PROJECTIVE SCHUR GROUPS OF HENSELIAN FIELDS

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Abstract.

One of the open questions that has emerged in the study of the projective Schur group $PS(F)$ of a field F is whether or not $PS(F)$ is an algebraic relative Brauer group over F, i.e. does there exist an algebraic extension L/F such that $PS(F) = Br(L/F)$? We show that the same question for the Schur group of a number field has a negative answer. For the projective Schur group, no counterexample is known. In this paper we prove that $PS(F)$ is an algebraic relative Brauer group for all Henselian valued fields F of equal characteristic whose residue field is a local or global field. For this, we first show how $PS(F)$ is determined by $PS(k)$ for an equicharacteristic Henselian field with arbitrary residue field k.

1. Introduction

Let F be a field, $Br(F)$ its Brauer group. The *Schur group* $S(F)$ of F is the subgroup of $Br(F)$ consisting of classes represented by *Schur algebras* over F. A finite dimensional central simple F-algebra A is called Schur over F if it is a homomorphic image of a group algebra FG with G finite. Equivalently, A is Schur over F if it is spanned as a vector space over F by a finite subgroup $\mathcal{G}$ of the group A^* of invertible elements of A. In 1978 [LO], Lorenz and Opolka introduced projective analogues to these notions. They defined the *projective Schur group* $PS(F)$ of F to be the subgroup of $Br(F)$ consisting of classes represented by *projective Schur algebras* over F. A finite dimensional central simple F-algebra A is projective Schur over F if it is spanned as a vector space over F by a subgroup $\mathcal{G}$ of the group A^* of invertible elements of A which is finite modulo F^*, i.e. $\mathcal{G}F^*/F^*$ is finite. In either case, when a subgroup $\mathcal{G}$ of A^* spans A over F, we write $A = F(\mathcal{G})$.

In view of the fact that these two subgroups of $Br(F)$ are defined in the language of algebras, we can ask for a natural characterization of them in the language of Galois cohomology, just as $Br(F)$ is a Galois cohomology group $H^2(G_F, F_s^*)$, where F_s denotes the separable closure of F. In the case of the Schur group, the Brauer-Witt theorem can be viewed as a positive answer to this question: Let F_{cyc} be the maximal cyclotomic extension

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of F and let μ denote the group of all roots of unity in F_s. Then $S(F)$ is the image of the canonical map $H^2(\mathcal{G}(F_{cyc}/F), \mu) \longrightarrow H^2(G_F, F_s^*) \cong Br(F)$ (cf. [Y, Cor. 3.11]).

In the case of $PS(F)$, all known examples of projective Schur algebras are Brauer equivalent to *radical abelian algebras*, defined as follows: Let $A = (L/F, G, f)$ be a crossed product algebra, where L is a finite Galois extension field of F, G is the Galois group $\mathcal{G}(L/F)$, and $f \in H^2(G, L^*)$. Then A is said to be a *radical algebra* if $L = F(T)$ for some subgroup T of L^* containing F^* such that T/F^* is finite and f is represented by a 2-cocycle with values in T. This A is called a radical abelian algebra if in addition $\mathcal{G}(L/F)$ is abelian. It is easy to see that every radical algebra over F lies in $PS(F)$. The first two authors have conjectured that all projective Schur algebras are Brauer equivalent to radical algebras, and even to radical abelian algebras. The radical algebra conjecture is equivalent to the conjecture that $PS(F)$ is the image in $Br(F) = H^2(\mathcal{G}(F_s/F), F_s^*)$ of $H^2(\mathcal{G}(F(T_s)/F), T_s)$, where T_s is the subgroup of F_s^* consisting of elements of finite order modulo F^*. This would provide an analogue for $PS(F)$ of the Brauer-Witt theorem for $S(F)$. The radical abelian algebra conjecture has an analogous homological interpretation. The radical abelian algebra conjecture has been proved for all fields of nonzero characteristic [AS_4, Cor. 1.5]. In characteristic 0 only partial results are known [AS_1], [AS_3], [AS_4], [AS_6].

Another way of describing some subgroups of $Br(F)$ is as algebraic relative Brauer groups. Let M/F be a field extension. The *relative Brauer group* $Br(M/F)$ is the kernel of the restriction map $\text{res}_{M/F}\colon Br(F) \to Br(M)$. A subgroup H of $Br(F)$ is called an *algebraic relative Brauer group* if there exists an *algebraic* extension M/F such that $Br(M/F) = H$. It is known that every subgroup of $Br(F)$ is a relative Brauer group, by taking M to be an iterated generic splitting field of the division algebras in H, cf. [FS, Th. 1]; but it is not true in general that every subgroup is an algebraic relative Brauer group (e.g., let H be any nontrivial finite subgroup of $Br(F)$, if F is a global field [FKS, Cor. 4]). Of course $Br(F)$ itself is an algebraic relative Brauer group by definition. We ask if $S(F)$ and $PS(F)$ are algebraic relative Brauer groups. The answer is negative for $S(F)$ even for F a number field, as we will show in §6 below. For $PS(F)$ this question has an obvious affirmative answer for local and global fields F since in that case $PS(F) = Br(F)$ by [LO, Satz 3], or see [AS_2, p. 531]. There is no good reason to believe that $PS(F)$ is an algebraic relative Brauer group for every field F, but so far no counterexample has been found.

This paper is concerned with the radical (abelian) algebra conjecture and the algebraic relative Brauer group question for $PS(F)$ for fields F with Henselian valuation such that the residue field k has the same characteristic as F. We show in Cor. 4.6 that the radical (resp. radical abelian) conjecture holds for F if it holds for k. We prove in §5 that if k is a local or global field, then $PS(F)$ is an algebraic relative Brauer group.

The proofs of our main results require detailed information about the Brauer group of a Henselian valued field F, which we give in §3. Beyond the known results, which we recall, we construct explicit splitting maps for the inertially split part of $Br(F)$ and for the tame part of $Br(F)$. The splitting maps are used in §4 to show exactly how $PS(F)$ is built from $PS(k)$, where k is the residue field of the Henselian valuation on F (assuming $\operatorname{char}(k) = \operatorname{char}(F)$). This generalizes to arbitrary equicharacteristic Henselian fields results in [AS$_6$] for iterated power series fields.

Whenever a projective Schur algebra $F(\mathcal{G})$ has *abelian* finite group $\mathcal{G}F^*/F^*$, there is an associated symplectic pairing on $\mathcal{G}F^*/F^*$ given by commutators. We will show in §2 how such pairings and their associated Lagrangians can elucidate the structure of an arbitrary reduced projective Schur algebra. This provides a unified approach to a number of previous results on projective Schur algebras, as well as being needed for the analysis of the Henselian situation in §4.

We point out in passing that the algebraic relative Brauer group question has been studied for the m-torsion subgroups ${}_mBr(F)$ of $Br(F)$. In general ${}_mBr(F)$ is not an algebraic relative Brauer group. Counterexamples exist for F a power series field $k((t))$, with k a local field [AS$_5$, Sec. 4]. On the other hand, for F a global field ${}_mBr(F)$ is an algebraic relative Brauer group for every m, see [AS$_5$], [KS$_1$], [Po], [KS$_2$].

We will use the following notation throughout the paper: If C is a torsion abelian group, we write $\exp(C)$ for the exponent of C; ${}_nC$ for the n-torsion subgroup of C; and $C(p)$ for the p-primary component of C. For $c \in C$, $o(c)$ denotes the order of c. If C is associated to a field F, C' denotes the prime-to-p part of C if $\operatorname{char}(F) = p \neq 0$, while $C' = C$ if $\operatorname{char}(F) = 0$. We write $\mu(F)$ for the group of roots of unity in a field F; we write μ_n for the group of n n-th roots of unity. If S is a central simple algebra over F, $\deg(S) = \sqrt{\dim_F(S)}$ is the degree of S, and $\exp(S)$ is the exponent of S, which is the order of the class $[S]$ of S in the Brauer group $Br(F)$. If $\mu_n \subseteq F$ and $a, b \in F^*$ we write $(a, b; F)_n$ for the symbol algebra of dimension n^2 over F with generators i, j and relations $i^2 = a$, $j^2 = b$, and $ij = \omega ji$, where ω is some primitive n-th root of unity in F.

2. Projective Schur algebras of abelian type and Langrangians

A projective Schur algebra $A = F(\mathcal{A})$ is said to be *of abelian type* if the finite group $\mathcal{A}F^*/F^*$ is abelian. Associated to such an A is a nonsingular symplectic pairing on $\mathcal{A}F^*/F^*$. We will first recall some properties of such algebras, which can be seen easily by using this pairing. The data about the abelian case are relevant for more general projective Schur algebras because we will see in Prop. 2.2 below that every reduced projective Schur algebra admits after scalar extension a decomposition into a Schur algebra and a projective Schur algebra of abelian type. Furthermore, we will show that Lagrangian subgroups of $\mathcal{A}F^*/F^*$ with respect to the symplectic pairing yield useful refinements of such tensor decompositions. The results in this section provide a unified approach to arguments in several papers by the first two authors.

Proposition 2.1. *Let $A = F(\mathcal{A})$ be a projective Schur algebra of abelian type, where $F^* \subseteq \mathcal{A} \subseteq A^*$ (so $\mathcal{A}/F^*$ is a finite abelian group). Then,*

(a) *There is a well-defined pairing $B_{\mathcal{A}} \colon \mathcal{A}/F^* \times \mathcal{A}/F^* \to F^*$ given by $(aF^*, bF^*) \mapsto aba^{-1}b^{-1}$.*

(b) *The pairing $B_{\mathcal{A}}$ is nondegenerate, bimultiplicative and symplectic, and $\mathrm{im}(B_{\mathcal{A}})$ is a finite (cyclic) subgroup of $\mu(F)$.*

(c) $|\mathcal{A}/F^*| = \dim_F(A)$ *and* $\exp(\mathcal{A}/F^*) = |\mathrm{im}(B_{\mathcal{A}})|$.

(d) *$A \cong S_1 \otimes_F \cdots \otimes_F S_m$, where each S_i is a symbol algebra, with* $\exp(\mathcal{A}/F^*) = \underset{1\le i\le m}{\mathrm{lcm}} \deg(S_i)$.

Proof. This mostly known, cf. [LO, Hilfsatz 1] and [AS$_3$, Th. 1.1], but we will show how the use of the pairing $B_{\mathcal{A}}$ facilitates the proof. For $a \in \mathcal{A}$, let $\widetilde{a} = aF^* \in \mathcal{A}/F^*$. The pairing $B_{\mathcal{A}}$ is well-defined because F^* is central in A^*. That $B_{\mathcal{A}}$ is symplectic means that $B_{\mathcal{A}}(\widetilde{a}, \widetilde{a}) = 1$ for all $a \in \mathcal{A}$; this is evident here. Let $[a, b] = aba^{-1}b^{-1}$. Since $B_{\mathcal{A}}$ maps into a central subgroup of A^*, the commutator identities $[a, bc] = [a, b]b[a, c]b^{-1}$ and $[ab, c] = a[b, c]a^{-1}[a, c]$ show that $B_{\mathcal{A}}$ is bimultiplicative. Let $e = \exp(\mathcal{A}/F^*)$. The bimultiplicativity of $B_{\mathcal{A}}$ shows that $\mathrm{im}(B_{\mathcal{A}}) \subseteq \mu_e$. The nondegeneracy of $B_{\mathcal{A}}$ means that for any $a \in \mathcal{A}$, if $B_{\mathcal{A}}(\widetilde{a}, \widetilde{b}) = 1$ for all $b \in \mathcal{A}$, then $a \in F^*$; this holds as $Z(F(\mathcal{A})) = F$. Because $B_{\mathcal{A}}$ is nondegenerate, for any $a \in \mathcal{A}$ the function $f_{\widetilde{a}} \colon \mathcal{A}/F^* \to F^*$ given by $f_{\widetilde{a}}(\widetilde{b}) = B_{\mathcal{A}}(\widetilde{a}, \widetilde{b})$ must have the same order in the group $\mathrm{Hom}(\mathcal{A}/F^*, F^*)$ as the order $o(\widetilde{a})$ of $\widetilde{a}$ in $\mathcal{A}/F^*$. Hence if we take $\widetilde{a} \in \mathcal{A}/F^*$ with $o(\widetilde{a}) = e$ we see that $\mathrm{im}(f_{\widetilde{a}}) = \mu_e$; thus $\mathrm{im}(B_{\mathcal{A}}) = \mu_e$; so $|\mathrm{im}(B_{\mathcal{A}})| = e = \exp(\mathcal{A}/F^*)$. Since we have a nondegenerate bimultiplicative symplectic pairing defined on a finite abelian group, it is known (see, e.g., [Wa, Th. (3)]) that there is a symplectic base of $\mathcal{A}/F^*$, i.e., a generating set $\widetilde{a}_1, \ldots, \widetilde{a}_m, \widetilde{b}_1, \ldots, \widetilde{b}_m$ of $\mathcal{A}/F^*$, such that $B(\widetilde{a}_i, \widetilde{b}_i) = \omega_i$, where $o(\omega_i) = o(\widetilde{a}_i) = o(\widetilde{b}_i)$ for each i, and $B_{\mathcal{A}}(\widetilde{a}_i, \widetilde{a}_j) = B_{\mathcal{A}}(\widetilde{b}_i, \widetilde{b}_j) = 1$ for all i, j, and $B_{\mathcal{A}}(\widetilde{a}_i, \widetilde{b}_j) = 1$ whenever $i \neq j$. Let $n_i = o(\widetilde{a}_i) = o(\widetilde{b}_i) = o(\omega_i)$. Then we have $a_i^{n_i}, b_i^{n_i} \in F^*$ and $a_i b_i = \omega_i b_i a_i$ for all i, and $a_i a_j = a_j a_i$, $b_i b_j = b_j b_i$ for all i, j, and $a_i b_j = b_j a_i$ whenever $j \neq i$. Let $c_i = a_i^{n_i}$ and $d_i = b_i^{n_i} \in F^*$. Then the relations just listed among all the a_i and b_i show that $F(\mathcal{A})$ is a homomorphic image of $S = \bigotimes_{i=1}^{m} (c_i, d_i; F)_{n_i}$. Since S is simple, the map of S onto A must be an isomorphism. So $\dim_F(A) = \dim_F(S) = n_1^2 \cdots n_m^2 = |\mathcal{A}/F^*|$, since $\mathcal{A}/F^*$ is the direct product of its cyclic subgroups generated by the $\widetilde{a}_i$ and the $\widetilde{b}_i$. This direct product decomposition shows that $\exp(\mathcal{A}/F^*) = \mathrm{lcm}(n_1, ..., n_m) = \underset{1\le i\le m}{\mathrm{lcm}} (\deg(S_i))$, where $S_i = (c_i, d_i; F)_{n_i}$. □

A *radical abelian* extension of a field F is an abelian Galois field extension K of F such that $K = F(U)$, where U is a subgroup of K^* with $U \supseteq F^*$ and U/F^* finite.

