Birational geometry of quadrics in characteristic 2

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The theory of quadratic forms can be regarded as studying an important special case of the general problem of birational classification of algebraic varieties. A typical example of a Fano fibration in minimal model theory is a quadric bundle, which can be viewed without loss of information as a quadric hypersurface over the function field of the base variety. This description feeds the problem into the theory of quadratic forms, which seeks to classify quadrics over an arbitrary field up to birational equivalence. One can hope to extend the successes of quadratic form theory to broader areas of birational geometry.

In the other direction, birational geometry emphasizes the interest of particular problems in the theory of quadratic forms. In the study of Fano fibrations in positive characteristic, one cannot avoid fibrations whose generic fiber is nowhere smooth over the function field of the base variety, such as "wild conic bundles" [17]. That can happen even when the total space is smooth. In this paper, we solve two of the main problems of birational geometry for quadrics which are nowhere smooth over a field. Such quadrics are called quasilinear; they exist only in characteristic 2.

First, the "quadratic Zariski problem" [18] has a positive answer for quasilinear quadrics: for any quasilinear quadrics X and Y of the same dimension over a field such that there are rational maps from X to Y and from Y to X, X and Y must be birational (Theorem 6.5). Next, a quasilinear quadric over a field k is ruled, that is, birational to $Z \times \mathbf{P}^1$ for some variety Z over k, if and only if its first Witt index is greater than 1 (Theorem 6.4). At the same time, we find that if a quasilinear quadric is ruled, then it is ruled over a lower-dimensional quadric Z. All these statements are conjectured for arbitrary quadrics, but they are wide open for other classes of quadrics, for example in characteristic not 2.

The proofs begin by extending Karpenko and Merkurjev's theorem on the essential dimension of quadrics to characteristic 2 [13]. The extension has already been made for smooth quadrics in the book by Elman, Karpenko, and Merkurjev [3]. Here we cover all quadrics in characteristic 2, smooth or not. As in other advances on quadratic forms over the past decade, the proofs use the Chow groups of algebraic cycles on quadrics and products of quadrics.

Karpenko and Merkurjev's theorem is a major achievement of the theory of quadratic forms. It includes Karpenko's theorem that an anisotropic quadric with first Witt index equal to 1 is not ruled [11, Theorem 6.4], which we also extend here to characteristic 2. This is a strong nontriviality statement about the birational geometry of quadrics.

As an application, we extend Kollár's results on the birational geometry of conics (including non-smooth conics in characteristic 2) [14, section 4] to quadrics of any dimension (Corollary 3.3). Finally, besides proving the conjectured characterization of ruledness (Conjecture 6.1) for all quasilinear quadratic forms, we check it for non-quasilinear quadratic forms in characteristic 2 of dimension at most 6, using the results of Hoffmann and Laghribi (section 7). It is known for quadratic forms of dimension at most 9 in characteristic not 2 [20].

The inspiration for this work was Hoffmann and Laghribi's series of papers showing that the classical theory of quadratic forms (usually restricted to characteristic not 2, or at most to nonsingular quadratic forms in characteristic 2) admits a rich generalization to singular quadratic forms in characteristic 2, including the extreme case of quasilinear

forms [5, 6, 7].

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1 Notation

A quadratic form q means a homogeneous polynomial of degree 2 on a finite-dimensional vector space over a field. We call q *nonsingular* if the associated bilinear form $b(x, y) = q(x+y)-q(x)-q(y)$ is nondegenerate. A quadratic form q over a field k of characteristic 2 is called *quasilinear* (or totally singular) if the associated bilinear form is identically zero, or equivalently if q has the form $a_1x_1^2 + \cdots + a_nx_n^2$ in coordinates. Equivalently, a nonzero quadratic form q is quasilinear if only if the associated projective quadric $Q = \{q = 0\}$ becomes isomorphic to the double hyperplane $x^2 = 0$ over the algebraic closure of k. For any quadratic form q on a vector space V, the *radical* of q (called the quasilinear part $\mathrm{ql}(q)$ in characteristic 2) is the linear subspace of the elements of V orthogonal to all of V. The non-smooth locus of the associated projective quadric Q over k is the subquadric associated to the radical of q. Two quadratic forms over k are called *similar* if one is isomorphic to a scalar multiple of the other. A *subform* of a quadratic form means any linear subspace with the restricted quadratic form. This differs from Hoffmann-Laghribi's terminology, where a form α is called a subform of β if it is an orthogonal summand of β [7, Definition 2.8].

A quadratic form q over a field is called *isotropic* if there is a nonzero vector x such that $q(x) = 0$. The function field $k(q)$ means the field $k(Q)$ of rational functions on the associated projective quadric Q. The *total index* $i_t(q)$ is defined to be the maximum dimension of an isotropic subspace (a linear subspace on which q is identically zero). (For nonsingular forms, the total index coincides with the "Witt index", the maximum number of orthogonal copies of the hyperbolic plane that a form contains [7, Definition 2.4].) We define the *first Witt index* $i_1(q)$ of an anisotropic quadratic form q of dimension at least 2 over a field to be the total index of q over $k(q)$. We sometimes write $i_1(Q)$ for $i_1(q)$. We always have $i_1(q) \geq 1$. Section 7 helps to show the meaning of the first Witt index for forms of low dimension in characteristic 2.

Springer's theorem says that an anisotropic quadratic form over a field k remains anisotropic over any field extension of odd degree. The usual proof [16, Theorem VII.2.7] works in any characteristic.

We make some elementary observations which may be helpful. An anisotropic quadratic form is automatically nonsingular in characteristic not 2, but not necessarily in characteristic 2. For one thing, a nonsingular quadratic form in characteristic 2 has even dimension (because the associated bilinear form is a nondegenerate alternating form), and so a codimension-1 subform of a nonsingular anisotropic form must have a 1-dimensional quasilinear part. Moreover, there are plenty of anisotropic forms with larger quasilinear part over suitable fields: for example, for any field k_0 of characteristic 2, the quasilinear form $\langle t_1, \ldots, t_n\rangle = t_1x_1^2 + \cdots + t_nx_n^2$ is anisotropic over the function field $k_0(t_1, \ldots, t_n)$, as one easily checks. Over a perfect field of characteristic 2, an anisotropic quasilinear form has dimension at most 1. But it would be wrong to restrict oneself to perfect fields in characteristic 2, since that excludes all function fields of positive dimension.

2 Basic properties of quasilinear forms

In this section we give quick proofs of some known properties of quasilinear quadratic forms, for later use. These were proved earlier by Hoffmann and Laghribi [5, Proposition 5.3], [15, Corollaire 3.3], [7, Corollary 2.7(2)].

