabolic subgroup $P \subset G$ over L of type S_2 such that $H' = P \cap \tau(P)$ is a reductive part of P. It is a reductive subgroup of G over F whose semisimple part $[H', H']$ has type A_5 and is generated by roots $\alpha_1, \alpha_3, \ldots, \alpha_6$ in a proper ordering of the corresponding root system of G. The centralizer $H'' = C_G([H', H'])$ is an F-defined subgroup of G of type A_1 containing the central torus of H'. It is generated by the root subgroups $U_{\pm\tilde{\alpha}}$ where $\tilde{\alpha}$ is the highest root.

Consider $H = H'' \cdot [H', H']$. It is an F-defined semisimple subgroup of G of type $\mathsf{A}_1 + \mathsf{A}_5$ whose component of type A_1 is split over L and is of the form $\mathrm{SL}_1(D)$ where D is a quaternion algebra over F. It follows that $L \in \mathcal{A}(SB(D))$. Conversely, any splitting field of D is contained in $\mathcal{A}(X_{S_2})$.

Let L_1 be another quadratic extension of F such that $L_1 \in \mathcal{A}(X_{S_2})$. We are going to show that $L_1 \in \mathcal{A}(SB(D))$. Theorem 7.2 then implies $L_1 \sim_{SB(D)} L$ and hence $L_1 \sim_{X_{S_2}} L$.

Applying the above construction to L_1 we can construct an F-subgroup H_1 of G of type $\mathsf{A}_1 + \mathsf{A}_5$. Its component of type A_1 correspond to quaternion algebras D_1 splitting over L_1 and it suffices to show that $D_1 = D$.

H_1 and H are related to subsystems of type $\mathsf{A}_1 + \mathsf{A}_5$ in the root system E_6. Since every two roots in the root system of type E_6 are conjugate by an element in the corresponding Weyl group $W(\mathsf{E}_6)$, so are every two root subsystems of type $\mathsf{A}_1 + \mathsf{A}_5$. Then H_1, H are conjugate over F_{sep}. Let $H_1 = gHg^{-1}$ where $g \in G(F_{sep})$. Fixing F-defined maximal tori T_1, T in H_1, H respectively, we may assume additionally that $T_1 = gTg^{-1}$. As H_1, H, T_1, T are F-defined, for each $\sigma \in \mathrm{Gal}(F_{sep}/F)$ we have $a_\sigma = g^{-\sigma+1} \in N_G(H) \cap N_G(T)$. It follows that $H_1 = {}^\lambda H$ where $\lambda = (a_\sigma)$; in particular we have $D_1 = {}^\lambda D$.

The torus T can be decomposed as $T = T'T''$ where T', T'' are intersections of T with simple components of H of types A_1 and A_5 respectively. Since groups of type A_1 have no nontrivial outer automorphisms, one checks that a_σ can be decomposed as $a_\sigma = a'_\sigma a''_\sigma$ where a'_σ and a''_σ are contained in the normalizers of T' and T'' in the simple components of H respectively. Thus twisting of D is given actually by the cocycle $\lambda' = (a'_\sigma)$. This cocycle takes values in $\mathbf{SL}_1(D)$ and hence twisting does not change D since any cocycle in $Z^1(F, \mathbf{SL}_1(D))$ is equivalent to a cocycle with coefficients in the center μ_2 of $\mathbf{SL}_1(D)$.

16.7. **Basic type** S_3**.** Let $L_1, L_2 \in \mathcal{A}(X_{S_3})$ be two quadratic extensions of F contained in $\mathcal{A}(X_{S_3})$. As $\mathcal{A}(X_{S_3}) \subset \mathcal{A}(X_{S_2})$, the above construction applied to L_1 or L_2 gives us an F-defined subgroup H of G of type $\mathsf{A}_1 + \mathsf{A}_5$. Its simple components, say H' and H'', are of the form $H' = \mathbf{SL}_1(D)$ and $H'' = \mathbf{SU}(A, \tau)$ where D is a quaternion algebra over F splitting over L_1, L_2 and A is a central simple algebra of degree 6 over K equipped with an involution τ of the second kind.

Any field extension L/F splitting D makes H'' a strongly inner form of a quasi-split group of type A_5, by Theorem 3.3, hence it also splits A. Then

$[A] = [D \otimes_F K]$ in the Brauer group $\mathrm{Br}(K)$ and hence $H'' \simeq \mathbf{SU}(T, h)$ where $T = D \otimes_F K$ and h is a skew-hermitian form of dimension 3 given on a vector space V over T equipped with the standard involution, say σ, of the second kind.