A projective Schur algebra $A = F(\mathcal{G})$ is said to be *reduced* if for every subgroup $\mathcal{H}$ of $\mathcal{G}$ with $\mathcal{H} \supseteq \mathcal{G}'$ (the derived group of $\mathcal{G}$) and every subfield L, $F \subseteq L \subseteq A$ such that $\mathcal{G}$ acts on L by conjugation, the subalgebra $L(\mathcal{H})$ is simple. Recall from [AS$_1$, Th. 1.4] that every projective Schur algebra is Brauer equivalent to a reduced projective Schur algebra. We now collect basic properties of subalgebras $F(\mathcal{H})$ of a reduced projective Schur algebra $F(\mathcal{G})$, where $\mathcal{G}' \subseteq \mathcal{H} \subseteq \mathcal{G}$.

Let $A = F(\mathcal{G})$ be a projective Schur algebra with $\mathcal{G} \supseteq F^*$. Assume A is reduced. Let $\mathcal{H}$ be any subgroup of $\mathcal{G}$ such that $\mathcal{H} \supseteq \mathcal{G}'$. Let

$B = F(\mathcal{H})$, a simple algebra, as A is reduced;

$L = Z(B)$, a field;

$\widehat{\mathcal{H}} = C_{\mathcal{G}}(L)$, a normal subgroup of $\mathcal{G}$;

$E = F(\widehat{\mathcal{H}})$;

$\mathcal{T} = C_{\widehat{\mathcal{H}}B^*}(B)$;

$T = L(\mathcal{T})$.

Proposition 2.2. *In the setting just described,*

(a) *L is abelian Galois over F with Galois group $\mathcal{G}(L/F) \cong \mathcal{G}/\widehat{\mathcal{H}}$, and L lies in a radical abelian extension of F.*
(b) *E is a simple algebra with $Z(E) = L$ and $E = C_A(L) = B \otimes_L T$.*
(c) *$L^* \subseteq \mathcal{T}$ and $\mathcal{T}/L^* \cong \widehat{\mathcal{H}}/(\widehat{\mathcal{H}} \cap B^*)$, a finite abelian group.*
(d) *T is a projective Schur algebra of abelian type over L.*
(e) *Let $G = \mathcal{G}/\widehat{\mathcal{H}}$, a finite abelian group. Then $A = E * G$, a ring-theoretic crossed product.*
(f) *If $e = \exp(\mathcal{T})$, then $\mu_e \subset F^*$.*

Proof. Since $\mathcal{G}' \subseteq \mathcal{H}$, we have $\mathcal{H} \lhd \mathcal{G}$, so $\mathcal{G}$ acts by conjugation on B. Hence B is simple as A is reduced. So L is a field with $[L : F] < \infty$, and $\mathcal{G}$ acts by conjugation on L. Clearly, $F \subseteq L^{\mathcal{G}} \subseteq A^{\mathcal{G}} = F$, so L is Galois over F and $\mathcal{G}$ maps onto $\mathcal{G}(L/F)$ with kernel $\widehat{\mathcal{H}}$. Therefore, $\mathcal{G}(L/F) \cong \mathcal{G}/\widehat{\mathcal{H}}$, which is abelian as $\widehat{\mathcal{H}} \supseteq \mathcal{H} \supseteq \mathcal{G}'$. Since $F(\mathcal{H})$ is a simple F-algebra with $\mathcal{H}/F^*$ finite and $L = Z(F(\mathcal{H}))$ is Galois over F, by [AS$_2$, Th. 1.3] L lies in a radical extension of F. Because L is also abelian Galois over F, [AS$_2$, Prop. 2.1] shows that L lies in a radical abelian extension of F, proving (a).

As for (e), note first that since $\widehat{\mathcal{H}} \supseteq \mathcal{G}'$ and A is reduced, $E = F(\widehat{\mathcal{H}})$ is simple. Let $n = |\mathcal{G}/\widehat{\mathcal{H}}| = [L : F]$, and let $g_1, ..., g_n$ be a set of coset representatives of $\widehat{\mathcal{H}}$ in $\mathcal{G}$. Then $\mathcal{G} \subseteq \sum\limits_{i=1}^{n} Eg_i$, so $\sum\limits_{i=1}^{n} Eg_i = A$. Hence $\dim_F(E) \geq \dim_F(A)/n$. Clearly $E \subseteq C_A(L)$. So using the Double Centralizer Theorem,

$$\dim_F(E) \le \dim_F(C_A(L)) = \dim_F(A)/[L:F] \le \dim_F(E).$$

So, equality holds throughout, which shows that $E = C_A(L)$; hence $Z(E) = L$. The dimension equality shows that the sum $A = \sum_{i=1}^{n} Eg_i = A$ is direct. Hence $A = E * (\mathcal{G}/\widehat{\mathcal{H}})$, a ring-theoretic crossed product, proving (e). We have also proved the first three assertions of (b).

Now consider $\mathcal{T}$. Since $\widehat{\mathcal{H}}$ normalizes $\mathcal{H}$ (as $\mathcal{H} \triangleleft \mathcal{G}$), $\widehat{\mathcal{H}}$ also normalizes B^*. So $\widehat{\mathcal{H}}B^*$ is a subgroup of A^*, and the definition of $\mathcal{T}$ as $C_{\widehat{\mathcal{H}}B^*}(B)$ makes sense. Observe also that $B = F(\mathcal{H}) \subseteq F(\widehat{\mathcal{H}}) = E$; so, $\mathcal{T} \subseteq E$. Since $L^* \subseteq \mathcal{T} \cap B^* \subseteq Z(B^*) = L^*$, we have $\mathcal{T} \cap B^* = L^*$. Also, for any $\widehat{h} \in \widehat{\mathcal{H}}$, conjugation by $\widehat{h}$ gives an L-linear automorphism of B. By Skolem-Noether, there is a $b \in B^*$ conjugation by which produces the same automorphism of B. Then $\widehat{h}b^{-1} \in \mathcal{T}$. This shows that $\widehat{\mathcal{H}} \subseteq \mathcal{T}B^*$; since $\mathcal{T} \subseteq \widehat{\mathcal{H}}B^*$ by definition, we have $\widehat{\mathcal{H}}B^* = \mathcal{T}B^*$. Then,

$$\mathcal{T}/L^* = \mathcal{T}/(\mathcal{T} \cap B^*) \cong \mathcal{T}B^*/B^* = \widehat{\mathcal{H}}B^*/B^* \cong \widehat{\mathcal{H}}/(\widehat{\mathcal{H}} \cap B^*).$$

Because $F^*\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap B^*$, $\widehat{\mathcal{H}}/(\widehat{\mathcal{H}} \cap B^*)$ is finite (as $\widehat{\mathcal{H}}/F^*$ is finite) and abelian (as $\widehat{\mathcal{H}}/\mathcal{H}$ is abelian), proving (c).

Next, consider $T = L(\mathcal{T})$. We have $E = F(\widehat{\mathcal{H}}) \subseteq BT \subseteq E$, since we just saw $\widehat{\mathcal{H}} \subseteq B^*\mathcal{T} \subseteq E$. Since $E = B \otimes_L C_E(B)$ by the Double Centralizer Theorem and $T \subseteq C_E(B) = E$, we have $E = BT = B \otimes_L T \subseteq B \otimes_L C_E(B) = E$. So $T = C_E(B)$. This completes the proof of (b) and shows also that $L = Z(T)$. Since $T = L(\mathcal{T})$ is a central simple L-algebra, part (c) shows that T is a projective Schur algebra of abelian type, proving (d).

Because $\mathcal{T}/L^*$ is abelian, we have the pairing $B_{\mathcal{T}} \colon \mathcal{T}/L^* \times \mathcal{T}/L^* \to L^*$ described in Prop. 2.1. The key to proving (f) is to see that $\operatorname{im}(B_{\mathcal{T}}) \subseteq F^*$. For this, observe that $\mathcal{G}$ acts by conjugation on $\mathcal{H}$, B, L, and $\widehat{\mathcal{H}}$, so on $\widehat{\mathcal{H}}B^*$, so on $\mathcal{T} = C_{\widehat{\mathcal{H}}B^*}(B)$ and on $\mathcal{T}/L^*$. Take any $g \in \mathcal{G}$ and $t \in \mathcal{T}$; write $t = \widehat{h}b$ with $\widehat{h} \in \widehat{\mathcal{H}}$ and $b \in B^*$. Let $c = gtg^{-1}t^{-1}$. Then $c \in \mathcal{T}$, as $gtg^{-1} \in \mathcal{T}$; likewise, $gbg^{-1}b^{-1} \in B^*$. So

$$c = [g\widehat{h}g^{-1}\widehat{h}^{-1}]\widehat{h}[gbg^{-1}b^{-1}]\widehat{h}^{-1} \in \mathcal{G}'\widehat{h}B^*\widehat{h}^{-1} \subseteq B^*.$$

Hence $c \in \mathcal{T} \cap B^* = L^*$. Since $gtg^{-1} = ct$, this shows that $\mathcal{G}$ acts trivially on $\mathcal{T}/L^*$. Because the pairing $B_{\mathcal{T}}$ is clearly compatible with the $\mathcal{G}$-action, $\mathcal{G}$ must also act trivially on $\operatorname{im}(B_{\mathcal{T}})$, i.e. $\operatorname{im}(B_{\mathcal{T}}) \subseteq L^{*\mathcal{G}} = F^*$, as claimed. We have $\operatorname{im}(B_{\mathcal{T}}) = \mu_\ell$, where $\ell = |\operatorname{im}(B_{\mathcal{T}})|$ (see Prop. 2.1(b)); so $\mu_\ell \subseteq F^*$. Let $e = \exp(\mathcal{T})$. Then by Prop. 2.1(d) and (c), $e | \exp(\mathcal{T}/L^*) = \ell$. Thus $\mu_e \subseteq \mu_\ell \subseteq F^*$, proving (f). $\square$

We digress to show that the preceding results yield a considerably simplified proof of the exponent reduction theorem that was the main result of [AS$_4$]. For this, let F_{cyc} be the maximal cyclotomic extension of F.

Proposition 2.3 [AS$_4$, Th. 1.3]**.** *Let A be a projective Schur algebra over F, and let $e = \exp(A \otimes_F F_{cyc})$. Then $\mu_e \subseteq F^*$.*

Proof. Let $A = F(\mathcal{G})$ with $\mathcal{G} \supseteq F^*$. We may assume that A is reduced. We apply Prop. 2.2 with $\mathcal{H} = \mathcal{G}'$. Recall (see [Sc, p. 443]) that since $\mathcal{G}$ has a central subgroup F^* with $\mathcal{G}/F^*$ finite, $|\mathcal{G}'| < \infty$. Hence $B = F(\mathcal{G}')$ is a Schur algebra. By the Brauer splitting theorem [CR, pp. 385, 418], B is split by F_{cyc} and $L = Z(B) \subseteq F_{cyc}$. Now $E = C_A(L) \sim A \otimes_F L$ in $Br(L)$. Since $E = B \otimes_L T$ as in Prop. 2.2, we have $A \otimes_F F_{cyc} \sim E \otimes_L F_{cyc} \sim T \otimes_L F_{cyc}$. Let $d = \exp(T)$. Then

$$e = \exp(A \otimes_F F_{cyc}) = \exp(T \otimes_L F_{cyc}) \bigm| \exp(T) = d.$$

Since $\mu_d \subseteq F^*$ by Prop. 2.2(f), this shows $\mu_e \subseteq F^*$. $\square$

We now return to the setup of Prop. 2.2, with $A = F(\mathcal{G})$, a reduced projective Schur algebra ($F^* \subseteq \mathcal{G}$), and $\mathcal{H}$ a subgroup of $\mathcal{G}$ with $\mathcal{G}' \subseteq \mathcal{H}$, and the objects associated to $\mathcal{H}$ described there, including the projective Schur algebra of abelian type $T = L(\mathcal{T})$. We will show how to use subgroups $\mathcal{L}/L^*$ of $\mathcal{T}/L^*$ to build new subgroups $\mathcal{H}_1$ of $\mathcal{G}$ containing $\mathcal{H}$ so that the objects associated to $\mathcal{H}_1$ by Prop. 2.2 have a nice description in terms of $\mathcal{H}$ and the corresponding objects for $\mathcal{H}$. This will be needed in §4. It also provides a unified approach to constructions that were given in [AS$_1$],[AS$_2$],[AS$_4$],[AS$_6$]. In each of those papers an $\mathcal{H}_1$ is chosen after starting with some $\mathcal{H}$, so that $\mathcal{H}_1 \supseteq \mathcal{H}$ and $F(\mathcal{H}_1)$ is maximal with respect to some property. In examining these constructions, one sees that what was needed was primarily an $\mathcal{H}_1$ so that its associated T as in Prop. 2.2 is trivial and $F(\mathcal{H}_1) = F(\mathcal{H}) \otimes_L Z(F(\mathcal{H}_1))$. We will see that this occurs whenever $\mathcal{L}/L^*$ is a Lagrangian of $\mathcal{T}/L^*$.

For the projective Schur algebra of abelian type $T = L(\mathcal{T})$ of Prop. 2.2 (with $L^* \subseteq \mathcal{T}$), we have the nondegenerate symplectic pairing $B_{\mathcal{T}}\colon \mathcal{T}/L^* \times \mathcal{T}/L^* \to L^*$ described in Prop. 2.1. Take any subgroup Λ of $\mathcal{T}/L^*$, and let $\mathcal{L}$ be the inverse image of Λ in $\mathcal{T}$. Then $L(\mathcal{L})$ is an L-subalgebra of T with $\dim_L(L(\mathcal{L})) = |\Lambda|$ (see [TW, Ex. 2.4(c)]). Let $\Lambda^\perp = \{\gamma \in \mathcal{T}/L^* \mid B_{\mathcal{T}}(\gamma,\lambda) = 1 \text{ for all } \lambda \in \Lambda\}$, a subgroup of $\mathcal{T}/L^*$. Because $B_{\mathcal{T}}$ is nondegenerate, we have $|\Lambda||\Lambda^\perp| = |\mathcal{T}/L^*|$ (cf. [TW, (2.2)]), so $\Lambda^{\perp\perp} = \Lambda$. Abusing notation slightly, let $\mathcal{L}^\perp$ denote the inverse image of $\Lambda^\perp$ in $\mathcal{T}$. It is easy to check that $L(\mathcal{L}^\perp) = C_T(L(\mathcal{L}))$ (see [TW, Lemma 2.5(i)]). So $Z(L(\mathcal{L})) = L(\mathcal{L}) \cap L(\mathcal{L}^\perp) = L(\mathcal{L} \cap \mathcal{L}^\perp)$. In particular, $L(\mathcal{L})$ is commutative iff $\Lambda \subseteq \Lambda^\perp$, iff $B_{\mathcal{T}}$ is trivial on $\Lambda \times \Lambda$. The subgroup Λ is called a *Lagrangian* of $\mathcal{T}/L^*$ with respect to $B_{\mathcal{T}}$ if $\Lambda^\perp = \Lambda$. It is easy to see that if Λ is any subgroup of $\mathcal{T}/L^*$ with $\Lambda \subseteq \Lambda^\perp$, then Λ lies in some Lagrangian of $\mathcal{T}/L^*$. Thus the algebras $L(\mathcal{L})$ for $\mathcal{L}/L^*$ a Langrangian are the maximal commutative subalgebras of T of the form $L(\Lambda)$ for $\mathcal{L} \subseteq \mathcal{T}$.