Lemma 2.1 *An anisotropic quasilinear form over a field k of characteristic 2 remains anisotropic over any separable extension of k.*

Proof. It suffices to show that an anisotropic quasilinear quadratic form q over a field k remains anisotropic over a finite Galois extension l/k. The point is that the set of isotropic vectors for q over l is an l-linear subspace which is invariant under the Galois group. Therefore it comes from a k-linear subspace. QED

Lemma 2.2 *Let X and Y anisotropic quadrics over a field k of characteristic 2 such that Y becomes isotropic over the function field $k(X)$. If Y is quasilinear, then X is quasilinear.*

Proof. By Lemma 2.1, since Y has no k-points, it has no rational points over the separable closure of k. If X is not quasilinear, then a dense open subset of X is smooth over k, and so the rational points of X over the separable closure of k are dense in X. Therefore there is no rational map from X to Y over k; that is, Y is anisotropic over $k(X)$. QED

Lemma 2.3 *Let q be a anisotropic quadratic form over a field k of characteristic 2. If q is not quasilinear, then $q_{k(q)}$ is the orthogonal sum of $i_1(q)$ hyperbolic planes with an anisotropic form.*

Proof. By Lemma 2.2, the quasilinear part $\mathrm{ql}(q)$ remains anisotropic over $k(q)$. By definition of the first Witt index $i_1(q)$, the maximum dimension of an isotropic subspace in $q_{k(q)}$ is $i_1(q)$. Since $\mathrm{ql}(q_{k(q)})$ is anisotropic, for each nonzero isotropic vector x in $q_{k(q)}$ there is a vector in the space which is not orthogonal to x. This allows us to split off hyperbolic planes as we want. QED

3 Many quadrics are not ruled

Karpenko showed that an anisotropic quadric with first Witt index 1 over a field of characteristic not 2 is not ruled [11, Theorem 6.4]. In this section, we extend Karpenko's theorem to characteristic 2.

For any n-dimensional variety X over a field k, we define the group of mod 2 correspondences from X to itself to be the Chow group $CH_n(X \times X)/2$ of n-dimensional algebraic cycles on $X \times X$ modulo rational equivalence. For the basic properties of Chow groups, we refer to Chapter 1 of Fulton's book [4]. Note that correspondences are often defined only for X smooth and proper over k, in which case one can compose correspondences. Nonetheless, we can use the group of correspondences even for non-smooth quadrics in characteristic 2. For any correspondence $\alpha \in CH_n(X \times X)/2$, define the degrees $\deg_1(\alpha)$ and $\deg_2(\alpha)$ in $\mathbf{Z}/2$ to be the pushforwards of α via the two projections to $CH_n(X)/2 \cong \mathbf{Z}/2$.

Here is our extension of Karpenko's theorem [11, Theorem 6.4] to any characteristic. The proof follows the original one but is modified in order to avoid the restriction to smooth quadrics. For quasilinear quadrics, the original proof does not work at all, but there is a simple direct argument. The extension to *smooth* quadrics in characteristic 2 was proved recently by Elman, Karpenko, and Merkurjev [3, Theorem 72.3]. We say that a variety X over a field k is *ruled* if it is birational to $Y \times \mathbf{P}^1$ for some variety Y over k.

Theorem 3.1 *Let X be an anisotropic quadric (not necessarily smooth) of dimension n over a field k, and suppose that the first Witt index $i_1(X)$ is equal to 1. For any correspondence $\alpha \in CH_n(X \times X)/2$, we have $deg_1(\alpha) = deg_2(\alpha)$ in $\mathbf{Z}/2$.*

Corollary 3.2 *Let X be an anisotropic quadric (not necessarily smooth) over a field k. Suppose that X has first Witt index equal to 1. Then X is not ruled over k.*

Proof (Corollary 3.2). Every rational map from X to itself determines a correspondence by taking the closure of the graph. Thus Theorem 3.1 implies that every rational map from X to itself has odd degree. If X were birational to a product $Y \times \mathbf{P}^1$ over k, then projecting to Y and then including $Y \cong Y \times \{a\}$ for some point $a \in \mathbf{P}^1(k)$ into $Y \times \mathbf{P}^1$ would give a rational endomorphism of X with degree 0. (As written, this proof assumes that k is infinite, which is reasonable since the situation is trivial for finite fields: every anisotropic quadric over a finite field is smooth of dimension at most 0.) QED

Proof (Theorem 3.1). We first prove this for k of characteristic 2 and q quasilinear. This case differs from all others in that X is nowhere smooth over k and X does not become rational over any extension field. Nonetheless, we can argue as follows. Since q has first Witt index equal to 1, the quadric X over $k(X)$ becomes the projective cone over an anisotropic quadric X'. For any variety S of the same dimension n as X with a morphism $S \to X$ of odd degree, X' remains anisotropic over $k(S)$ by Springer's theorem. Therefore X has only one rational point over $k(S)$, the base of the cone. Equivalently, there is only one rational map from S to X over k. Thus, if S is a prime correspondence (a subvariety of $X \times X$ of dimension n) with $\deg_1(S) = 1 \pmod 2$, then the second projection $S \to X$ must be the same map as the first projection. In particular, $\deg_2(S) = 1 \pmod 2$ and the theorem is proved (since the same proof shows that $\deg_2(S)$ odd implies $\deg_1(S)$ odd). Clearly it suffices to consider prime correspondences, as we have done.

We now consider the remaining case, where k can have any characteristic but q is not quasilinear. For any correspondence $\alpha \in CH_n(X \times X)/2$, we will show that $\deg_1(\alpha) = \deg_2(\alpha)$ in $\mathbf{Z}/2$.

The assumption that $i_1(q) = 1$ means that over the function field $k(q)$, the form $q_{k(q)}$ has total index 1. Since q is not quasilinear, the obvious "generic" rational point of X over $k(q)$ is not orthogonal to the whole vector space V where q is defined. Therefore $q_{k(q)}$ can be written as an orthogonal sum $H \perp q'$, where H is the hyperbolic plane. Since $q_{k(q)}$ has total index 1, the form q' is anisotropic. From now on, write X for the quadric associated to $q_{k(q)}$ and X' for the codimension-2 subquadric associated to q'.

Let $A = X - X'$. It is immediate that A is smooth over $k(q)$. The trick is to observe that the open variety A behaves like a compact variety "modulo 2", in the sense that we have a well-defined degree map

$$\deg : CH_0(A)/2 \to \mathbf{Z}/2.$$

To prove that, we use the standard exact sequence for Chow groups [4, Proposition 1.8],

$$CH_0X' \to CH_0X \to CH_0A \to 0.$$

So it suffices to show that every zero-cycle on X' has even degree. That is immediate from Springer's theorem, since X' is an anisotropic quadric. Likewise, let us now show that we get well-defined numbers $\deg_i(\alpha)$ for any correspondence $\alpha \in CH_n(A \times A)/2$ and $i = 1$ or 2. Since X is rational over $k(q)$, X' remains anisotropic over the function field of X. So every n-dimensional cycle in $X \times X'$ has even degree over the first factor X, and trivially every n-dimensional cycle in $X' \times X$ has zero degree over the first factor X. So the standard exact sequence for Chow groups shows that $\deg_1(\alpha)$ is defined in $\mathbf{Z}/2$ for any correspondence α in $CH_n(A \times A)$. The same goes for $\deg_2(\alpha)$.