We are also given that H'' is isotropic over L_1, L_2, i.e. h represents zero over L_1, L_2. Since L_1, L_2 splits T and have degree 2 over F, there are 1-dimensional T-submodules U_1, U_2 in V such that h restricted to U_1, U_2 is isotropic over L_1 and L_2 respectively. This fact can be reformulated as follows. Let $v_1 \in U_2$ and $v_2 \in U_2$ be two nontrivial vectors. Let σ_1, σ_2 be involutions on T corresponding to the elements $a_1 = h(v_1, v_1)$ and $a_2 = h(v_2, v_2)$, i.e. σ_1, σ_2 are compositions of σ with the inner conjugation given by a_1, a_2. Then $\mathbf{SU}(T, \sigma_1)$ and $\mathbf{SU}(T, \sigma_2)$ are split over L_1 and L_2 respectively.

Clearly $\mathbf{SU}(T, \sigma_1)$ and $\mathbf{SU}(T, \sigma_2)$ are subgroups in $H'' = \mathrm{SU}(T, h)$ and being groups of type A_1 they are of the form $\mathbf{SL}_1(D_1)$ and $\mathbf{SL}_1(D_2)$ where D_1, D_2 are quaternion algebras over F contained in T such that $D_1 \otimes K = D_2 \otimes K = T$. Recall that the pure part D_i° of D_i is given by

$$D_i^\circ = \{\, x \in T \mid \sigma_i(x) = -x\} = \{\, x \in D^\circ \otimes K \mid a_i \sigma(x) a_i^{-1} = -x\}.$$

We now note that v_i is defined up to $t_i \in T$, hence D_i is defined up to conjugation in T. Since by our construction, D_i and D are split over L_i, we can modify v_i, a_i and D_i (by replacing them with $t_i v_i$, $t_i a_i \sigma(t_i)$ and $t_i D_i t_i^{-1}$ for a proper element $t_i \in T$) such that $D \cap D_i$ contains a common subfield isomorphic to L_i.

More precisely, this common subfield can be obtained as follows. As a_i is skew with respect to σ we can write it in the form $a_i = a_i' + a_i''$ where $a_i' \in K$ and $a_i'' \in D^\circ$. Modifying a_i, if necessary, we may assume that $a_i'' \neq 0$. Then we have

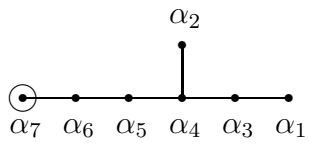
$$D^\circ \cap D_i^\circ = \{\, d \in D^\circ \mid \sigma_i(d) = -d\} = \{\, d \in D^\circ \mid a_i d a_i^{-1} = d\}.$$

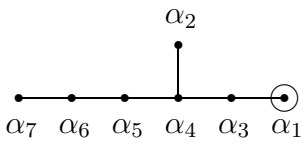
It follows that $D^\circ \cap D_i^\circ$ has dimension 1 and is generated by a_i''.

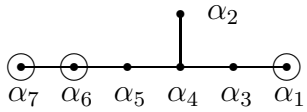
We are ready to finish the proof. Consider a vector $v(t) = (1-t)v_1 + tv_2$ where t is an indeterminate. Let $a(t) = h(v(t), v(t))$. Write $a(t) = a(t)' + a(t)''$ as above. Here $a(t)' \in K \otimes_F F(t)$ and $a(t)'' \in D^\circ \otimes_F F(t)$. Clearly, $a(t)''$ generate a maximal subfield $L(t)$ in $D \otimes_F F(t)$ splitting $D \otimes_F F(t)$ and making H'' isotropic. This implies that $L(t) \in \mathcal{A}(X_{S_3})$. Since $v(0) = v_1$ and $v(1) = v_2$, arguing as in Theorem 9.2 we see that $L(t)$ has two discrete valuations with residue fields isomorphic to L_1, L_2.

17. Type E_7

By Tits classification [28], if G is isotropic over an extension L/F but not split, then its Tits index is one of the following:

a) 

b) 

c) 

Thus we need to consider the following basic types only:

$$S_1 = \{\alpha_1, \dots, \alpha_6\}, \quad S_2 = \{\alpha_2, \dots, \alpha_7\}, \quad S_3 = \{\alpha_2, \dots, \alpha_5\}, \quad S_4 = \emptyset.$$

If $p = 3$, then we can repeat the same argument as in the case of type ${}^1\mathsf{E}_6$. Consider the case $p = 2$.