Now fix a subgroup Λ of $\mathcal{T}/L^*$, let $\mathcal{L}$ be its inverse image in $\mathcal{T}$. Let $\Lambda = \{a_1L^*, ..., a_mL^*\}$, with $a_i \in \mathcal{L} \subseteq \mathcal{T} \subseteq \widehat{\mathcal{H}}B^*$. Write each $a_i = \widehat{h}_i b_i$ with $\widehat{h}_i \in \widehat{\mathcal{H}}$ and $b_i \in B^*$. Let $\mathcal{H}_1 =$

$\langle \mathcal{H}, \widehat{h}_1, ..., \widehat{h}_m \rangle$, a subgroup of $\mathcal{G}$ with $\mathcal{G}' \subseteq \mathcal{H} \subseteq \mathcal{H}_1 \subseteq \widehat{\mathcal{H}}$. Associated to $\mathcal{H}_1$ we have the objects of Prop. 2.2: $B_1 = F(\mathcal{H}_1)$, $L_1 = Z(B_1)$, $\widehat{\mathcal{H}}_1 = C_{\mathcal{G}}(L_1)$, $E_1 = F(\widehat{\mathcal{H}}_1) = B_1 \otimes_{L_1} T_1$, where $T_1 = L_1(\mathcal{T}_1)$ with $\mathcal{T}_1 = C_{\widehat{\mathcal{H}}_1 B_1^*}(B_1)$.

Proposition 2.4. *In the situation described in the preceding paragraph,*

(a) $B_1 = B \otimes_L L(\mathcal{L})$; $L_1 = L(\mathcal{L} \cap \mathcal{L}^\perp)$, *which is a Kummer extension field of L, and lies in a radical abelian extension of F;* $\mathcal{H} \subseteq \mathcal{H}_1 \subseteq \widehat{\mathcal{H}}_1 \subseteq \widehat{\mathcal{H}}$; $E_1 = C_A(L_1) = B \otimes_L L(\mathcal{L}\mathcal{L}^\perp) \subseteq E$; *and* $T_1 = L(\mathcal{L}^\perp) \subseteq T$.

(b) $[T_1 : L_1] = |\mathcal{L}^\perp / (\mathcal{L} \cap \mathcal{L}^\perp)|$

(c) *If Λ is a Lagrangian of $\mathcal{T}/L^*$, then* $L_1 = L(\mathcal{L}) = T_1$ *and* $B_1 = E_1$.

Proof. For each generator a_i of $\mathcal{L}$, we have $a_i = \widehat{h}_i b_i$, with each $b_i \in B = F(\mathcal{H}) \subseteq F(\mathcal{H}_1) = B_1$, and $\widehat{h}_i \in F(\mathcal{H}_1) = B_1$; so $a_i \in B_1$. Also, $L \subseteq B \subseteq B_1$. Thus $B \cdot L(\mathcal{L}) \subseteq B_1$. On the other hand, each $\widehat{h}_i = a_i b_i^{-1} = b_i^{-1} a_i \in B \cdot L(\mathcal{L})$, and $\mathcal{H} \subseteq B$; so $B_1 = F(\mathcal{H}_1) \subseteq B \cdot L(\mathcal{L})$. Hence $B_1 = B \cdot L(\mathcal{L})$. But, $B \cdot L(\mathcal{L}) = B \otimes_L L(\mathcal{L})$ because $L(\mathcal{L}) \subseteq T$ and $E = B \otimes_L T$ by Prop. 2.2(b); so, $B_1 = B \otimes_L L(\mathcal{L})$. Since $C_T(L(\mathcal{L})) = L(\mathcal{L}^\perp)$, as noted above, we have $Z(L(\mathcal{L})) = L(\mathcal{L}) \cap L(\mathcal{L}^\perp) = L(\mathcal{L} \cap \mathcal{L}^\perp)$. Hence $L_1 = Z(B_1) = Z(B \otimes_L L(\mathcal{L})) = Z(B) \otimes_L Z(L(\mathcal{L})) = L \otimes_L L(\mathcal{L} \cap \mathcal{L}^\perp) = L(\mathcal{L} \cap \mathcal{L}^\perp)$. This L_1 is a field as A is reduced and $B_1 = F(\mathcal{H}_1)$ with $\mathcal{G}' \subseteq \mathcal{H} \subseteq \mathcal{H}_1$. Moreover, L_1 is a Kummer extension of L since $\exp((\mathcal{L} \cap \mathcal{L}^\perp)/L^*) \mid \exp(\mathcal{T}/L^*) = |\mathrm{im}(B_{\mathcal{T}})|$ and $\mathrm{im}(B_{\mathcal{T}}) \subseteq \mu(L)$ by Prop. 2.1(b),(c). Prop. 2.2(a), applied with $\mathcal{H}_1$ in place of $\mathcal{H}$, says that L_1 lies in a radical abelian extension of F. Because $L_1 \supseteq L$, we have $\widehat{\mathcal{H}}_1 = C_{\mathcal{G}}(L_1) \subseteq C_{\mathcal{G}}(L) = \widehat{\mathcal{H}}$. The other inclusions in $\mathcal{H} \subseteq \mathcal{H}_1 \subseteq \widehat{\mathcal{H}}_1 \subseteq \widehat{\mathcal{H}}$ are clear. Now by Prop. 2.2(b), $E_1 = C_A(L_1) \subseteq C_A(L) = E$, as $L \subseteq L_1$. Therefore, $T_1 = C_{E_1}(B_1) \subseteq C_E(B_1) = C_{B \otimes_L T}(B \otimes_L L(\mathcal{L})) = C_T(L(\mathcal{L})) = L(\mathcal{L}^\perp)$. But also, $L(\mathcal{L}^\perp) = C_T(L(\mathcal{L})) \subseteq C_A(L(\mathcal{L} \cap \mathcal{L}^\perp)) = E_1$. Since $L(\mathcal{L}^\perp)$ centralizes B and $L(\mathcal{L})$, it centralizes $B \otimes_L L(\mathcal{L}) = B_1$. Thus, $L(\mathcal{L}^\perp) \subseteq T_1$; so equality holds. Finally, $E_1 = B_1 \otimes_{L_1} T_1 = B \otimes_L L(\mathcal{L}) \otimes_{L(\mathcal{L} \cap \mathcal{L}^\perp)} L(\mathcal{L}^\perp) = B \otimes_L L(\mathcal{L}\mathcal{L}^\perp)$, completing the proof of (a).

For (b), note that

$$\begin{aligned}[T_1 : L_1] &= [L(\mathcal{L}^\perp) : L(\mathcal{L} \cap \mathcal{L}^\perp)] = [L(\mathcal{L}^\perp) : L]/[L(\mathcal{L} \cap \mathcal{L}^\perp) : L] \\ &= |\Lambda^\perp|/|\Lambda \cap \Lambda^\perp| = |\mathcal{L}^\perp / \mathcal{L} \cap \mathcal{L}^\perp|.\end{aligned}$$

Part (c) follows immediately from (a) and (b), since Λ is a Lagrangian just when $\Lambda = \Lambda^\perp$, so $\mathcal{L} = \mathcal{L}^\perp$. $\square$

3. Splitting maps for the Brauer group of a Henselian field

In this section we give the properties of division algebras over Henselian fields that will be needed for the analysis of the projective Schur groups of such fields. We first recall some known results, then give analogues for an arbitrary Henselian valuation to Witt's direct sum decomposition theorem for the Brauer group of a complete discretely valued field.

Let F be a field with a valuation $v\colon F^* \to \Gamma_F$, where Γ_F is the value group of v, a totally ordered abelian group, written additively. Let V_F be the valuation ring of v and M_F the unique maximal ideal of V_F; let $\overline{F} = V_F/M_F$, the residue field of v; and let $U_F = V_F - M_F$, the group of valuation units. Let Δ be the divisible hull of Γ_F; so $\Delta \supseteq \Gamma_F$, and the ordering on Γ_F extends uniquely to Δ making Δ an ordered abelian group. Note that $\Delta \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$. If L is a field algebraic over F and w is any extension of V to L, then there are canonical injections which we view as inclusions $\overline{F} \hookrightarrow \overline{L}$, $\Gamma_F \hookrightarrow \Gamma_L$, and $\Gamma_L \hookrightarrow \Delta$. Recall the Fundamental Inequality [E, (13.10)], which says that whenever $[L : F] < \infty$, we have

$$[\overline{L} : \overline{F}]\,|\Gamma_L : \Gamma_F| \le [L : F] \tag{3.1}$$

Assume now and throughout the rest of this section that the valuation v on F is Henselian. This means that Hensel's Lemma holds for v, or equivalently (see, e.g., [E, Cor. 16.6]), that v has a unique extension to each field $L \supseteq F$ with L algebraic over F. Thus the extension of v to any such L, which we again denote by v, is also Henselian.

Examples 3.1. (a) If v is a valuation on a field F with Γ_F embedding in $\mathbb{R}$ (equivalently, if V_F has Krull dimension 1) and F is complete with respect to v, then v is Henselian [E, (16.7)].

(b) Suppose F_0 is a field with Henselian valuation v_0. Let F be the Laurent series field $F = F_0((x))$. Then the canonical extension v of v_0 to F (given by $v(\sum\limits_{i=k}^{\infty} c_i x^i) = (v_0(c_k), k) \in \Gamma_{F_0} \times \mathbb{Z}$ if $c_k \neq 0$) is Henselian, with $V_F = V_0 + xF_0[[x]]$, $\overline{F} = \overline{F_0}$, and $\Gamma_F = \Gamma_{F_0} \times \mathbb{Z}$. The ordering on Γ_F is the right-to-left lexicographical ordering, in which $(\gamma, i) \le (\delta, j)$ just when $i < j$ or both $i = j$ and $\gamma \le \delta$. That v is Henselian is a special case of the fact that composites of Henselian valuations are Henselian [Rib, p. 211, Prop. 10].

Let L be an algebraic extension of the Henselian field F. If $[L : F] < \infty$, we say that L is *unramified* over F if $[\overline{L} : \overline{F}] = [L : F]$ and $\overline{L}$ is separable over $\overline{F}$. When this occurs, the Fundamental Inequality shows that $\Gamma_L = \Gamma_F$. At the other extreme, we say that L is *totally ramified* over F if $|\Gamma_L : \Gamma_F| = [L : F]$. Also, L is said to be *tamely ramified* over F if $\operatorname{char}(\overline{F}) \nmid |\Gamma_L : \Gamma_F|$, $\overline{L}$ is separable over $\overline{F}$, and $[\overline{L} : \overline{F}]\,|\Gamma_L : \Gamma_F| = [L : F]$. If L is algebraic over F but $[L : F] = \infty$, we say that L is unramified (resp. totally ramified, tamely ramified) over F if L is a union of finite degree extensions of F each of which is unramified (resp. totally ramified, tamely ramified) over F.

Let F_s be a fixed separable closure of our Henselian valued field F. Let F_{nr} denote the *maximal unramified extension* of F in F_s. That is, F_{nr} is the inertia field for the extension of v from F to F_s. It is known (see [E, (19.10), (19.11)]) that for fields L with $F \subseteq L \subseteq F_s$, L is unramified over F iff $L \subseteq F_{nr}$. Furthermore (see [E, (19.12), (19.8), (19.6)]), $\overline{F_{nr}} \cong \overline{F}_s$, and F_{nr} is Galois over F, with Galois group $\mathcal{G}(F_{nr}/F) \cong \mathcal{G}(\overline{F}_s/\overline{F})$, which is the absolute Galois group of $\overline{F}$, also denoted $G_{\overline{F}}$. From the isomorphism of Galois groups, one can see that there is a one-to-one correspondence $L \mapsto \overline{L}$ between unramified field extensions L of F (with $L \subseteq F_s$) and separable algebraic field extensions of $\overline{F}$. If E is a separable extension of $\overline{F}$, we call the field $L \supseteq F$ with $\overline{L} = E$ the *inertial lift* of E over $\overline{F}$; we will often write $F(E)$ for this inertial lift of E.

When the valuation on F is complete and discrete, Witt gave a description of the Brauer group $Br(F)$. We now give generalizations of Witt's theorem, which are valid for any Henselian valued field. The basic exact sequence (3.3) below was derived in [JW, pp. 154-156], but the splitting maps in Th. 3.2 and Prop. 3.3 were not given there. We will review the derivation of (3.3), since it is essential to this paper, and we need to know the actual maps in it, not just the existence of an exact sequence. Let $\Gamma = \Gamma_F$, $\Delta = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$, and let $G = \mathcal{G}(F_{nr}/F) \cong G_{\overline{F}}$. We first describe

$$SBr(F) = Br(F_{nr}/F)$$

which is called the *inertially split part* of $Br(F)$.

The short exact sequence of discrete G-modules

$$1 \longrightarrow U_{F_{nr}} \longrightarrow F_{nr}^* \xrightarrow{v} \Gamma \longrightarrow 0$$

induces an exact sequence in cohomology

$$H^1(G, \Gamma) \longrightarrow H^2(G, U_{F_{nr}}) \longrightarrow H^2(G, F_{nr}^*) \longrightarrow H^2(G, \Gamma). \tag{3.2}$$

We interpret the terms in the sequence. We have $H^1(G, \Gamma) = \mathrm{Hom}_c(G, \Gamma) = 0$ (continuous homomorphisms; there are none, as G is profinite and Γ is torsion-free). It is known (see [JW, Th. 5.6(a)]) that the residue map $U_{F_{nr}} \longrightarrow \overline{F_{nr}}^* \cong \overline{F}_s^*$ induces an isomorphism $H^2(G, U_{F_{nr}}) \cong H^2(G, \overline{F}_s^*) \cong Br(\overline{F})$. The next term in (3.2) is $SBr(F)$. Since Δ is uniquely divisible, $H^1(G, \Delta) = H^2(G, \Delta) = 0$, so $H^2(G, \Gamma) \cong H^1(G, \Delta/\Gamma) = \mathrm{Hom}_c(G, \Delta/\Gamma)$. We shall give a splitting map which shows that the last map in (3.2) is onto, so (3.2) becomes the short exact sequence

$$0 \longrightarrow Br(\overline{F}) \longrightarrow SBr(F) \xrightarrow{\beta} \mathrm{Hom}_c(G, \Delta/\Gamma) \longrightarrow 0 \tag{3.3}$$

It is easy to see that (3.3) is functorial with respect to algebraic field extensions, i.e., for any field $L \supseteq F$ with L algebraic over F, and for the unique Henselian extension of v to

L, the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Br(\overline{F}) & \longrightarrow & SBr(F) & \longrightarrow & \mathrm{Hom}_c(G_{\overline{F}}, \Delta/\Gamma_F) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & Br(\overline{L}) & \longrightarrow & SBr(L) & \longrightarrow & \mathrm{Hom}_c(G_{\overline{L}}, \Delta/\Gamma_L) & \longrightarrow & 0 \end{array} \tag{3.4}$$

Here the right vertical map arises from the canonical inclusion $G_{\overline{L}} \hookrightarrow G_{\overline{F}}$ and the surjection $\Delta/\Gamma_F \to \Delta/\Gamma_L$.