Let X^{sm} denote the smooth locus of X. Define a homomorphism $\delta : CH_n(X \times X)/2 \to \mathbf{Z}/2$ as the composition

$$CH_n(X \times X)/2 \to CH_n(X^{\mathrm{sm}} \times X^{\mathrm{sm}})/2 \to CH_0(X^{\mathrm{sm}})/2 \to \mathbf{Z}/2,$$

where the second map is pullback by the diagonal map $X^{\mathrm{sm}} \to X^{\mathrm{sm}} \times X^{\mathrm{sm}}$, which is a local complete intersection map since X^{sm} is smooth over $k(q)$ [4], and the third map is our degree map. Since the degree map is well-defined on the open subset A of X^{sm}, $\delta(\alpha)$ only depends on the restriction of α to $CH_n(A \times A)/2$.

In coordinates, the quadric X over $k(q)$ is defined by an equation $x_1x_2 + q' = 0$, and A is the open subset of X where x_1 and x_2 are not both zero. Let A_1 and A_2 be the hypersurfaces in A defined by $x_1 = 0$ and $x_2 = 0$, respectively. Clearly A_1 and A_2 are disjoint, and $A - A_1$ and $A - A_2$ are isomorphic to open subsets of affine space. So homotopy invariance [4, Theorem 3.3(a)] and the standard exact sequence for Chow groups imply that

$$CH_n((A - A_1) \times A) = CH_0 A$$

and

$$CH_n(A \times (A - A_2)) = CH_0 A.$$

Since the quadric X has a rational point $p = [1, 0, \ldots, 0]$ in the smooth locus, we have $CH_0(X)/2 \cong \mathbf{Z}/2 \cdot [p]$ and hence, using our degree map, $CH_0(A)/2 \cong \mathbf{Z}/2 \cdot [p]$. We can then use the Mayer-Vietoris sequence for the open covering of $A \times A - A_1 \times A_2$ by $(A - A_1) \times A$ and $A \times (A - A_2)$ to compute that

$$CH_n(A \times A - A_1 \times A_2)/2 \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2,$$

where the isomorphism is given by mapping α to $(\deg_1(\alpha), \deg_2(\alpha))$.

We now complete the proof. Let α be any correspondence from X to itself defined over k. Then $\delta(\alpha) = 0$ in $\mathbf{Z}/2$, since this is defined by restricting α to a zero-cycle on the diagonal $X^{\mathrm{sm}} \subset X^{\mathrm{sm}} \times X^{\mathrm{sm}}$ and every zero-cycle on X^{sm} over k has even degree (as X is anisotropic over k). Then extend scalars to $k(q)$, where we have a rational point $p \in A \subset X$, and consider the restriction of α to $CH_n(A \times A)/2$. Since $\deg_1(A \times p) = 1$, the cycle

$$\widetilde{\alpha} := \alpha - \deg_1(\alpha)(A \times p) - \deg_2(\alpha)(p \times A)$$

maps to zero under the above homomorphism to $\mathbf{Z}/2 \oplus \mathbf{Z}/2$. Therefore, $\widetilde{\alpha}$ is the image of some element of $CH_n(A_1 \times A_2)$. Since A_1 and A_2 are disjoint, this element restricts to zero on the diagonal of A, and so we have $\delta(\widetilde{\alpha}) = 0$ in $\mathbf{Z}/2$. On the other hand, it is clear that

$$\delta(\widetilde{\alpha}) \equiv \delta(\alpha) - \deg_1(\alpha) - \deg_2(\alpha) \pmod 2,$$

and so $\deg_1(\alpha) \equiv \deg_2(\alpha) \pmod 2$. QED

Let us record a corollary on the birational geometry of quadrics in any characteristic, slightly stronger than non-ruledness. The statement here extends Kollár's results on conics (which specifically included non-smooth conics in characteristic 2) [14, section 4]. We say that a *uniruling* of an n-dimensional variety X over a field k is a dominant rational map $Y \times \mathbf{P}^1 \dashrightarrow X$ for some $(n-1)$-dimensional variety Y over k.

Corollary 3.3 *An anisotropic quadric X over any field k with first Witt index equal to 1 has no odd degree uniruling. If in addition X is quasilinear, then it has no separable uniruling. By contrast, every non-quasilinear quadric (in any characteristic) has a degree 2 separable uniruling.*

Proof. Let X be an anisotropic quasilinear quadric with $i_1(X) = 1$ over a field k of characteristic 2. Suppose we have a separable uniruling $Y \times \mathbf{P}^1 \dashrightarrow X$. We know that $X_{k(X)}$ is the projective cone over an anisotropic quadric X'. The function field $k(Y \times \mathbf{P}^1)$ is a separable extension of $k(X)$, and so the quasilinear quadric X' remains anisotropic over $k(Y \times \mathbf{P}^1)$ by Lemma 2.1. So $X_{k(Y \times \mathbf{P}^1)}$ is the projective cone over an anisotropic quadric and hence has exactly one rational point. That is, there is only one rational map from $Y \times \mathbf{P}^1$ to X over k, which must be the given dominant map. That is

a contradiction, since we could also project $Y \times \mathbf{P}^1$ to $Y \times \{a\}$ for some $a \in \mathbf{P}^1(k)$ and then compose with our map to X. Thus X is not separably uniruled. (In particular, X has no odd degree uniruling.)

Next, let X be an anisotropic quadric over any field k such that $i_1(X) = 1$ and X is not quasilinear. Let n be the dimension of X over k. If there is an odd degree separable uniruling $f : Y \times \mathbf{P}^1 \dashrightarrow X$, let p be a general point in $\mathbf{P}^1(k)$ and let π be the projection map from $Y \times \mathbf{P}^1$ to $Y \times \{p\}$. Consider the rational map from $Y \times \mathbf{P}^1$ to $X \times X$ given by $(f, f\pi)$ and take the closure of the image. This gives an n-dimensional cycle on $X \times X$ with odd degree over the first factor and zero degree over the second factor, contradicting Theorem 3.1. So X has no odd degree uniruling.

Finally, let $X \subset \mathbf{P}^{n+1}$ be a non-quasilinear quadric over a field k of any characteristic. We repeat the argument from Kollár [14, Remark 4.1.7]. A dense open subset of X is smooth over k. A general line in $\mathbf{P}^{n+1}$ intersects X transversely, and so the intersection points are in k or in a degree 2 separable extension E of k. Thus X is rational over E. A fortiori, X has a degree 2 separable uniruling over k. QED

4 The Chow group of zero-cycles on a quadric

In this section we extend two steps in Karpenko-Merkurjev's proof to arbitrary quadrics in characteristic 2. First, we show that the Chow group of zero-cycles injects into the integers by the degree map. In characteristic not 2, this is due to Swan [19] and Karpenko [10, Prop. 2.6]. We will extend Kahn's proof [9, Théorème 3.2], a simple variant of the proof of Springer's theorem, to characteristic 2. Later in the section, we determine the first Witt index of "generic" subforms of any quadratic form.