Lemma 17.1. *Let F be 2-special. Then we have $n(S_1) = n(S_2) = n(S_3) = 2$.*

Proof. By [31], G contains an F-defined subgroup H of type E_6. It is a strongly inner form of a quasi-split group of type E_6, since F is 2-special. We know that any such a group (and hence G) has rank at least 2 over a proper quadratic extension E/F. The above diagrams show that $X_{S_i}(E) \neq \emptyset$ for $i = 1, 2, 3$. □

By Lemma 15.3, if E/F is a quadratic extension such that $X_{S_1}(E) \neq \emptyset$, then $X_{S_3}(E) \neq \emptyset$. Hence it suffices to consider types S_2, S_3 only.

17.1. **Basic type S_2.** Repeating verbatim the argument in 16.6 we find that G contains an F-defined subgroup H of type $\mathsf{A}_1 + \mathsf{D}_6$ such that $\mathcal{A}(X_{S_2}) = \mathcal{A}(SB(D))$ where D is a quaternion algebra over F corresponding to the simple component of H of type A_1. The connectedness of $\mathcal{A}(S_2)$ follows from Theorem 7.2.

17.2. **Basic type S_3.** Let L/F be a quadratic extension belonging to $\mathcal{A}(X_{S_3})$. Let H_1 and H_2 be the simple components of types A_1 and D_6 respectively of the group H constructed in 17.1. We have $H_1 \simeq \mathbf{SL}_1(D)$ where D is a quaternion algebra over F. The above Tits diagrams show that the anisotropic L-kernel of H_2 is of the form $\mathbf{Spin}(f)$ where f is a 4-Pfister form. It follows that up to isogeny H is of the form $\mathbf{SU}(T, h)$ where T is a quaternion algebra over F equipped with the standard symplectic involution τ and h is a skew hermitian form with respect to τ defined over a 6-dimensional vector space V over T.

Lemma 17.2. *If T is nontrivial, then we have $D = T$.*

Proof. If E/F is an extension splitting D, then H_2 viewed over E is a strongly inner form of a split group of type D_6. Hence H_2 is of the form $\mathbf{Spin}(g)$ where g is a 6-dimensional quadratic form from I^3. It follows that T is split over E.

As this is true for all splitting fields of D, the quaternion algebras D and T coincide. $\square$

Let Y be the involution variety corresponding to $\mathrm{SO}(D,h)$. By our construction, $L \in \mathcal{A}(Y)$. Conversely, let $E \in \mathcal{A}(Y)$. We noted above that H_2 viewed over E is related to the quadratic form g of dimension 12 from I^3. Since g is isotropic over E and since 10-dimensional quadratic forms from I^3 are isotropic, we conclude that E-rank of H_2 is at least 2. This implies that E-rank of G is at least 2. An inspection of the above Tits diagrams shows that $E \in \mathcal{A}(X_{S_3})$. Thus we showed that $\mathcal{A}(Y) = \mathcal{A}(X_{S_3})$ and the connectedness of $\mathcal{A}(Y)$ follows from Theorem 10.3.

18. Two examples of a nontrivial $\overline{\mathrm{CH}}_0(X)$

In this section we give two examples of projective homogeneous varieties X with nontrivial group $\overline{\mathrm{CH}}_0(X)$.

18.1. **First Example.** We owe to Vishik the observation that [9] essentially contains an example. Let F be a field of characteristic different from 2 and let $a, b \in F^\times$. Let $Q_{a,b}$ be the quadric of the 2-fold Pfister form $\langle\langle a, b\rangle\rangle = \langle 1, -a, -b, ab\rangle$. In the split case (over a quadratic extension E/F), $Q_{a,b}$ is isomorphic to $\mathbb{P}^1_F \times \mathbb{P}^1_F$, in particular, there are two families of lines on $Q_{a,b}$. Let $l_{a,b}$ be one of these lines and denote by $u_{a,b} \in \mathrm{CH}_1(Q_{a,b})$ the push-forward of the class of $l_{a,b}$ under the morphism $(Q_{a,b})_E \to Q_{a,b}$.

Let $X_{a,b}$ be the variety of isotropic planes of $\langle\langle a, b\rangle\rangle$. Over the field $E \in \mathcal{A}(X_{a,b})$, the variety $X_{a,b}$ is the disjoint union of two copies of the projective line. We can view $l_{a,b}$ as a rational point of $X_{a,b}$ over E. Denote the push-forward of the class of $l_{a,b}$ in $\mathrm{CH}_0(X_{a,b})$ by $x_{a,b}$, so $x_{a,b}$ is a 0-cycle of degree 2. The incidence correspondence between $Q_{a,b}$ and $X_{a,b}$ induces a map $\mathrm{CH}_0(X_{a,b}) \to \mathrm{CH}_1(Q_{a,b})$ taking $u_{a,b}$ to $x_{a,b}$.