Theorem 3.2. *For any Henselian valued field F, there is a homomorphism $f\colon \mathrm{Hom}_c(G, \Delta/\Gamma) \to SBr(F)$ splitting the β of* (3.3). *Hence*

$$SBr(F) \cong Br(\overline{F}) \oplus \mathrm{Hom}_c(G_{\overline{F}}, \Delta/\Gamma_F)$$

Proof. The abelian torsion group Δ/Γ has its canonical primary decomposition $\Delta/\Gamma = \bigoplus_{p \text{ prime}} (\Delta/\Gamma)(p)$, with $(\Delta/\Gamma)(p) = \bigcup_{j=1}^{\infty} p^{-j}\Gamma/\Gamma$. Fix a prime number p. Choose $\{\gamma_i\}_{i\in I_p} \subseteq \Gamma$ so that the γ_i map to a $\mathbb{Z}/p\mathbb{Z}$-vector space base of $\Gamma/p\Gamma$. Then for each $n \in \mathbb{N}$, the γ_i also map to a base of $\Gamma/p^n\Gamma$ as a free $\mathbb{Z}/p^n\mathbb{Z}$-module. So, $\{p^{-n}\gamma_i\}_{i\in I_p}$ maps to a base of the free $\mathbb{Z}/p^n\mathbb{Z}$-module $p^{-n}\Gamma/\Gamma$. Let $p^{-\infty}\gamma_i/\Gamma$ denote $\bigcup_{j=1}^{\infty} \langle p^{-j}\gamma_i + \Gamma\rangle \subseteq \Delta/\Gamma$. Then, $(\Delta/\Gamma)(p) = \bigoplus_{i\in I_p} p^{-\infty}\gamma_i/\Gamma$.

Let γ be any of the γ_i, and take any $t \in F^*$ with $v(t) = \gamma$. We use t to define a homomorphism from $\mathrm{Hom}_c(G, p^{-\infty}\gamma/\Gamma)$ to the Brauer group. For this, note that since $\mathbb{Z}[1/p]\gamma \cong \mathbb{Z}[1/p]$ and $\mathbb{Z}[1/p]\gamma \cap \Gamma = \mathbb{Z}\gamma$ (as $\gamma \notin p\Gamma$), we have $p^{-\infty}\gamma/\Gamma \cong \mathbb{Z}[1/p]\gamma/\mathbb{Z}\gamma \cong \mathbb{Z}[1/p]/\mathbb{Z}$. Thus there is a homomorphism

$$\theta\colon H^1(G, p^{-\infty}\gamma/\Gamma) \longrightarrow H^1(G, \mathbb{Z}[1/p]/\mathbb{Z}) \xrightarrow{\delta} H^2(G, \mathbb{Z}) \tag{3.5}$$

where δ is the connecting homomorphism arising from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}[1/p]/\mathbb{Z} \to 0$ of trivial G-modules. We use the cup product pairing $H^0(G, F_{nr}^*) \cup H^2(G, \mathbb{Z}) \to H^2(G, F_{nr}^*) \cong SBr(F)$ to define our map

$$f_t\colon \mathrm{Hom}_c(G, p^{-\infty}\gamma/\Gamma) \longrightarrow SBr(F) \quad \text{by} \quad \chi \mapsto (t) \cup \theta(\chi). \tag{3.6}$$

Then f_t is a group homomorphism, as the cup product is $\mathbb{Z}$-bilinear. To describe $f_t(\chi)$ as an algebra, let $N = \ker(\chi)$, an open normal subgroup of G, and let K be the fixed field of N. So $F \subseteq K \subseteq F_{nr}$ and $\mathcal{G}(K/F) \cong \mathrm{im}(\chi)$, which is a finite subgroup of $p^{-\infty}\gamma/\Gamma$; thus $\mathrm{im}(\chi)$ is the cyclic group $p^{-r}\gamma/\Gamma$, where $[K : F] = p^r$. Choose $\sigma \in \mathcal{G}(F_{nr}/F) = G$ with $\chi(\sigma) = p^{-r}\gamma + \Gamma$; then $\sigma|_K$ is a generator of $\mathcal{G}(K/F)$. It is easy to check that $\theta(\chi)$

is represented by the cocycle $z \in Z^2(G, \mathbb{Z})$ given by: for $\tau, \rho \in G$, if $\tau|_K = (\sigma|_K)^i$ and $\rho|_K = (\sigma|_K)^j$ with $0 \le i, j \le p^r - 1$,

$$z(\tau, \rho) = \begin{cases} 0 & \text{if} \quad i + j < p^r, \\ 1 & \text{if} \quad i + j \ge p^r. \end{cases}$$

Then f_t is represented by $z' \in Z^2(G, F_{nr}^*)$ given by

$$z'(\tau, \rho) = \begin{cases} 1 & \text{if} \quad i + j < p^r, \\ t & \text{if} \quad i + j \ge p^r, \end{cases}$$

(cf. [Se, p. 176, Lemma 1]). From the formula for z' it is easy to see that for the β of (3.3), $\beta(f_t(\chi)) = \chi$, as $v(t) = \gamma$. Also we can read off from the formula for z' that the algebra in $SBr(F)$ represented by $f_t(\chi)$ is the cyclic algebra $(K/F, \sigma, t)$.

Our splitting map f is obtained by aggregating such f_t: For each prime p, we have $\{\gamma_i\}_{i \in I_p} \subseteq \Gamma$ mapping to a $\mathbb{Z}/p\mathbb{Z}$-base of $\Gamma/p\Gamma$. For each $i \in I_p$ choose some $t_i \in F^*$ with $v(t_i) = \gamma_i$. We have $\mathrm{Hom}_c(G, \Delta/\Gamma)(p) = \bigoplus_{i \in I_p} \mathrm{Hom}_c(G, p^{-\infty}\gamma_i/\Gamma)$. Define $f_p \colon \mathrm{Hom}_c(G, \Delta/\Gamma)(p) \longrightarrow SBr(F)$ by $f_p = \bigoplus_{i \in I_p} f_{t_i}$. Then define $f \colon \mathrm{Hom}_c(G, \Delta/\Gamma) \to SBr(F)$ by $f = \bigoplus_{p \text{ prime}} f_p$. Since each $\beta \circ f_{t_i}$ is the identity on $\mathrm{Hom}_c(G, p^{-\infty}\gamma_i/\Gamma)$, we have $\beta \circ f$ is the identity on $\mathrm{Hom}_c(G, \Delta/\Gamma)$, so f is a splitting map for (3.3). $\square$

Note that the splitting map f of Th. 3.2 depends on the choice of the t_i. Similarly, in Witt's theorem for complete discrete valuations, the splitting map depends on the choice of a uniformizing parameter. The reason for using the primary decomposition of Δ/Γ in the proof of Th. 3.2 is that the index sets I_p could have different sizes for different primes p. (By definition, $|I_p| = \dim_{\mathbb{Z}/p\mathbb{Z}}(\Gamma/p\Gamma)$.) Of course if Γ were a free $\mathbb{Z}$-module, we could choose a family $\{t_i\} \subseteq F^*$ mapping to a $\mathbb{Z}$-base of Γ. We could then use the same t_i for each prime p, which would simplify the description of the splitting map.

We now show that the splitting of $SBr(F)$ is compatible with suitably chosen scalar extensions. In the proof of Th. 3.2, we chose $\{\gamma_i\}_{i \in I_p} \subseteq \Gamma$ and $\{t_i\}_{i \in I_p} \subseteq F^*$ with $v(t_i) = \gamma_i$, for any fixed prime p. To allow for consideration of all primes at once, we write $\gamma_{p,i}$ for the earlier γ_i and $t_{p,i}$ for t_i.

Proposition 3.3. *Let L be an algebraic extension of the Henselian valued field F.*

(a) *Suppose L is unramified over F. Then $\Gamma_L = \Gamma_F$, and we can use the same $t_{p,i}$ for the splitting map of $SBr(L)$ as for $SBr(F)$. Then we have a commutative diagram*

$$\begin{tikzcd} Br(\overline{F}) \oplus \mathrm{Hom}_c(G_{\overline{F}}, \Delta/\Gamma_F) \arrow[r, "\cong"] \arrow[d, "\mathrm{res}\oplus\mathrm{res}"'] & SBr(F) \arrow[d, "\mathrm{res}"'] \\ Br(\overline{L}) \oplus \mathrm{Hom}_c(G_{\overline{L}}, \Delta/\Gamma_L) \arrow[r, "\cong"] & SBr(L) \end{tikzcd}$$

(b) *With the $t_{p,i}$ chosen for the splitting map f of Th. 3.2, take any subset $\{j_{p,i} |\ p$ prime, $i \in I_p\} \subseteq \mathbb{N}$. For each such p and i, let $s_{p,i}$ be some $p^{j_{p,i}}$th root of $t_{p,i}$, and suppose $L = F(s_{p,i} |\ p$ prime, $i \in I_p)$. Then L is totally ramified over F, $\{v(s_{p,i}) |\ i \in I_p\}$ maps to a $\mathbb{Z}/p\mathbb{Z}$-base of Δ/Γ_L, and if we use the $t_{p,i}$ for the splitting of $SBr(F)$ and the $s_{p,i}$ for the splitting of $SBr(L)$, then there is a commutative diagram*

$$\begin{array}{ccc} Br(\overline{F}) \oplus \mathrm{Hom}_c(G_{\overline{F}}, \Delta/\Gamma_F) & \xrightarrow{\cong} & SBr(F) \\ {\scriptstyle \mathrm{res}\oplus\mathrm{can}}\big\downarrow & & {\scriptstyle \mathrm{res}}\big\downarrow \\ Br(\overline{L}) \oplus \mathrm{Hom}_c(G_{\overline{L}}, \Delta/\Gamma_L) & \xrightarrow{\cong} & SBr(L) \end{array}$$

where can *is induced by the canonical surjection $\Delta/\Gamma_F \to \Delta/\Gamma_L$.*

Proof. (a) Assume L is unramified over F, so $\Gamma_L = \Gamma_F$. Take any prime p, any $\gamma \in \Gamma_F$, $\gamma \notin p\Gamma_F$, and any $t \in F^*$ with $v(t) = \gamma$. We have an evidently commutative diagram

$$\begin{array}{ccccc} H^1(G_{\overline{F}}, p^{-\infty}\gamma/\Gamma_F) & \xrightarrow{\theta} & H^2(G_{\overline{F}}, \mathbb{Z}) & \xrightarrow{(t)\cup\cdot} & SBr(F) \\ {\scriptstyle \mathrm{res}}\big\downarrow & & {\scriptstyle \mathrm{res}}\big\downarrow & & {\scriptstyle \mathrm{res}}\big\downarrow \\ H^1(G_{\overline{L}}, p^{-\infty}\gamma/\Gamma_L) & \xrightarrow{\theta} & H^2(G_{\overline{L}}, \mathbb{Z}) & \xrightarrow{(t)\cup\cdot} & SBr(L) \end{array} \tag{3.7}$$

where the horizontal maps θ are as in (3.5), the right horizontal maps are cup product with (t), and the first two vertical restriction maps arise from the canonical inclusion $G_{\overline{L}} \hookrightarrow G_{\overline{F}}$. The composition in the top row is the map f_t of (3.6) for F, and in the bottom row is the f_t for L. Since the splitting map $f\colon \mathrm{Hom}_c(G_{\overline{F}}, \Delta/\Gamma_F) \to SBr(F)$ is a direct sum of such f_t and likewise for L replacing F (using the same family of t's) the commutativity of (3.7) for each t yields the commutativity of the diagram in (a).

(b) For each p, i let $\delta_{p,i} = v(s_{p,i}) = p^{1/j_{p,i}}\gamma_{p,i}$. The Fundamental Inequality (3.1) shows that the field $L_{p,i} = F(s_{p,i})$ is totally ramified over F with $\Gamma_{L_{p,i}} = \langle\delta_{p,i}\rangle + \Gamma_F$. Since the relative value groups $\Gamma_{L_{p,i}}/\Gamma_F$ are all independent in Δ/Γ_F, our field L, which is the compositum of all the $L_{p,i}$, is totally ramified over F with $\Gamma_L = \sum\limits_{p,i}\langle\delta_{p,i}\rangle + \Gamma_F$; so $\overline{L} = \overline{F}$. We have $\Gamma_L/\Gamma_F = \bigoplus\limits_{p,i} (\langle\delta_{p,i}\rangle + \Gamma_F)/\Gamma_F$ with each summand $(\langle\delta_{p,i}\rangle + \Gamma_F)/\Gamma_F \subseteq p^{-\infty}\gamma_{p,i}/\Gamma_F$. Since $\Delta/\Gamma_F = \bigoplus\limits_{p,i} p^{-\infty}\gamma_{p,i}/\Gamma_F$, we have a compatible decomposition $\Delta/\Gamma_L \cong \bigoplus\limits_{p,i} (p^{-\infty}\gamma_{p,i}/\Gamma_F)/(\langle\delta_{p,i}\rangle + \Gamma_F) \cong \bigoplus\limits_{p,i} p^{-\infty}\delta_{p,i}/\Gamma_L$. Hence $\{\delta_{p,i} |\ i \in I_p\}$ maps to a $\mathbb{Z}/p\mathbb{Z}$-base of $\Gamma_L/p\Gamma_L$, so the $s_{p,i}$, with $v(s_{p,i}) = \delta_{p,i}$, are a valid set to use for the splitting map of $SBr(L)$. Because the canonical map $\Delta/\Gamma_F \to \Delta/\Gamma_L$ sends $p^{-\infty}\gamma_{p,i}/\Gamma_F$ onto $p^{-\infty}\delta_{p,i}/\Gamma_L$, it suffices for (b) to verify the commutativity of the diagrams

$$\begin{array}{ccc} \mathrm{Hom}_c(G_{\overline{F}},\ p^{-\infty}\gamma_{p,i}/\Gamma_F) & \xrightarrow{f_{p,i}} & SBr(F) \\ \downarrow & & \mathrm{res}\downarrow \\ \mathrm{Hom}_c(G_{\overline{L}},\ p^{-\infty}\delta_{p,i}/\Gamma_L) & \xrightarrow{f'_{p,i}} & SBr(L) \end{array} \tag{3.8}$$

where the maps $f_{p,i}$ and $f'_{p,i}$ are as in (3.6). Of course $G_{\overline{L}} = G_{\overline{F}}$ here, as $\overline{L} = \overline{F}$. To analyze (3.8), take any $\chi \in \mathrm{Hom}_c(G_{\overline{F}},\ p^{-\infty}\gamma_{p,i}/\Gamma_F)$, and let ψ be the image of χ in $\mathrm{Hom}_c(G_{\overline{L}},\ p^{-\infty}\delta_{p,i}/\Gamma_L)$. We have the diagram

$$\begin{array}{ccccc} H^1(G_{\overline{F}},\ p^{-\infty}\gamma_{p,i}/\Gamma_F) & \longrightarrow & H^2(G_{\overline{F}}, \mathbb{Z}\gamma_{p,i}) & \longrightarrow & H^2(G_{\overline{F}}, \mathbb{Z}) \\ \downarrow & & \downarrow & & p^{j_{p,i}}\downarrow \\ H^1(G_{\overline{L}},\ p^{-\infty}\delta_{p,i}/\Gamma_L) & \longrightarrow & H^2(G_{\overline{L}}, \mathbb{Z}\delta_{p,i}) & \longrightarrow & H^2(G_{\overline{L}}, \mathbb{Z}) \end{array} \tag{3.9}$$

where the middle vertical map arises from the inclusion $\mathbb{Z}\gamma_{p,i} \hookrightarrow \mathbb{Z}\delta_{p,i}$, and the right vertical map is multiplication by $p^{j_{p,i}}$. In (3.9), the left rectangle is commutative since the horizontal maps are connecting homomorphisms arising from compatible short exact sequences of $G_{\overline{F}}$-modules. The right rectangle in (3.9) is evidently commutative. The composition of the top (resp. bottom) maps in (3.9) is the θ of (3.5) for $\gamma_{p,i}$ (resp. $\delta_{p,i}$). So the commutativity of (3.9) shows that $p^{j_{p,i}}\theta(\chi) = \theta(\psi)$. Hence $\mathrm{res}_{L/F}(f_{p,i}(\chi)) = \mathrm{res}_{L/F}((t_{p,i}) \cup \theta(\chi)) = (s_{p,i}^{p^{j_{p,i}}}) \cup \theta(\chi) = (s_{p,i}) \cup p^{j_{p,i}}\theta(\chi) = (s_{p,i}) \cup \theta(\psi) = f'_{p,i}(\psi)$, proving the commutativity of (3.8), as desired. $\square$

There is a further well-described part of $Br(F)$ for F Henselian, which is what we get when we add in the tame totally ramified division algebras. A central division algebra T over our Henselian field F is said to be *tame and totally ramified* (TTR) if (with respect to the unique extension of the valuation v on F to T) $|\Gamma_T : \Gamma_F| = [T : F]$ and $\mathrm{char}(\overline{F}) \nmid [T : F]$. The theory of such division algebras is described in [TW]. In particular, it is known (see [Dr, Th. 1]) that every such T is isomorphic to a tensor product of symbol algebras (so lies in $PS(F)$), and that $\exp(T) = \exp(\Gamma_T/\Gamma_F)$ (by [TW, Ex. 4.4(ii)]). The possible TTR algebras are thus constrained by the roots of unity in F. Here is a typical example of a TTR symbol algebra:

Example 3.4. Suppose $\mu_n \subseteq F^*$, $\mathrm{char}(\overline{F}) \nmid n$, and $s, t \in F^*$ such that $v(s)$ and $v(t)$ generate a group of order n^2 in $\Gamma_F/n\Gamma_F$. Then the symbol algebra $T = (s, t; F)_n$ is TTR, with $\Gamma_T = \langle \frac{1}{n}v(s), \frac{1}{n}v(t)\rangle + \Gamma_F$ (see [TW, Prop. 3.5]).