Lemma 4.1 *Let Q be any anisotropic quadric over a field k, or any isotropic quadric of dimension greater than 0. Then the degree map gives an injection from the Chow group of zero-cycles on Q to the integers.*

The conclusion clearly fails for a smooth isotropic quadric of dimension 0 (which consists of two points).

Proof. This is known in characteristic not 2, and so we assume that k has characteristic 2 (the proof in characteristic not 2 is in fact one step shorter). First suppose that $Q \subset \mathbf{P}^{n+1}$ has a k-rational point p and Q has dimension greater than 0. If Q is a union of two hyperplanes over k, then the conclusion is clear (since the two hyperplanes intersect, by our dimension assumption). So assume that Q is not a union of two hyperplanes. Then Q does not contain all tangent lines to Q at p (if Q is non-smooth at p, we just mean by this that Q does not contain all lines through p). Therefore, by choosing a general tangent line to Q at p, we see that a line L in $\mathbf{P}^{n+1}$ maps by the pullback map $CH_1\mathbf{P}^{n+1} \to CH_0Q$ [4, Proposition 2.6(a)] to $2p$. For any other k-rational point q in Q, the line through p and q is either contained in Q, in which case q is rationally equivalent to p, or it meets Q only in p and q, in which case again $q = L|_Q - p = p$ in CH_0Q. Any closed point of Q can be viewed as the pushforward of an F-point q of Q for some finite extension F/k by the proper map $\pi : Q_F \to Q$. By what we have said, q is rationally equivalent to an integer m times the point p on Q_F, and so its pushforward is rationally equivalent to $[F : k]m$ times the point p on Q.

If Q has no k-rational point, we will show that that the pullback map $CH_1\mathbf{P}^{n+1} = \mathbf{Z} \cdot L \to CH_0Q$ is surjective. Any closed point on Q is the pushforward of an F-point q of Q for some finite extension F/k by the proper map $\pi : Q_F \to Q$. We will show that $\pi_*(q)$ is rationally equivalent to a multiple of $L|_C$ by induction on $[F : k]$.

If F is not generated by one element over k, then F is not a separable extension of k, and so we can find an intermediate field $k \subset E \subset F$ with $[E : F] = 2$. Then the pushforward of q from Q_F to Q_E is an effective zero-cycle of degree 2, which spans a

line in $\mathbf{P}^{n+1}$ over E. If this line is not contained in Q_E, then this zero-cycle of degree 2 on Q_E is the intersection of a line with Q_E, and so its pushforward to Q over k is $[E:k]L|_C$, as we want. On the other hand, if this line is contained in Q_E, then our zero-cycle of degree 2 on Q_E is rationally equivalent to 2 times an E-point, and so we are done by our induction on $[F:k]$.

If the field F is generated by one element over k, say $F = k(\alpha)$, then q lies on a rational curve C (the image of a map $\mathbf{P}^1 \to \mathbf{P}^{n+1}$ over k) of degree d which is less than $[F:k]$. This curve is found by writing the coordinates in F as polynomials in α. This curve is not contained in Q; otherwise Q would have a k-point. So the pullback of C to Q is on the one hand rationally equivalent to $dL|_C$ and on the other is equal to $\pi_*(q)+z$, where z is an effective zero-cycle of degree $2d-[F:k] < [F:k]$. Thus, by our induction on $[F:k]$, z is rationally equivalent to a multiple of $L|_C$, and so the same goes for $\pi_*(q)$. QED

We now extend another step in Karpenko-Merkurjev's argument [13, Proposition 1.2] to arbitrary quadrics in characteristic 2: the determination of the first Witt index of "generic" subforms of a given form. These subforms are defined over a purely transcendental extension of the ground field.

Lemma 4.2 *Let q be an anisotropic quadratic form over a field k, and let j be an integer with $0 \le j \le \dim q - 2$. Then there is a purely transcendental extension F/k and a codimension-j subform s of q_F such that*

$$i_1(s) = \begin{cases} i_1(q) - j & \text{if } i_1(q) > j; \\ 1 & \text{if } i_1(q) \le j. \end{cases}$$

Proof. We assume that k has characteristic 2, the result being known for characteristic not 2. It suffices to prove the lemma for $j = 1$. Let F be the function field of the projective space of hyperplanes in the vector space of q; clearly F is a purely transcendental extension of k. We have a canonical codimension-1 subform s of q_F (the "generic" subform). We want to compute the total index (maximal dimension of an isotropic subspace) of s over the function field E/F of the corresponding projective quadric S. Clearly E is the function field over k of the variety of pairs (p, H) where $p \in Q$ and H is a hyperplane in $\mathbf{P}^{n+1}$ that contains p, where $Q \subset \mathbf{P}^{n+1}$ is the projective quadric associated to q. So (by choosing p first and then H, so to speak) we can also view E as the function field over $k(q)$ of the variety of hyperplanes in $\mathbf{P}^{n+1}$ that contain the canonical ("generic") point p_0 in $Q(k(q))$. In particular, E is a purely transcendental extension of $k(q)$. From this point of view, it becomes easy to compute the total index of s over the field E. Clearly the quadric S_E is a codimension-1 subquadric of Q_E that contains the rational point p_0, and so s_E has total index at least 1.

The form $q_{k(q)}$ has total index equal to $i_1(q)$, by definition. Since E is a purely transcendental extension of $k(q)$, q_E also has total index equal to $i_1(q)$. If $i_1(q)$ is equal to 1, then s_E has total index at most 1 just because it is a subform of q_E. Hence s_E has total index equal to 1, and we are done.

It remains to consider the case where $r := i_1(q)$ is at least 2. Clearly the total index of s_E is at least $r-1$ (as s_E is a codimension-1 subform of q_E), and we want to show that equality holds; here s_E is the "generic" codimension-1 subform of $q_{k(q)}$ that contains the isotropic vector p_0. It suffices to find one codimension-1 subform of q over $k(q)$, or a purely transcendental extension thereof, which contains p_0 and has total index $r-1$ (rather than r). That in turn follows if we can find one codimension-r subform of q over $k(q)$, or a purely transcendental extension thereof, which is anisotropic. This is easy for q quasilinear; in that case, the maximal isotropic subspace of $q_{k(q)}$ (of dimension r) is unique, and we can take the desired codimension-r subform to be any complementary linear subspace over $k(q)$ to the maximal isotropic subspace.