Let $q = \langle 1, -a, -b, ab, -c, -d, cd\rangle$ for some $a, b, c, d \in F^\times$ and let Q be the corresponding 5-dimensional quadric. We have natural embeddings $Q_{a,b} \hookrightarrow Q$ and $Q_{c,d} \hookrightarrow Q$. Denote the images of $u_{a,b}$ and $u_{c,d}$ in $\mathrm{CH}_1(Q)$ by the same symbols.

It is shown in [9, Cor. 5.2] that in the case $-1 \in F^{\times 2}$, the classes $u_{a,b}$ and $u_{c,d}$ are equal if and only if the Pfister form $\langle\langle a, b, c, d\rangle\rangle$ is isotropic.

Denote by X the variety of isotropic planes of q. The images of $x_{a,b}$ and $x_{c,d}$ in $\mathrm{CH}_0(X)$ will be denoted by the same symbols. We have the incidence correspondence between Q and X that induces a homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_1(Q)$ taking $x_{a,b} - x_{c,d}$ to $u_{a,b} - u_{c,d}$.

If F is a field such that $-1 \in F^{\times 2}$ and the Pfister form $\langle\langle a, b, c, d\rangle\rangle$ is anisotropic, the class $u_{a,b} - u_{c,d}$ is nontrivial and therefore $x_{a,b} - x_{c,d} \neq 0$. As $\deg(x_{a,b} - x_{c,d}) = 0$, the class $x_{a,b} - x_{c,d}$ represents a nontrivial element of $\overline{\mathrm{CH}}_0(X)$. Note that X corresponds to the subset $\{\alpha_1, \alpha_3\}$ of the set of vertices of the Dynkin diagram B_3.

18.2. **Second Example.** We shall give an example of a 3-dimensional projective homogeneous variety X with a nontrivial group $\overline{\mathrm{CH}}_0(X)$. Note that this is an example of the smallest possible dimension in view of Proposition 4.5.

Let C_i, $i = 1, 2, 3$, be three conic curves corresponding to quaternion algebras Q_i over F and let $X = C_1 \times C_2 \times C_3$. For every subset $S \subset \{1, 2, 3\}$ let Q_S be the tensor product of Q_i for all $i \in S$ (in particular, $Q_\emptyset = F$). We define the complex

$$\coprod_{S\neq\emptyset} K_1(Q_S) \xrightarrow{\alpha} (F^\times)^3 \xrightarrow{\beta} H^3(F, \mathbb{Z}/2\mathbb{Z}) \tag{18.1}$$

by the following formulas:

$$\alpha(x) = (a_1, a_2, a_3), \quad a_i = \prod_{S\ni i} \mathrm{Nrd}_S(x_S),$$

$$\beta(a_1, a_2, a_3) = (a_1) \cup [Q_1] + (a_2) \cup [Q_2] + (a_3) \cup [Q_3],$$

where $\mathrm{Nrd}_S : K_1(Q_S) \to K_1(F) = F^\times$ is the reduced norm homomorphism, (a) denotes the class in $H^1(F, \mathbb{Z}/2\mathbb{Z})$ corresponding to $a \in F^\times$ under the Kummer isomorphism and $[Q]$ denotes the class of Q in $H^2(F, \mathbb{Z}/2\mathbb{Z})$.

Proposition 18.2. *The group $\overline{\mathrm{CH}}_0(X)$ is isomorphic to the homology group of the complex* (18.1).

Proof. In the spectral sequence (4.4) for X the only possibly non-trivial differential coming to $E_*^{3,-3}$ is $d_2 : A^1(X, K_2) = E_2^{1,-2} \to E_2^{3,-3} = \mathrm{CH}_0(X)$ and all the differentials coming to $E_*^{1,-2}$ are trivial. Therefore, the sequence

$$K_1(X)^{(1)} \to A^1(X, K_2) \xrightarrow{d_2} \mathrm{CH}_0(X) \xrightarrow{e'} K_0(X)$$

is exact. As the image of d_2 coincides with $\mathrm{Ker}(e') = \overline{\mathrm{CH}}_0(X)$, it is sufficient to identify the cokernel of the first homomorphism in the sequence with the homology group of the complex (18.1).