A central division algebra D over a Henselian field F is said to be *tame* if D is split by the *maximal tamely ramified extension* F_{tr} of F. This field F_{tr} is the ramification field of the extension of v from F to F_s. It has the property that for any field L with $F \subseteq L \subseteq F_s$, L is tamely ramified over F iff $L \subseteq F_{tr}$ (see [E, (20.7), (20.16)]). It is known [JW, Lemma 6.2] that if D is tame, then $D \sim S \otimes_F T$ in $Br(F)$, where S is inertially split and T is TTR. (But the S and T are not uniquely determined, and it is not in general possible to express $D \cong S \otimes_F T$ with S inertially split and T TTR.) Let the tame part of $Br(F)$ be denoted by

$$TBr(F) = \{[D] \in Br(F) |\ D \text{ is tame}\} = Br(F_{tr}/F).$$

For the primary components of $TBr(F)$ we have (see [HW, Prop. 4.3]) for every prime p,

$$TBr(F)(p) = \begin{cases} Br(F)(p) & \text{if} \quad p \neq \operatorname{char}(\overline{F}), \\ SBr(F)(p) & \text{if} \quad p = \operatorname{char}(\overline{F}). \end{cases} \tag{3.10}$$

There is a noncanonical splitting for the inclusion of $SBr(F)$ in $TBr(F)$, expressed in terms of the primary components as follows. Take any prime $p \neq \operatorname{char}(\overline{F})$, and as above take $\{t_i \mid i \in I_p\} \subseteq F^*$ with $\{v(t_i)\}$ mapping to a $\mathbb{Z}/p\mathbb{Z}$-base of $\Gamma_F/p\Gamma_F$. Fix some total ordering on the index set I_p. For any $n \in \mathbb{N}$ with $\mu_{p^n} \subseteq F^*$, let T_{p^n} be the subgroup of $TBr(F)$ generated by the symbol algebras $\{(t_i, t_j; F)_{p^n} |\ i, j \in I_p,\ i < j\}$. As the proof of Prop. 3.5 below shows, T_{p^n} is a free $\mathbb{Z}/p^n\mathbb{Z}$-module, and these symbol algebras are a base. If $\mu_{p^n} \subseteq F^*$ for all $n \in \mathbb{N}$, then let $T_{p^\infty} = \bigcup_{n=1}^{\infty} T_{p^n}$.

Proposition 3.5. *Fix any Henselian valued field F and any prime $p \neq \operatorname{char}(\overline{F})$. If $r \in \mathbb{N}$ is maximal such that $\mu_{p^r} \subseteq F^*$, then $Br(F)(p) = SBr(F)(p) \oplus T_{p^r}$. If $\mu_{p^n} \subseteq F^*$ for every n, then $Br(F)(p) = SBr(F)(p) \oplus T_{p^\infty}$.*

Proof. We have the exact sequence

$$0 \longrightarrow SBr(F)(p) \longrightarrow Br(F)(p) \xrightarrow{\rho} Br(F_{nr})(p), \tag{3.11}$$

where ρ is the restriction map. Assume first that r is maximal such that $\mu_{p^r} \subseteq F^*$. For $D \in Br(F)(p)$, since $p \neq \operatorname{char}(\overline{F})$, D is tame (see (3.10)). So $D \sim S \otimes_F T$ in $Br(F)$, where $S \in SBr(F)$ and T is TTR, with S and T both p-primary. It follows that the division algebra T has degree a power of p. In a tensor decomposition of T into symbol algebras, each symbol algebra factor has degree p^m for some m with $\mu_{p^m} \subseteq F^*$, so $m \leq r$. Hence,

$$\exp(D \otimes_F F_{nr}) = \exp(T \otimes_F F_{nr}) \ \big|\ \exp(T) \ \big|\ p^r.$$

That is, $\operatorname{im}(\rho) \subseteq {}_{p^r}Br(F_{nr})$.

We claim that ${}_{p^r}Br(F_{nr})$ is a free $\mathbb{Z}/p^r\mathbb{Z}$-module with base $\mathcal{B} = \{(t_i, t_j; F_{nr})_{p^r} \mid i, j \in I_p,\ i < j\}$. For this, note first that every division algebra $D \in {}_{p^r}Br(F_{nr})$ is tame (see

(3.10)), so TTR, since $SBr(F_{nr}) = 0$; so D is a tensor product of TTR symbol algebras. Since $F_{nr}^*/F_{nr}^{*p^r} \cong \Gamma/p^r\Gamma$, $\{t_i \mid i \in I_p\}$ maps to a base of the free $\mathbb{Z}/p^r\mathbb{Z}$-module $F_{nr}^*/F_{nr}^{*p^r}$. Hence, $\mathcal{B}$ is a generating set for ${}_{p^r}Br(F_{nr})$. To see $\mathbb{Z}/p^r\mathbb{Z}$-independence of $\mathcal{B}$ it suffices to check independence for $\{(t_i, t_j; F_{nr})_{p^r} \mid i, j \in J,\ i < j\}$ for any finite subset $J \subseteq I_p$. Say $J = \{i_1, \ldots, i_\ell\}$ with $i_1 < i_2 < \ldots < i_\ell$. Take any $u = t_{i_1}^{m_1} \ldots t_{i_{\ell-1}}^{m_{\ell-1}}$ with not all $m_j \in p^r\mathbb{Z}$, and let $A = (u, t_{i_\ell}; F_{nr})_{p^r}$. By induction on ℓ, it suffices to check that $[A]$ is not in the subgroup S of ${}_{p^r}Br(F)$ generated by $\{(t_i, t_j; F_{nr})_{p^r} \mid i, j \in \{i_1, \ldots, i_{i_{\ell-1}}\},\ i < j\}$. But, if s is maximal such that p^s divides each of the m_j, then $s < r$ and $u = u_0^{p^s}$ with $u_0 \notin F_{nr}^{*p}$. Then in $Br(F_{nr})$, $A \sim A_0 = (u_0, t_{i_\ell}; F_{nr})_{p^{r-s}}$. Ex. 3.4 shows that A_0 is a TTR division algebra with $\Gamma_{A_0} = \frac{1}{p^{r-s}}\langle v(u_0), v(t_{i_\ell})\rangle + \Gamma_F$. On the other hand, by [TW, Ex. 4.4(i)], every division algebra in S has value group lying in $\sum_{j=1}^{\ell-1} \left\langle \frac{1}{p^r} v(t_{i_j})\right\rangle + \Gamma_F$, and this group does not contain $\frac{1}{p^{r-s}} v(t_{i_\ell})$. Thus, $[A] \notin S$, and the claim is proved.

Because ${}_{p^r}Br(F_{nr})$ is a free $\mathbb{Z}/p^r\mathbb{Z}$-module with base $\mathcal{B}$, there is a well-defined group homomorphism $h_r \colon {}_{p^r}Br(F_{nr}) \to {}_{p^r}Br(F)$ given by $(t_i, t_j; F_{nr})_{p^r} \mapsto (t_i, t_j; F)_{p^r}$. Clearly $\operatorname{im}(h_r) = T_{p^r}$ and $\rho \circ h_r = \mathrm{id}$ on ${}_{p^r}Br(F_{nr})$; combined with what we have proved above, this shows that $\operatorname{im}(\rho) = {}_{p^r}Br(F_{nr})$ and that h_r is a splitting map for (3.11). Thus $Br(F)(p) = \ker(\rho) \oplus \operatorname{im}(h_r) = SBr(F)(p) \oplus T_{p^r}$.

Now assume instead that $\mu_{p^n} \subseteq F^*$ for every $n \in \mathbb{N}$. For each n, we have a homomorphism $h_n \colon {}_{p^n}Br(F_{nr}) \to {}_{p^n}Br(F)$ defined as above. Since $h_n \,|_{\,{}_{p^m}Br(F_{nr})} = h_m$ for $m < n$, these maps are compatible, and we can take their union, obtaining $h \colon Br(F_{nr})(p) \to Br(F)(p)$ with $\rho \circ h = \mathrm{id}$. Thus h is a splitting map for (3.11), and $Br(F)(p) = \ker(\rho) \oplus \operatorname{im}(h) = SBr(F)(p) \oplus T_{p^\infty}$, as desired. $\square$

Remark 3.6. F. Chang tells us that for p odd, all the division algebras in T_{p^r} are TTR, whenever $\mu_{p^r} \subseteq F^*$. A proof of this is given in [C, Th. 2.3.2]. But this is not true in general for $p = 2$. For example if $|I_2| \geq 3$, let T be the underlying division algebra of $(t_1, t_2; F)_2 \otimes (t_1, t_3; F)_2 \otimes (t_2, t_3; F)_2$. Then $T \sim (-1, t_2; F)_2 \otimes (t_1 t_2, t_2 t_3; F)_2$ in $Br(F)$. If $\mu_4 \not\subseteq F^*$, then this equivalence is an isomophism, and T is not TTR. But if $\mu_4 \subseteq F^*$, then $T \cong (t_1 t_2, t_2 t_3; F)_2$ which is TTR.

4. Projective Schur groups of Henselian valued fields

Let F be an equicharacteristic Henselian valued field with residue field k. (Equicharacteristic means $\operatorname{char}(k) = \operatorname{char}(F)$.) In this section we show how $PS(F)$ is related to $PS(k)$. We show that if every projective Schur algebra over k is a radical (resp. radical abelian) algebra, then the same is true for projective Schur algebras over F. The results here generalize those in [AS_6], which treated the case where $F = k((t_1))...((t_n))$.

Let F be any field, and let K be a finite radical field extension of F, i.e., $K = F(U)$ where U is a subgroup of K^* with $U \supseteq F^*$ and U/F^* is finite. If L is an intermediate field, $F \subseteq L \subseteq K$, and L is abelian Galois over F, then it is shown in [AS_2, Prop. 2.1] that $L \subseteq M$ where M is a compositum of a finite cyclotomic extension of F and a finite Kummer extension of F. In particular, if L is a radical abelian extension of F, then L lies in such an M. Let F_{ra} denote the maximal radical abelian extension of F, i.e., the compositum $F_{ra} = F_{cyc} \cdot F_{kum}$, where F_{cyc} is the maximal cyclotomic extension of F and F_{kum} is the maximal Kummer extension of F. (The notation F_{radab} was used for F_{ra} in [AS_6].)

Now, suppose F has a Henselian valuation, with residue field k. As in §3 we write F_{nr} for the maximal unramified extension of F; $SBr(F)$ for $Br(F_{nr}/F)$; and $F(k_{ra})$ for the unramified extension of F with residue field k_{ra}. We have,

$$F(k_{ra}) \;=\; F_{nr} \cap F_{ra}\,. \tag{4.1}$$

To see this, note first that the maximal unramified extension of F in F_{kum} is $F(k_{kum})$. Therefore, $F_{nr} \cap F_{ra} = F_{nr} \cap (F_{cyc} \cdot F_{kum}) = F_{cyc} \cdot (F_{nr} \cap F_{kum}) = F(k_{cyc}) \cdot F(k_{kum}) = F(k_{ra})$, where the second equality is immediate from the corresponding equality of associated subgroups of the absolute Galois group of F, as F_{nr} is Galois over F.

Proposition 4.1. (cf. [AS_6, Th. 2.3]) *Let F be a Henselian valued field with residue field k, and assume* $\mathrm{char}(F) = \mathrm{char}(k)$. *Then $PS(F) \cap SBr(F) \subseteq Br(F(k_{ra})/F)$.*

Proof. Let $A \in PS(F) \cap SBr(F)$, say $A = F(\mathcal{G})$, where $\mathcal{G}$ is a subgroup of A^* spanning A as an F-vector space, with $F^* \subseteq \mathcal{G}$ and $|\mathcal{G}/F^*| < \infty$. Assume A is reduced. Let $\mathcal{H} = \mathcal{G}'$ (the derived group of $\mathcal{G}$, a finite group), and as in Prop. 2.2 let $B = F(\mathcal{H})$, $L = Z(B)$, $\widehat{\mathcal{H}} = C_{\mathcal{G}}(L)$, and $E = F(\widehat{\mathcal{H}}) = C_A(L) = B \otimes_L T$, where $T = C_E(B) = L(\mathcal{T})$, where $\mathcal{T} = C_{\widehat{\mathcal{H}}B^*}(B)$. So, L is a field, and as B is a Schur algebra, by the Brauer splitting theorem ([CR, pp. 385, 418]) $L \subseteq F_{cyc}$ and F_{cyc} splits B. Note that $F_{cyc} = F(k_{cyc}) \subseteq F_{nr}$, since F is Henselian and $\mathrm{char}(k) = \mathrm{char}(F)$, so that F and k contain "the same" roots of unity. Because $A \in SBr(F)$ and $E = C_A(L) \sim L \otimes_F A$ in $Br(L)$, we have $E \in SBr(L)$. Since $E = B \otimes_L T$ and $B \in SBr(L)$, we must also have $T \in SBr(L)$. By Prop. 2.2(d), $T = L(\mathcal{T})$ is a projective Schur algebra of abelian type over L. Therefore, if we let $\Lambda = \mathcal{T}/L^*$, Prop. 2.1 shows that we have the nondegenerate symplectic pairing $B_{\mathcal{T}}\colon \Lambda \times \Lambda \to \mu(L)$ induced by commutators of elements of $\mathcal{T}$. Let $n = \exp(\Lambda)$. Since $\mu_n = \mathrm{im}(B_{\mathcal{T}}) \subseteq F^*$, we have $\mathrm{char}(F) \nmid n$. Therefore, $\mathrm{char}(k) = \mathrm{char}(F) \nmid |\Lambda| = \dim_L(T)$, using Prop. 2.1(c).