So we can assume q is not quasilinear. Then $q_{k(q)}$ is the orthogonal sum of r copies of the hyperbolic plane together with an anisotropic form φ by Lemma 2.3. Therefore, over the purely transcendental extension $k(q)(t_1, \ldots, t_r)$ of $k(q)$, q contains a codimension-r subform which is the orthogonal sum of φ and the quasilinear form $\langle t_1, \ldots t_r \rangle$. One can check by hand that this subform is anisotropic. That is what we needed. QED

5 Essential dimension of quadrics

We now have all the tools needed to extend Karpenko and Merkurjev's theorem on the essential dimension to arbitrary quadrics in characteristic 2. This has been done by Elman, Karpenko, and Merkurjev for *smooth* quadrics in characteristic 2 [3, Theorem 73.1].

For any anisotropic projective quadric X over a field k, define the *essential dimension* $\dim_{\mathrm{es}}(X)$ to be $\dim(X) - (i_1(X) - 1)$.

Theorem 5.1 *Let X be an anisotropic projective quadric (not necessarily smooth) over any field k, and let Y be a proper variety over k with all closed points of even degree. Suppose that Y has a closed point of odd degree over $k(X)$. Then*

(1) $\dim_{es}(X) \leq \dim(Y)$*;*

(2) if, moreover, $\dim_{es}(X) = \dim(Y)$*, then X is isotropic over $k(Y)$.*

Proof. For k of characteristic not 2, this is Theorem 3.1 in Karpenko-Merkurjev [13]. In characteristic 2, the same proof works, using the following new tools: the calculation of the Chow group of zero-cycles on a quadric (Lemma 4.1), the calculation of the first Witt index of a generic subquadric (Lemma 4.2, which extends Proposition 1.2 in [13]), and Theorem 3.1 on correspondences of a quadric with first Witt index 1, which extends Karpenko's theorem (Theorem 3.2 in [13]).

In more detail, here is the proof. The assumption means that there is a correspondence $\alpha : X \rightsquigarrow Y$ of odd degree over X. We can assume that α is a prime correspondence, that is, an irreducible subvariety Z of $X \times Y$ of dimension $\dim(X)$. By Springer's theorem, statement (2) follows if we can find a correspondence $Y \rightsquigarrow X$ of odd degree over Y. Also, it will suffice to show that X is isotropic over $k(Y)$ after a purely transcendental extension of the field k.

Assume first that $i_1(X) = 1$, so that $\dim_{\mathrm{es}}(X) = \dim(X)$. We prove (1) and (2) simultaneously by induction on $n = \dim(X) + \dim(Y)$.

For $n = 0$, X and Y are varieties of dimension zero over k, with X of degree 2 and Y of even degree over k. We are given another 0-dimensional variety Z of odd degree over X with a map to Y over k. Since Z has degree 2 times an odd number over k, it has odd degree over Y. Thus Z is a correspondence from Y to X of odd degree over Y, and the theorem is proved.

So let $n > 0$ and assume the theorem for smaller values of n. We first prove (2). Thus, we assume that $\dim(Y) = \dim(X) > 0$. It suffices to show that α has odd degree over Y. Suppose that α has even degree over Y. By adding a suitable multiple of $x \times Y$ to α, where x is a closed point of degree 2 on X, we can assume that α has degree zero over Y. Since $CH_0(X_{k(Y)})$ injects into the integers by the degree map (Lemma 4.1), there is a dense open subset $U \subset Y$ such that the restriction of α to $X \times U$ is rationally equivalent to zero. By the standard exact sequence of Chow groups, α is rationally equivalent to a correspondence $\alpha' : X \rightsquigarrow Y'$ for some lower-dimensional closed subset Y' of Y, still of odd degree over X. This contradicts statement (1) (for a lower value of $n = \dim(X) + \dim(Y)$).

We now prove (1). Thus, assume that $\dim(Y) < \dim(X)$. We will obtain a contradiction. Replacing Y by the image of the given prime correspondence α in Y, we can assume that Z maps onto Y.

By Lemma 4.2, after replacing k by a purely transcendental extension, X has a subquadric X' of the same dimension as Y such that $i_1(X') = 1$. The hypotheses on X and Y still hold over the new field k. We can pull back α to $X' \times Y$ since X' is a local complete intersection subscheme of X [4, section 6.2], yielding a correspondence $X' \rightsquigarrow Y$ of odd degree over X'. By induction, property (2) holds for X' and Y; that is, there is a rational map from Y to X' over k, and hence a rational map $\beta : Y \dashrightarrow X$. We can compose the dominant map $Z \to Y$ with the rational map β to get a rational map $Z \dashrightarrow X$. Thus Z gives a correspondence from X to itself, of odd degree over the first factor and of zero degree over the second factor (because it maps into the proper subquadric X' of X). But this contradicts Theorem 3.1.

The induction is complete: Theorem 5.1 is proved for $i_1(X) = 1$. Now let $i_1(X)$ be arbitrary. After replacing k by a purely transcendental extension, which does not change the properties assumed of X and Y, X has a subquadric X' of dimension equal to $\dim_{\mathrm{es}}(X)$ such that $i_1(X') = 1$, by Lemma 4.2. The correspondence $\alpha : X \rightsquigarrow Y$ pulls back to a correspondence from X' to Y, still of odd degree over X'. By the first part of the proof, $\dim_{\mathrm{es}}(X) = \dim(X') \leq \dim(Y)$. If $\dim(X') = \dim(Y)$, then the first part of the proof shows that X' and hence X have a rational point over $k(Y)$. QED

In particular, an anisotropic quadric X cannot be "compressed" below $\dim_{\mathrm{es}}(X)$, in the sense that there is no rational map from X to any variety of dimension less than $\dim_{\mathrm{es}}(X)$ with all closed points of even degree.

We now consider the most natural situation, where the variety Y is also a quadric.

Theorem 5.2 *Let X and Y be anisotropic quadrics (not necessarily smooth) over any field k. Suppose that Y is isotropic over $k(X)$. Then*

(1) $\dim_{es}(X) \leq \dim_{es}(Y)$*;*

(2) moreover, the equality $\dim_{es}(X) = \dim_{es}(Y)$ *holds if and only if X is isotropic over $k(Y)$.*

Proof. For k of characteristic not 2, this is Theorem 4.1 in Karpenko-Merkurjev [13]. Their proof works in characteristic 2, as follows. Let Y' be a subquadric of Y with $\dim(Y') = \dim_{\mathrm{es}}(Y)$. The definition of $i_1(Y)$ implies that Y' becomes isotropic over the function field $k(Y)$. For any anisotropic quadrics A, B, C over any field k with A isotropic over $k(B)$ and B isotropic over $k(C)$, A is isotropic over $k(C)$ [3, Proposition 22.16]. Since Y is isotropic over $k(X)$, Y' is isotropic over $k(X)$. By Theorem 5.1, $\dim_{\mathrm{es}}(X) \leq \dim(Y')$, which proves statement (1).