The group $A^1(X, K_2)$ was computed in [12]. If all Q_i are split, we have

$$A^1(X, K_2) = K_1(F) \otimes \mathrm{CH}^1(X) \simeq F^\times \otimes \mathbb{Z}^3 = (F^\times)^3,$$

where we identify $\mathrm{CH}^1(X)$ with $\mathrm{Pic}(X) = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \oplus \mathbb{Z}h_3 \simeq \mathbb{Z}^3$ (here h_i is the class of a rational point in C_i). In the general case the group $A^1(X, K_2)$ is canonically isomorphic to a subgroup of $(F^\times)^3$, namely to the kernel of the homomorphism β.

The group $K_1(X)$ was computed in [21, §8, Th. 4.1]:

$$K_1(X) = \coprod_S K_1(Q_S). \tag{18.3}$$

The first term of the topological filtration $K_1(X)^{(1)}$ is the kernel of the natural homomorphism $K_1(X) \to K_1F(X)$. We claim that for every S, the corresponding homomorphism $K_1(Q_S) \to K_1F(X)$ is the reduced norm map Nrd_S followed by the natural homomorphism $K_1F \to K_1F(X)$. Since the group K_1 of a field injects when the field gets extended, we may assume that all Q_i are

split. In this case the reduced norm Nrd_S is an isomorphism and (18.3) identifies $K_1(X)$ with the direct sum of $K_1(F)L_S$, where L_S is the tensor product of the canonical line bundles L_i on the $C_i \simeq \mathbb{P}^1_F$ such that $i \in S$. As L_S maps to 1 in $K_0F(X)$, the claim follows.

We have shown that $K_1(X)^{(1)}$ consists of all $x = (x_S)$ such that $\sum \mathrm{Nrd}_S(x_S) = 0$. We claim that the image of x in $A^1(X, K_2) \subset (F^\times)^3$ is equal to (a_1, a_2, a_3), where a_i is the product of $\mathrm{Nrd}_S(x_S)$ for all $S \ni i$. We can assume that all Q_i are split. Under the identification of $K_1(Q_S)$ with $K_1(F)$, the element x corresponds to $\sum b_S L_S = \sum_{S\neq\emptyset} b_S(L_S - 1)$, where $b_S = \mathrm{Nrd}_S(x_S)$. Note that the image of $L_S - 1$ under the natural map $K_0(X)^{(1)} \to \mathrm{CH}^1(X)$ is equal to the first Chern class $c_1(L_S) = \sum_{i\in S} c_1(L_i) = \sum_{i\in S} h_i$. Therefore, the image of x in $A^1(X, K_2) = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \oplus \mathbb{Z}h_3$ is equal to $a_1h_1 \oplus a_2h_2 \oplus a_3h_3$, where a_i is the product of b_S for all $S \ni i$.

It follows from the claim that the image of the homomorphism $K_1(X)^{(1)} \to A^1(X, K_2) \subset (F^\times)^3$ is equal to $\mathrm{Im}\,\alpha$. □

Example 18.4. Let $F = k(\!(t)\!)$ be the Laurent power series field over a field k of characteristic different from 2, $Q_i = (b_i, -t)_F$ for some $b_i \in k^\times$. Let b_S be the product of b_i for all $i \in S$, $k_S = k(b_S^{1/2})$ and $\mathrm{Norm}_S : k_S^\times \to k^\times$ be the norm homomorphism. Since $(t^n, t^m, t^p) \in \mathrm{Im}(\alpha)$ for all n, m, p, $H^p(F, \mathbb{Z}/2\mathbb{Z})$ is equal to $H^p(k, \mathbb{Z}/2\mathbb{Z}) \oplus H^{p-1}(k, \mathbb{Z}/2\mathbb{Z}) \cup (-t)$ and $\mathrm{Im}\,\mathrm{Nrd}_S = \mathrm{Im}\,\mathrm{Norm}_S \cdot F^{\times 2} \cup \mathrm{Im}\,\mathrm{Norm}_S \cdot t \cdot F^{\times 2}$, the homology group of (18.1) is isomorphic to the homology group of the complex

$$\coprod_{S\neq\emptyset} k_S^\times \xrightarrow{\alpha'} (k^\times)^3 \xrightarrow{\beta'} H^2(k, \mathbb{Z}/2\mathbb{Z}),$$

where

$$\alpha'(x) = (a_1, a_2, a_3), \quad a_i = \prod_{S\ni i} \mathrm{Norm}_S(x_S),$$

$$\beta'(a_1, a_2, a_3) = (a_1) \cup (b_1) + (a_2) \cup (b_2) + (a_3) \cup (b_3).$$

There is a field k and the elements b_i so that the complex is not acyclic (see [24, §5]) and therefore $\overline{\mathrm{CH}}_0(X) \neq 0$ in this case.

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