Let $v\colon F^* \longrightarrow \Gamma$ be our Henselian valuation on F, where Γ is the value group of v, and let $\Delta = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$, the divisible hull of Γ. Because commutators of elements of $\mathcal{T}$ are roots of unity, the valuation v induces a well-defined group homomorphism $w\colon \Lambda \longrightarrow \Delta/\Gamma$ given by $w(tL^*) = \frac{1}{n}v(t^n) + \Gamma$, where $n = \exp(\Lambda)$. Let $\Lambda_0 = \ker(w) \subseteq \Lambda$. Because $T = L(\mathcal{T})$ is a projective Schur algebra of abelian type, the same is true for $L_{nr} \otimes_L T = L_{nr}(\widetilde{\mathcal{T}})$, where $\widetilde{\mathcal{T}}/L_{nr}^*$ is the image of $\mathcal{T}/L^*$ under the canonical injection $T^*/L^* \hookrightarrow (L_{nr} \otimes_L T)^*/L_{nr}^*$. Let $\widetilde{\Lambda} = \widetilde{\mathcal{T}}/L_{nr}^*$, and let $B_{\widetilde{\mathcal{T}}}\colon \widetilde{\Lambda} \times \widetilde{\Lambda} \to \mu(L_{nr}^*)$ be the associated nondegenerate pairing. Since L_{nr} is unramified over L, and hence over F, the value group of L_{nr} is Γ. Let $\widetilde{w}\colon \widetilde{\Lambda} \to \Delta/\Gamma$ be the homomorphism defined the same way as w, and let $\widetilde{\Lambda}_0 = \ker(\widetilde{w})$. The isomorphism $j\colon \Lambda \to \widetilde{\Lambda}$ is clearly an isometry between the pairings $B_{\mathcal{T}}$ and $B_{\widetilde{\mathcal{T}}}$; also, clearly $w = \widetilde{w} \circ j$. By [TW, Th. 4.3] $L_{nr} \otimes_L T$ is similar to a tame totally ramified division algebra D over L_{nr} with relative value group $\Gamma_D/\Gamma \cong \widetilde{\Lambda}_0^\perp/(\widetilde{\Lambda}_0 \cap \widetilde{\Lambda}_0^\perp)$. But, we saw above that $T \in SBr(L)$. Hence, $D = L_{nr}$, so $\widetilde{\Lambda}_0^\perp/(\widetilde{\Lambda}_0 \cap \widetilde{\Lambda}_0^\perp)$ must be trivial, i.e., $\widetilde{\Lambda}_0^\perp \subseteq \widetilde{\Lambda}_0$. From the isometry between $B_{\mathcal{T}}$ and $B_{\widetilde{\mathcal{T}}}$ it follows that $\Lambda_0^\perp \subseteq \Lambda_0$. Therefore, by the comments preceding Prop. 2.4, there is a Lagrangian Λ_1 of Λ with $\Lambda_0^\perp \subseteq \Lambda_1$, so $\Lambda_1 = \Lambda_1^\perp \subseteq \Lambda_0^{\perp\perp} = \Lambda_0$. Let $L_1 = L(\mathcal{L}_1)$, where $\Lambda_1 = \mathcal{L}_1/L^* \subseteq \mathcal{T}/L^*$. By Prop. 2.4(a), L_1 is a field which lies in a radical abelian extension of F, so $L_1 \subseteq F_{ra}$. Note that for any $t \in \mathcal{L}_1$, we have $t^n \in L^*$ as $\exp(\Lambda) = n$, and $v(t^n) \in n\Gamma$ as $tL^* \in \Lambda_1 \subseteq \Lambda_0 = \ker(w)$. So, there is $\ell \in L^*$ with $v((t\ell)^n) = 0$. Since L_1 is generated over L by valuation units of n-th roots of elements of L and $\operatorname{char}(k) \nmid n$, we have L_1 is a compositum of unramified extensions of L, hence L_1 is unramified over L. Because L is unramified over F, it follows that $L_1 \subseteq F_{nr}$. Thus, $L_1 \subseteq F_{nr} \cap F_{ra} = F(k_{ra})$, with the last equality given by (4.1). Prop. 2.4(a) shows that for $E_1 = C_A(L_1)$ we have $E_1 = B \otimes_L L_1$. So E_1 is a Schur algebra over L_1, since B is a Schur algebra over L. Hence, there is a root of unity ω such that $L_1(\omega)$ splits E_1. Then $L_1(\omega)$ splits A as $L_1 \otimes_F A \sim E_1$ in $Br(L_1)$; also $L_1(\omega) \subseteq F(k_{ra})$ since $L_1 \subseteq F(k_{ra})$. So, $A \in Br(F(k_{ra})/F)$, as desired. $\square$

In order to relate unramified phenomena over a Henselian field to corresponding phenomena over the residue field, one often uses the valuation ring V as a bridge. To employ that bridge here for projective Schur algebras, we need to know about the structure of "tame" twisted group rings over V. The next proposition gives what is needed. It may be of some interest in its own right.

Proposition 4.2. *Let V be a Henselian valuation ring, M its maximal ideal, F its quotient field, and $k = V/M$ its residue field. Let V^zG be a twisted group ring over V, where $z \in H^2(G, V^*)$ and G is a finite group acting trivially on V. Suppose $\operatorname{char}(k) \nmid |G|$. Then $V^zG = \bigoplus_{i=1}^m S_i$, where each S_i is an Azumaya algebra over $Z(S_i)$, and $Z(S_i)$ is a valuation ring unramified over V. For the corresponding twisted group ring $k^{\overline{z}}G$ over k, we have $k^{\overline{z}}G \cong \bigoplus_{i=1}^m S_i/MS_i$, with each S_i/MS_i a simple algebra with center the residue field of $Z(S_i)$.*

Proof. Let $R = V^zG$. We have $R/MR \cong k^{\overline{z}}G$, where $\overline{z}$ is the image of z in $H^2(G, k^*)$. Because $\operatorname{char}(k) \nmid |G|$ it is known by [P, p. 30, Th. 4.2] that $k^{\overline{z}}G$ is a semisimple k-algebra. So, $Z(k^{\overline{z}}G) = \bigoplus_{i=1}^{m} L_i$, where each L_j is a field with $[L_j : k] < \infty$. Moreover, we can see that each L_j must be separable over k. For, $L_j \otimes_k k^{\overline{z}}G \cong L_j^{\overline{z}}G$, which is semisimple; so its center $\bigoplus_{i=1}^{m} L_j \otimes_k L_i$ is a direct sum of fields. Since $L_j \otimes_k L_j$ is a direct sum of fields, L_j must be separable over k. Consequently, $k^{\overline{z}}G$ is a separable algebra over k. Because R/MR is a separable V/M-algebra, it follows by [DI, p. 72, Th. 7.1] that R is a separable V-algebra. Let $Z = Z(R)$. Then, R is a central separable algebra (= Azumaya algebra) over Z and Z is separable over V by [DI, p. 55, Th. 3.8]. Hence, Z is a direct summand of R as a Z-module, so as a V-module, by [DI, p. 51, Lemma 3.1]. Therefore, as R is a free V-module of rank $|G| < \infty$, Z is a finitely generated projective V-module, hence a free V-module as V is local. Now, $R/MR \cong R \otimes_Z (Z/MZ)$, which is a central separable Z/MZ-algebra (see [DI, p. 61, Lemma 5.1]); so $Z/MZ \cong Z(R/MR) \cong \bigoplus_{i=1}^{m} L_i$.

Suppose first that Z/MZ is a field, i.e., $m = 1$. Now, $Z \otimes_V F$ is a commutative separable F-algebra by [DI, p. 44, Cor. 1.7], so $Z \otimes_V F = \bigoplus_{j=1}^{n} K_j$, where each K_j is a field separable over F. Let W_j be the unique (as V is Henselian) valuation ring of K_j extending V; so, W_j is the integral closure of V in K_j. Let $T = \bigoplus_{j=1}^{n} W_j$, which is the integral closure of V in $\bigoplus_{j=1}^{n} K_j$. When we view $Z \subseteq Z \otimes_V F = \bigoplus_{j=1}^{n} K_j$, we have $Z \subseteq T$, since Z is integral over V. Because MZ was assumed to be a maximal ideal of Z, we have $MT \cap Z = MZ$, so $Z/MZ \subseteq T/MT$. Thus,

$$\begin{aligned}\dim_k(T/MT) &\geq \dim_k(Z/MZ) = \operatorname{rk}_V(Z) = \dim_F(Z \otimes_V F)\\ &= \sum_{j=1}^{n} [K_j : F] \geq \sum_{j=1}^{n} \dim_k(W_j/MW_j) = \dim_k(T/MT).\end{aligned}$$

See [Bou, Ch. VI, § 8, No. 5, Cor. to Prop. 5] for the last inequality here. Therefore, equality holds throughout. Hence, $T/MT = Z/MZ \cong L_1$, which is a field separable over k. So, $n = 1$, $T = W_1$, and $W_1/MW_1 = T/MT \cong L_1$. So, MW_1 is the maximal ideal of W_1, and $\dim_k(W_1/MW_1) = [K_1 : F]$. Hence, W_1 is unramified over V. So, W_1 is a finitely generated V-module (see [Bou, Ch. VI, § 8, No. 5, Th. 2]); since $\dim_k(Z/MZ) = \dim_k(W_1/MW_1)$, Nakayama's Lemma shows that $Z = W_1$, as desired.

Now drop the assumption that Z/MZ is a field. We still have $Z/MZ \cong \bigoplus_{i=1}^{m} L_i$, where each L_i is a field separable over k. Let $\widetilde{e}_1, \dots, \widetilde{e}_m$ be the primitive orthogonal idempotents of Z/MZ, with the $\widetilde{e}_i$ numbered so that each $\widetilde{e}_i(Z/MZ) \cong L_i$. Because Z is a finitely generated module over the Henselian local ring V, the $\widetilde{e}_i$ lift to pairwise orthogonal idempotents $e_1, \dots, e_m \in Z$ with each $\widetilde{e}_i$ the image of e_i in Z/MZ, and $e_1 + \ldots + e_m = 1$, by [Az, Th. 24] or [MMU, p. 180, Th. A.18]. Then, $Z = \bigoplus_{i=1}^{m} Z_i$, where each $Z_i = e_i Z$. Since each Z_i is a direct summand of Z, Z_i is a finitely generated projective, hence free, V-module. Moreover, each Z_i is a separable V-algebra with $Z_i/MZ_i = e_iZ/Me_iZ \cong \widetilde{e}_i(Z/MZ) \cong L_i$, which is a field separable over k. Therefore, the argument of the preceding paragraph, applied to Z_i, shows that Z_i is a valuation ring unramified over V. The e_i are orthogonal central idempotents of R; so $R = \bigoplus_{i=1}^{m} S_i$, where $S_i = e_iR$. Since R is central separable over Z, each S_i is central separable over $e_iZ = Z_i$, i.e., S_i is an Azumaya algebra over Z_i, which is a valuation ring unramified over V. Also, $k^{\overline{z}}G \cong R/MR = \bigoplus_{i=1}^{m} S_i/MS_i$. Each $S_i/MS_i \cong S_i \otimes_{Z_i} (S_i/MZ_i)$, so S_i/MS_i is an Azumaya algebra over the field Z_i/MZ_i, i.e., a simple algebra with center Z_i/MZ_i, which is the residue field of Z_i. $\square$

For the rest of this section, we adopt the following *standing hypotheses*:

F is a field with Henselian valuation v with residue field k, with $\mathrm{char}(k) = \mathrm{char}(F)$.

We write V for the valuation ring of v; $\Gamma = \Gamma_F$ for the value group of v; and $\Delta = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$. We routinely identify $Br(k)$ with its canonical image in $Br(F)$ via the map of (3.3) above. Also, recall that C' denotes the prime to p part of a torsion abelian group C associated to F if $\mathrm{char}(F) = p \neq 0$, while $C' = C$ if $\mathrm{char}(F) = 0$.

Proposition 4.3. (cf. [AS$_6$, Prop. 2.6]) $PS(F)' \cap Br(k) = PS(k)'$.

Proof. Recall how the canonical inclusion $Br(k) \hookrightarrow Br(F)$ can be obtained (cf. [JW, Th. 5.6(a), Th. 2.8]): The map $\theta\colon Br(V) \to Br(k)$ given by $[A] \mapsto [A \otimes_V k]$ is an isomorphism, as V is Henselian. The map $\varphi\colon Br(V) \to Br(F)$ given by $[A] \mapsto [A \otimes_V F]$ is injective as V is a valuation ring. The inclusion $Br(k) \hookrightarrow Br(F)$ is $\varphi \circ \theta^{-1}$. Of course if V contains a coefficient field, which is a subfield $k_0 \subseteq V$, which maps isomorphically onto k under the composition $k_0 \hookrightarrow V \to V/M = k$ (with M the maximal ideal of V), then we can identify k with k_0, and the map $Br(k) \hookrightarrow Br(F)$ is given by scalar extension $[A] \mapsto [A \otimes_k F]$. If $\mathrm{char}(k) = 0$, then there always is a coefficient field, as V is Henselian. This is provable in the same way as for a complete discrete valuation ring (see e.g. [ZS, p. 280, Cor. 2]), since only Hensel's Lemma is used. If $\mathrm{char}(k) = \mathrm{char}(F) = p \neq 0$, then there may not always exist a coefficient field; if k is separably generated over its prime field, then there is a coefficient field.

The inclusion $PS(k)' \subseteq PS(F)' \cap Br(k)$ is now clear if $\mathrm{char}(k) = 0$, since then V contains a coefficient field. On the other hand, if $\mathrm{char}(k) = p \neq 0$, then we have the description in [AS$_4$, Th. 1.4] of algebras in $PS(k)'$ as tensor products of symbol algebras together with a cyclic algebra $(\ell/k, \sigma, a)$, with $\ell \subseteq k_{cyc}$. Every such algebra has an obvious lift to an Azumaya algebra of the same type over V, which then extends by $- \otimes_V F$ to a central simple algebra of the same type over F. So again it is clear that $PS(k)' \subseteq PS(F)' \cap Br(k)$.

For the reverse inclusion, the argument is based on that in [AS$_6$, Prop. 2.6], but adapted to work even if V does not contain a coefficient field, and to apply for an arbitrary value group.

Let $\alpha \in Br(k)'$ with image $\widetilde{\alpha}$ in $Br(F)'$ which lies in $PS(F)'$. Let $\widetilde{A} = F(\mathcal{G})$ be a projective Schur algebra over F representing $\widetilde{\alpha}$, where $\mathcal{G}$ is a subgroup of $\widetilde{A}^*$ with $F^* \subseteq \mathcal{G}$, $\mathcal{G}$ spans $\widetilde{A}$ as an F-vector space, and $|\mathcal{G}/F^*| < \infty$. Let $G = \mathcal{G}/F^*$. By [AS$_6$, Lemma 2.5], we may assume that $|G|$ is prime to p if $\operatorname{char}(F) = p \neq 0$. We have the central extension

$$1 \longrightarrow F^* \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow 1.$$

Denote by $z \in H^2(G, F^*)$ the cohomology class of this extension (with G acting trivially on G^*). There is a corresponding surjective F-algebra homomorphism,

$$\eta \colon F^z G \longrightarrow F(\mathcal{G}),$$

where $F^z G$ is the group algebra twisted by z.