If $\dim_{\mathrm{es}}(X) = \dim_{\mathrm{es}}(Y)$, then Theorem 5.1 implies that X is isotropic over $k(Y')$. Since Y' is isotropic over $k(Y)$, X is isotropic over $k(Y)$. Conversely, if X is isotropic over $k(Y)$, then applying statement (1) with X and Y reversed shows that $\dim_{\mathrm{es}}(Y) \leq \dim_{\mathrm{es}}(X)$, and hence the equality holds. Thus (2) is proved. QED

Laghribi pointed out the following application (Corollary 5.3), of which the last part is new for quasilinear forms. It was proved by Izhboldin in characteristic not 2 [8, Theorem 0.2], as part of his construction of fields with u-invariant 9. Later Corollary 5.3 in characteristic not 2 became an easy consequence of the Karpenko-Merkurjev theorem. It was proved by Hoffmann-Laghribi for non-quasilinear forms in characteristic 2 [7, Theorem 1.3], but Theorem 5.2 allows an easier approach.

Corollary 5.3 *Let q and r be anisotropic quadratic forms over any field k such that* $\dim r > 2^n$ *for some integer n. If q has dimension at most 2^n, then q is not isotropic over $k(r)$. If q has dimension $2^n + 1$, then q is isotropic over $k(r)$ if and only if r is isotropic over $k(q)$.*

Proof. This is immediate from Theorem 5.2 together with Hoffmann-Laghribi's theorem that $i_1(r) \leq \dim(r) - 2^n$ [7, Lemma 4.1]. QED

6 Quasilinear forms and the quadratic Zariski problem

Karpenko's theorem, extended in this paper to characteristic 2 (Corollary 3.2), asserts that every anisotropic quadratic form with first Witt index equal to 1 is not ruled (that is, the corresponding projective quadric is not ruled). We conjecture the converse, in any characteristic (Conjecture 6.1). In this section, we prove the conjecture for quasilinear forms of any dimension (Theorem 6.4). At the same time, we solve the quadratic Zariski problem for quasilinear quadrics [18]. That is, two anisotropic quasilinear quadrics X and Y of the same dimension such that X is isotropic over $k(Y)$ and Y is isotropic over $k(X)$ must be birational (Theorem 6.5). In this respect, birational geometry is understood better for quasilinear quadrics than for any other class of quadrics, for example those of characteristic not 2. The section concludes by describing which quasilinear quadrics are regular schemes.

To see that birational geometry is nontrivial for quasilinear quadrics, note that two quasilinear quadratic forms whose associated quadrics are birational need not be similar. We can imitate Ahmad-Ohm's example in characteristic not 2 [1, Example 1.5]. Let k_0 be any field of characteristic 2, let $k = k_0(a, b, c)$, and consider the quasilinear forms $\langle 1, a, b, ab, c\rangle$ and $\langle 1, a, c, ac, b\rangle$ over k. They are not similar, as one checks by hand (write down the properties that a similarity factor in k must satisfy, and use that the quasi-Pfister form $\langle\langle a, b, c\rangle\rangle := \langle 1, a, b, c, ab, ac, bc, abc\rangle$ is anisotropic over k). But the 3-dimensional quadrics associated to these two forms are birational over k by Theorem 6.5, using that the two forms are neighbors of the same quasi-Pfister form $\langle\langle a, b, c\rangle\rangle$. (See section 7 for the definitions.)

Here is the proposed converse to Karpenko's theorem.

Conjecture 6.1 *Let q be an anisotropic quadratic form over a field k. If the first Witt index of q is greater than 1, then the associated projective quadric is ruled over k.*

We begin with a special case of the quadratic Zariski problem (Lemma 6.2). The following lemmas are not stated in the strongest possible form: for instance, the assumption that Y is isotropic over $k(X)$ can be omitted from Lemma 6.2, in view of Theorem 5.2. The lemmas merely serve the purpose of proving Theorems 6.4 and 6.5.

Lemma 6.2 *Let X and Y be anisotropic quasilinear quadrics over a field k with X isotropic over $k(Y)$ and Y isotropic over $k(X)$. Suppose that X and Y have the same dimension and that both have first Witt index equal to 1. Then X and Y are birational over k.*

Proof. Let q be a quadratic form whose associated quadric is X. Since q is quasilinear and $i_1(q) = 1$, the set of isotropic vectors for q over $k(X)$ is a 1-dimensional vector space over $k(X)$. Equivalently, there is a unique rational map from X to itself over k, which must be the identity map. The same goes for Y. We are assuming that there are rational maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ over k. Both maps are dominant by Theorem 5.1 since X and Y have first Witt index 1. Therefore we can compose them. We must have $fg = 1_Y$ and $gf = 1_X$. QED

Lemma 6.3 *Let X be an anisotropic quadric over a field k. Let Y be a subquadric of codimension $i_1(X) - 1$. Let A be any anisotropic quadric over k. Then A has the same total index over $k(X)$ as over $k(Y)$.*

Proof. By definition of the first Witt index, Y is isotropic over $k(X)$. That is, there is a rational map $X \dashrightarrow Y$ over k. The map is dominant by Theorem 5.1. Using this map, we can view $k(X)$ as an extension field of $k(Y)$, and so the inequality $i_t(A_{k(Y)}) \leq i_t(A_{k(X)})$ is clear.

For the opposite inequality, it suffices to show that $i_t(A_{k(Z)}) \geq i_t(A_{k(X)})$ for every codimension-1 subquadric Z in X. Since X is anisotropic, so is Z. Therefore Z is irreducible and reduced, and so the scheme Z is regular at its generic point. Since Z is a Cartier divisor in X, it follows that X is regular at the generic point of Z. In other words, the local ring of X at the generic point of Z is a discrete valuation ring. By the valuative criterion of properness, any rational map from X to a proper variety B over k is defined at the generic point of Z. Let B be the Grassmannian of isotropic linear subspaces of dimension $i_t(A_{k(X)})$ in a quadratic form associated to A. Then there is a rational map from X to B over k, and hence a rational map from Z to B over k. That is, $i_t(A_{k(Z)}) \geq i_t(A_{k(X)})$. QED

The following statement proves Conjecture 6.1 for quasilinear quadrics.

Theorem 6.4 *Let X be an anisotropic quasilinear quadric over a field k with first Witt index r. Let Y be a subquadric of codimension $r-1$. Then Y has first Witt index 1, and X is birational to $Y \times \mathbf{P}^{r-1}$ over k.*

Proof. Clearly X is isotropic over $k(Y)$. By definition of the first Witt index, Y is isotropic over $k(X)$. So there is a rational map $\pi : X \dashrightarrow Y$ over k, which is dominant by Theorem 5.1. We have $i_1(Y) = 1$ by Theorem 5.2.

Apply Lemma 6.3 to $A = X$. We find that X has the same total index r over $k(Y)$ as over $k(X)$. Let q be a quadratic form whose associated quadric is X. Since q is quasilinear, the set of isotropic vectors for q over $k(Y)$ is a vector space over $k(Y)$, which we know has dimension r. Let $s_1, \ldots, s_r$ be a basis; we can view $s_1, \ldots, s_r$ as rational maps from Y to the affine quadric $\{q = 0\}$. Define a rational map φ from $Y \times \mathbf{P}^{r-1}$ to X over k by $(y, [a_1, \ldots, a_r]) \mapsto [a_1 s_1(y) + \ldots + a_r s_r(y)]$.

Think of $k(Y)$ as a subfield of $k(X)$ via the map π. Since X has the same total index over $k(Y)$ as over $k(X)$, the (r-dimensional) vector space of isotropic vectors for the quadratic form q over $k(X)$ is the vector space of isotropic vectors for q over $k(Y)$ tensored over $k(Y)$ with $k(X)$. The identity map on X corresponds to some element of the first vector space. Therefore there are rational functions f_i on X such that $[f_1(s_1 \circ \pi) + \cdots + f_r(s_r \circ \pi)]$ is the identity map from X to itself. Define a rational map ψ from X to $Y \times \mathbf{P}^{r-1}$ by $\psi(x) = (\pi(x), [f_1(x), \ldots, f_r(x)])$; then $\varphi\psi$ is the identity on X. So ψ has degree 1. That is, ψ is birational. QED

The following statement solves the quadratic Zariski problem for quasilinear quadrics.

Theorem 6.5 *Let X and Y be anisotropic quasilinear quadrics over a field k such that X is isotropic over $k(Y)$ and Y is isotropic over $k(X)$. Then there is a subquadric X' of X such that X is birational to $X' \times \mathbf{P}^{i_1(X)-1}$ and Y is birational to $X' \times \mathbf{P}^{i_1(Y)-1}$ over k.*

Proof. Let $r = i_1(X)$. Let X' be any subquadric of codimension $r-1$ in X. By Theorem 6.4, $i_1(X') = 1$ and X is birational to $X' \times \mathbf{P}^{r-1}$. Likewise, let $s = i_1(Y)$ and let Y' be any subquadric of codimension $s-1$ in Y; then $i_1(Y') = 1$ and Y is birational to $Y' \times \mathbf{P}^{s-1}$. By Theorem 5.2, X' and Y' have the same dimension. By Lemma 6.2, X' and Y' are birational. QED

The proof suggests the question of which quasilinear quadrics are regular schemes. Such quadrics arise geometrically as the generic fiber of a nowhere smooth quadric fibration with smooth total space. The following lemma gives a complete answer, essentially due to Hoffmann: a quasilinear quadric is a regular scheme if and only the coefficients are generic in a strong sense, or again if and only if the splitting pattern is the generic one. In particular, if a quasilinear quadric is regular, then it is anisotropic and has first Witt index 1; but the converse is false, as shown by the form $\langle 1, a, b, ab, c \rangle$ over $k = k_0(a, b, c)$.

We define the splitting pattern of a quasilinear quadratic form q over a field k as follows. Let $k_0 = k$, $q_0 = q_{\mathrm{an}}$ (the anisotropic part of q), $k_{j+1} = k_j(q_j)$, and $q_{j+1} = ((q_j)_{k_{j+1}})_{\mathrm{an}}$, where we stop with q_h such that $\dim(q_h) \leq 1$. Then the splitting pattern of q is defined to be the sequence $(\dim(q_0), \ldots, \dim(q_h))$. We refer to section 7 for the definition of quasi-Pfister forms.

Lemma 6.6 *Let q be a nonzero quasilinear quadric over a field k. After scaling q, we can assume that it has the form $q = a_1x_1^2 + \cdots + a_{n-1}x_{n-1}^2 + x_n^2$. The following are equivalent.*

(1) The projective quadric associated to q is a regular scheme.

(2) $da_1, \ldots, da_{n-1}$ are linearly independent over k in the module of differentials Ω_k^1 $(= \Omega_{k/\mathbf{F}_2}^1)$.

(3) $da_1 \wedge \cdots \wedge da_{n-1} \neq 0$ in Ω_k^{n-1}.

(4) The quasi-Pfister form $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ is anisotropic over k; that is, $1, a_1, \ldots, a_n, a_1a_2, \ldots, a_{n-2}\,a_{n-1}, \ldots, a_1 \cdots a_{n-1}$ in k are linearly independent over k^2.

(5) The form q is anisotropic and has splitting pattern $(n, n-1, \ldots, 1)$.

Proof. Properties (2) and (3) are equivalent by the definition $\Omega_k^{n-1} = \Lambda^{n-1}\Omega_k^1$ (exterior power over k). Hoffmann showed the equivalence of (3), (4), and (5) [5, Lemma 8.1(ii), Theorem 7.25].

It remains to show the equivalence of (1) and (2). Since the quadric Q is a hypersurface in the regular scheme $\mathbf{P}_k^{n-1}$, the non-regular locus of Q is defined by the equations $q = 0$ and $dq = 0$, where dq lies in $\Omega_{k[x_1,\ldots,x_n]}^1$. Since $q = a_1x_1^2 + \ldots, a_{n-1}x_{n-1}^2 + x_n^2$, we have $dq = (da_1)x_1^2 + \ldots + (da_{n-1})x_{n-1}^2$, which lies in the subgroup $\Omega_k^1 \otimes_k k[x_1, \ldots, x_n]$. Thus the subscheme of $\mathbf{P}_k^{n-1}$ defined by $dq = 0$ is the intersection of a family of quasilinear quadrics. We read off that the subscheme defined by $q = 0$ and $dq = 0$ is empty if and only if $da_1, \ldots, da_{n-1}$ are linearly independent in Ω_k^1. QED

7 Ruled quadrics in characteristic 2

Conjecture 6.1 proposes that every anisotropic quadratic form with first Witt index greater than 1 is ruled (that is, the corresponding projective quadric is ruled). The conjecture is true for all quasilinear forms (Theorem 6.4) and for all quadratic forms of dimension at most 9 in characteristic not 2 [20]. In this section, we check Conjecture 6.1 for all quadratic forms of dimension at most 6 in characteristic 2, using the results of Hoffmann and Laghribi.

Throughout this section we will assume that the base field k has characteristic 2.