Suppose for now that z lies in the image of the map $H^2(G, V^*) \to H^2(G, F^*)$. Let $V^z G$ be the corresponding twisted group ring over V. We have $V^z G = \bigoplus_{i=1}^{m} S_i$, as in Prop. 4.2. So, $F^z G \cong F \otimes_V V^z G \cong \bigoplus_{i=1}^{m} (F \otimes_V S_i)$. Each $F \otimes_V S_i$ is an Azumaya algebra (= central simple algebra) over the field $F \otimes_V Z(S_i)$, which is the quotient field of the valuation ring $Z(S_i)$. The surjection η above shows that one of the simple summands $F \otimes_V S_j$ of $F^z G$ is isomorphic to $F(\mathcal{G})$. By comparing centers, we find $F \otimes_V Z(S_j) = F$, so $Z(S_j) = Z(S_j) \cap F = V$. Let $A = k \otimes_V S_j \cong S_j / M S_j$. Then, A is a central simple k-algebra which by Prop. 4.2 is a direct summand of $k^z G$. Hence, A is a projective Schur algebra over k. Since $[A]$ maps to $[\widetilde{A}]$ under the injective map $Br(k) \to Br(F)$, we have $[A] = \alpha \in Br(k)$, showing that $\alpha \in PS(k)'$, as desired.

So far we have assumed that z was in the image of $H^2(G, V^*) \longrightarrow H^2(G, F^*)$. Now drop this assumption. The short exact sequence of trivial G-modules $1 \to V^* \to F^* \to \Gamma \to 1$ yields the exactness of $H^2(G, V^*) \to H^2(G, F^*) \to H^2(G, \Gamma)$. Since Δ is a uniquely divisible group, we have $H^2(G, \Gamma) \cong H^1(G, \Delta/\Gamma) = \operatorname{Hom}(G, \Delta/\Gamma)$ (cf. the comments preceding (3.3) above). Hence we have an exact sequence $H^2(G, V^*) \to H^2(G, F^*) \to \operatorname{Hom}(G, \Delta/\Gamma)$. This is clearly functorial in F, i.e., for any field K algebraic over F, we have a commutative diagram with exact rows:

$$\begin{array}{ccccc}
H^2(G, V^*) & \longrightarrow & H^2(G, F^*) & \longrightarrow & \operatorname{Hom}(G, \Delta/\Gamma) \\
\downarrow & & \downarrow & & \downarrow \\
H^2(G, V_K^*) & \longrightarrow & H^2(G, K^*) & \longrightarrow & \operatorname{Hom}(G, \Delta/\Gamma_K)
\end{array}$$

Let $\psi \in \mathrm{Hom}(G, \Delta/\Gamma)$ be the image of $z \in H^2(G, F^*)$. Then, $\mathrm{im}(\psi)$ is a finite subgroup of Δ/Γ. Let K be a totally ramified finite degree extension of F such that $\mathrm{im}(\psi) \subseteq \Gamma_K/\Gamma$. Then ψ maps to 0 in $\mathrm{Hom}(G, \Delta/\Gamma_K)$, so the commutative diagram shows that the image z_1 of z in $H^2(G, K^*)$ lies in the image of $H^2(G, V_K^*)$. Thus the preceding argument applies over K (we work with $K^{z_1}G = K \otimes_F F^z G$, which maps to the central simple K-algebra $K(\mathcal{G}_1) = K \otimes_F F(\mathcal{G})$, where $\mathcal{G}_1 = K^*\mathcal{G}$). The argument shows that $\widetilde{\alpha}_K$ lies in the image of $PS(\overline{K})'$ in $Br(K)$, where $\overline{K}$ is the residue field of V_K. This $\widetilde{\alpha}_K$ is the image in $Br(K)$ of the image $\alpha_{\overline{K}}$ of α in $Br(\overline{K})$, by the commutative diagram (3.4) with K replacing L; so $\alpha_{\overline{K}} \in PS(\overline{K})'$. But $\overline{K} \cong \overline{F} = k$, as K is totally ramified over F. Hence $\alpha \in PS(k)'$, as desired. $\square$

Theorem 4.4. *Assuming the standing hypotheses stated above, let $SBr(F)$ denote the inertially split part of $Br(F)$ and let $SPS(F) = PS(F) \cap SBr(F)$. Then there is a split exact sequence:*

$$0 \longrightarrow PS(k)' \xrightarrow{j} SPS(F)' \xrightarrow{\eta} \mathrm{Hom}_c(\mathcal{G}(k_{ra}/k), \Delta/\Gamma)' \longrightarrow 0$$

If k is perfect, then the above sequence is split exact without $(-)'$.

Proof. Since the valuation v is Henselian, $F(k_{ra}) \subseteq F_{ra}$, and v extends uniquely to $F(k_{ra})$. Moreover since $F(k_{ra})$ is an unramified extension of F, the value groups $\Gamma_{F(k_{ra})}$ and Γ_F are equal. Applying Th. 3.2 to F and $F(k_{ra})$ we obtain (using the functoriality noted in (3.4)) a commutative and exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Br(k_{ra})' & \xrightarrow{i_{ra}} & SBr(F(k_{ra}))' & \xrightarrow{\pi_{ra}} & \mathrm{Hom}_c(G_{k_{ra}}, \Delta/\Gamma)' & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{res}}\uparrow & & {\scriptstyle \mathrm{res}}\uparrow & & {\scriptstyle \mathrm{res}}\uparrow & & \\
0 & \longrightarrow & Br(k)' & \xrightarrow{i} & SBr(F)' & \xrightarrow{\pi} & \mathrm{Hom}_c(G_k, \Delta/\Gamma)' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Br(k_{ra}/k)' & \xrightarrow{\mathrm{res}(i)} & SBr(F(k_{ra})/F)' & \xrightarrow{\mathrm{res}(\pi)} & \mathrm{Hom}_c(\mathcal{G}(k_{ra}/k), \Delta/\Gamma)' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The third row is split exact because the first two rows are split exact with compatible splitting maps by the proof of Prop. 3.3(a). Now $PS(k)' \subseteq Br(k_{ra}/k)'$ [AS$_2$, Cor. 2.3], and by Prop. 4.1, $SPS(F)' \subseteq Br(F(k_{ra})/F)'$. Moreover by Prop. 4.3 we have $PS(k)' \subseteq PS(F)' \cap Br(k)'$, hence $PS(k)' \subseteq PS(F)' \cap SBr(F)' = SPS(F)'$. We therefore obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Br(k_{ra}/k)' & \xrightarrow{\text{res}(i)} & SBr(F(k_{ra})/F)' & \xrightarrow{\text{res}(\pi)} & \text{Hom}_c(\mathcal{G}(k_{ra}/k),\Delta/\Gamma)' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & PS(k)' & \xrightarrow{j} & SPS(F)' & \xrightarrow{\eta} & \text{Hom}_c(\mathcal{G}(k_{ra}/k),\Delta/\Gamma)' & \longrightarrow & 0 \end{array}$$

where the top row is split exact and the bottom row is exact at $PS(k)'$. Again by Prop. 4.3 we have $PS(k)' \supseteq PS(F)' \cap Br(k)' = PS(F)' \cap Br(k_{ra}/k)'$ so the bottom row is exact at $SPS(F)'$. To complete the proof of the first assertion of the theorem, we prove that the splitting map $f\colon \text{Hom}(\mathcal{G}(k_{ra}/k),\Delta/\Gamma)' \longrightarrow SBr(F(k_{ra})/F)'$ takes values in $PS(F)'$ hence in $SPS(F)'$. Indeed, for $\chi \in \text{Hom}_c(\mathcal{G}(k_{ra}/k),\Delta/\Gamma)'$, $f(\chi)$ is a tensor product of cyclic algebras of the form $(F(E)/F,\sigma,t)$, where E is a cyclic extension of k lying in k_{ra} with $[F(E):F]=[E:k]$ dividing the order of χ. Since $F(E)\subseteq F_{ra}$ each such cyclic algebra is a radical abelian algebra of degree not a multiple of $\text{char}(F)$, so lies in $SPS(F)'$.

For the second assertion of the theorem, we may assume $\text{char}(k)=p\neq 0$, and we need only prove the assertion for the p-primary components. By (3.3) we have an exact sequence

$$0 \longrightarrow Br(k) \longrightarrow SBr(F) \longrightarrow \text{Hom}_c(G_k,\Delta/\Gamma) \longrightarrow 0.$$

Since the p-power map is an automorphism of k (k is perfect of characteristic p), $Br(k)(p)=0$. Hence $SBr(F)(p)=\text{Hom}_c(G_k,\Delta/\Gamma)(p)$. Similarly, since k has no nontrivial Kummer p-extensions, $SBr(F(k_{ra}))(p) = SBr(F(k_{cyc}))(p) = \text{Hom}_c(G_{k_{cyc}},\Delta/\Gamma)(p)$, and we get a commutative diagram

$$\begin{array}{ccc} SBr(F)(p) & \xrightarrow{\cong} & \text{Hom}_c(G_k,\Delta/\Gamma)(p) \\ {\scriptstyle\text{res}}\downarrow & & {\scriptstyle\text{res}}\downarrow \\ SBr(F(k_{cyc}))(p) & \xrightarrow{\cong} & \text{Hom}_c(G_{k_{cyc}},\Delta/\Gamma)(p) \end{array}$$

It follows that the corresponding kernels of the restriction maps are isomorphic:

$$SBr(F(k_{cyc})/F)(p) \xrightarrow{\cong} \text{Hom}_c(G_{k_{cyc}},\Delta/\Gamma)(p).$$

But by Prop. 4.1, $SPS(F)(p)\subseteq SBr(F(k_{cyc})/F)(p)$. Furthermore, using the splitting map, we see that the map $SPS(F)(p)\longrightarrow \text{Hom}(G_{k_{cyc}},\Delta/\Gamma)(p)$ is surjective, and the result follows. □

For an equicharacteristic Henselian field F, for any prime $p\neq \text{char}(F)$, Th. 3.2 and Prop. 3.5 yield a direct sum decomposition

$$Br(F)(p) \;=\; Br(k)(p)\oplus f(\text{Hom}_c(G_k,\Delta/\Gamma)(p))\oplus T\,, \tag{4.2}$$

where $f\colon \text{Hom}_c(G_k,\Delta/\Gamma)\longrightarrow SBr(F)$ is the (injective) splitting map of Th. 3.2 above and $T=T_{p^r}$ if r is maximal such that $\mu_{p^r}\subseteq k$, and $T=T^{p^\infty}$ if there is no such r. We can now see that there is a compatible direct sum decomposition of $PS(F)(p)$. For this, let $k_{ra,p}$ be the maximal p-extension of k in the abelian Galois extension k_{ra}. Then,

Proposition 4.5. *For any prime* $p \neq \mathrm{char}(k)$,

$$PS(F)(p) \;=\; PS(k)(p) \oplus f(\mathrm{Hom}_c(\mathcal{G}(k_{ra,p}/k), \Delta/\Gamma)) \oplus T\,,$$

where f *and* T, *are as in* (4.2) *above.*

Proof. Since T is generated by symbol algebras, $T \subseteq PS(F)(p)$. Therefore, it suffices to see that $SPS(F)(p) = PS(k)(p) \oplus f(\mathrm{Hom}_c(\mathcal{G}(k_{ra,p}/k, \Delta/\Gamma))$. But this is clear from Th. 4.4 since $\mathcal{G}(k_{ra,p}/k)$ is the p-part of the abelian profinite group $\mathcal{G}(k_{ra}/k)$. □

Corollary 4.6. *Suppose* $\mathrm{char}(k) = \mathrm{char}(F) = 0$. *If every element of* $PS(k)$ *is represented by a radical (resp. radical abelian) algebra, then the same holds for* $PS(F)$.

Proof. We recall first [AS_3, Lemma 2.4] that the tensor product of radical algebras is Brauer equivalent to a radical algebra, and the tensor product of radical abelian algebras is Brauer equivalent to a radical abelian algebra. Now as $\mathrm{char}(k) = 0$ every element of $Br(F)$ is tame, so is represented by a tensor product $S \otimes_F T$, where $[S] \in SBr(F)$ and T is a tensor product of symbol algebras (which are clearly radical abelian algebras). We therefore need to prove the assertion for $PS(F) \cap SBr(F) = SPS(F)$. But this follows easily from Theorem 4.4 and the fact that every element of $\mathrm{Hom}_c(\mathcal{G}(k_{ra}/k), \Delta/\Gamma)$ is represented by a tensor product of cyclic radical abelian algebras. □

Corollary 4.7. *Suppose the residue field* k *is a local or global field. Then every projective Schur algebra over* F *is Brauer equivalent to a radical abelian algebra.*

Proof. If $\mathrm{char}(F) = 0$, the result follows from the preceding corollary since it holds for k, by [O,(4.3)]. (To complete the argument in [O] for the case $p = 2$ one can observe that for any number field k, the field $k(\mu_{2^\infty})$ obtained by adjoining to k all 2^n-th roots of unity for all n contains *nonreal* cyclic extensions of arbitrary 2-power degree.) If $\mathrm{char}(F) \neq 0$, the result holds because it holds for any field of characteristic not zero [AS_4, Cor. 1.5]. □

5. Projective Schur groups of Henselian fields as algebraic relative Brauer groups

In this section we prove

Theorem 5.1. *Let* F *be a field with Henselian valuation* v, *value group* $\Gamma = \Gamma_F$, *and residue field* k. *Assume* k *is a local or global field and that* $\mathrm{char}(k) = \mathrm{char}(F)$. *Then* $PS(F)$ *is an algebraic relative Brauer group* $Br(M/F)$ *with* M/F *a radical abelian extension.*

Proof. We consider the cases $k = \mathbb{R}$ and $k = \mathbb{C}$ separately. In the case $k = \mathbb{C}$, we have from Prop. 3.5 that for any prime p,

$$Br(F)(p) = SBr(F)(p) \oplus T_{p^\infty}\,,$$

and since $SBr(F)(p) = 0$, $Br(F)(p) = T_{p^\infty} \subseteq PS(F)$, so $Br(F) = PS(F) = Br(F_{kum}/F)$ and we are done.

We turn next to the case $k = \mathbb{R}$. Then the maximal unramified extension of F has residue field $\mathbb{C}$, so $F_{nr} = F(\sqrt{-1})$. Hence, $SBr(F) = Br(F(\sqrt{-1})/F)$, a 2-torsion group. By Prop. 3.5, for $p \neq 2$, $Br(F)(p) = 0$, so $Br(F) = Br(F)(2) \cong SBr(F)(2) \oplus T_2$. Each summand is generated by quaternion algebras, so $PS(F) = Br(F) = Br(F_{kum}/F)$ and we are done.

We now assume that k is either a nonarchimedean local field or a global field.

Let p be a prime number. We will prove the theorem for p-primary components. More precisely, we will prove that $PS(F)(p) = Br(M_p/F)$ with M_p/F a radical abelian p-extension. It then follows that $PS(F) = Br(M/F)$ with M equal to the composite of all the M_p.