Some evidence for Conjecture 6.1 is that a "stabilized" version is true (as in characteristic not 2). We can restrict here to non-quasilinear forms. Namely, if Q is a non-quasilinear anisotropic quadric with $r := i_1(Q)$ greater than 1, let Q' be any subquadric of codimension $r-1$. The definition of $i_1(Q)$ implies that Q' is isotropic over $k(Q)$; that is, there is a rational map π from Q to Q' over k. The quadric Q' is not quasilinear, for example by Lemma 2.2. The map π is dominant by Theorem 5.1. So Q' has a smooth point over $k(Q)$ and hence is rational over $k(Q)$. Likewise, there is also a rational map from Q' to Q (the inclusion) whose image meets the smooth locus of Q, and so Q is rational over $k(Q')$. Thus the product $Q \times Q'$ is birational both to $Q \times \mathbf{P}^a$ and to $Q' \times \mathbf{P}^{r-1+a}$ for some a. So it seems plausible (by analogy with the quadratic Zariski problem [18]) that Q should be birational to $Q' \times \mathbf{P}^{r-1}$ (where $r - 1 \geq 1$) and hence that Q should be ruled.

In order to prove Conjecture 6.1 in low dimensions, we will use Lemma 7.1 below, a sufficient condition for ruledness. The lemma is a characteristic 2 version of a result by Ahmad and Ohm [1, after 1.3]. In order to formulate the lemma, we need to define the most important classes of quadratic forms in characteristic 2.

An n-fold Pfister form over a field k of characteristic 2 is the nonsingular quadratic form defined as the tensor product

$$\langle\langle a_1, \ldots, a_{n-1}, c]] = \langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_{n-1}\rangle_b \otimes [1, c]$$

for some a_i in k^* and c in k [7, section 2.5]. Here $\langle 1, a\rangle_b$ denotes the bilinear form $x_1y_1 + ax_2y_2$ and $[1, c]$ is the quadratic form $x_1^2 + x_1x_2 + cx_2^2$. Explicitly, for any quadratic form q, the tensor product $\langle a_1, \ldots, a_n\rangle_b \otimes q$ is the quadratic form $a_1q \perp \cdots \perp a_nq$. An n-fold quasi-Pfister form is the quasilinear quadratic form defined as the tensor product

$$\langle\langle a_1, \ldots, a_n\rangle\rangle = \langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_n\rangle_b \otimes \langle 1\rangle$$

for some a_i in k [5, Definition 4.4]. A *Pfister neighbor* means a quadratic form q similar to a subform of a Pfister form φ with $\dim(q) > \dim(\varphi)/2$. *Quasi-Pfister neighbors* are defined analogously.

Lemma 7.1 *Let P be a Pfister form or quasi-Pfister form over a field k, P_1 a nonzero subform of P, and $b_1, \ldots, b_s$ any elements of k^*. Then the projective quadric associated to $b_1P \perp \cdots \perp b_{s-1}P \perp b_sP_1$ is birational to the product of a projective space with the quadric $b_1P \perp \cdots \perp b_{s-1}P \perp \langle b_s\rangle$; in particular, it is ruled if P_1 has dimension at least 2.*

Proof. A Pfister form or quasi-Pfister form P is strongly multiplicative: there is a "multiplication" xy on the vector space of P which is a rational function of x and linear in y such that $P(xy) = P(x)P(y)$ [16, Theorem X.2.11]. (For quasi-Pfister forms, the "multiplication" can even be taken to be bilinear, as Albert observed long ago [2, section 8].) Given a general point on the first quadric $b_1P(x_1) + \cdots + b_{s-1}P(x_{s-1}) + b_sP(x_s) = 0$ (where $x_s \in P_1$), we map it to a point on the second quadric by noting that $b_1P(x_sx_1) + \cdots + b_{s-1}P(x_sx_{s-1}) + b_sP(x_s)^2 = 0$. The general fibers of this rational map are linear spaces. QED

In particular, any quadratic form divisible by a binary quadratic form is ruled, and any *special Pfister neighbor* or *special quasi-Pfister neighbor* (that is, a form $b_1P \perp b_2P_1$ where P is a Pfister form or quasi-Pfister form and P_1 is a nonzero subform of P) is ruled if its dimension is not of the form $2^a + 1$. More generally, Theorem 6.4 implies that every quasi-Pfister neighbor is ruled if its dimension is not of the form $2^a + 1$, but Lemma 7.1 could be considered more explicit when it applies.

We now show that an anisotropic form q in characteristic 2 with $i_1(q) > 1$ must be ruled when q has dimension at most 6. Hoffmann-Laghribi showed that an anisotropic quadratic form q of dimension $2^a + m$ over any field, where $1 \le m \le 2^a$, has $i_1(q) \le m$ [7, Lemma 4.1]. In particular, a 3-dimensional anisotropic form q has $i_1(q) = 1$ and so there is nothing to prove.

So let q be a 4-dimensional anisotropic form with $i_1(q) > 1$, hence $i_1(q) = 2$ by Hoffmann-Laghribi. We can apply the following result of Hoffmann and Laghribi [7, Theorem 1.2(1)], [5, Corollary 7.29, Remark 7.30].

Theorem 7.2 *Let q be an anisotropic form over a field k of characteristic 2 such that $2^{n+1} - 2 \le \dim(q) \le 2^{n+1}$. If $i_1(q) = \dim(q) - 2^n$ (the largest possible), then q is either a Pfister neighbor or a quasi-Pfister neighbor.*

In particular, a 4-dimensional anisotropic form q with $i_1(q) > 1$ is similar to a Pfister or quasi-Pfister form. In both cases, q is ruled by Lemma 7.1.

For any 5-dimensional anisotropic form q, we have $i_1(q) = 1$ and so there is nothing to prove. So let q be a 6-dimensional form with $i_1(q)$ greater than 1. Then $i_1(q) = 2$, and q is a Pfister neighbor or quasi-Pfister neighbor by Theorem 7.2. If q is quasilinear, then it is ruled by Theorem 6.4, and so we can assume that q is a Pfister neighbor of

dimension 6. The radical of q has dimension 0 or 2. Laghribi described the forms of both types [15, Proposition 3.2 and its proof]. Namely, if q is nonsingular, then q can be written as $\langle u, v, w\rangle_b \otimes [1, \Delta]$ for some elements u, v, w in k^* and Δ in k, while if the radical of q has dimension 2, then q can be written as $(r\langle u, v\rangle_b \otimes [1, s]) \perp \langle u, v\rangle$. In both cases, we read off that q is a special Pfister neighbor, and so q is ruled.

For q of dimension 7, Conjecture 6.1 remains open. If $i_1(q) = 3$, then q is either a Pfister neighbor or a quasi-Pfister neighbor by Theorem 7.2. If q is quasilinear, then it is ruled by Theorem 6.4, and so we can assume that q is a Pfister neighbor. Then q is special by Witt cancellation [3, Theorem 8.3] (which shows that all codimension-1 subforms of a given Pfister form are similar). Therefore q is ruled if $i_1(q) = 3$. We can expect that there are no 7-dimensional anisotropic forms with $i_1(q) = 2$, as in characteristic not 2, but that seems to be unknown.

More generally, Karpenko found all possible values of the first Witt index for anisotropic forms of a given dimension in characteristic not 2 [12], [3, Theorem 75.13]. We can hope that exactly the same results hold for arbitrary anisotropic quadratic forms in characteristic 2.

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