For a given Galois extension E/k, we set $X(E/k) := \mathrm{Hom}_c(\mathcal{G}(E/k), \mathbb{Q}/\mathbb{Z})$, the character group of E/k, written additively. When $\mathcal{G}(E/k)$ is abelian there is an isomorphism between the lattice of intermediate fields K, $k \subseteq K \subseteq E$ and the lattice of subgroups of $X(E/k)$ given by $K \leftrightarrow X(K/k)$. If $m \in \mathbb{N}$, let $E^{(m)}$ denote the subfield of E corresponding to the subgroup $mX(E/k)$. So, $E^{(m)}$ is the smallest subfield of E containing k such that $\mathcal{G}(E/E^{(m)})$ is m-torsion. Since $k_{ra} = k_{cyc}k_{kum}$ (see §4), we have $X(k_{ra}/k) = X(k_{cyc}/k) + X(k_{kum}/k)$. It follows that $mX(k_{ra}/k) = mX(k_{cyc}/k) + mX(k_{kum}/k)$. Note that $mX(k_{cyc}/k) = X(k_{cyc}/k)$ when $\mathrm{char}(k) \neq 0$ since $X(k_{cyc}/k)$ is divisible in this case.

Let $m = p^r$ be the number of p-power roots of unity in k (or equivalently in F). Then m is finite. Denote by $k_{ra,p}$ the p-part of k_{ra}, i.e., the maximal p-extension of k contained in k_{ra}. Similarly let $k_{cyc,p}$ and $k_{kum,p}$ denote the p-parts of k_{cyc} and k_{kum}, respectively.

Proposition 5.2. *Let F be a field with Henselian valuation v, value group Γ, and residue field k. Let $p \neq \mathrm{char}(k)$ be a prime number. Set $m = p^r$ with r maximal such that k contains the p^rth roots of unity. Set $L := k_{ra,p}^{(m)}$. Assume $PS(k)(p) = Br(L/k)$. Then $PS(F)(p) = Br(M_p/F)$, where $M_p = F(L)(\sqrt[m]{t_i} \mid i \in I_p)$, where $v(t_i) = \gamma_i$ $(i \in I_p)$, and $\{\overline{\gamma_i} \mid i \in I_p\}$ is a $\mathbb{Z}/p\mathbb{Z}$-base of $\Gamma/p\Gamma$. ($\overline{\gamma_i} = \gamma_i + p\Gamma$.)*

Proof. As usual, let $\Delta = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$. For fields $E \supseteq K \supseteq k$ with E/K Galois, let $Y(E/K) = \mathrm{Hom}_c(\mathcal{G}(E/K), \Delta/\Gamma)$. Note that $Y(E/K)$ has similar functorial properties to $X(E/K)$. Indeed, since $\Delta/\Gamma(p) \cong \bigoplus_{i \in I_p} \mathbb{Q}/\mathbb{Z}(p)$, we have

$$Y(E/K)(p) = \mathrm{Hom}_c(\mathcal{G}(E/K), \Delta/\Gamma(p)) \cong \bigoplus_{i \in I_p} \mathrm{Hom}_c(\mathcal{G}(E/K), \mathbb{Q}/\mathbb{Z}(p)) = \bigoplus_{i \in I_p} X(E/K)(p).$$

The isomorphism holds because every continuous homomorphism from a profinite group to a discrete group has finite image. Clearly, the direct sum decomposition is compatible with the canonical inclusion $Y(N/K) \hookrightarrow Y(M/K)$ for fields $K \subseteq N \subseteq M$.

We compute $Br(M_p/F)$ using the decomposition

$$Br(F)(p) \ = Br(k)(p) \oplus f(Y(k_s/k)(p)) \oplus T_{p^r}$$

in (4.2). Clearly $T_{p^r} \subseteq Br(M_p/F)$ since each generator of T_{p^r} has a maximal subfield in M_p. Now, take any $\alpha \in Br(k(p))$ and $\chi \in Y(k_s/k)(p)$. By Prop. 3.3(a) (for $\mathrm{res}_{F(L)/F}$) and 3.3(b) (for $\mathrm{res}_{M_p/F(L)}$), we have $\alpha + f(\chi) \in Br(M_p/F)$ iff α and $f(\chi)$ each lie in $Br(M_p/F)$, iff $\alpha \in Br(L/k)$ (as $\overline{M_p} = L$) and χ maps to 0 in $\mathrm{Hom}_c(G_L, \Delta/\Gamma_{M_p})$. Since $\Gamma_{M_p}/\Gamma = \frac{1}{m}\Gamma/\Gamma$ is the m-torsion subgroup of Δ/Γ, the condition on χ is equivalent to: $0 = m\mathrm{res}_{L/k}(\chi) = \mathrm{res}_{L/k}(m\chi)$, i.e., $m\chi \in Y(L/k)$. Now, $Y(L/k) \cong \underset{i\in I_p}{\oplus} X(L/k) = \underset{i\in I_p}{\oplus} mX(k_{ra,p}/k) \cong mY(k_{ra,p}/k)$. Hence, $m\chi \in Y(L/k)$ iff $m\chi = m\psi$ with $\psi \in Y(k_{ra,p}/k)$. We claim that this last equality holds iff $\chi \in Y(k_{ra,p}/k)$. "If" is clear, taking $\psi = \chi$. Conversely, suppose $m\chi = m\psi$. Since $G_k/\ker(\chi - \psi) \cong \mathrm{im}(\chi - \psi)$ which is m-torsion, the fixed field of $\ker(\chi - \psi)$ is an m-Kummer extension of k. So, $\chi - \psi \in Y(k_{kum,p}/k) \subseteq Y(k_{ra,p}/k)$. Then, $\chi = \psi + (\chi - \psi) \in Y(k_{ra,p}/k)$, proving the claim. We have thus shown that $Br(M_p/F) = Br(L/k) \oplus f(Y(k_{ra,p}/k)) \oplus T_{p^r}$. Since $Br(L/k) = PS(k)(p)$ by hypothesis, Prop. 4.5 shows that $Br(M_p/F) = PS(F)(p)$. This completes the proof of Prop. 5.2. □

We now apply Prop. 5.2 to prove Th. 5.1. We will handle separately below an exceptional case when k is a number field which is not totally imaginary and $p = 2$. If k is a local field or a global field in the nonexceptional case and $p \neq \mathrm{char}(k)$, we will verify that the hypothesis $PS(k)(p) = Br(k^{(m)}_{ra,p}/k)$ is satisfied, proving Theorem 5.1. If k is a local or global field, then $PS(k) = Br(k)$ [LO, Satz 3]. We need to check that $L = k^{(m)}_{ra,p}$ splits every element of $Br(k)(p)$. If k is local, then since $X(k_{cyc,p}/k)$ contains a copy of $\mathbb{Q}/\mathbb{Z}(p)$, so does $mX(k_{cyc,p}/k)$, and $mX(k_{cyc,p}/k) \subseteq mX(k_{ra,p}/k) = X(L/k)$. Hence, L contains extensions of k of all p-power degrees; so $Br(L/k) = Br(k)(p)$. If k is global, then for any finite prime $\mathfrak{p}$ of k and any divisor $\mathfrak{P}$ of $\mathfrak{p}$ in L, we have $L_{\mathfrak{P}}$ contains $k^{(m)}_{\mathfrak{p},cyc,p}$, so $Br(L_{\mathfrak{P}}/k_{\mathfrak{p}}) = Br(k_{\mathfrak{p}})(p)$. Since p is odd or $\mathrm{char}(k) \neq 0$ or k is totally imaginary, the local global principle for $Br(k)$ [R, p. 276, Th. 32.11] shows that $Br(L/k) = Br(k)(p)$.

It remains to treat the exceptional case, for which we will need the following lemma:

Lemma 5.3. *Let k be a number field which is not totally imaginary. Then there exists a totally imaginary quadratic extension $\ell = k(\sqrt{\beta})$ of k such that ℓ/k does not embed into a cyclic degree 4 extension of k.*

Proof. By the approximation theorem there is $\beta \in k$ with $\beta < 0$ in each real completion of k. Then, $\ell = k(\sqrt{\beta})$ is totally imaginary. Since there is a real place of k, this β cannot be a sum of two squares in k. Therefore, by Albert's criterion [A, p. 207, Th. 11, p. 208, Ex. 1] ℓ cannot embed in a cyclic degree 4 extension of k. □

We now return to the proof of Th. 5.1 in the exceptional case. So, $p = 2$ and $m = 2$, and we replace the previous $L = k_{ra,2}^{(2)}$ by $L' = k_{ra,2}^{(2)}(\sqrt{\beta}) = L(\sqrt{\beta})$ with $k(\sqrt{\beta})$ as in Lemma 5.3, and replace M_2 by $M_2' = M_2(\sqrt{\beta})$. The earlier argument shows that $Br(L'_{\mathfrak{P}}/k_{\mathfrak{p}}) = Br(k_{\mathfrak{p}})(2)$ for each finite prime $\mathfrak{p}$ of k. Since L' has no real embeddings, it follows that $Br(L'/k) = Br(k)(2) = PS(k)(2)$. We compute $Br(M_2')$ by a variant of the proof of Prop. 5.2. For $\chi \in Y(k_s/k)(2)$, the condition for $\mathrm{res}_{M_2'/F}(f(\chi)) = 0$ is now that $2\chi \in Y(L'/k) = Y(L/k) + Y(k(\sqrt{\beta})/k) = 2Y(k_{ra,2}/k) + Y(k(\sqrt{\beta})/k)$. We claim that this holds iff $\chi \in Y(k_{ra,2}/k)$. Again, "if" is clear; for the converse, we suppose $2\chi = 2\psi + \varphi$ with $\psi \in Y(k_{ra,2}/k)$ and $\varphi \in Y(k(\sqrt{\beta})/k)$. We have

$$Y(k(\sqrt{\beta})/k) \cap 2Y(k_s/k)(2) \cong \bigoplus_{i \in I_2} \big(X(k(\sqrt{\beta})/k) \cap 2X(k_s/k)(2)\big) = 0,$$

since $k(\sqrt{\beta})$ lies in no cyclic extension of k of degree 4. But, $\varphi = 2(\chi - \psi)$ lies in this intersection, so $\varphi = 0$. Then, $2\chi = 2\psi$, and the rest of the argument to see that $PS(F)(2) = Br(M_2'/F)$ is the same as for Prop. 5.2.

It remains only to prove that $PS(F)(p) = Br(M_p/F)$ when $p = \mathrm{char}(k)$. For this, note first that for any field K with $\mathrm{char}(K) = p$, we have $Br(K_{cyc,p}/K) \subseteq PS(K)(p) \subseteq Br(K_{cyc,p}/K)$. The first inclusion holds as every finite subextension of $K_{cyc,p}/K$ is cyclic, so $Br(K_{cyc,p}/K)$ consists of cyclotomic cyclic algebras, which clearly lie in $PS(K)$. The second inclusion holds because $PS(K)(p) \subseteq Br(K_{ra,p}/K)$ by [AS$_2$, Cor. 2.3] (this is also deducible from Propositions 2.2(a) and 2.4(c) above) and $K_{ra,p} = K_{cyc,p}$ as K contains no pth roots of unity. Thus, $PS(K)(p) = Br(K_{cyc,p}/K)$. For the fields in the proof of Th. 5.1 with $p = \mathrm{char}(k)$, we have $m = 1$, so $L = k_{ra,p} = k_{cyc,p}$ and $M_p = F(L) = F(k_{cyc,p}) = F_{cyc,p}$. Hence, $Br(M_p/F) = Br(F_{cyc,p}/F) = PS(K)(p)$. This completes the proof of Theorem 5.1. $\square$

6. The Schur group case

For F a number field, $PS(F) = Br(F)$, so trivially $PS(F)$ is the algebraic relative Brauer group $Br(F_s/F)$. On the other hand, we now show that this need not be the case for the classical Schur group $S(F)$. We are grateful to Allan Herman for suggesting that $S(F)$ is not an algebraic relative Brauer group, because the local invariants of an element of the Schur group are uniformly distributed [BS, Th. 1]. Here is an explicit example: Let ζ_n be a primitive n-th root of unity in $\mathbb{Q}_{cyc}$. Let $F = \mathbb{Q}(\zeta_{12})$, and let $M = F(\zeta_{13})$. Let B be the cyclic algebra $(M/F, \sigma, \zeta_{12})$. This B is a Schur (division) algebra, of exponent 12 over F as one can check by looking at it over the completion $F_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime of F over (13). (Indeed, tensoring B with $F_{\mathfrak{p}}$, we get an algebra $B_{\mathfrak{p}}$ and it suffices to show that it has order 12. This is equivalent to ζ_{12} having order 12 mod norms from $M_{\mathfrak{p}}$. But this is the case by local class field theory because $M_{\mathfrak{p}}/F_{\mathfrak{p}}$ is totally and tamely ramified: the norm group is generated by a local parameter and the one-units.) (13) splits completely in F into a product of four primes, and the local invariant of B at each of these is of order 12 (in fact these are the only nontrivial invariants of B, so they must be 1/12, 5/12, 7/12, 11/12 by [BS, Th. 1]). Any splitting field must have local degree divisible by 12 at these four places. Now take an algebra with local invariants say 1/2, 1/2, 0, 0 at these four places and 0 everywhere else. It is not in the Schur group because these local invariants are not uniformly distributed [BS, Th. 1] (in fact, since these four invariants are not of the same order, it is enough to invoke [B, Th. 1]) and is split by any field that splits B.

One can also prove that $S(F)$ is not an algebraic relative Brauer group for F a formal power series field $k((t))$ over certain fields k. Here is a sketch of the proof. If $A = F(\mathcal{G})$ is a Schur algebra over F with $\mathcal{G}$ finite, then $A_0 := k(\mathcal{G})$ is a Schur algebra over k and $A = A_0 \otimes_k F$. It follows that $S(F) = \mathrm{res}_{F/k}(S(k))$. Recall that $S(k)(p) = 0$ if $\mathrm{char}(k) = p \neq 0$; so $S(k) \subseteq Br(k)'$. Discretely valued fields have no TTR algebras (as $|\Gamma/p\Gamma| < p^2$ for $\Gamma = \mathbb{Z}$, see the definition of T_{p^r} in §3). Thus, Prop. 3.5 and Th. 3.2 above reduce to the classical Witt decomposition

$$Br(k((t)))' = SBr(k((t)))' \cong \; Br(k)' \oplus \mathrm{Hom}_c(G_k, \mathbb{Q}/\mathbb{Z})'$$

Here, $S(F)$ sits in the first component. For the sake of simplicity let $k = \mathbb{R}$ (but the same type of argument works for any field k with $S(k)$ nontrivial and G_k pronilpotent). Suppose $S(F)$ were an algebraic relative Brauer group $Br(L/F)$ with L/F algebraic. Let L_0/F be the maximal unramified subextension of L/F, and let $\ell = \overline{L_0}$. Then, l is either $\mathbb{R}$ or $\mathbb{C}$. If $l = \mathbb{C}$, then L will split the nontrivial quaternion algebra $(-1, t)$. This algebra belongs to the second component in the Witt decomposition hence is not in $S(F)$, contradiction. It follows that L/F is totally ramified, hence its residue field is k. So, (see (3.4)) $Br(k)$ injects into $Br(L)$, hence L does not split the nontrivial quaternion Schur algebra $(-1, -1)$, another contradiction.

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